Mind and Interference Effects in Computation

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ABSTRACT. The quantum mechanical description of nature as a duality between an univisualizable dynamics and a visualizable concretization, the latter corresponding to reduction of the state vector when a measurement is made is shown to have a correspondence in computation. Namely, a conceptual structure called the field (of real numbers, say) and its operations, and the concretization of that structure by means of a measuring apparatus called a (digital) computer. The probabilistic, computable state reduction operator of quantum mechanics is replaced by a deterministic noncomputable operation, an extension of rounding. States, wavefunction, dynamics, observation, uncertainty and nonlocality are shown to have their counterparts in the new model, an example of a new physics as forecast by Penrose (upon which to base a study of mind). The algorithmic counterpart of the double slit experiment to validate the existence of interference in the new framework is defined and performed. A notion of spin in computation is introduced.

1. INTRODUCTION

Roger Penrose projects the need for a new understanding in physics to explain how the physical brain gives rise to the mind [15,16,17]. He points out that mind (consciousness) and understanding have (at least) a logical connection. Appealing to mathematical understanding, which, according to the Gödel-Turing theorem, is not computable, he infers that noncomputability in some aspects of consciousness strongly suggests that it should be a feature of all consciousness. Continuing, he speculates that the new physics, should also be noncomputable. Penrose suggests that the new physical ideas might well arise as a deterministic replacement (called OR for 'objective reduction') of the probabilistic measurement process (called R for 'reduction of the state-vector' or 'collapse of the wave function') which accompanies conventional quantum mechanics when effects in the latter are 'magnified up to the classical level' for the purpose of observation. Indeed the probabilistic quality of R comes from the interaction of the macroscopic environment with the quantum level behavior which accompanies observation.

R is where nonlocality enters into our present physics, and Penrose expects that aspect of R to persist when it is replaced by OR. He goes on to argue that OR is likely

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to be a gravitational phenomenon, and that the isolation from the environment needed for the subtle quantum effects is provided by the microtubules which are found in almost all eukariotic cells along with the tubular dimers which comprise the microtubule walls. (See also Hameroff [10].)

Great controversy attends these ideas. It is hardly universally accepted that the Gödel incompleteness theorem implies the inability to simulate human intelligence in a computer. See Putnam [18]. Many dispute the possibility that quantum effects are either of the correct scale or have the appropriate isolated environment in the astronomically complex warm and wet arena of a human brain.

We claim that these issues may be left aside, for we shall show that interference and nonlocal quantum effects are characteristics of the computation process itself. A device that computes may exhibit them, the subtle hardware needs of quantum physics notwithstanding. (This claim will have an impact on strong AI as well as other of the philosophical issues of the mind/brain problem.)

Quantum mechanics attributes a dual aspect to nature. This consists of an unvisualizable dynamics when the system is not observed and a visualizable concretization of the system (corresponding to the collapse of the wave function) when it is measured. The interplay between these aspects causes the anomalies which prompted the invention of quantum mechanics in the first part of this century [4], and continues to motivate studies of the meaning of the subject [1,6].

There is a dualism in computation which echoes that of nature (mechanics). Namely, a conceptual structure called the field (of the real or complex numbers, say) and its operations, and the concretization of that structure by means of a measuring apparatus called a (digital) computer. Anomalies between the conceptual and the concrete aspects of computation have long been observed. Among them are the errors in digital computation caused by the finite precision of the digital computer itself. An example of one of the difficulties which arise from this is the loss of the associative law of addition. Since these errors are usually given the characterization, "cancellations and rounding errors", they have thus far failed to stimulate the development of a quantum theory of numbers and computation. There are interference effects in computation; they are in the information and its processing.

The reduction operation between constituents of this computational duality is an appropriate extension of rounding (denoted operationally by $\Box$) and (unlike R of traditional physics but) like OR of the projected new physics, $\Box$ is noncomputable and deterministic. (In a manner of speaking, 'computation is noncomputable'.) We shall also see nonlocality (interference) in this picture.

Quantum effects and computation are already studied in quite different contexts than what is developed here. One branch of this work deals with the exploitation of quantum effects as a way to encode and process binary information [8,9,13]. Another branch deals with using the states and the dynamics of quantum theory as the states and the process of computation [2,3,19,20].

In Section 2, we introduce our notions and discuss noncomputability. States, wavefunctions, dynamics, observation, uncertainty and nonlocality are also discussed. In

Section 3, we define the algorithmic equivalent of the double slit experiment and give the results of performing it. In Section 4, we show how spin may be accommodated in this framework. An appendix contains some comments and some details.

2. Notions

In this section, we introduce the notions and constructs of a quantum theory of numbers and computation. These include screens, roundings, noncomputability, states, wavefunctions, dynamics, observation, uncertainty and nonlocality.

Floating-point screen and operations: We introduce a floating-point screen $\mathbf{S}$ of numbers x with base b [12].

$$x = \mu b^e (= \mu_x b^{e_x}),$$

where the mantissa μ has length ℓ and the exponent $e \in [e_{\min}, e_{\max}]$. Let $m_p(x)$ be the midpoint of a floating-point number x and its predecessor floating-point number (that is, the largest floating-point number less than x), and let $m_s(x)$ be the midpoint of x and its successor. For the convenience, we shall consider the rounding to nearest, so that for $x \in \mathbf{S}$,

$$x = \Box z, \ \forall z \in \mathbb{R} | m_p(x) < z \leq m_s(x).$$

The rounding operator $\Box$ is defined componentwise on vectors.

Arithmetic operations in $\mathbf{S}$, denoted $\boxdot$, $\circ \in \{+, -, \times, \div\}$, are defined in terms of their counterparts in $\mathbb{R}$ by semimorphism [12], viz

$$a \boxdot b = \Box(a \circ b)$$

for all $a, b \in \mathbf{S}$.

Noncomputability of $\Box$: A real number is specified by the class of sequences of rationals which converge to it. If such a sequence $\{a_j\}_{j=0}^{\infty}$, converging to a, say, is given by one closed form means or another, each screen number $\Box a_j$ could be computed, but it is not clear that $\Box a$ could be determined by computation. Certainly if the sequence $\{a_j\}$ is delivered one term at a time from a block box, we could never determine by computation or another means what $\Box a$ is.

Consider the special case where the screen $\mathbf{S} = \mathbf{S}_I$ consists of the integers, and consider the rational sequences $\{a_j\}$ converging to $1/2$. The corresponding sequences $\{\Box a_j\}$ are in $1-1$ correspondence with all the real numbers (encoded in binary) in $(0, 1]$. Thus except for the enumerable subset of the sequences $\{\Box a_j\}$ which have a finite number at zeros or a finite number of ones, the limit of $\{\Box a_j\}$ is not determinable. For example, the sequence (in binary) $.11, .101, .1001, \ldots$ converges to $.1$ (to $1/2$ in decimal) whose round in $\mathbf{S}_I$ is zero, but $\Box a_j = 1$ for every j.

Let us consider an example depending on the twin primes. Let

$$a_j^{-1} = j\text{th prime } - (j-1)\text{st prime}.$$

Since the asymptotic distribution of the primes is $j \log j$, this sequence has an accumulation point at zero. If there are infinitely many twin primes, this sequence has an accumulation point at $\frac{1}{2}$ as well. In fact the existence of a limit of this sequence, hence the determination of $\Box a$ is at present unknown by computational or other means.
Wave function: The wave function or state corresponding to the vector $x = (x_1, \ldots, x_n)$, $x \in \mathbf{S}^n$, is

$$\psi(z) = \sum_{k=1}^{n} B_k e^{i\frac{z}{\hbar}}.$$

B_k is a step function of z with support on the interval $(m_p(x_k), m_s(x_k)]$. It has the value $B_k = (m_s(x_k) - m_p(x_k))^{-1/2} = .5b^{(1-\ell-e_k)/2}$ on that interval.
Observation: An observation corresponds to a Hermitian operator S and has the value $(\psi, S\psi) \equiv \int_{-\infty}^{\infty} \psi(S\psi)^* dz$, where the asterisk denotes the complex conjugate. [14].

The state corresponding to $x \in \mathbf{S}$ is a single term in the sum here. To observe the 'location' of this state we define the operator L (corresponding to location) as multiplication by z. We claim that the location of the state $B_1 e^{\frac{iz}{\hbar}}$ is $x_1 = \Box z$, $z \in (m_p(x), m_s(x)]$. Indeed a calculation shows that

$$(\psi, L\psi) = \int_{-\infty}^{\infty} \psi z \psi^* dz = \int_{-\infty}^{\infty} B_1^2 z dz = \frac{m_p(x_1) + m_s(x_1)}{2}.$$

Collapse of the wave function: Thus we may define the reduction operation as rounding by formally extending rounding from intervals of reals to wave functions, viz

$$x = \Box z \equiv \Box\psi(z) = (\psi(z), z\psi(z)).$$

Note that the $\Box$ here (since it operates on states) is different than the original $\Box$ which operates on intervals. (For convenience, we use the same symbol for both operations.) The original $\Box$ operates on step functions, the characteristic functions of intervals while the new $\Box$ operates on the wave function, in the present case a carrier wave superposed on the step function.
Summation: The summation operator is given by

$$S\psi(z) = \hbar \int_{-\infty}^{z} \zeta\psi(\zeta) d\zeta.$$

Using the $\psi(z)$, a calculation (see the Appendix, A1) gives

$$s = (\psi, S\psi) = \frac{1}{2}\sum_{k=1}^{n} \sigma_k + O(\hbar),$$

where

$$\sigma_k = m_s(x_k) + m_p(x_k).$$

That is, this observation S yields the sum of the components of x with an error which is $O(\hbar)$. (See the Appendix, A2 for a comment on the value of $\hbar$.) We see that upon measurement, the wave function will have collapsed into $\psi(z) = Be^{\frac{i}{\hbar}z}$, where B is a step function which is proportional to s on the interval $z \in (m_p(\Box s), m_s(\Box s)]$ and is zero otherwise. This law of measurement is seemingly different than the customary R (projection and renormalization) in quantum mechanics itself. In fact, it is deterministic and "environment" independent, and as we have seen, noncomputable. Thus it satisfies the properties of the object reduction (OR) process forecast by Penrose.

Uncertainty principle: The uncertainty principle follows from the Heisenberg inequality [5], a general mathematical relationship involving functions $f(x)$ and their Fourier transforms $\hat{f}(w)$, equivalently, involving operators and their complements. We shall give two examples of the uncertainty principle. The first deals with the summation operator S and the second with the location operator L.

Summation. Using the identity $(2\pi i\omega)^{-1}\frac{d}{dw}\hat{f}(w) = -2\pi i(\int_{-\infty}^{x}\xi f(\xi)d\xi)\hat{}$, we find the complementary operator D to the summation operator S, viz

$$D = \frac{i\hbar}{4\pi^2 x}\frac{d}{dx}.$$

For the commutator, we have $[S, D] = \hbar^2/2\pi^2$. Then the Heisenberg inequality applied to D and S states that

$$\int_{-\infty}^{\infty}[x - i\hbar\int_{-\infty}^{\infty}\psi^*(z)\int_{-\infty}^{z}\xi\psi(\xi)d\xi dz]^2|\psi(x)|^2 dx$$
$$\times\int_{-\infty}^{\infty}[\omega - \frac{i\hbar}{4\pi^2}\int_{-\infty}^{\infty}\frac{d\psi(z)}{dz}\frac{\psi^*(z)}{z}dz]^2|\hat{\psi}(\omega)|^2 d\omega \geq (16\pi^2)^{-1}.$$

To compute $(\psi, D\psi)$ it suffices to consider the single term $\psi(z) = Be^{i\frac{z}{\hbar}}$. Then a computation gives

$$(\psi, D\psi) = \frac{1}{4\pi^2}\frac{\log m_s - \log m_p}{m_s - m_p}.$$

An additional computation gives (see the Appendix, A3)

$$(\psi, D\psi) = \frac{1}{4\pi^2 x}(1 + O(b^{-\ell})).$$

Thus the uncertainty in observing S and D (simultaneously) corresponds to an uncertainty computing in the mean and the harmonic mean of the components of a vector. In the case of scalars, the uncertainty is in the simultaneous determination of the scalar itself and its reciprocal. Note the correspondence of this to electrical resistance in series and in parallel.

Location. Using the identity $i\frac{d}{dw}\hat{f}(w) = (xf(x))\hat{}$, we find the complimentary operator V ("velocity") to L, viz

$$V = i\frac{d}{dx}.$$

For the commutator we have $[L, V] = -i$. To compute $(\psi, V\psi)$, we use $\psi = Be^{i\frac{z}{\hbar}}$, once again to find

$$(\psi, V\psi) = -\hbar^{-1}.$$

We interpret this by saying that the average velocity of a scalar ("a free particle") corresponds to a drift to the left.

Dynamics: Time is clocked in cycles, $t = 0, 1, \ldots$, and an initial state

$$\varphi(0) = a_0 B_0 e^{i\frac{z}{\hbar}}$$

evolves through multiplication,

$$\varphi(t) = a_t B_t \varphi(t-1), \quad t = 1, 2, \ldots .$$

The B_t are step functions which specify the evolution from $t-1$ to t, and the a_t are normalization factors. In particular,

$$a_t = \frac{\|B_{t-1}B_{t-2}\cdots B_0\|}{\|B_t B_{t-1}\cdots B_0\|}, \quad t = 0, 1, \ldots ,$$

where the empty product is unity, and

$$\|C\|^2 = \int |Ce^{i\frac{z}{\hbar}}|^2 dz.$$

Nonlocality: With a notion of dynamics in hand, we may display nonlocality by duplicating the canonical interferometer experiment. In the present context the passage through the interferometer consists of executing an algorithm on the state $a_0 B_0 e^{i\frac{z}{\hbar}}$ which 'enters' it. This algorithm gives the output state $B_{out}e^{iz/h} = \prod_{j=0}^{3} a_j B_j e^{iz/h}$, where (neglecting normalizations for convenience)

$$\begin{aligned} B_1 &= (1+i)B_0, \\ B_2 &= i\ B_0, \\ B_3 &= (1+i)B_0. \end{aligned}$$

Then

$$B_{out} = -2B_0.$$

To see that this consists of the entry state following two separate interfering algorithms, we simply review the canonical interferometer experiment comparing it with these algorithmic steps. For convenience, we shall denote the entry state $B_0 e^{iz/\hbar} = (m_s(x_0) - m_p(x_0))^{-1/2} e^{iz/\hbar}$ which represents the number x_0, by the Dirac symbol $|x_0>_t$. The subscript t refers to the time clocked in cycles.

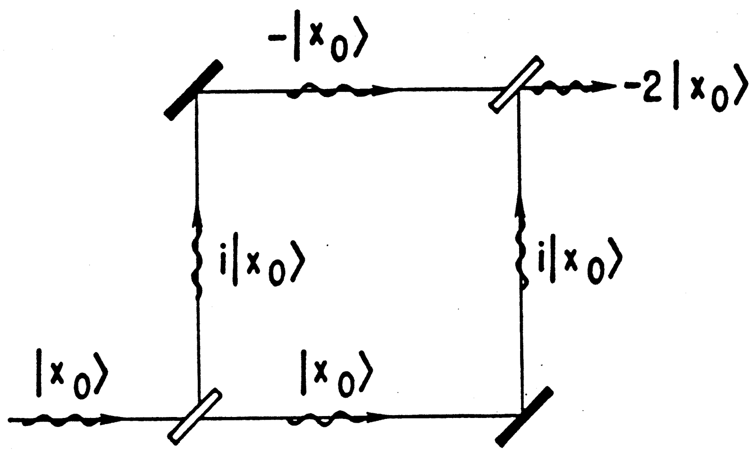


Diagram illustrating nonlocality

Referring to the diagram (in which depicts a "half silvered mirror" and a "fully solved mirror"), the state $|x_0>$ enters the interferometer at $t = 0$. We define the first dynamic step (an algorithm) by

$$|x_0>_0 \rightarrow |x_1>_1 = |x_0> + i|x_0>,$$

indicating that $|x_0>$ has simultaneously become the sum of unity times itself and i times itself, the latter denoting the path upward ("reflection from the half silvered mirror"), the former denoting the path rightward (passage through that mirror). Continuing, we next have "reflections of the two fully silvered mirrors" giving

$$|x_1>_1 \rightarrow |x_2>_2 = i(|x_0> + i|x_0>) = i|x_0> - |x_0> .$$

Finally, the encounter with the "half silvered mirror on the upper right" gives

$$\begin{aligned} |x_2>_2 \rightarrow |x_3>_3 &= i(|x_0> + i|x_0>) - x_0> - i|x_0> \\ &= -2|x_0> . \end{aligned}$$

This indicates that the state emerges from the interferometer (upper right in the diagram) as follows. The two (nonlocal) aspects of the state during this dynamical process have interfered with each other. With respect to components of the state, those with imaginary amplitude (the exit upward) have interfered perfectly destructively, giving a null value. The exiting real components (the rightward exit) have interfered perfectly constructively.

3. Algorithmic Double Slit Experiment

In this section, we validate the wave approach to computation, by demonstrating through an experiment, the existence of interference in computation. We do this by defining and then performing the algorithmic analog of the double slit experiment.

The use of $(\psi, S\psi)$ to represent an observation implies the use of a probability amplitude (as opposed to a probability density) in this theory. One way to justify this choice is to perform an algorithmic version of the double slit experiment [7].

In the computer version of this experiment, the passage of a data state (information) through an algorithm takes the roll of the passage of a physical state along a trajectory. For convenience, we specialize to triples of real numbers,

$$p = p(a, b, c).$$

Let

$$m(p) = a + b + c,$$

and let

$$\overline{u} \equiv \Box u, \ \forall u \in \mathbb{R}.$$

Experiment A (the case of "interference")

Choose triples p at random and compute $m(p)$ in $\mathbf{S}$. That is, compute

$$\overline{m}(p) = \overline{a} \boxplus \overline{b} \boxplus \overline{c}.$$

There are three methods of summation (three algorithms):

$$\begin{aligned} i) &\quad (\overline{a} \boxplus \overline{b}) \boxplus \overline{c} \\ ii) &\quad \overline{a} \boxplus (\overline{b} \boxplus \overline{c}) \\ iii) &\quad (\overline{a} \boxplus \overline{c}) \boxplus \overline{b}. \end{aligned}$$

For each triple $p(a, b, c)$, choose a summation method i), ii) or iii) at random. Next compute the quantity

$$e(p) = 1 - \frac{\overline{m}(p)}{m(p)}$$

and make a scatter plot of $e(p)$. The latter is the computer analog of the density of photons hitting the screen in the conventional double slit experiment. We expect $e(p)$ to exhibit an interference pattern (possibly periodic).

Experiment B (the case of "non-interference")

Here we examine each randomly chosen triple prior to the computation of $\overline{m}(p)$ (so that as in the conventional double slit experiment, the interference should disappear). Here is how we proceed. Let

$$|u| = \max\{|a|, |b|, |c|\},$$

and let

$$\{v, w\} = \{a, b, c\} - \{u\}.$$

Now compute, using ii)

$$m_{ii}(p) = \overline{u} \boxplus (\overline{v} \boxplus \overline{w}),$$

and

$$e_{ii}(p) = 1 - \frac{m_{ii}(p)}{m(p)}.$$

Make a scatter plot of $e_{ii}(p)$. We expect $e_{ii}(p)$ to be a unimodal curve exhibiting no (little) interference.

Experimental Results: The actual experiment was performed as follows.

(i) k numbers are chosen at random in $[0, 1] : n'_1, \ldots, n'_k$

(ii) They are scaled to be in [-range, range]: $n_1, \ldots, n_k$

(iii) A sum S_1 (representing $\bar{m}(p)$) is computed in double precision, viz

$$S_1 = \boxed{\sum_{i=1}^{k}} n_i$$

(iv) The exact sum (representing $m(p)$) is computed in two ways. These are called $S_2(1)$ and $S_2(2)$, resp.

1. For $S_2(1)$ we simulate full precision arithmetic and compute the sum exactly. This sum is then rounded to double precision, viz

$$S_2(1) = \Box \sum_{i=1}^{k} n_i.$$

2. The $n_1, \ldots, n_k$ are sorted to give a sequence

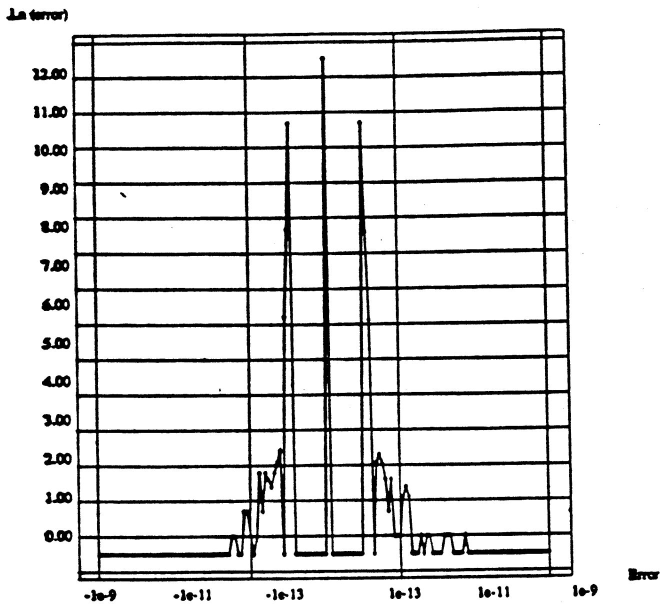
$$m_1 \leq m_2 \leq \cdots \leq m_{k_1} \leq 0 < p_1 \leq p_2 \leq \cdots p_{k_2}, \quad k_1 + k_2 = k.$$

The sum $S_2(2)$ is computed in double precision, viz

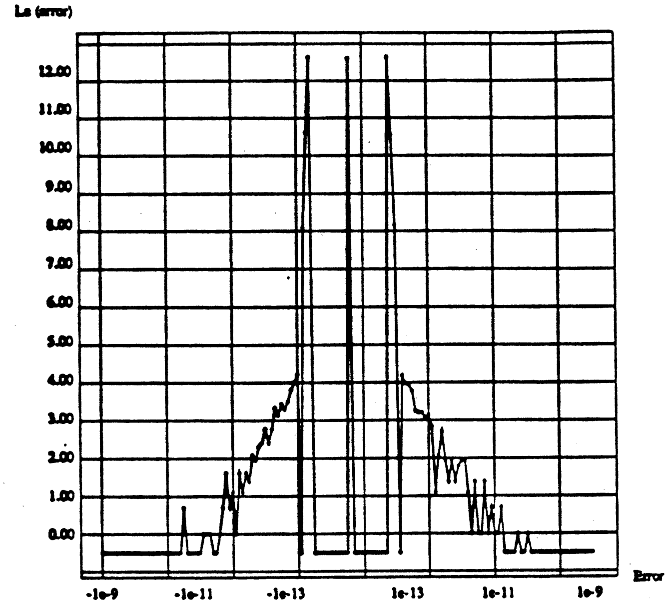
$$\begin{aligned} t_1 &= (\cdots((p_1 \boxplus p_2) \boxplus p_3) \cdots \boxplus p_{k_2}), \\ t_2 &= (\cdots((m_{k_1} \boxplus m_{k_1 - 1}) \boxplus m_{k_1 - 2}) \cdots \boxplus m_1), \\ S_2(2) &= t_1 \boxplus t_2. \end{aligned}$$

(v) The error is computed in two ways. 1. $e(k) = 1 \boxminus S_1 \boxdiv S_2(i)$, $i = 1, 2$. That is, in double precision. 2. $e(k) = 1 - S_1 \div S_2(i)$, $i = 1, 2$. That is, with (simulated) full precision.

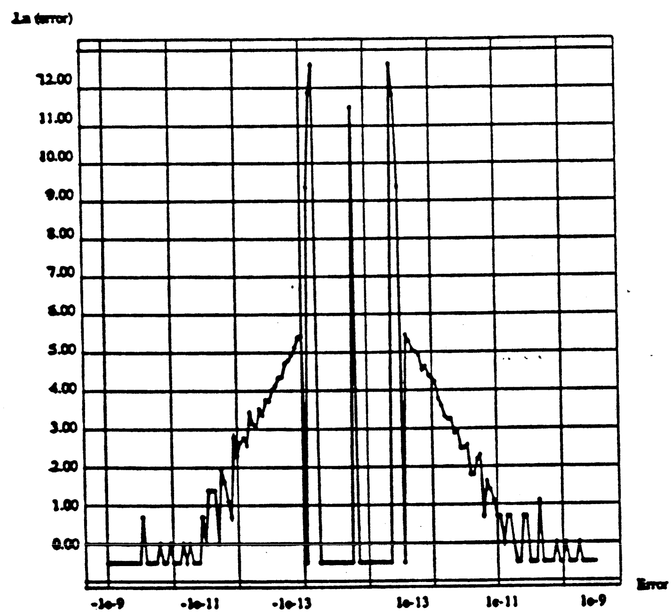
In the figures we make a scatter plot of $\log e(k)$ versus $e(k)$ for the cases $k = 3, 10, 50, 100$. In each case, one million sample sums were chosen at random, and range $= 10^{20}$. To the accuracy of the graphical representation, there is no difference in the results of all the experimental variations.



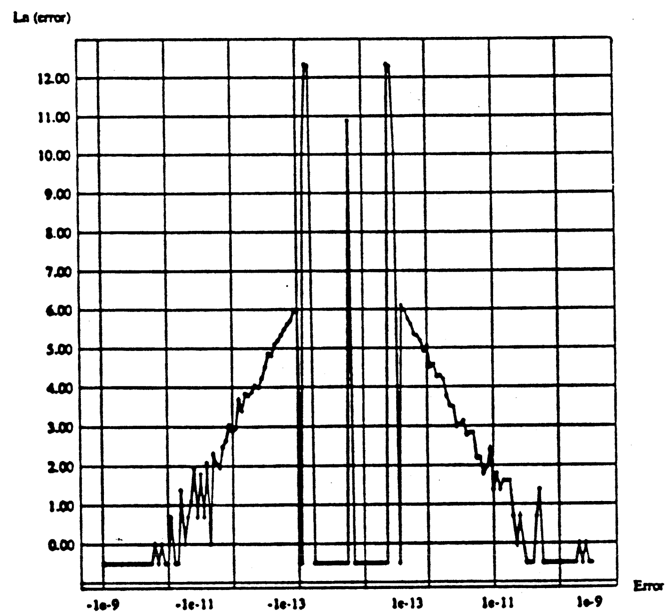
k=3



k=10



k=50



k=100

Results of the Algorithmic Double Slit Experiment

Examples of states and observables: The triple $p(a, b, c)$ is the computer analog of a particle in mechanics (more properly, a state in quantum mechanics), and $m(p)$ is the analog of an observable. It might be convenient to think of it as the mass or location (when measured) of the state. As an indication of the possibilities of computer states and observables in wave computing we offer two illustrations.

Polynomials: The state is an $(n + 2)$-tuple

$$p = p(x, a_0, \ldots, a_n),$$

and the observable is

$$m(p) = \sum_{j=1}^{n} a_j x^{n-j}.$$

An observation is $\overline{m}(p)$, computer evaluation of $m(p)$.

Fourier series: The state is an infinite sequence,

$$p = p(a_0, a_{\pm 1}, \ldots),$$

and the observable is

$$m(p) = \sum_{-\infty}^{\infty} a_j e^{ij\theta}.$$

An observation is $\overline{m}(p)$, a computer evaluation of a truncated version of this series [11].

4. Spin

We introduce a notion of spin in computation by means of measurements (roundings) of complex numbers (complex intervals). The rounding of the real and imaginary parts are treated differently, indeed both the rounding and the screen are duplex: $\square = (\square_r, \square_i)$ and $\mathbf{S} = (\mathbf{S}_r, \mathbf{S}_i)$.

With $z = x + iy$ (intervals) we have

$$\square z \equiv \square_r x + i\sigma(y)\square_i y.$$

Here $\square_r x \in \mathbf{S}_r$ and $\square_i y \in \mathbf{S}_i$, and

$$\sigma(y) = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & sgn\square_i y \geq 0, \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & sgn\square_i y \leq 0. \end{cases}$$

The conventional case of spin in quantum mechanics is obtained by taking

$$\mathbf{S}_i = \{1, -1\},$$

or

$$\mathbf{S}_i = \{0, \pm 1, \pm 2, \ldots\},$$

etc. However, we see that many other possibilities exist.

An arithmetic operation $\boxdot$, $\circ \in \{+, -, \times, \div\}$ on two floating point complex numbers (two rounded (i.e. measured) complex intervals) is defined by semimorphism [12], viz

$$\begin{aligned}\Box z_1 \boxdot \Box z_2 &\equiv \Box[\Box_r x_1 \circ \Box_r x_2 + i(\sigma(y_1)\Box_i y_1 \circ \sigma(y_2)\Box_i y_2)] \\ &= \Box_r(\Box_r x_1 \circ \Box_r x_2) + i\Box_i(\sigma(y_1)\Box_i y_1 \circ \sigma(y_2)\Box_i y_2),\end{aligned}$$

where (recall that) the round of a vector is taken componentwise.

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Appendix

A1: Evaluation of $(\psi, S\psi)$ for $S\psi(z) = \frac{i}{\hbar}\int_{-\infty}^{\infty}\zeta\psi(\zeta)d\zeta$ and $\psi(z) = \sum_{k=1}^{n} B_k e^{i\frac{z}{\hbar}}$ with $B_k(z) = 0$ except on the interval $(m_s(x_n), m_p(x_k)]$ where $B_k(z) = (m_s(x_k) - m_p(x_k))^{-1/2}$ proceeds as follows.

Without loss of generality, we may suppose that the $x_k, k = 1, \ldots, n$ are different floating point numbers. Now integrating directly gives

$$\begin{aligned} S\psi(z) &= \sum_{k=1}^{n} B_k[(i\hbar + \min(z, m_s(x_k))e^{\frac{i}{\hbar}\min(z,m_s(x_k))} \\ &\qquad - (i\hbar + \min(z, m_p(x_n))e^{\frac{i}{\hbar}\min(z,m_p(x_k))}] \\ &\equiv \sum_{k=1}^{n} B_k C_k(z). \end{aligned}$$

Using this, we have

$$\int_{-\infty}^{\infty} \bar{\psi}(z)S\psi(z)dz = \sum_{\ell=1}^{n}\sum_{k=1}^{n}\int_{m_p(x_\ell)}^{m_s(x_\ell)} B_\ell B_k e^{-i\frac{z}{\hbar}} C_k(z)dz.$$

On the interval of integration, all $B_k(z)$ vanish except the one for which $k = \ell$. Moreover, on this interval the minima may be resolved, viz

$$\min(z, m_s(x_\ell)) = z \quad \text{and} \quad \min(z, m_p(x_\ell)) = m_p(x_\ell).$$

Using these observations and performing the integration here gives

$$\begin{aligned} (\psi, S\psi) &= \sum_{k=1}^{n} B_{x_k}^2 \left[\frac{(z + i\hbar)^2}{2} - i\hbar e^{-i\frac{z}{\hbar}} e^{\frac{i}{\hbar}m_p(x_k)}(m_p(x_k) + i\hbar)\right]_{m_p(x_k)}^{m_s(x_k)} \\ &= \frac{1}{2}\sum_{k=1}^{n}\sigma_k + i\hbar\sum_{k=1}^{n}[1 - i\hbar\frac{(m_p(x_k) + i\hbar)}{\Delta_k}(m_s(x_k)e^{-\frac{i}{\hbar}\Delta_k} - m_p(x_k))]. \end{aligned}$$

Here

$$\Delta_k = m_s(x_k) - m_p(x_k).$$

A2: An obvious choice for $\hbar$ is "the machine epsilon", $b^{-\ell+e_{\min}}$. For this choice, the last sum here is $O(b^{-\ell})$. However, this choice causes the loss of the universality of $\hbar$ which prevails in mechanics. This suggests that whereas in nature there may be a unique observable reality, the collection of different classes of computers provide a collection of computational realities.

A3: The evaluation of $(\log m_s - \log m_p)/(m_s - m_p)$ proceeds as follows. Using Taylor's theorem with remainder, we have

$$\log m_s = \log x + (m_s - x)\frac{1}{x} - \frac{(m_s - x)^2}{2}\frac{1}{x_+^2}$$

and

$$\log m_p = \log x + (m_p - x)\frac{1}{x} - \frac{(m_p - x)^2}{2}\frac{1}{x_-^2}.$$

Here $x_+ \in (x, m_s)$ and $x_- \in (m_p, x)$. Then subtracting, we find

$$\frac{\log m_s - \log m_p}{m_s - m_p} = \frac{1}{x}\left[1 - \frac{1}{4}\left(\frac{m_s - x}{x_+}\frac{x}{x_+} - \frac{m_p - x}{x_-}\frac{x}{x_-}\right)\right].$$

Continuing, we have

$$\begin{aligned} \frac{m_s - x}{x_+}\frac{x}{x_+} &\sim \frac{b^{e_x}(m + \frac{1}{2}b^{-\ell}) - b^{e_x}m}{e^x(m + \frac{1}{4}b^{-\ell})}\frac{mb^{e_x}}{e^x(m + \frac{1}{4}b^{-\ell})} \\ &= \frac{1}{2}b^{-\ell}(1 + O(b^{-\ell})). \end{aligned}$$

Using this result and the corresponding one with m_p and x_- replacing m_s and x_+, resp., we find

$$\frac{\log m_s - \log m_p}{m_s - m_p} = \frac{1}{x}(1 + O(b^{-\ell})).$$

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