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The Order of Monotone Piecewise Cubic Interpolation

by

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Abstract

Fritsch and Carlson [3] developed an algorithm which produces a monotone C^1 piecewise cubic interpolant to a monotone function. We show that the algorithm yields a third-order approximation, while a modification is fourth-order accurate.

1. Introduction.

In addition to being a good approximation to the function, it is often desirable that an interpolant reproduces such properties as nonnegativity, monotonicity, and convexity. In this paper, we analyze three algorithms which produce monotone C^1 piecewise cubic interpolants to a monotone function.

Since the interpolant is a piecewise cubic, one would hope that such an algorithm would yield a third- or fourth-order L_∞ approximation whenever the function interpolated is sufficiently smooth. However, if the algorithm (considered as a map from the set of monotone functions to the set of monotone C^1 piecewise cubics) is linear, then it is at best first-order accurate (see de Boor and Swartz [2]). Consequently, if greater accuracy is desired, the algorithm must be nonlinear.

Fritsch and Carlson [3] proposed such an algorithm. Given an initial C^1 piecewise cubic interpolant, they modify the derivative values of that interpolant (where necessary) to produce a monotone C^1 piecewise cubic interpolant. Since the modification process is nonlinear, one might hope that the Fritsch-Carlson Algorithm is more than first-order accurate.

In Section 2, we review the Fritsch-Carlson Algorithm and present two modifications, the Two-Sweep and Extended Two-Sweep Algorithms, which also produce monotone C^1 piecewise cubic interpolants. In Section 3, we prove that all three algorithms yield third-order L_∞ approximations to a C^3 monotone function. However, in Section 4, we demonstrate that neither the Fritsch-Carlson Algorithm nor the Two-Sweep Algorithm is a fourth-order method, where, in the case of the latter algorithm, we assume that the initial approximate derivative values are not fourth-order accurate. On the other hand, the Extended Two-Sweep Algorithm is a fourth-order method if the initial approximate derivative values are third-order accurate. Finally, some numerical examples are presented in Section 5.

For brevity and simplicity, we assume that the function interpolated is monotone increasing throughout the remainder of the paper. The extension to decreasing functions is trivial.

2. Algorithms.

In this section, we review the Fritsch-Carlson Algorithm and present two modifications, the Two-Sweep and Extended Two-Sweep Algorithms.

The basis of the Fritsch-Carlson Algorithm is a technique for determining whether a cubic polynomial $p(x)$ is monotone on the interval $[x_i, x_{i+1}]$. Central to this technique is the closed region $\underline{M}$ (see Figure 2-1[1]) bounded by the axes and the 'upper half' of the ellipse

$$x^2 + y^2 + xy - 6x - 6y + 9 = 0. \tag{2.1}$$

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[1] Also shown in Figure 2-1 are the closed regions $\underline{A}, \ldots, \underline{E}$ used in the expression of the algorithms. A segment of the line $x + y = 4$ forms the border between the regions $\underline{A}$ and $\underline{B}$ and also between the regions $\underline{D}$ and $\underline{E}$. The region $\underline{C}$ is bounded by the lines $x = 3$ and $y = 3$.

Fritsch and Carlson [3] show that p(x) is monotone on $[x_i, x_{i+1}]$ if and only if $(p'(x_i), p'(x_{i+1})) \varepsilon \underline{M}_i$,[2] where

$$\underline{M}_i = \underline{M} \cdot \Delta_i = \{ (x\Delta_i, y\Delta_i) : (x,y) \varepsilon \underline{M} \},$$
$$\Delta_i = [p(x_{i+1}) - p(x_i)]/h_i, \quad h_i = x_{i+1} - x_i.$$

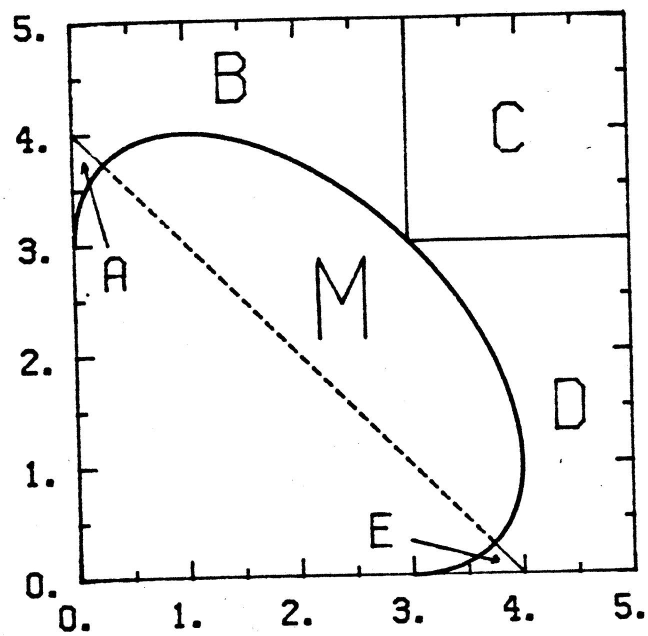

Figure 2-1: The monotonicity region $\underline{M}$ and associated exterior regions $\underline{A}, \ldots, \underline{E}$. All regions are closed.

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[2] We also scale the regions $\underline{A}, \ldots, \underline{E}$ by Δ_i and refer to them as $\underline{A}_i, \ldots, \underline{E}_i$, respectively. However, if $\Delta_i = 0$, we extend this convention by taking $\underline{C}_i$ to be the whole first quadrant; all other regions contract to either points or lines in the obvious way.

Thus, starting with a set of function values $\{f(x_i)\}$ and approximate derivative values $\{d_i\}$, it is easy to determine whether the cubic Hermite interpolant of these values is monotone. Moreover, if the initial interpolant is not monotone, then the condition on p' indicates how the values $\{d_i\}$ should be modified to make it monotone.

Figure 2-2 presents a three step meta-algorithm[3] for finding a monotone cubic Hermite interpolant. Only Step 2 is specified completely. In Step 1, any technique for computing the initial approximate derivative values $\{d_i\}$ is acceptable, although the accuracy of the initial values is one of the prime factors in determining the accuracy of the interpolant. Three possible implementations of Step 3 are developed in the remainder of this section.

Step 1: Compute the initial approximate derivative values $\{d_i\}$.

Step 2: Ensure that each d_i is nonnegative.

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FOR i := 1 STEP 1 UNTIL n DO
    d_i := max{d_i,0};
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Step 3: Modify $\{d_i\}$ so that each ordered pair $(d_i,d_{i+1}) \in \underline{M}_i$.

Figure 2-2: Preliminary Algorithm.

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3 Although Steps 2 and 3 can be combined easily saving one pass through the data, considering these two steps separately simplifies the analysis.

If Step 3 terminates, then the algorithm produces a set of approximate derivative values which, together with the function values $\{f(x_i)\}$, determine a monotone cubic Hermite interpolant of f. The difficulty in implementing Step 3 is that modifying one derivative value d_i affects both of the ordered pairs (d_{i-1},d_i) and (d_i,d_{i+1}). Because of the shape of $\underline{M}$, decreasing the magnitude of d_i in moving (d_i,d_{i+1}) into $\underline{M}_i$ may force (d_{i-1},d_i) out of $\underline{M}_{i-1}$, and vice versa.

For this reason, Fritsch and Carlson base their algorithm on a region $\underline{S}$ properly contained in $\underline{M}$ with the following important property:

If $(x,y) \in \underline{S}$ and $0 \leq x^+ \leq x$ and $0 \leq y^+ \leq y$, then $(x^+,y^+) \in \underline{S}$.

The Fritsch-Carlson Algorithm consists of Steps 1 and 2 of the Preliminary Algorithm together with Step 3 as shown in Figure 2-3.[4]

Alternatively, any technique for projecting the points (d_i,d_{i+1}) into $\underline{M}_i$ which is guaranteed to terminate could be used in Step 3. One such method, the Two-Sweep Algorithm, is shown in Figure 2-4.

On the Forward Sweep, only the second component of each ordered pair is altered, so that modifying (d_i,d_{i+1}) does not affect (d_j,d_{j+1}) for $j<i$. Consequently, it is easy to see that $(d_i,d_{i+1}) \in \underline{M}_i \cup \underline{D}_i \cup \underline{E}_i$ after the

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[4] Here, again, we have used the notation $\underline{S}_i$ to stand for $\underline{S}\cdot\Delta_i$.

Step 3: Modify $\{d_i\}$ so that each ordered pair $(d_i, d_{i+1}) \in \underline{M}_i$.

FOR i := 1 STEP 1 UNTIL n-1 DO

IF $(d_i, d_{i+1}) \notin \underline{S}_i$ THEN

Compute d_i^+ and d_{i+1}^+ so that

(a) $0 \leq d_i^+ \leq d_i$,

(b) $0 \leq d_{i+1}^+ \leq d_{i+1}$, and

(c) $(d_i^+, d_{i+1}^+) \in \underline{S}_i$;

$d_i := d_i^+$; $d_{i+1} := d_{i+1}^+$;

Figure 2-3: Step 3 of the Fritsch-Carlson Algorithm.

Step 3: Modify $\{d_i\}$ so that each ordered pair $(d_i, d_{i+1}) \in \underline{M}_i$.

Forward Sweep - modify the second component only.

FOR i := 1 STEP 1 UNTIL n-1 DO

IF $(d_i, d_{i+1}) \in C_i$ THEN

$d_{i+1} := 3\Delta_i$;

ELSE IF $(d_i, d_{i+1}) \in \underline{A}_i \cup \underline{B}_i$ THEN

Decrease d_{i+1} to project (d_i, d_{i+1}) onto the boundary of $\underline{M}_i$;

Backward Sweep - modify the first component only.

FOR i := n-1 STEP -1 UNTIL 1 DO

IF $(d_i, d_{i+1}) \in \underline{D}_i \cup \underline{E}_i$ THEN

Decrease d_i to project (d_i, d_{i+1}) onto the boundary of $\underline{M}_i$;

Figure 2-4: Step 3 of the Two-Sweep Algorithm.

Forward Sweep.

On the Backward Sweep, only the first component of each ordered pair is altered, so that modifying (d_i,d_{i+1}) does not affect (d_j,d_{j+1}) for $j>i$. Moreover, decreasing the magnitude of d_i ensures that the neighboring point (d_{i-1},d_i) remains in $\underline{M}_{i-1} \cup \underline{D}_{i-1} \cup \underline{E}_{i-1}$, so that (d_{i-1},d_i) can be projected into $\underline{M}_{i-1}$ by decreasing the magnitude of d_{i-1} on the next pass through the loop. Therefore, after the Backward Sweep is completed, $(d_i,d_{i+1}) \varepsilon \underline{M}_i$ and the associated cubic Hermite interpolant is monotone.

The major short-coming of the Two-Sweep Algorithm is that it may move a point (d_i,d_{i+1}) much farther than necessary when projecting it into $\underline{M}_i$. This problem is most acute in the regions $\underline{A}$ and $\underline{E}$ close to the points (0,3) and (3,0), respectively, where the boundary of $\underline{M}$ is tangent to the axes (see Section 4). Therefore, we now consider the Extended Two-Sweep Algorithm described in Figure 2-5.

If the ordered pair of approximate derivative values (d_i,d_{i+1}) does not lie in $\underline{M}_i$, then this algorithm allows the magnitude of d_i to be increased on the Forward Sweep and the magnitude of d_{i+1} to be increased on the Backward Sweep. However, the amount by which they can be increased is constrained by the requirement that, on the Forward Sweep, the preceding ordered pair (d_{i-1},d_i) must remain in $\underline{M}_{i-1} \cup \underline{D}_{i-1} \cup \underline{E}_{i-1}$ and, on the Backward Sweep, (d_{i+1},d_{i+2}) must remain in $\underline{M}_{i+1}$. Because of these constraints, it is clear that $(d_i,d_{i+1}) \varepsilon \underline{M}_i$ after the two sweeps of the extended algorithm have been completed. Consequently, the associated cubic Hermite interpolant is monotone.

Step 3: Modify $\{d_i\}$ so that each ordered pair $(d_i,d_{i+1}) \in \underline{M}_i$.

Forward Sweep - modify the second component only unless the ordered pair lies in $\underline{A}_i$.

FOR i := 1 STEP 1 UNTIL n-1 DO

CASE $(d_i,d_{i+1}) \in C_i$:

$d_{i+1} := 3\Delta_i$;

CASE $(d_i,d_{i+1}) \in \underline{B}_i$:

Decrease d_{i+1} to project (d_i,d_{i+1}) onto the boundary of $\underline{M}_i$;

CASE $(d_i,d_{i+1}) \in \underline{A}_i$:

Increase d_i until either

(a) (d_i,d_{i+1}) reaches the boundary of $\underline{A}_i$, or

(b) (d_{i-1},d_i) reaches the boundary of $\underline{M}_{i-1} \cup \underline{D}_{i-1} \cup \underline{E}_{i-1}$ (if $i > 1$);

IF $(d_i,d_{i+1}) \notin \underline{M}_i$ THEN

Decrease d_{i+1} to project (d_i,d_{i+1}) onto the boundary of $\underline{M}_i$;

Backward Sweep - modify the first component only unless the ordered pair lies in $\underline{E}_i$.

FOR i := n-1 STEP -1 UNTIL 1 DO

CASE $(d_i,d_{i+1}) \in \underline{D}_i$:

Decrease d_i to project (d_i,d_{i+1}) onto the boundary of $\underline{M}_i$;

CASE $(d_i,d_{i+1}) \in \underline{E}_i$:

Increase d_{i+1} until either

(a) (d_i,d_{i+1}) reaches the boundary of $\underline{E}_i$, or

(b) (d_{i+1},d_{i+2}) reaches the boundary of $\underline{M}_{i+1}$ (if $i < n-1$);

IF $(d_i,d_{i+1}) \notin \underline{M}_i$ THEN

Decrease d_i to project (d_i,d_{i+1}) onto the boundary of $\underline{M}_i$;

Figure 2-5: Step 3 of the Extended Two-Sweep Algorithm.

3. Third-Order Convergence.

In this section, we prove that each of the algorithms presented in Section 2 yields a third-order L_∞ approximation to a C^3 monotone function, provided that the initial approximate derivative values are second-order accurate and, in the case of the Fritsch-Carlson Algorithm, that $\underline{S}$ is suitably chosen.

We begin by considering what restrictions on the region $\underline{S}$ are necessary for the Fritsch-Carlson Algorithm to be third-order accurate. To this end, the following result is useful.

Lemma 3.1: If $p_1(x)$ and $p_2(x)$ are two polynomials of degree three or less that satisfy

$$p_1(x_i) = p_2(x_i) \quad \text{and} \quad p_1(x_{i+1}) = p_2(x_{i+1}),$$

then

$$\max \{ |p_1(x) - p_2(x)| : x_i \leq x \leq x_{i+1} \} \tag{3.1}$$
$$\geq \frac{h_i}{6\sqrt{3}} \max \{ |p_1'(x_i) - p_2'(x_i)|, |p_1'(x_{i+1}) - p_2'(x_{i+1})| \}.$$

Proof: Evaluating

$$p_1(x) - p_2(x) = (x-x_i)\left[\frac{x-x_{i+1}}{h_i}\right]^2 [p_1'(x_i) - p_2'(x_i)]$$
$$+ (x-x_{i+1})\left[\frac{x-x_i}{h_i}\right]^2 [p_1'(x_{i+1}) - p_2'(x_{i+1})]$$

at the points

$$y_i = x_i + [\tfrac{1}{2} - \tfrac{1}{\sqrt{12}}]h_i \quad \text{and} \quad z_i = x_i + [\tfrac{1}{2} + \tfrac{1}{\sqrt{12}}]h_i$$

yields

$$p_1(y_i) - p_2(y_i) = \frac{h_i}{6} \{ [\tfrac{1}{2} + \tfrac{1}{\sqrt{12}}] [p_1'(x_i) - p_2'(x_i)] \qquad (3.2)$$

$$- [\tfrac{1}{2} - \tfrac{1}{\sqrt{12}}] [p_1'(x_{i+1}) - p_2'(x_{i+1})] \}$$

and

$$p_1(z_i) - p_2(z_i) = \frac{h_i}{6} \{ [\tfrac{1}{2} - \tfrac{1}{\sqrt{12}}] [p_1'(x_i) - p_2'(x_i)] \qquad (3.3)$$

$$- [\tfrac{1}{2} + \tfrac{1}{\sqrt{12}}] [p_1'(x_{i+1}) - p_2'(x_{i+1})] \},$$

respectively. If

$$|p_1'(x_i) - p_2'(x_i)| \geq |p_1'(x_{i+1}) - p_2'(x_{i+1})|,$$

then, from (3.2),

$$|p_1(y_i) - p_2(y_i)| \geq \frac{h_i}{6} \{ [\tfrac{1}{2} + \tfrac{1}{\sqrt{12}}] |p_1'(x_i) - p_2'(x_i)|$$

$$- [\tfrac{1}{2} - \tfrac{1}{\sqrt{12}}] |p_1'(x_{i+1}) - p_2'(x_{i+1})| \}$$

$$\geq \frac{h_i}{6\sqrt{3}} |p_1'(x_i) - p_2'(x_i)|,$$

which implies (3.1). On the other hand, if

$$|p_1'(x_i) - p_2'(x_i)| \leq |p_1'(x_{i+1}) - p_2'(x_{i+1})|,$$

then (3.1) follows from (3.3). Q.E.D.

Unless $(1,1) \varepsilon \underline{\bar{S}}$ (the closure of $\underline{S}$), the Fritsch-Carlson Algorithm is at best first-order accurate. Consider the approximation to $f(x) = x$ on a uniform mesh. In this case,

$$f'(x_i) = f'(x_{i+1}) = \Delta_i = 1, \quad \text{for} \quad i=1,\ldots,n-1.$$

Consequently, for each i, one of d_i and d_{i+1} must be bounded away from 1, and the result follows from Lemma 3.1.

Similarly, unless $\underline{T} \subset \bar{\underline{S}}$, where $\underline{T}$ is the closed triangle with vertices (0,0), (2,0), (0,2), the Fritsch-Carlson Algorithm is at best second-order accurate. Assume some point (s,2-s), $0 \leq s < 1$, on the 'upper half' of the hypotenuse of $\underline{T}$ is not in $\bar{\underline{S}}$ and consider the approximation to $f(x) = (x-a)^2$ on the interval [a,b]. For any $h \leq H_s \equiv \frac{2(1-s)}{2-s}(b-a)$, choose a set of knots $\{x_i\}$ and an integer j such that $x_j = a + \frac{sh}{2(1-s)}$ and $h_j = h = \max\{h_i\}$. With this choice of x_j and h_j, $x_{j+1} \leq b$,

$$\frac{f'(x_j)}{\Delta_j} = \frac{2(x_j-a)}{2(x_j-a)+h_j} = s, \quad \text{and} \quad \frac{f'(x_{j+1})}{\Delta_j} = \frac{2(x_j-a)+2h_j}{2(x_j-a)+h_j} = 2-s.$$

Moreover, $\Delta_j \geq h_j = h$. Therefore, when the Fritsch-Carlson Algorithm terminates, at least one of the approximate derivative values $\{d_i\}$ must satisfy

$$|f'(x_i) - d_i| \geq ch,$$

for some constant $c > 0$, and, by Lemma 3.1, the associated cubic Hermite interpolant is at best second-order accurate. A similar result holds for the 'lower half' of the hypotenuse of $\underline{T}$.

On the other hand, if $\underline{T} \subset \underline{S}$, then the Fritsch-Carlson Algorithm is third-order accurate.[5] Before proving this result, we state the following useful lemma.

Lemma 3.2: If $f \in C^1[a,b]$ is monotone increasing, then, for any of the algorithms described in Section 2,

$$d_i^+ \geq 0 \quad \text{and} \quad |f'(x_i) - d_i^+| \leq |f'(x_i) - d_i|, \qquad i=1,\ldots,n,$$

where d_i and d_i^+, respectively, are the approximate derivative values before and after the execution of Step 2.

Proof: If d_i is modified in Step 2, then $d_i < 0$ and $d_i^+ = 0$ (see Figure 2-2). Hence, since $f'(x_i) \geq 0$,

$$|f'(x_i) - d_i^+| = |f'(x_i)| < |f'(x_i) - d_i|.$$

On the other hand, if d_i is not modified, then $d_i^+ = d_i \geq 0$. Q.E.D.

Theorem 3.3: Assume that

1. $f \in C^3[a,b]$ is monotone increasing;

2. the initial derivative approximations $\{d_i\}$ satisfy

$$|f'(x_i) - d_i| \leq ch^2, \qquad i=1,\ldots,n,$$

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[5] The four regions $\underline{S}_1,\ldots,\underline{S}_4$ considered in [3] all contain the triangle $\underline{T}$.

for some constant c;

3. $\underline{T} \subset \underline{S}$; and

4. whenever a point (d_i, d_{i+1}) is projected into $\underline{S}_i$, the new point (d_i^+, d_{i+1}^+) satisfies

$$2\Delta_i \leq d_i^+ + d_{i+1}^+,$$

i.e., the point is not moved 'much farther' than necessary.

Then the modified approximate derivative values $\{d_i^*\}$ produced by the Fritsch-Carlson Algorithm satisfy

$$|f'(x_i) - d_i^*| \leq [c + \tfrac{1}{6}\|f^{(3)}\|_\infty]h^2, \quad i=1,\ldots,n. \tag{3.4}$$

Consequently, the associated monotone cubic Hermite interpolant is a third-order L_∞ approximation to f.

Proof: From Lemma 3.2,

$$d_i \geq 0 \quad \text{and} \quad |f'(x_i) - d_i| \leq ch^2 \tag{3.5}$$

at the termination of Step 2.

Assume that d_i is modified in Step 3 when (d_{i-1}^+, d_i) is projected to $(d_{i-1}^*, d_i^+) \in \underline{S}_{i-1}$.[6] The values d_{i-1}^+, d_{i-1}^*, and d_i^+ may differ from the

[6] d_{i-1}^+ and d_{i-1}^* are approximate derivative values that have been modified either once or twice, respectively.

initial values d_{i-1} and d_i satisfying (3.5), but

$$0 \le d^*_{i-1} \le d^+_{i-1} \le d_{i-1} \quad \text{and} \quad 0 \le d^+_i \le d_i.$$

If $f'(x_i) \le d^+_i$, then

$$0 \le d^+_i - f'(x_i) \le d_i - f'(x_i) \le ch^2.$$

Therefore, assume that $f'(x_i) \ge d^+_i$. Note that

$$2\Delta_{i-1} = f'(x_{i-1}) + f'(x_i) - \frac{1}{6}f^{(3)}(y_{i-1})h^2_{i-1}$$

for some $y_{i-1} \varepsilon [x_{i-1},x_i]$. From Assumption 4,

$$2\Delta_{i-1} \le d^*_{i-1} + d^+_i,$$

so that

$$f'(x_i) - d^+_i \le d^*_{i-1} - f'(x_{i-1}) + \frac{1}{6}f^{(3)}(y_{i-1})h^2_{i-1}.$$

Therefore,

$$\begin{aligned} 0 &\le f'(x_i) - d^+_i \\ &\le d^*_{i-1} - f'(x_{i-1}) + \frac{1}{6}f^{(3)}(y_{i-1})h^2_{i-1} \\ &\le d_{i-1} - f'(x_{i-1}) + \frac{1}{6}f^{(3)}(y_{i-1})h^2_{i-1} \\ &\le [c + \frac{1}{6}\|f^{(3)}\|_\infty]h^2 \end{aligned}$$

by Assumption 2.

If d^+_i is decreased to d^*_i to project (d^+_i,d_{i+1}) into $\underline{S}_i$ on the next pass through the loop, then a similar argument shows that inequality (3.4) remains valid. Q.E.D.

Essentially the same argument shows that the Two-Sweep Algorithm is third-order accurate. However, the Extended Two-Sweep Algorithm may increase some approximate derivative values. Therefore, we adopt a different approach based upon the following lemma.[7]

Lemma 3.4: Assume that

1. $f \in C^3[a,b]$ is monotone increasing; and,

2. for some $\alpha > 0$, $(f'(x_{i-1}), f'(x_i)) \notin \underline{T}^{\alpha}_{i-1} = \underline{T}^{\alpha} \cdot \Delta_{i-1}$, where $\underline{T}^{\alpha}$ is the closed triangle with vertices $(0,0)$, $(2+\alpha,0)$, $(0,2+\alpha)$.

Then

$$\Delta_{i-1} < \frac{1}{6\alpha} \| f^{(3)} \|_{\infty} h^2_{i-1}$$

and

$$f'(x_{i-1}) + f'(x_i) < \left[\frac{1}{6} + \frac{1}{3\alpha}\right] \| f^{(3)} \|_{\infty} h^2_{i-1}.$$

Proof: If $(f'(x_{i-1}), f'(x_i)) \notin \underline{T}^{\alpha}_{i-1}$, then

$$(2+\alpha)\Delta_{i-1} < f'(x_{i-1}) + f'(x_i).$$

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[7] In passing, note that this lemma can also be used to prove a different version of Theorem 3.3: if Assumptions 3 and 4 are replaced by

$\tilde{3}$. $\underline{T}^{\alpha} \subset \underline{S}$ for some $\alpha > 0$,

then the Fritsch-Carlson Algorithm is still third-order accurate.

However,

$$2\Delta_{i-1} = f'(x_{i-1}) + f'(x_i) - \frac{1}{6}f^{(3)}(y_{i-1})h_{i-1}^2 \qquad (3.6)$$

for some $y_{i-1} \varepsilon [x_{i-1},x_i]$, so that

$$\Delta_{i-1} < \frac{1}{6\alpha}f^{(3)}(y_{i-1})h_{i-1}^2 \leq \frac{1}{6\alpha}\|f^{(3)}\|_\infty h_{i-1}^2.$$

Finally, using (3.6),

$$f'(x_{i-1}) + f'(x_i) < \left[\frac{1}{6} + \frac{1}{3\alpha}\right]\|f^{(3)}\|_\infty h_{i-1}^2.$$

Q.E.D.

Theorem 3.5: Assume that

1. $f \varepsilon C^3[a,b]$ is monotone increasing; and

2. the initial derivative approximations $\{d_i\}$ satisfy

$$|f'(x_i) - d_i| \leq ch^2, \quad i=1,\ldots,n$$

for some constant c.

Then the modified approximate derivative values $\{d_i^*\}$ produced by either the Two-Sweep or the Extended Two-Sweep Algorithm satisfy

$$|f'(x_i) - d_i^*| \leq \max\{c, \tfrac{1}{2}\|f^{(3)}\|_\infty\}h^2, \quad i=1,\ldots,n. \qquad (3.7)$$

Consequently, the associated monotone cubic Hermite interpolant is a third-order L_∞ approximation to f.

Proof: By Lemma 3.2, the approximate derivative values satisfy

$$d_i \geq 0 \quad \text{and} \quad |f'(x_i) - d_i| \leq ch^2$$

at the completion of Step 2 of either algorithm. Therefore, they also satisfy (3.7). Below, we show that, if all the approximate derivative values satisfy (3.7) when one is modified in Step 3, then the modified value also satisfies (3.7). Thus, the theorem follows by induction.

In the Extended Two-Sweep Algorithm, d_i is modified in Step 3 only if

1. (d_{i-1},d_i) is projected downwards in the Forward Sweep,

2. (d_i,d_{i+1}) is projected to the right in the Forward Sweep,

3. (d_i,d_{i+1}) is projected to the left in the Backward Sweep, or

4. (d_{i-1},d_i) is projected upwards in the Backward Sweep.

For the Two-Sweep Algorithm, only Cases 1 and 3 are applicable. Therefore, proving (3.7) for the Extended Two-Sweep Algorithm also shows that this inequality is valid for the Two-Sweep Algorithm.

Consider Case 1 first: (d_{i-1},d_i) is projected downwards in the Forward Sweep. If $f'(x_i) \leq d_i^+$, then

$$0 \leq d_i^+ - f'(x_i) \leq d_i - f'(x_i) \leq \max\{c, \tfrac{1}{2}\|f^{(3)}\|_\infty\}h^2,$$

since $d_i^+ \leq d_i$. Therefore, assume that $f'(x_i) \geq d_i^+$. If $(f'(x_{i-1}),f'(x_i)) \in \underline{T}_{i-1}^1$, then

$$f'(x_i) \leq 3\Delta_{i-1} \leq d_i^+,$$

a contradiction. Thus, $(f'(x_{i-1}),f'(x_i)) \notin \underline{T}_{i-1}^1$, whence

$$f'(x_{i-1}) + f'(x_i) < \tfrac{1}{2}\|f^{(3)}\|_\infty h_{i-1}^2$$

by Lemma 3.4. Since $f'(x_i) \geq d_i^+ \geq 0$ and both $f'(x_{i-1})$ and $f'(x_i)$ are nonnegative,

$$0 \leq f'(x_i) - d_i^+ \leq f'(x_i) \leq f'(x_{i-1}) + f'(x_i) \leq \tfrac{1}{2}\|f^{(3)}\|_\infty h^2.$$

Next consider Case 2: (d_i, d_{i+1}) is projected to the right in the Forward Sweep. If $d_i^+ \leq f'(x_i)$, then

$$0 \leq f'(x_i) - d_i^+ \leq f'(x_i) - d_i \leq \max\{c, \tfrac{1}{2}\|f^{(3)}\|_\infty\}h^2,$$

since $d_i \leq d_i^+$. Therefore, assume that $d_i^+ \geq f'(x_i)$. If $(f'(x_i), f'(x_{i+1})) \notin \underline{T}_i^{1/2}$, then

$$\Delta_i \leq \tfrac{1}{3}\|f^{(3)}\|_\infty h^2$$

by Lemma 3.4. But $d_i^+ \leq \frac{1}{2}\Delta_i$ since $(d_i^+, d_{i+1}) \in \underline{A}_i$, so that

$$0 \leq d_i^+ - f'(x_i) \leq d_i^+ \leq \tfrac{1}{2}\Delta_i \leq \tfrac{1}{6}\|f^{(3)}\|_\infty h^2.$$

On the other hand, if $(f'(x_i), f'(x_{i+1})) \in \underline{T}_i^{1/2}$, then

$$f'(x_i) + f'(x_{i+1}) \leq \tfrac{5}{2}\Delta_i \leq d_{i+1} - \tfrac{1}{2}\Delta_i,$$

since $(d_i, d_{i+1}) \in \underline{A}_i$ implies that $3\Delta_i \leq d_{i+1}$. Re-arranging terms,

$$\tfrac{1}{2}\Delta_i + f'(x_i) \leq d_{i+1} - f'(x_{i+1}),$$

whence

$$d_i^+ + f'(x_i) \leq d_{i+1} - f'(x_{i+1}),$$

since $d_i^+ \leq \frac{1}{2}\Delta_i$. Therefore,

$$0 \le d_i^+ - f'(x_i) \le d_i^+ + f'(x_i) \le d_{i+1} - f'(x_{i+1})$$
$$\le \max\{c, \tfrac{1}{2}\|f^{(3)}\|_\infty\}h^2.$$

Cases 3 and 4 are handled in a similar manner. **Q.E.D.**

4. Fourth-Order Convergence.

In this section, we demonstrate that neither the Fritsch-Carlson Algorithm nor the Two-Sweep Algorithm is a fourth-order method, where, in the case of the latter algorithm, we assume that the initial approximate derivative values are less than fourth-order accurate. On the other hand, the Extended Two-Sweep Algorithm is a fourth-order method if the initial approximate derivative values are third-order accurate.

To see that the Fritsch-Carlson Algorithm is not a fourth-order method, consider the function $f(x) = (x-1)^3$ on the interval $[0,3]$. For any positive integer m, let the knots be

$$x_i = 3ih, \quad i = 0,1,\ldots,3m+2, \quad \text{where} \quad h = \frac{3}{3m+2}.$$

A simple computation shows that

$$f'(x_m) = \tfrac{4}{3}h^2, \quad f'(x_{m+1}) = \tfrac{1}{3}h^2, \quad \text{and} \quad \Delta_m = \tfrac{1}{3}h^2,$$

whence

$$\left(\frac{f'(x_m)}{\Delta_m}, \frac{f'(x_{m+1})}{\Delta_m}\right) = (4,1)$$

is on the boundary of $\underline{M}$. On the other hand, any region $\underline{S}$ used in Step 3 of the Fritsch-Carlson Algorithm must be contained in the region $\underline{S}_1$, the square with vertices (0,0) (0,3), (3,3), (3,0), so that the modified derivative approximation d_m^* must satisfy $d_m^* \leq 3\Delta_m$. Thus,

$$f'(x_m) - d_m^* \geq f'(x_m) - 3\Delta_m = \frac{1}{3}h^2,$$

and, from Lemma 3.1, the Fritsch-Carlson Algorithm yields at best a third-order approximation to f.

To see that the Two-Sweep Algorithm is not a fourth-order method if the initial approximate derivative values are less than fourth-order accurate, once again consider the function $f(x) = (x-1)^3$ on the interval [0,3]. For $2 \leq p \leq 4$, choose the knots $\{x_i\}$ such that, for some j, $x_j = 1-h^{p/2}$ and $h_j = h \equiv \max\{h_i\}$. Hence,

$$f'(x_j) = 3h^p, \qquad f'(x_{j+1}) = 3[h^2 - 2h^{1+p/2} + h^p],$$

and

$$\Delta_j = h^2 - 3h^{1+p/2} + 3h^p.$$

It is easy to check that $(f'(x_j),f'(x_{j+1}))$ is on the boundary between $\underline{M}_j$ and $\underline{A}_j$ and that $(f'(x_i),f'(x_{i+1})) \in \underline{S}_1 \cdot \Delta_i$ for $i \neq j$. Let $d_j = 0$ and $d_i = f'(x_i)$ for $i \neq j$. Then d_j is a p^{th}-order approximation to $f'(x_j)$ and all other d_i are exact. In addition, since $d_j < f'(x_j)$, it follows that $(d_j,d_{j+1}) \in \underline{A}_j \backslash \underline{M}_j$ and $(d_i,d_{i+1}) \in \underline{S}_1 \cdot \Delta_i$ for $i \neq j$. Consequently, the only approximate derivative value that is modified by the Two-Sweep Algorithm is d_{j+1} and it is set to $d_{j+1}^+ = 3\Delta_j$ on the Forward Sweep. Hence,

$$f'(x_{j+1}) - d_{j+1}^+ = 3h^{1+p/2} - 6h^p,$$

and, by Lemma 3.1, the Two-Sweep Algorithm yields at best an order $2+\frac{p}{2}$ approximation to f. In particular, if the Two-Sweep Algorithm is used to modify the derivative values of a cubic spline interpolant, then the resulting monotone C^1 piecewise cubic interpolant may be of order $3\frac{1}{2}$, rather than 4, since the initial approximate derivative values are only third-order accurate.

However, for both the Fritsch-Carlson and Two-Sweep Algorithms, this degradation in the order of the approximation arises only under very special circumstances. If the region $\underline{S}$ associated with the Fritsch-Carlson Algorithm contains a triangle $\underline{T}^{\alpha}$ for some $\alpha > 0$, then, using an argument similar to the one employed in the proof of Theorem 4.1, one can show that the degradation in the order of either of these two algorithms occurs only in intervals immediately adjacent to an interval containing a root of f' of exact multiplicity two. Moreover, for the Two-Sweep Algorithm, the degradation occurs only if, as $h \to 0$, there are infinitely many grids each containing an interval $[x_i,x_{i+1}]$ and a point t in that interval at which f' has a root of exact multiplicity two and the distance between t and one of the endpoints of the interval is less than c_1h_i but greater than $c_2h_i^2$ for all positive constants c_1 and c_2.

Another point about all three algorithms should be emphasized: whenever h is sufficiently small, most of the initial derivative approximations are not changed by any of the algorithms. Thus, if the initial derivative approximations are third-order accurate, then the interpolant produced by any of the algorithms is locally a fourth-order approximation on most intervals. Moreover, if the initial interpolant is a cubic spline, then this additional smoothness is lost only at the knots where the derivative values are modified.

We end this section with a convergence result for the Extended Two-Sweep Algorithm.

Theorem 4.1: Assume that

1. $f \in C^4[a,b]$ is monotone increasing;

2. whenever $f'(x) = f''(x) = f^{(3)}(x) = f^{(4)}(x) = 0$, there is a $\delta > 0$ such that, if $y \in [x,x+\delta) \cap [a,b]$, then either

   a. $f'(y) = 0$ or

   b. there exist constants m_1, m_2, and r such that

   $$m_1(y-x)^r \le f'(y) \le m_2(y-x)^r,$$

   where $\frac{10}{11} \le \frac{m_2}{m_1} \le \frac{11}{10}$ and $r \ge 3$;

   and, if $y \in (x-\delta,x] \cap [a,b]$, then either

   a. $f'(y) = 0$ or

   b. there exist constants m_3, m_4, and s such that

   $$m_3(x-y)^s \le f'(y) \le m_4(x-y)^s$$

   where $\frac{10}{11} \le \frac{m_4}{m_3} \le \frac{11}{10}$ and $s \ge 3$; and

3. the initial derivative approximations $\{d_i\}$ satisfy

$$|f'(x_i) - d_i| \le ch^3, \quad i=1,\ldots,n.$$

Then, for h sufficiently small, the modified approximate derivative values $\{d_i^*\}$ produced by the Extended Two-Sweep Algorithm satisfy

$$|f'(x_i) - d_i^*| \le \tilde{c}h^3, \quad i=1,\ldots,n, \tag{4.1}$$

where

$$\tilde{c} = \max\{8\|f^{(4)}\|_\infty, \frac{99}{32}\|f^{(4)}\|_\infty + \frac{173}{8}c\}. \tag{4.2}$$

Consequently, the associated monotone cubic Hermite interpolant is a fourth-order L_∞ approximation to f.[8]

Proof: To prove this result, we combine a compactness argument with induction. The essence of the proof is outlined below; the details, which are straightforward but tedious, are in the Appendix.

For each $t \in [a,b]$, we choose a $\delta_t > 0$ that determines an open interval $I_t = (t-\delta_t, t+\delta_t)$, where δ_t depends upon f in a neighborhood of t. Since $\{I_t\}$ forms an open covering of the compact interval [a,b], there exists a finite subcovering of [a,b]. Moreover, for $h = \max\{h_i\}$

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[8] The proof of this result requires Assumption 2, although we suspect that the theorem remains valid for any monotone $C^4[a,b]$ function. It is also worth noting that Assumption 2 holds for any piecewise analytic function.

sufficiently small, each interval $[x_{i-1},x_{i+1}] \subset I_t$, one of the intervals of the subcovering. The proof relies heavily upon exploiting the local properties of f on each interval of the finite subcovering.

The actual induction hypothesis used is slightly stronger than (4.1):

1. If $[x_{i-1},x_{i+1}] \subset I_t$, $f'(t) = f''(t) = 0$, $f^{(3)}(t) \neq 0$, and $t \in (x_{i-1},x_i]$, then

$$|f'(x_i) - d_i| \leq \left[\frac{5}{6}\|f^{(4)}\|_\infty + 6.5c\right]h^3.$$

2. If $[x_{i-1},x_{i+1}] \subset I_t$, $f'(t) = f''(t) = 0$, $f^{(3)}(t) \neq 0$, and $t \in [x_i,x_{i+1})$, then

$$|f'(x_i) - d_i| \leq \left[\frac{99}{32}\|f^{(4)}\|_\infty + \frac{173}{8}c\right]h^3.$$

3. Otherwise,

$$|f'(x_i) - d_i| \leq \max\{c,\ 8\|f^{(4)}\|_\infty\}h^3.$$

By Lemma 3.2,

$$d_i \geq 0 \quad \text{and} \quad |f'(x_i) - d_i| \leq ch^3$$

at the termination of Step 2. Consequently, the induction hypothesis is satisfied at the beginning of Step 3. In the Appendix, we show that, if all the approximate derivative values satisfy the hypothesis when one is modified in Step 3, then the modified value also satisfies the hypothesis. Thus, the theorem follows by induction. Q.E.D.

5. Numerical Results.

In this section, we compare the piecewise cubic interpolants produced by CUBSPL [1], the Fritsch-Carlson Algorithm, and the Extended Two-Sweep Algorithm for the two sets of monotone data given in Section 5 of [3].

In the case of CUBSPL, we used the 'not-a-knot' boundary conditions to complete the specification of the cubic spline interpolant. Since CUBSPL is based upon a fourth-order linear algorithm, it does not, in general, produce a monotone approximation to a set of monotone data.

We implemented the Fritsch-Carlson Algorithm described in [3] and, following their suggestion, we took the region $\underline{S}$ required in Step 3 to be $\underline{S}_2$, the intersection of the disk of radius three centered at the origin with the first quadrant. The results in Sections 3 and 4 above show that this method is third-order, but not fourth-order, accurate.

We used the derivative of the cubic spline interpolant produced by CUBSPL for the initial derivative approximations required in Step 1 of the Extended Two-Sweep Algorithm. Since these approximate derivative values are third-order accurate, the monotone interpolant produced by the Extended Two-Sweep Algorithm is fourth-order accurate.

Figure 5-1 shows the interpolants produced by CUBSPL and the Extended Two-Sweep Algorithm for the first data set (AKIMA 3) in [3]. Figure 5-2 shows the interpolants produced by the Fritsch-Carlson Algorithm and the Extended Two-Sweep Algorithm for the same data set. Figures 5-3 and 5-4 show the interpolants generated by the same two pairs of methods, but for the second data set (RPN 14) in [3].

The interpolant produced by CUBSPL is clearly not monotone for either data set and does not yield a 'visually pleasing' approximation in either case.

For the first data set, the interpolants produced by the Fritsch-Carlson and Extended Two-Sweep Algorithms differ significantly on the interval [11,15]. Because the Extended Two-Sweep Algorithm projects approximate derivative values onto the boundary of $\underline{M}$, it produces an interpolant with a zero slope in this interval. This is not the case for the Fritsch-Carlson Algorithm, since it projects approximate derivative values into the interior of $\underline{M}$. We leave the subjective question of which approximation is 'visually more pleasing' to the reader.

For the second data set, the interpolants produced by the Fritsch-Carlson and Extended Two-Sweep Algorithms are virtually indistinguishable at the resolution of these plots: monotonicity imposes a severe constraint in this example.

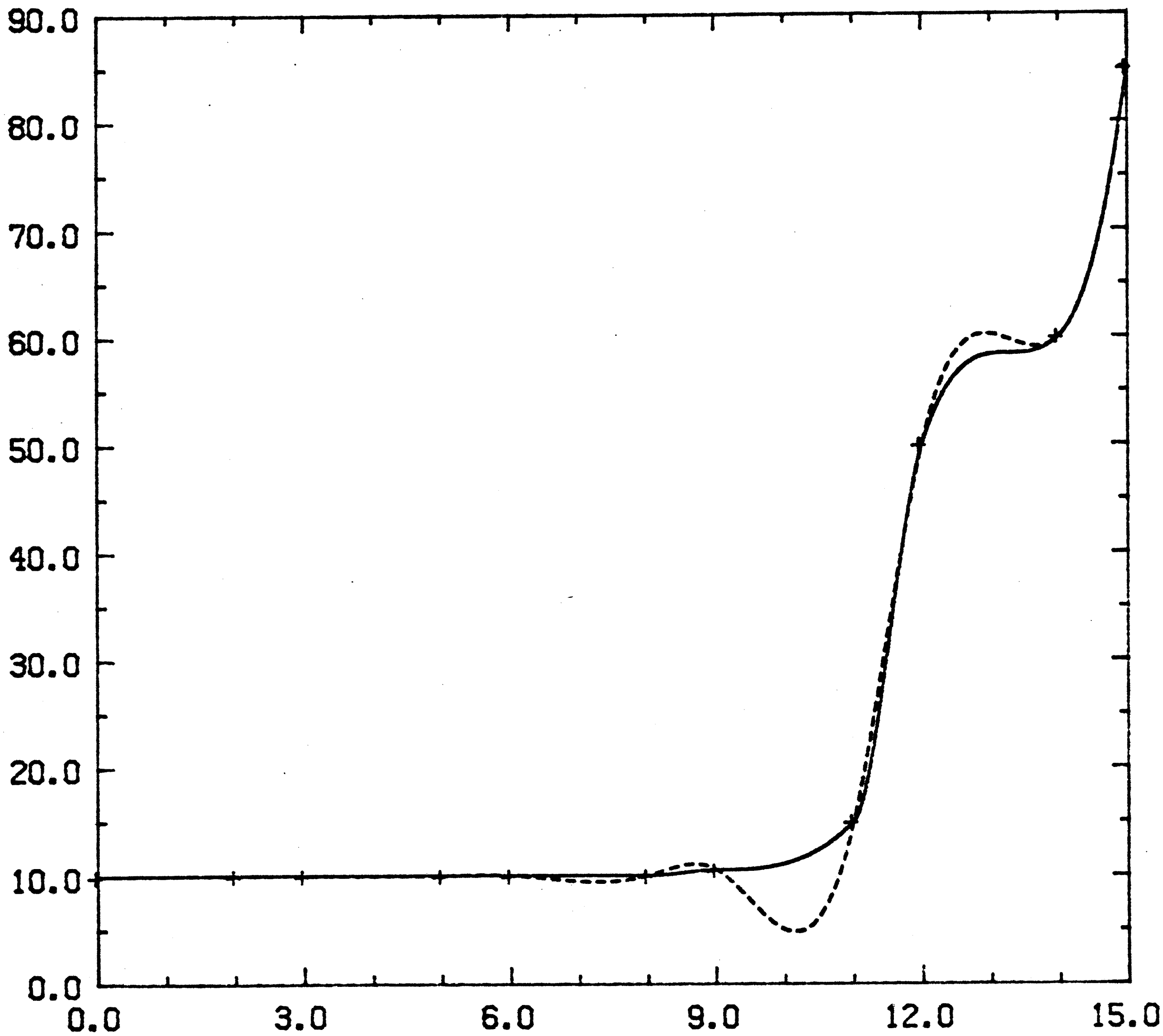

Figure 5-1: A plot of the interpolants produced by CUBSPL (dotted curve) and the Extended Two-Sweep Algorithm (solid curve) for the data set AKIMA 3.

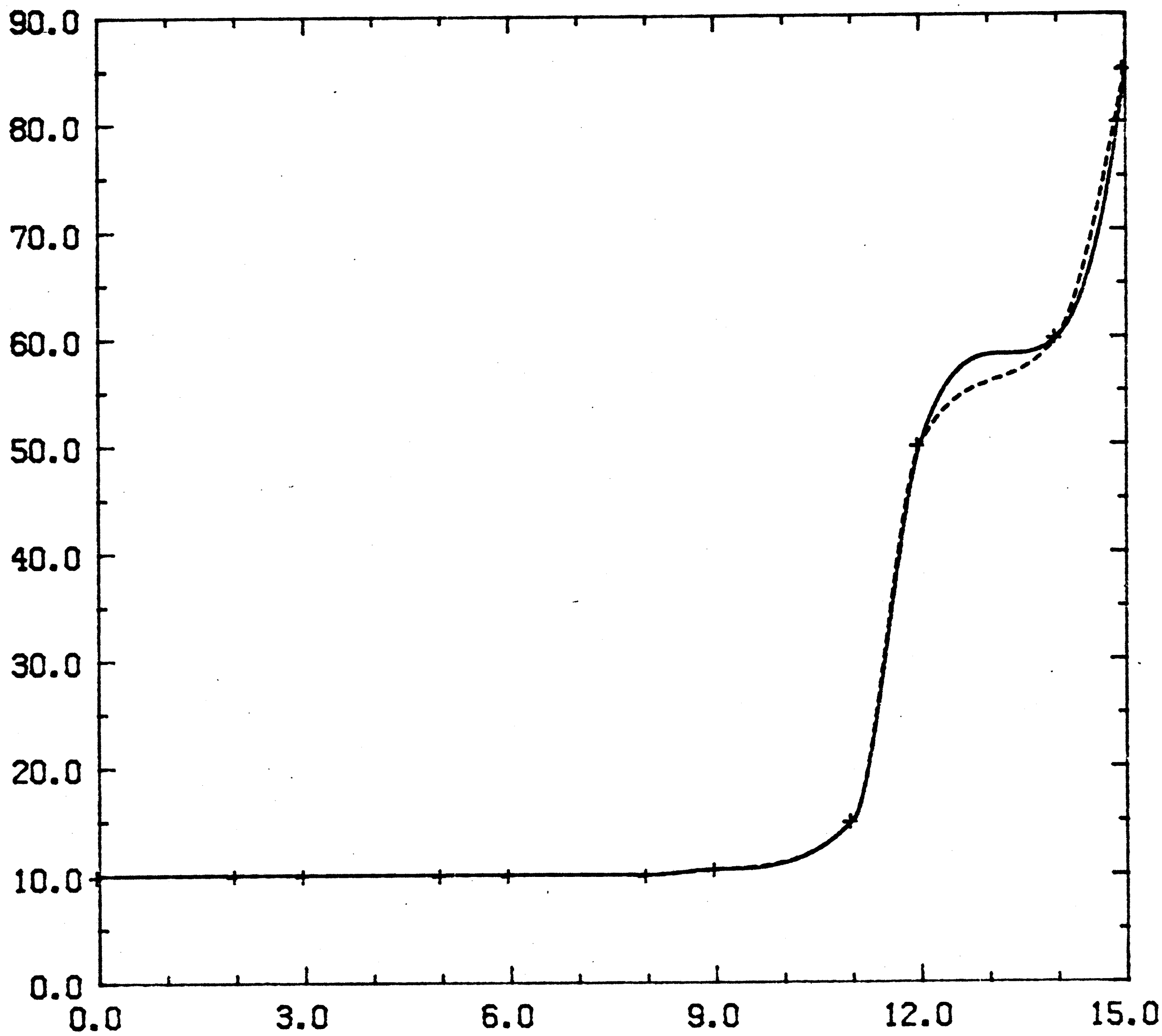

Figure 5-2: A plot of the interpolants produced by the Fritsch-Carlson Algorithm (dotted curve) and the Extended Two-Sweep Algorithm (solid curve) for the data set AKIMA 3.

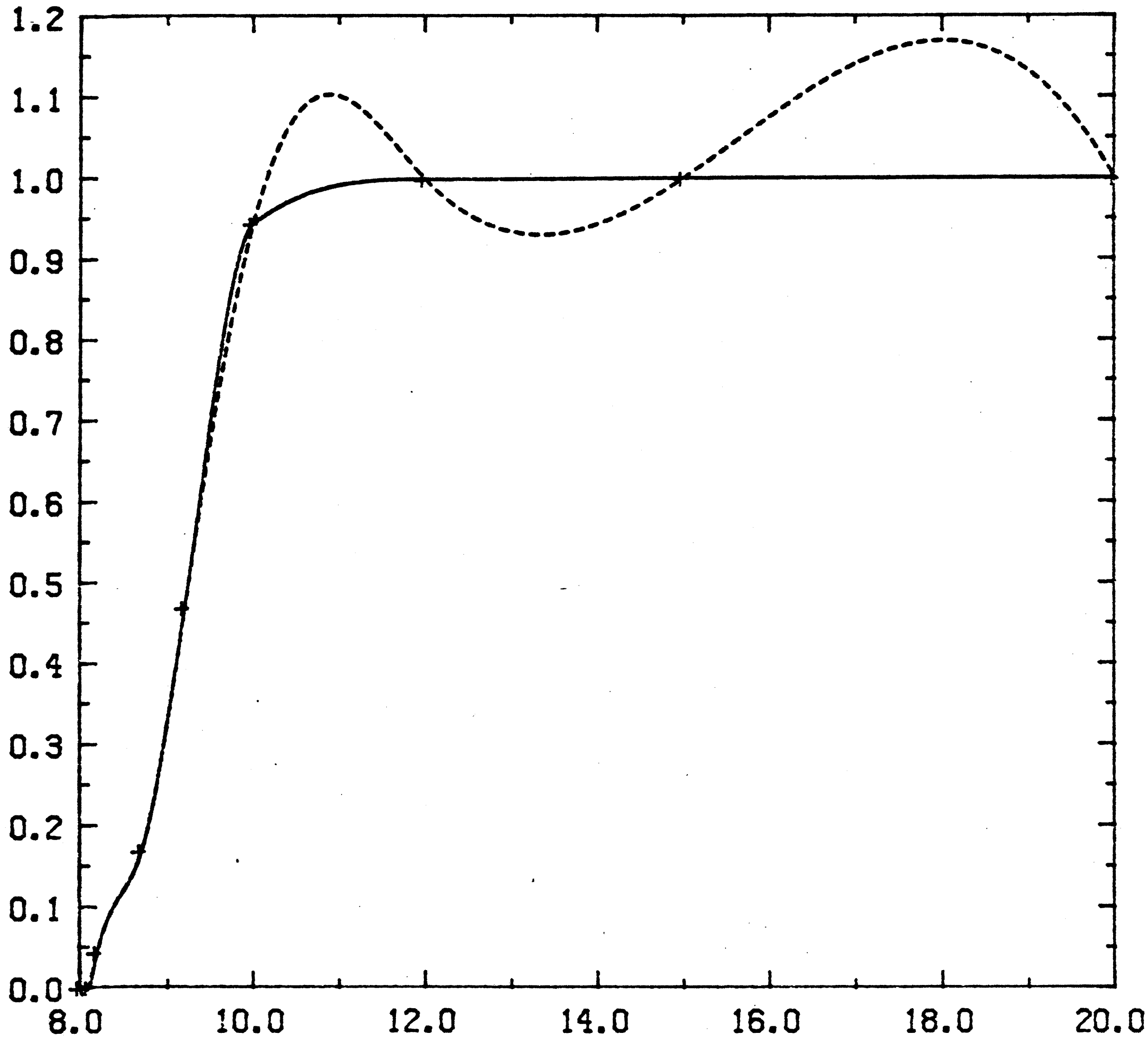

Figure 5-3: A plot of the interpolants produced by CUBSPL (dotted curve) and the Extended Two-Sweep Algorithm (solid curve) for the data set RPN 14.

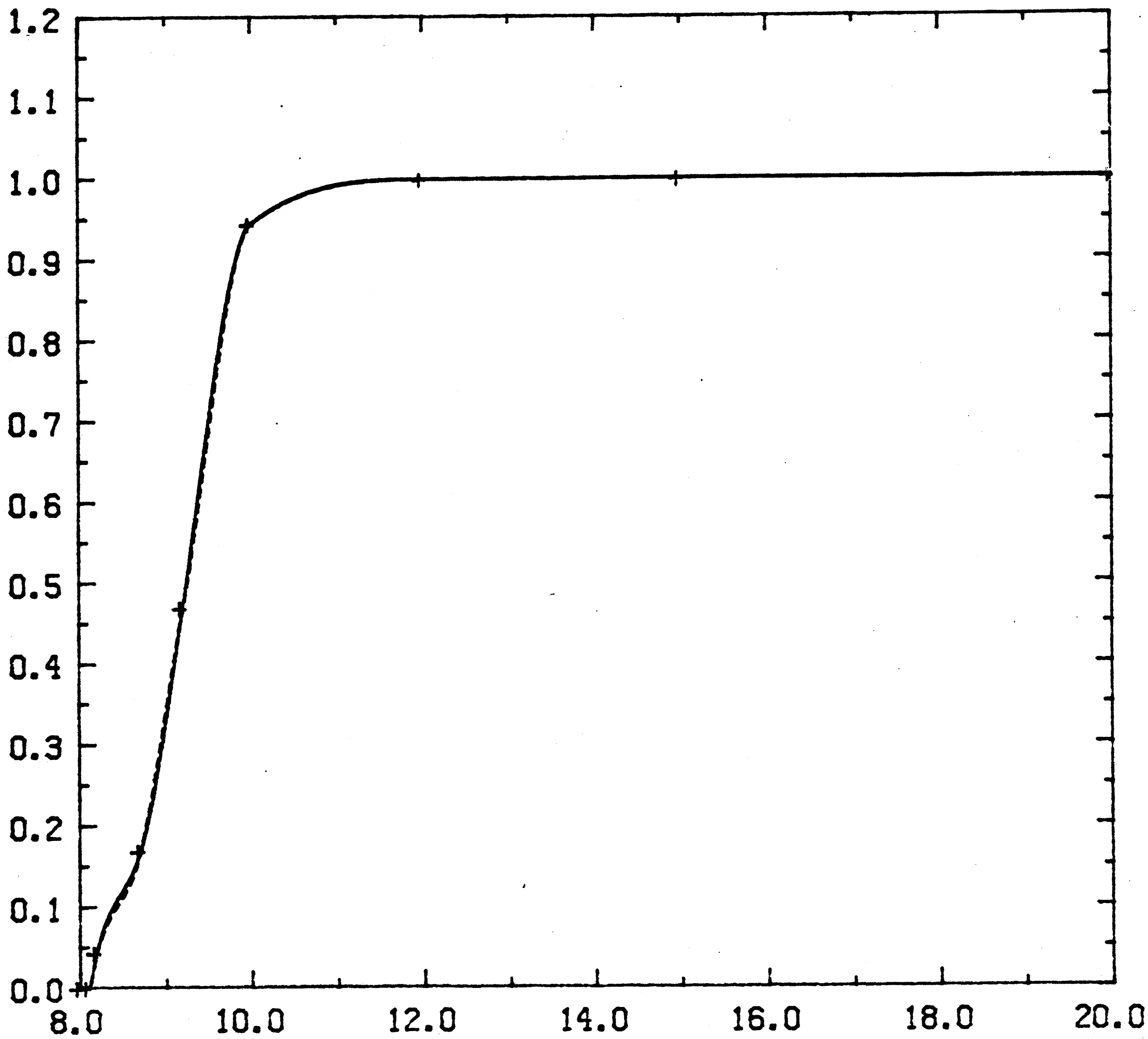

Figure 5-4: A plot of the interpolants produced by the Fritsch-Carlson Algorithm (dotted curve) and the Extended Two-Sweep Algorithm (solid curve) for the data set RPN 14.

References

[1] C. de Boor. A Practical Guide to Splines. Springer-Verlag, New York, 1978.

[2] C. de Boor and B. Swartz. Piecewise monotone interpolation. J. Approx. Th. 21:411-416, 1977.

[3] F. N. Fritsch and R. E. Carlson. Monotone piecewise cubic interpolation. SIAM J. Numer. Anal. 17:238-246, 1980.

Appendix

I. Proof of Theorem 4.1.

In this appendix, we complete the proof of Theorem 4.1. To begin, we state and prove two useful lemmas.

Lemma 5.1: If $f \in C^4[a,b]$ and $f'(t) = f''(t) = 0$ but $f^{(3)}(t) \neq 0$ for some $t \in [x_i,x_{i+1}]$, then

$$|f'(x_i) - \frac{3\gamma^2}{1-3\gamma+3\gamma^2}\Delta_i| \leq \frac{7}{24}\|f^{(4)}\|_\infty \gamma^2 h_i^3 \tag{5.1}$$

and

$$|f'(x_{i+1}) - \frac{3(1-\gamma)^2}{1-3\gamma+3\gamma^2}\Delta_i| \leq \frac{7}{24}\|f^{(4)}\|_\infty (1-\gamma)^2 h_i^3, \tag{5.2}$$

where $\gamma = (t-x_i)/h_i$. Moreover, the locus of points

$$\{ (\frac{3\gamma^2}{1-3\gamma+3\gamma^2} , \frac{3(1-\gamma)^2}{1-3\gamma+3\gamma^2}) : 0 \leq \gamma \leq 1 \} \tag{5.3}$$

is the elliptical boundary of $\underline{M}$.

Proof: Inequalities (5.1) and (5.2) follow from the Taylor series expansions

$$f'(x_i) = \frac{1}{2}f^{(3)}(t)\gamma^2 h_i^2 - \frac{1}{6}f^{(4)}(y_1)\gamma^3 h_i^3,$$

$$f'(x_{i+1}) = \frac{1}{2}f^{(3)}(t)(1-\gamma)^2 h_i^2 + \frac{1}{6}f^{(4)}(y_2)(1-\gamma)^3 h_i^3, \quad \text{and}$$

$$\Delta_i = \frac{1}{6}f^{(3)}(t)[(1-\gamma)^3+\gamma^3]h_i^2 + \frac{1}{24}[f^{(4)}(y_3)(1-\gamma)^4-f^{(4)}(y_4)\gamma^4]h_i^3,$$

for some y_1, y_2, y_3, $y_4 \in [x_i,x_{i+1}]$. The validity of (5.3) is established easily from (2.1). **Q.E.D.**

Lemma 5.2: Assume that

1. $f \in C^4[a,b]$ is monotone increasing;

2. $f'(t) = f''(t) = 0$ but $f^{(3)}(t) \neq 0$ for some $t \in [x_i, x_{i+1}]$;

3. $(d_i, d_{i+1}) \in \underline{A}_i$; and

4. the initial derivative approximations satisfy

$$f'(x_i) \leq d_i \quad \text{and} \quad |f'(x_{i+1}) - d_{i+1}| \leq ch^3$$

for some constant c.

Then, for the unique d^+_{i+1} such that $(d_i, d^+_{i+1}) \in \underline{M}_i \cap \underline{A}_i$,

$$|f'(x_{i+1}) - d^+_{i+1}| \leq \max\{c, \tfrac{7}{12}\|f^{(4)}\|_\infty\}h^3. \tag{5.4}$$

A similar result holds for $(d_i, d_{i+1}) \in \underline{E}_i$.

Proof: Throughout this proof, we use inequalities (5.1) and (5.2) of Lemma 5.1 without explicit reference.

Consider two cases depending upon whether $d^+_{i+1} > f'(x_{i+1})$.

Case 1: If $d^+_{i+1} > f'(x_{i+1})$, then

$$0 < d^+_{i+1} - f'(x_{i+1}) \leq d_{i+1} - f'(x_{i+1}) \leq ch^3,$$

since $d^+_{i+1} \leq d_{i+1}$.

Case 2: If $d_{i+1}^{+} \leq f'(x_{i+1})$, then consider two subcases depending upon whether

$$\frac{3\gamma^2}{1-3\gamma+3\gamma^2}\Delta_i < d_i. \tag{5.5}$$

Case 2.1: If (5.5) is valid, then

$$\frac{3(1-\gamma)^2}{1-3\gamma+3\gamma^2}\Delta_i < d_{i+1}^{+}, \tag{5.6}$$

since the segment of the curve (5.3) that forms the boundary between $\underline{M}_i$ and $\underline{A}_i$ is an increasing function of γ in both the x and y co-ordinates. Consequently,

$$0 \leq f'(x_{i+1}) - d_{i+1}^{+} \leq f'(x_{i+1}) - \frac{3(1-\gamma)^2}{1-3\gamma+3\gamma^2}\Delta_i \leq \frac{7}{24}\|f^{(4)}\|_\infty h^3.$$

Case 2.2: If (5.5) is not valid, then consider two subcases depending upon whether $\gamma > \frac{1}{3}$.

Case 2.2.1: If $\gamma > \frac{1}{3}$, then

$$\frac{3\gamma^2}{1-3\gamma+3\gamma^2} > 1.$$

This bound together with the observation that $d_i \leq \frac{1}{2}\Delta_i$ (since $(d_i, d_{i+1}) \in \underline{A}_i$) shows that

$$\frac{1}{2}\Delta_i \leq \frac{3\gamma^2}{1-3\gamma+3\gamma^2}\Delta_i - d_i \leq \frac{3\gamma^2}{1-3\gamma+3\gamma^2}\Delta_i - f'(x_i) \leq \frac{7}{24}\|f^{(4)}\|_\infty \gamma^2 h^3.$$

In addition,

$$3\Delta_i \leq d_{i+1}^{+} \leq f'(x_{i+1}) \leq 4\Delta_i + \frac{7}{24}\|f^{(4)}\|_\infty (1-\gamma)^2 h^3,$$

whence,

$$0 \leq f'(x_{i+1}) - d_{i+1}^{+} \leq \frac{7}{12}\|f^{(4)}\|_{\infty}h^3.$$

Case 2.2.2: Alternatively, if $0 \leq \gamma \leq \frac{1}{3}$, then there exists a unique $\xi \in [0,\gamma]$ such that

$$d_i = \frac{3\xi^2}{1-3\xi+3\xi^2}\Delta_i, \tag{5.7}$$

since, by assumption,

$$0 \leq d_i \leq \frac{3\gamma^2}{1-3\gamma+3\gamma^2}\Delta_i$$

and the right side of this inequality is a strictly increasing function of γ for $0 \leq \gamma \leq \frac{1}{3}$. Moreover, since $(d_i, d_{i+1}^{+}) \in \underline{M}_i \cap \underline{A}_i$,

$$d_{i+1}^{+} = \frac{3(1-\xi)^2}{1-3\xi+3\xi^2}\Delta_i$$

by (5.3). Therefore,

$$\begin{aligned} 0 &\leq f'(x_{i+1}) - d_{i+1}^{+} \\ &= f'(x_{i+1}) - \frac{3(1-\gamma)^2}{1-3\gamma+3\gamma^2}\Delta_i + \frac{3(1-\gamma)^2}{1-3\gamma+3\gamma^2}\Delta_i - \frac{3(1-\xi)^2}{1-3\xi+3\xi^2}\Delta_i \\ &\leq \frac{7}{24}\|f^{(4)}\|_{\infty}h^3 + 9(\gamma-\xi)\Delta_i, \end{aligned} \tag{5.8}$$

since, for $0 \leq \xi \leq \gamma \leq \frac{1}{3}$,

$$0 \leq \frac{3(1-\gamma)^2}{1-3\gamma+3\gamma^2} - \frac{3(1-\xi)^2}{1-3\xi+3\xi^2} \leq 9(\gamma-\xi).$$

To bound $9(\gamma-\xi)\Delta_i$, note that, for $0 \leq \xi \leq \gamma \leq \frac{1}{3}$ and $f'(x_i) - d_i \leq 0$,

$$0 \le 3\gamma(\gamma - \xi)\Delta_i \le 3(\gamma^2 - \xi^2)\Delta_i$$

$$\le \frac{3\gamma^2}{1-3\gamma+3\gamma^2}\Delta_i - \frac{3\xi^2}{1-3\xi+3\xi^2}\Delta_i$$

$$= \frac{3\gamma^2}{1-3\gamma+3\gamma^2}\Delta_i - f'(x_i) + f'(x_i) - d_i$$

$$\le \frac{7}{24}\|f^{(4)}\|_\infty \gamma^2 h^3,$$

whence

$$0 \le 9(\gamma - \xi)\Delta_i \le \frac{7}{24}\|f^{(4)}\|_\infty h^3.$$

Combining this with (5.8), we get that

$$0 \le f'(x_{i+1}) - d_{i+1}^{+} \le \frac{7}{12}\|f^{(4)}\|_\infty h^3.$$

Q.E.D.

Proof of Theorem 4.1: As stated in Section 4, we combine a compactness argument with induction to proof this result.

For each point $t \in [a,b]$, we choose a $\delta_t > 0$ that determines an open interval $I_t = (t-\delta_t, t+\delta_t)$. Since $\{I_t\}$ forms an open covering of the compact interval $[a,b]$, there exists a finite subcovering of $[a,b]$. Moreover, for $h = \max\{h_i\}$ sufficiently small, each interval $[x_{i-1}, x_{i+1}] \subset I_t$, one of the intervals of the subcovering. The proof relies heavily upon exploiting the local properties of f on each interval of the finite subcovering.

In choosing δ_t, we consider four cases.

1. If $f'(t) \neq 0$, then choose $\delta_t > 0$ such that

$$0 < f'(x) < 3f'(y)$$

for all $x, y \in I_t \cap [a,b]$.

2. If $f'(t) = 0$ but $f''(t) \neq 0$, then choose $\delta_t > 0$ such that

$$0 < f''(x) < 1.5f''(y)$$

for all $x, y \in I_t \cap [a,b]$.

3. If $f'(t) = f''(t) = 0$ but $f^{(3)}(t) \neq 0$, then choose $\delta_t > 0$ such that

$$0 < f^{(3)}(x) < 1.1f^{(3)}(y)$$

for all $x, y \in I_t \cap [a,b]$.

4. If $f'(t) = f''(t) = f^{(3)}(t) = 0$, then choose δ_t such that, for all $y \in [t,t+\delta_t) \cap [a,b]$, either

   a. $f'(y) = 0$ or

   b. for some constants m_1, m_2, and r,

$$m_1(y-t)^r \leq f'(y) \leq m_2(y-t)^r,$$

   where $\frac{10}{11} \leq \frac{m_2}{m_1} \leq \frac{11}{10}$ and $r \geq 3$,

   and, for all $y \in (t-\delta_t,t] \cap [a,b]$, either

   a. $f'(y) = 0$ or

   b. for some constants m_3, m_4, and s,

$$m_3(t-y)^s \leq f'(y) \leq m_4(t-y)^s,$$

where $\frac{10}{11} \leq \frac{m_4}{m_3} \leq \frac{11}{10}$ and $s \geq 3$.

It is possible to choose δ_t to satisfy Cases 1-3 because the first three derivatives of f are continuous. If $f^{(4)}(t) \neq 0$, then Case 4 follows from the continuity of $f^{(4)}$. Otherwise, it follows directly from Assumption 2 of Theorem 4.1.

To prove that the induction hypothesis (stated in the abbreviated proof of Theorem 4.1 in Section 4) remains valid when an approximate derivative value d_i is modified in Step 3 of the Extended Two-Sweep Algorithm, we consider a number of cases depending upon the properties of f at t, where $[x_{i-1},x_{i+1}] \subset I_t$ is the interval under consideration. We prove the last case in the induction hypothesis first.

Case 1: Assume that $[x_{i-1},x_{i+1}] \subset I_t$ and $f'(t) \neq 0$.

Case 1.1: Assume that $(d_{i-1},d_i) \in \underline{A}_{i-1} \cup \underline{B}_{i-1} \cup \underline{C}_{i-1}$ and d_i is decreased to d_i^+ on the Forward Sweep. Hence, $d_i \geq d_i^+ \geq 3\Delta_{i-1}$. Since $\Delta_{i-1} = f'(y)$ for some $y \in [x_{i-1},x_i]$, it follows from the choice of I_t that $f'(x_i) \leq 3\Delta_{i-1}$. Therefore,

$$0 \leq d_i^+ - f'(x_i) \leq d_i - f'(x_i) \leq ch^3.$$

Case 1.2: Assume that $(d_i,d_{i+1}) \in \underline{A}_i$ and d_i is increased to d_i^+ on the Forward Sweep. If $d_i^+ \leq f'(x_i)$, then

$$0 \leq f'(x_i) - d_i^+ \leq f'(x_i) - d_i \leq ch^3.$$

On the other hand, if $d_i^+ \geq f'(x_i)$, then

$$0 \leq d_i^+ - f'(x_i) \leq d_i^+.$$

To bound d_i^+, note that $f'(x_{i+1}) \leq 3\Delta_i$ by the choice of I_t. Therefore,

$$3\Delta_i \leq d_{i+1} \leq 3\Delta_i + ch^3.$$

This inequality together with the observation that the curve $x = (y-3)^2$ is contained in $\underline{M}$ for $3 \leq y \leq 4$, shows that $d_i^+ \leq (ch^3)^2$, since $(d_i^+, d_{i+1}) \in \underline{A}_i$. Hence, for h sufficiently small,

$$|f'(x_i) - d_i^+| \leq ch^3.$$

Case 1.3: Since $f'(x_{i-1}) \leq 3\Delta_{i-1}$ and $f'(x_i) \leq 3\Delta_i$, a similar argument shows that

$$|f'(x_i) - d_i^*| \leq ch^3$$

after the Backward Sweep.

Case 2: Assume that $[x_{i-1}, x_{i+1}] \subset I_t$ and $f'(t) = 0$, but $f''(t) \neq 0$. (In this case, t must be one of the endpoints of the interval [a,b], since otherwise f would not be monotone.) As in Case 1, the choice of I_t ensures that

$$f'(x_{i-1}) \leq 3\Delta_{i-1}, \qquad f'(x_i) \leq 3\Delta_{i-1},$$
$$f'(x_i) \leq 3\Delta_i, \qquad f'(x_{i+1}) \leq 3\Delta_i.$$

Therefore, a similar argument shows that

$$|f'(x_i) - d_i^*| \leq ch^3$$

at the termination of Step 3 in this case as well.

Case 3: Assume that $[x_{i-1},x_{i+1}] \subset I_t$, $f'(t) = f''(t) = 0$, $f^{(3)}(t) \neq 0$ and $x_{i+1} \leq t$.

Case 3.1: Assume that $(d_{i-1},d_i) \in \underline{A}_{i-1} \cup \underline{B}_{i-1} \cup \underline{C}_{i-1}$ and d_i is decreased to d_i^+. From the choice of I_t, it follows that $f''(x) < 0$ for $x \in I_t$ and $x < t$. Therefore, $f'(x_i) \leq \Delta_{i-1}$. Hence, as in Case 1.1,

$$0 \leq d_i^+ - f'(x_i) \leq d_i - f'(x_i) \leq ch^3.$$

Case 3.2: Assume that $(d_i,d_{i+1}) \in \underline{A}_i$ and d_i is increased to d_i^+. Again, since $f''(x) < 0$ for $x \in I_t$ and $x < t$, it follows that $f'(x_{i+1}) \leq \Delta_i$. Consequently, the argument used in Case 1.2 shows that, for h sufficiently small,

$$|f'(x_i) - d_i^+| \leq ch^3$$

in this case as well.

Case 3.3: Assume that $(d_i,d_{i+1}) \in \underline{D}_i \cup \underline{E}_i$ and d_i is decreased to d_i^+. If $f'(x_i) \leq d_i^+$, then

$$0 \leq d_i^+ - f'(x_i) \leq d_i - f'(x_i) \leq ch^3.$$

Therefore, assume that $f'(x_i) > d_i^+$, and let $\gamma = (t-x_{i+1})/h_i$. Since $\Delta_i = f'(z)$ for some $z \in [x_i,x_{i+1}]$ and $f'(x) \leq 1.1f'(y)$ for all x, $y \in I_t$, it follows from the Taylor series expansions of $f'(x_i)$ and $f'(z)$ about t that

$$f'(x_i)/\Delta_i = f^{(3)}(x)(x_i-t)^2/f^{(3)}(y)(z-t)^2 \leq 1.1(\gamma+1)^2/\gamma^2.$$

Consequently, if $\gamma \geq 1/(\omega-1)$, where $\omega^2 = \frac{30}{11}$, then $f'(x_i) \leq 3\Delta_i \leq d_i^+$, a contradiction. Therefore, $0 \leq \gamma \leq 1/(\omega-1)$. A simple calculation, similar to the one used in the proof of Lemma 5.1, shows that

$$3\Delta_i \leq d_i^+ \leq f'(x_i) \leq 3\Delta_i + 6\|f^{(4)}\|_\infty h^3,$$

whence

$$0 \leq f'(x_i) - d_i^+ \leq 6\|f^{(4)}\|_\infty h^3$$

in accordance with the induction hypothesis.

Case 3.4: Assume that $(d_{i-1},d_i) \in \underline{E}_{i-1}$ and d_i is increased to d_i^+. If $d_i^+ \leq f'(x_i)$, then

$$0 \leq f'(x_i) - d_i^+ \leq f'(x_i) - d_i \leq \max\{c, 6\|f^{(4)}\|_\infty\}h^3.$$

On the other hand, if $d_i^+ \geq f'(x_i)$, then an argument similar to the one used in Case 3.3 together with the induction hypothesis shows that

$$d_{i-1} \leq 3\Delta_{i-1} + 6\|f^{(4)}\|_\infty h^3 + \tilde{c}h^3,$$

where $\tilde{c}$ is given in (4.2). Therefore, since $y = (x-3)^2$ is contained in $\underline{M}$ for $3 \leq x \leq 4$ and $(d_{i-1},d_i^+) \in \underline{E}_{i-1}$, it follows that

$$0 \leq d_i^+ - f'(x_i) \leq d_i^+ \leq (6\|f^{(4)}\|_\infty + \tilde{c})^2 h^6,$$

which, for h sufficiently small, satisfies the induction hypothesis.

Case 4: Assume that $[x_{i-1},x_{i+1}] \subset I_t$, $f'(t) = f''(t) = 0$, $f^{(3)}(t) \neq 0$ and $t \leq x_{i-1}$. An argument similar to the one used in Case 3 shows that the induction hypothesis holds in this case as well.

Case 5: Assume that $[x_{i-1},x_{i+1}] \subset I_t$, $f'(t) = f''(t) = f^{(3)}(t) = 0$ and $x_{i+1} \leq t$. In this case, either $f'(y) = 0$ for all $y \leq t$ in I_t or $f'(y)$ satisfies the bound in Condition 4b on I_t. If $f'(y) = 0$, then both Δ_{i-1} and Δ_i are zero. Hence, if $d_i \neq 0$, then $(d_{i-1},d_i) \in \underline{C}_{i-1}$ and d_i is set to zero on the Forward Sweep of the Extended Two-Sweep Algorithm. Furthermore, since d_i is not modified again, $d_i = f'(x_i) = 0$ at the termination of the Step 3. Therefore, assume that $f'(y)$ satisfies the bound in Condition 4b on I_t throughout the remainder of this case.

Case 5.1: Assume that $(d_{i-1},d_i) \in \underline{A}_{i-1} \cup \underline{B}_{i-1} \cup \underline{C}_{i-1}$ and d_i is decreased to d_i^+. Then, since $\Delta_{i-1} = f'(y)$ for some $y \in [x_{i-1},x_i]$, it follows from Condition 4b on I_t that

$$f'(x_i)/\Delta_{i-1} \leq m_4(t-x_i)^s/m_3(t-y)^s \leq 1.1.$$

Therefore, since $d_i \geq d_i^+ \geq 3\Delta_{i-1}$,

$$0 \leq d_i^+ - f'(x_i) \leq d_i - f'(x_i) \leq ch^3.$$

Case 5.2: Assume that $(d_i,d_{i+1}) \in \underline{A}_i$ and d_i is increased to d_i^+. An argument similar to the one above shows that $f'(x_{i+1}) \leq 1.1\Delta_i$. Consequently, as in Case 1.2,

$$|f'(x_i) - d_i^+| \leq ch^3$$

for h sufficiently small.

Case 5.3: Assume that $(d_i,d_{i+1}) \in \underline{D}_i \cup \underline{E}_i$ and d_i is decreased to d_i^+.

Hence, if $f'(x_i) \leq d_i^+$, then

$$0 \leq d_i^+ - f'(x_i) \leq d_i - f'(x_i) \leq ch^3.$$

Therefore, assume that $f'(x_i) > d_i^+$, whence

$$0 \leq f'(x_i) - d_i^+ \leq f'(x_i).$$

To bound $f'(x_i)$, let $\gamma = (t-x_{i+1})/h_i$. Then, since $\Delta_i = f'(z)$ for some $z \in [x_i, x_{i+1}]$,

$$f'(x_i)/\Delta_i \leq m_4(x_i-t)^s/m_3(z-t)^s \leq 1.1(\gamma+1)^s/\gamma^s$$

by Condition 4b. Consequently, if $\gamma \geq 1/(\omega-1)$, where $\omega^s = \frac{30}{11}$, then $f'(x_i) \leq 3\Delta_i \leq d_i^+$, a contradiction. Therefore, $0 \leq \gamma \leq 1/(\omega-1)$. Hence, if $s > 3$, then $f'(x_i) = o(h^3)$, and the induction hypothesis holds for h sufficiently small. On the other hand, if $s = 3$, then expanding $f'(x_i)$ as a Taylor series about t shows that

$$f'(x_i) = \frac{1}{6}f^{(4)}(x)(x_i-t)^3 \leq 8\|f^{(4)}\|_\infty h^3,$$

as required.

Case 5.4: Assume that $(d_{i-1}, d_i) \in \underline{E}_{i-1}$ and d_i is increased to d_i^+. Then, if $d_i^+ \leq f'(x_i)$,

$$0 \leq f'(x_i) - d_i^+ \leq f'(x_i) - d_i \leq \max\{c, 8\|f^{(4)}\|_\infty\}h^3.$$

On the other hand, if $d_i^+ \geq f'(x_i)$, let $\gamma = (t-x_i)/h_{i-1}$. Then, an argument similar to the one above together with the induction hypothesis shows that

$$d_{i-1} \leq 3\Delta_{i-1} + \tilde{c}h^3$$

for $\gamma \geq 1/(\omega-1)$, where $\omega^s = \frac{30}{11}$. Hence, we again have that

$$0 \leq d_i^+ - f'(x_i) \leq d_i^+ \leq (\tilde{c}h^3)^2.$$

Conversely, if $0 \leq \gamma \leq 1/(\omega-1)$, then, for $s > 3$,

$$\Delta_{i-1} = f'(z) \leq m_4(t-z)^s \leq m_4(\gamma+1)^s h^s = o(h^3),$$

while, for $s = 3$,

$$\Delta_{i-1} = f'(z) = \frac{1}{6}f^{(4)}(y)(z-t)^3 \leq 8\|f^{(4)}\|_\infty h^3.$$

In either case,

$$0 \leq d_i^+ - f'(x_i) \leq d_i^+ \leq \frac{1}{2}\Delta_{i-1} \leq 4\|f^{(4)}\|_\infty h^3,$$

for h sufficiently small.

Case 6: Assume that $[x_{i-1},x_{i+1}] \subset I_t$, $f'(t) = f''(t) = f^{(3)}(t) = 0$ and $t \leq x_{i-1}$. A similar argument to the one used in Case 5 shows that the induction hypothesis holds in this case as well.

Case 7: Assume that $[x_{i-1},x_{i+1}] \subset I_t$, $f'(t) = f''(t) = f^{(3)}(t) = 0$ and $t \in (x_{i-1},x_{i+1})$. The proof of the induction hypothesis follows easily from the observation that $f'(x_i)$, Δ_{i-1} and Δ_i are each bounded by $\|f^{(4)}\|_\infty h^3$.

This completes the proof of the third case of the induction hypothesis. We now consider the first two cases.

Case 8: Assume that $[x_{i-1},x_{i+1}] \subset I_t$, $f'(t) = f''(t) = 0$,

$f^{(3)}(t) \neq 0$ and $t \varepsilon (x_{i-1},x_i]$. First note that, for h sufficiently small, $[x_{i-2},x_i] \subset I_t$. Hence, an argument similar to the one presented in Case 3.1 shows that $f'(x_{i-1}) \leq \Delta_{i-2}$, from which it follows that, if $(d_{i-2},d_{i-1}) \varepsilon \underline{A}_{i-2} \cup \underline{B}_{i-2} \cup \underline{C}_{i-2}$, then

$$0 \leq d^+_{i-1} - f'(x_{i-1}) \leq d_{i-1} - f'(x_{i-1}) \leq ch^3.$$

Consequently, d_{i-1} satisfies (3.5) at the start of the Forward Sweep for d_i.

Case 8.1.1: Assume that $(d_{i-1},d_i) \varepsilon \underline{A}_{i-1}$. Note that d_i is decreased to d^+_i only if d_{i-1} has been increased to d^+_{i-1} and either

1. (d_{i-2},d^+_{i-1}) is on the boundary of $\underline{M}_{i-2} \cup \underline{D}_{i-2} \cup \underline{E}_{i-2}$, or

2. (d^+_{i-1},d_i) is on the boundary between $\underline{A}_{i-1}$ and $\underline{B}_{i-1}$.

In the first case, $d^+_{i-1} \geq \Delta_{i-2}$. But, as previously mentioned, $f'(x_{i-1}) \leq \Delta_{i-2}$, whence $f'(x_{i-1}) \leq d^+_{i-1}$. Therefore, by Lemma 5.2,

$$|f'(x_i) - d^+_i| \leq \max\{c, \frac{7}{12}\|f^{(4)}\|_\infty\}h^3.$$

On the other hand, if (d^+_{i-1},d_i) is on the boundary between $\underline{A}_{i-1}$ and $\underline{B}_{i-1}$, then the following case applies after noting that (d^+_{i-1},d_i) is closer to the boundary of $\underline{M}_i$ than (d_{i-1},d_i) was.

Case 8.1.2: Assume that $(d_{i-1},d_i) \varepsilon \underline{B}_{i-1} \cup \underline{C}_{i-1}$ and d_i is decreased to d^+_i. A simple calculation shows that the vertical distance from (d_{i-1},d_i) to the boundary of $\underline{M}_{i-1} \cup \underline{D}_{i-1}$ is less than or equal to 2.75 times the minimum distance from (d_{i-1},d_i) to the boundary of $\underline{M}_{i-1}$. From inequalities (5.1), (5.2) and the error bounds on d_{i-1} and d_i, it follows that the distance from (d_{i-1},d_i) to the boundary of $\underline{M}_{i-1}$ is less than or equal to

$$(2c + \frac{7}{24}\|f^{(4)}\|_\infty)h^3.$$

Consequently,

$$|f'(x_i) - d_i^+| \leq (6.5c + \frac{5}{6}\|f^{(4)}\|_\infty)h^3.$$

Case 8.2: Assume that $(d_i,d_{i+1}) \in \underline{A}_i$ and that d_i is increased to d_i^+. If $d_i^+ \leq f'(x_i)$, then we again have that

$$0 \leq f'(x_i) - d_i^+ \leq f'(x_i) - d_i \leq (6.5c + \frac{5}{6}\|f^{(4)}\|_\infty)h^3.$$

On the other hand, if $d_i^+ \geq f'(x_i)$, then

$$0 \leq d_i^+ - f'(x_i) \leq d_i^+.$$

Because $t \in (x_{i-1},x_i]$, an argument similar to the one presented in Case 3.2 shows that

$$d_{i+1} \leq f'(x_{i+1}) + ch^3 \leq 3\Delta_i + 6\|f^{(4)}\|_\infty h^3 + ch^3$$

and

$$d_i^+ \leq (6\|f^{(4)}\|_\infty + c)^2h^6,$$

which completes the analysis of this case.

Case 8.3: Assume that $(d_i,d_{i+1}) \in \underline{D}_i \cup \underline{E}_i$ and that d_i is decreased to d_i^+. Therefore, $d_i \geq d_i^+ \geq 3\Delta_i$. However, since $t \in (x_{i-1},x_i]$, $f'(x_i) \leq \Delta_i$. Hence,

$$0 \leq d_i^+ - f'(x_i) \leq d_i - f'(x_i) \leq (6.5c + \tfrac{5}{6}\|f^{(4)}\|_\infty)h^3.$$

Case 8.4: Assume that $(d_{i-1},d_i) \in \underline{E}_{i-1}$ and that d_i is increased to d_i^+. If $d_i^+ \leq f'(x_i)$, then

$$0 \leq f'(x_i) - d_i^+ \leq f'(x_i) - d_i \leq (6.5c + \tfrac{5}{6}\|f^{(4)}\|_\infty)h^3.$$

Therefore, assume that $d_i^+ \geq f'(x_i)$. In addition, note that, if $(d_{i-1},d_i) \in \underline{E}_{i-1}$, then we could not have had $(d_{i-1},d_i) \in \underline{A}_{i-1}$ on the Forward Sweep. Therefore, the bound

$$|d_{i-1} - f'(x_{i-1})| \leq ch^3$$

established at the beginning of Case 8 still holds. Moreover, since the slope of the curve that forms the boundary between $\underline{M}$ and $\underline{E}$ is less than or equal to one,

$$0 \leq d_i^+ - f'(x_i) \leq (c + \tfrac{7}{24}\|f^{(4)}\|_\infty)h^3.$$

Case 9: Assume that $[x_{i-1},x_{i+1}] \subset I_t$, $f'(t) = f''(t) = 0$, $f^{(3)}(t) \neq 0$ and $t \in [x_i,x_{i+1})$.

Case 9.1: Assume that $(d_{i-1},d_i) \in \underline{A}_{i-1} \cup \underline{B}_{i-1} \cup \underline{C}_{i-1}$ and d_i is decreased to d_i^+. Therefore, $d_i \geq d_i^+ \geq 3\Delta_{i-1}$. However, $f'(x_i) \leq \Delta_{i-1}$ by the choice of I_t. Hence,

$$0 \leq d_i^+ - f'(x_i) \leq d_i - f'(x_i) \leq ch^3.$$

Case 9.2: Assume that $(d_i,d_{i+1}) \in \underline{A}_i$ and d_i is increased to d_i^+. If $d_i^+ \leq f'(x_i)$, then

$$0 \leq f'(x_i) - d_i^+ \leq f'(x_i) - d_i \leq ch^3.$$

Therefore, assume that $d_i^+ \geq f'(x_i)$. Note that the inverse of the slope of the curve that forms the boundary between $\underline{M}$ and $\underline{A}$ is less than or equal to one. Therefore, as in Case 8.4,

$$0 \leq d_i^+ - f'(x_i) \leq (c + \frac{7}{24}\|f^{(4)}\|_\infty)h^3.$$

Case 9.3: Assume that $(d_i,d_{i+1}) \in \underline{D}_i \cup \underline{E}_i$. Since $d_i \geq 3\Delta_i$, d_i could not have been modified in Case 9.2. Hence, d_i must still satisfy (3.5). Consider the following two subcases.

Case 9.3.1: Assume that $(d_i,d_{i+1}) \in \underline{E}_i$. Note that d_i is decreased to d_i^+ only if d_{i+1} was increased to d_{i+1}^+ and either

1. (d_{i+1}^+,d_{i+2}) is on the boundary of $\underline{M}_{i+1}$, or

2. (d_i,d_{i+1}^+) is on the boundary between $\underline{D}_i$ and $\underline{E}_i$.

In the first case, $d_{i+1}^+ \geq \Delta_{i+1}$. In addition, $f'(x_{i+1}) \leq \Delta_{i+1}$ by the choice of I_t. Therefore, $f'(x_{i+1}) \leq d_{i+1}^+$ and

$$|f'(x_i) - d_i^+| \leq \max\{c, \frac{7}{12}\|f^{(4)}\|_\infty\}h^3$$

by Lemma 5.2. On the other hand, if (d_i,d_{i+1}^+) is on the boundary between $\underline{D}_i$ and $\underline{E}_i$, then the following case applies after noting that (d_i,d_{i+1}^+) is closer to the boundary of $\underline{M}_i$ than (d_i,d_{i+1}) was.

Case 9.3.2: Assume that $(d_i,d_{i+1}) \in \underline{D}_i$ and that d_i is decreased to d_i^+. As in Case 8.1.2, note that the horizontal distance from (d_i,d_{i+1}) to the boundary of $\underline{M}_i$ is less than or equal to 2.75 times the minimum distance from (d_i,d_{i+1}) to the boundary of $\underline{M}_i$. Moreover, inequalities (5.1), (5.2) and the induction hypothesis on the error in d_{i+1} imply that the distance from (d_i, d_{i+1}) to the boundary of $\underline{M}_i$ is less than or equal to

$$(7.5c + \frac{9}{8}\|f^{(4)}\|_\infty)h^3.$$

Consequently,

$$|f'(x_i) - d_i^+| \leq (\frac{99}{32}\|f^{(4)}\|_\infty + \frac{173}{8}c)h^3.$$

Case 9.4: Assume that $(d_{i-1}, d_i) \varepsilon \underline{E}_{i-1}$ and d_i is increased to d_i^+. If $d_i^+ \leq f'(x_i)$, then we again have that

$$0 \leq f'(x_i) - d_i^+ \leq f'(x_i) - d_i \leq (\frac{99}{32}\|f^{(4)}\|_\infty + \frac{173}{8}c)h^3.$$

On the other hand, if $d_i^+ \geq f'(x_i)$, then

$$0 \leq d_i^+ - f'(x_i) \leq d_i^+ \leq (6\|f^{(4)}\|_\infty + \max\{c,8\}\|f^{(4)}\|_\infty\})^2 h^6,$$

which follows from an argument similar to the one used in Case 3.4 after noting that

$$d_{i-1} \leq 3\Delta_{i-1} + (6\|f^{(4)}\|_\infty + \max\{c,8\}\|f^{(4)}\|_\infty\})h^3.$$

Q.E.D.