Abstract: Domain decomposition is a class of techniques that are designed to solve elliptic problems on irregular domains and on multiprocessor systems. Typically, a domain is decomposed into many smaller regular subdomains and the capacitance system governing the interface unknowns is solved by some version of the preconditioned conjugate gradient method. In this paper, we show that for a simple model problem — Poisson's equation on a rectangle decomposed into two smaller rectangles — the capacitance system can be inverted exactly by fast Fourier transform. ***No iteration is needed.*** An exact eigen-decomposition of the capacitance matrix also makes possible a comparison of various preconditioners that have been proposed in the literature. For example, we show that in the limit as the aspect ratio of the two rectangles tend to infinity, the preconditioner proposed by Golub-Mayers becomes exact, but the one proposed by Dryja does not. Both preconditioners, however, are poor when the aspect ratio is small.

Analysis of Preconditioners for Domain Decomposition

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1. Introduction

Domain decomposition is a class of techniques that is designed to solve elliptic problems on irregular domains and on multiprocessor systems. Typically, a domain is decomposed into many smaller regular subdomains and the capacitance system governing the interface unknowns is solved. This is a relatively old idea and can be traced to the Schwarz alternating procedure [9].

Such methods are attractive in many situations. In fact, the main reason for the resurgence of this old idea is its obvious advantage in implementation on multiprocessor systems. Even in a sequential computing environment, a natural partition of the computational domain often exists, such as in dividing a domain with irregular geometry into regular subregions for which fast solvers exist, or in dividing a problem with discontinous coefficients into subregions with constant coefficients. For this and other reasons, domain decomposition has received a lot of interest recently.

Since the capacitance system is expensive to evaluate and expensive to solve by direct methods, many of the methods proposed so far in the literature employ some form of preconditioned conjugate gradient (PCG) method for its solution. In each iteration, the product of the capacitance matrix and a given vector is required, which can be evaluated by solving problems on the subdomains. To minimize the number of subdomain solves, it is imperative to have a good preconditioner for the capacitance matrix. Dryja[5] is among the first to introduce such a preconditioner for two dimensional problems, which is in the form of a pseudo differential operator, namely, the square root of the one dimensional discrete Laplace operator. Later, Golub and Mayers[8] proposed a modification which significantly reduces the number of PCG iterations needed. Many other methods have been proposed along this approach, among which we mention [1, 2, 3, 6, 7, 8].

In this paper, we show that for a simple model problem — Poisson's equation on a rectangle decomposed into two smaller rectangles — the capacitance system can be inverted exactly by fast Fourier transform. *No PCG iteration is needed.* We derive an exact eigen-decomposition of the capacitance matrix which makes possible a comparison of various preconditioners that have been proposed in the literature. For example, we show that in the limit as the aspect ratio of the two rectangles tend to infinity, the preconditioner proposed by Golub-Mayers becomes exact, but the one proposed by Dryja does not. Both preconditioners, however, are poor when the aspect ratio is small.

In Section 2, we introduce the model problem and derive the capacitance system for the interface. The eigen-decomposition of the capacitance matrix is derived in Section 3. Comparisons of various preconditioners are discussed in Section 4 and we close in Section 5 with some remarks about extensions to irregular regions and divisions into more subdomains.

2. The Model Problem and the Interface System

Consider the following Poisson's equation :

$$\Delta u = f \qquad \text{on} \quad \Omega \tag{2.1}$$

with boundary condition

$$u = g \qquad \text{on} \quad \partial\Omega$$

and where the domain Ω is as illustrated in Fig. 1.

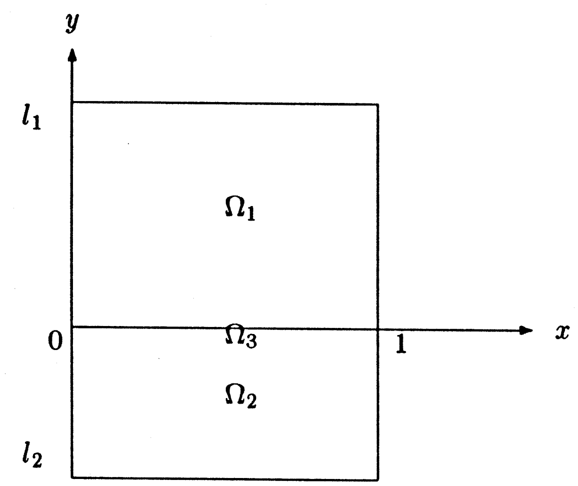

Fig. 1 The domain Ω and its partition

We partition Ω into two subdomains Ω_1 and Ω_2, with a common interface Ω_3. We use a uniform mesh with grid size h on Ω with n internal grid points in the x-direction, i.e.,

$$h = \frac{1}{(n+1)}.$$

We assume that l_1 and l_2 are integral multiples of h, with m_1 internal grid points in Ω_1 in the y-direction and m_2 internal grid points in Ω_2, i.e.,

$$l_1 = (m_1 + 1)h$$
$$l_2 = (m_2 + 1)h \quad .$$

Consider a standard 5-point centered difference approximation to (2.1). If we order the unknowns in Ω_1 first, then those in Ω_2 and finally those in Ω_3, then the discrete solution vector $u = (u_1, u_2, u_3)$ satisfies the following linear system

$$\begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{2.2}$$

where the matrices A_{11}, A_{22}, and A_{33} correspond to the discrete Laplacian on Ω_1, Ω_2 and Ω_3, and A_{13} and A_{23} correspond to the coupling between the unknowns in Ω_1 and Ω_2 with those in Ω_3.

Applying block Gaussian Elimination to (2.2), we obtain the following system for the interface unknowns u_3 :

$$Cu_3 = f_3 - A_{13}^T A_{11}^{-1} f_1 - A_{23}^T A_{22}^{-1} f_2 \tag{2.3}$$

where

$$C \equiv A_{33} - A_{13}^T A_{11}^{-1} A_{13} - A_{23}^T A_{22}^{-1} A_{23}. \tag{2.4}$$

The right hand side of (2.3) can be evaluated by solving two subdomain problems, one each on Ω_1 and Ω_2.

Note that C is expensive to evaluate because it requires $2n$ subdomain solves. Moreover, it is generally dense and therefore a direct method for solving (2.3) could be prohibitively expensive. The basic idea of domain decomposition is to solve the system (2.3) by preconditioned conjugate gradient (PCG) methods. In applying the PCG method, one needs to evaluate the matrix–vector product Cw for a given vector w. From (2.4), it is easily seen that each evaluation of Cw requires the solution of two subdomain problems. For example, the product $-A_{13}^T A_{11}^{-1} A_{13} w$ can be computed by solving the discretized version of (2.1) on Ω_1 with homogeneous right hand side (i.e., the Laplace equation) and the boundary condition $u = w$ on Ω_3, and then taking the solution on the first row of grid points above Ω_3.

3. The Eigen Decomposition of the Capacitance Matrix

In order to understand the performance of various preconditioners for C, it is necessary to first analyze the eigen–structure of C. It turns out that for the model problem, an exact eigen-decomposition of C can be derived by the use of Fourier analysis.

Define the vectors $w_j, j = 1, 2, \cdots, n$ by

$$w_j = \sqrt{2h}(\sin j\pi h, \sin 2j\pi h, \cdots, \sin nj\pi h)^T. \tag{3.1}$$

Consider the product

$$Cw_j \equiv A_{33}w_j - A_{13}^T A_{11}^{-1} A_{13} w_j - A_{23}^T A_{22}^{-1} A_{23} w_j \quad . \tag{3.2}$$

Let us consider the term $-A_{13}^T A_{11}^{-1} A_{13} w_j$ first. As mentioned earlier, this requires the solution of the discrete Laplace equation

$$\Delta_h v = 0 \qquad \text{on} \quad \Omega_1 \tag{3.3}$$

with boundary condition

$$\begin{aligned} v &= w_j & \text{on} \quad &\Omega_3 \\ \text{and} \quad v &= 0 & \text{on} \quad &\partial\Omega_1/\Omega_3 \quad . \end{aligned} \tag{3.4}$$

Consider a solution vector $v(x, y)$ of the form

$$v(ih, kh) = d_k(w_j)_i = d_k\sqrt{2h}\sin ij\pi h \quad , \tag{3.5}$$

where $0 \le i \le n+1$ and $0 \le k \le m_1 + 1$.

The boundary condition (3.4) implies that

$$d_0 = 1 \qquad \text{and} \quad d_{m_1+1} = 0. \tag{3.6}$$

Substituting (3.5) into (3.3), we get

$$(d_{k-1} - 2d_k + d_{k+1}) \sin ij\pi h + d_k(\sin(i-1)j\pi h - 2\sin ij\pi h + \sin(i+1)j\pi h) = 0.$$

It follows that the d_k's satisfy the following difference equation :

$$d_{k-1} - (2+\sigma_j)d_k + d_{k+1} = 0 \tag{3.7}$$

with the boundary condition (3.6) and where

$$\sigma_j \equiv 4\sin^2\frac{j\pi h}{2}.$$

The roots of the characteristic polynomial corresponding to (3.7) are

$$\begin{aligned} r_+ &= 1 + \frac{\sigma_j}{2} + \sqrt{\sigma_j + \frac{\sigma_j^2}{4}} \\ \text{and} \quad r_- &= 1 + \frac{\sigma_j}{2} - \sqrt{\sigma_j + \frac{\sigma_j^2}{4}}. \end{aligned} \tag{3.8}$$

The general solution to (3.7) is therefore given by

$$d_k = c_1 r_+^k + c_2 r_-^k \quad .$$

The constants c_1 and c_2 can be found by imposing the boundary conditions (3.6) which give

$$\begin{aligned} c_1 &= -\frac{r_-^{m_1+1}}{r_+^{m_1+1} - r_-^{m_1+1}} \\ \text{and} \quad c_2 &= \frac{r_-^{m_1+1}}{r_+^{m_1+1} - r_-^{m_1+1}} \quad . \end{aligned}$$

We therefore have

$$\begin{aligned} -A_{13}^T A_{11}^{-1} A_{13} w_j &\equiv d_1 w_j \\ &= \left(\frac{r_- - r_+\gamma_j^{m_1+1}}{1-\gamma_j^{m_1+1}}\right) w_j \quad , \end{aligned} \tag{3.9}$$

where

$$\gamma_j \equiv \frac{r_-}{r_+}. \tag{3.10}$$

By a similar computation, we have

$$-A_{23}^T A_{22}^{-1} A_{23} w_j = \left(\frac{r_- - r_+\gamma_j^{m_2+1}}{1-\gamma_j^{m_2+1}}\right) w_j. \tag{3.11}$$

Finally, it can easily be verified that

$$(A_{33}w_j)_i = \sin(i-1)j\pi h - 4\sin ij\pi h + \sin(i+1)j\pi h,$$

and therefore

$$A_{33}w_j = (-2-\sigma_j)w_j. \tag{3.12}$$

Combining (3.9), (3.11) and (3.12), we obtain

$$Cw_j = \lambda_j w_j$$

where

$$\lambda_j \equiv (-2 - \sigma_j + (\frac{r_- - r_+\gamma_j^{m_1+1}}{1-\gamma_j^{m_1+1}}) + (\frac{r_- - r_+\gamma_j^{m_2+1}}{1-\gamma_j^{m_2+1}})), \tag{3.13}$$

which after some simplification gives

$$\lambda_j = -(\frac{1+\gamma_j^{m_1+1}}{1-\gamma_j^{m_1+1}} + \frac{1+\gamma_j^{m_2+1}}{1-\gamma_j^{m_2+1}})\sqrt{\sigma_j + \frac{\sigma_j^2}{4}}. \tag{3.14}$$

Therefore (λ_j, w_j) is an eigenpair of C. Since the vectors $w_j, j = 1, 2, \cdots, n$ are orthonormal, we have the exact eigen–decomposition of C.

Theorem 3.1. *For the problem (2.1), the capacitance matrix C can be decomposed as*

$$C \equiv W \Lambda W^T$$

where

$$\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

and

$$W = (w_1, w_2, \cdots, w_n).$$

Furthermore, W is orthogonal.

Note that the products Cw and $C^{-1}w$ can be computed by the Fast Fourier Transform (FFT) (specifically, the sine transform). It follows that for the model problem, there is no need to use a preconditioned conjugate gradient method for the interface system: we can solve it directly by FFT.

4. Comparison of Preconditioners

Among the preconditioners proposed so far for C, two typical ones are :

$$M_D \equiv W diag(\lambda_1^D, \lambda_2^D, \cdots, \lambda_n^D)W^T \qquad \text{(Dryja [5])} \tag{4.1}$$

where

$$\lambda_j^D \equiv -2\sqrt{\sigma_j} \tag{4.2}$$

and

$$M_G \equiv W diag(\lambda_1^G, \lambda_2^G, \cdots, \lambda_n^G)W^T \qquad \text{(Golub–Mayers [8])} \tag{4.3}$$

where

$$\lambda_j^G \equiv -2\sqrt{\sigma_j + \frac{\sigma_j^2}{4}}. \tag{4.4}$$

Since we know the exact eigen–decomposition of C for the model problem, we can compare the performance of these two preconditioners. Specifically, we are interested in the *spectral* condition numbers $K(M_D^{-1}C)$ and $K(M_G^{-1}C)$, since these play a major role in the convergence rate of the PCG method.

It can easily be verified that

$$r_+ > 1, \qquad 0 < r_- < 1 \quad \text{and} \quad \gamma_j < 1. \tag{4.5}$$

Therefore, it follows immediately from (3.14) that

$$\lim_{\substack{m_1\to\infty\\ m_2\to\infty}} \lambda_j \equiv -2\sqrt{\sigma_j + \frac{\sigma_j^2}{4}}. \tag{4.6}$$

In other words, for fixed h, as the "aspect ratios" of Ω_1 and Ω_2 tend to infinity, $\lambda_j^G \to \lambda_j$. It follows that

$$\lim_{\substack{m_1\to\infty\\ m_2\to\infty}} K(M_G^{-1}C) = 1. \tag{4.7}$$

Next let us consider the case where l_1 and l_2 are fixed but $h \to 0$ (i.e., n, m_1 and $m_2 \to \infty$). Let the eigenvalues of $M_G^{-1}C$ be denoted by μ_j, where by (4.4) and (3.14)

$$\mu_j = \frac{1}{2}\left(\frac{1+\gamma_j^{m_1+1}}{1-\gamma_j^{m_1+1}} + \frac{1+\gamma_j^{m_2+1}}{1-\gamma_j^{m_2+1}}\right) \quad . \tag{4.8}$$

It can be verified that μ_j is a decreasing function of j, and therefore

$$K(M_G^{-1}C) \equiv \frac{\mu_1}{\mu_n} \quad . \tag{4.9}$$

Consider the term $\gamma_1^{m_1+1}$ in the limit as $h \to 0$ for l_1 and l_2 fixed. We have

$$\gamma_1^{m_1+1} = \left(1 - \frac{2\delta}{1+\delta}\right)^{l_1/h}$$

where

$$\delta \equiv \frac{\sqrt{\sigma_1 + \sigma_1^2/4}}{1+\sigma_1/2} \quad .$$

Making use of the fact that

$$\lim_{x\to 0}(1 + xf(x))^{1/x} = e^{\lim_{x\to 0} f(x)}, \tag{4.10}$$

we have

$$\lim_{h\to 0} \gamma_1^{m_1+1} = e^{-2\delta_0 l_1}$$

where

$$\begin{aligned}\delta_0 &= \lim_{h\to 0} \frac{\delta}{(1+\delta)h}\\ &= \lim_{h\to 0} \frac{2\sin\frac{\pi h}{2}}{h}\\ &= \pi \quad .\end{aligned}$$

Therefore, we have

$$\lim_{h\to 0} \mu_1 = \frac{1}{2}\left(\frac{1+e^{-2\pi l_1}}{1-e^{-2\pi l_1}} + \frac{1+e^{-2\pi l_2}}{1-e^{-2\pi l_2}}\right) \quad . \tag{4.11}$$

On the other hand, it is easy to verify that

$$\lim_{h \to 0} \sigma_n = 4$$

and

$$\lim_{h \to 0} \gamma_n = \frac{3 - 2\sqrt{2}}{3 + 2\sqrt{2}} < 1.$$

Therefore,

$$\lim_{h \to 0} \gamma_n^{m_1+1} = 0, \qquad \text{and} \quad \lim_{h \to 0} \gamma_n^{m_2+1} = 0$$

and so

$$\lim_{h \to 0} \mu_n = 1. \tag{4.12}$$

Combining (4.11) and (4.12) we have

Theorem 4.1. *For any l_1 and l_2,*

$$\lim_{h \to 0} K(M_G^{-1}C) = \frac{1}{2}\left(\frac{1 + e^{-2\pi l_1}}{1 - e^{-2\pi l_1}} + \frac{1 + e^{-2\pi l_2}}{1 - e^{-2\pi l_2}}\right).$$

Note that the limiting value of $K(M_G^{-1}C)$ is independent of h. Theorem 4.1 also shows that in the limit $h \to 0$, the rate with which $K(M_G^{-1}C) \to 1$ is exponential in l_1 and l_2. On the other hand, if l_1 and l_2 are small then the limiting value of $K(M_G^{-1}C)$ is given by

$$\lim_{\substack{h \to 0 \\ l_1, l_2 \to 0}} K(M_G^{-1}C) = \frac{1}{2\pi}\left(\frac{1}{l_1} + \frac{1}{l_2}\right),$$

and thus grows like $\frac{1}{l_1}$ and $\frac{1}{l_2}$.

Next, we consider $K(M_D^{-1}C)$. The eigenvalues μ_j of $M_D^{-1}C$ are given by

$$\mu_j = -\frac{1}{2}\left(\frac{1 + \gamma_j^{m_1+1}}{1 - \gamma_j^{m_1+1}} + \frac{1 + \gamma_j^{m_2+1}}{1 - \gamma_j^{m_2+1}}\right)\sqrt{1 + \frac{\sigma_j}{4}}. \tag{4.13}$$

Moreover, we have

$$1 \le \frac{\lambda_j^G}{\lambda_j^D} = \sqrt{1 + \frac{\sigma_j}{4}} \le \sqrt{2}.$$

The following result follows immediately.

Theorem 4.2. *For any h, l_1 and l_2*

$$\frac{1}{\sqrt{2}} \le \frac{K(M_D^{-1}C)}{K(M_G^{-1}C)} \le \sqrt{2} \quad ,$$

and

$$\lim_{\substack{l_1 \to \infty \\ l_2 \to \infty \\ h \to 0}} K(M_D^{-1}C) = \sqrt{2} \quad .$$

We thus see that the limiting value of $K(M_D^{-1}C)$ is also independent of h, but this value does not tend to unity even as l_1 and l_2 tend to infinity. This is a fundamental limitation of M_D.

Plots of $K(M_G^{-1}C)$ and $K(M_D^{-1}C)$ for various values of h and l in the case where $l_1 = l_2 = l$ are given in Figures 1 and 2. It is seen that when the aspect ratio is approximately one or larger, M_G is a pretty good preconditioner for C. But for small aspect ratios, both M_G and M_D deteriorate rapidly as preconditioners for C. It is also interesting to note that $K(M_G^{-1}C)$ tends to its asymptotic value even for very large values of h and that $K(M_D^{-1}C)$ has a smaller value for larger values of h.

Bjorstad and Widlund[1] proposed a preconditioner M_B (referred to as the "excellent method" in their paper) which corresponds to the capacitance matrix C assuming $l_1 = l_2$. By employing special symmetry in the case where $l_1 = l_2$, they derived a method for computing $CM_B^{-1}w$ exactly for a given vector w by solving Neumann problems on the subdomains. Obviously, this method is exact when $l_1 = l_2$ and in general it should be slightly better the Golub-Mayers preconditioner because it takes into account the aspect ratio of one of the subdomains.

5. Concluding Remarks

Our analysis provides some insight into the structure of the capacitance matrix system, which is central to the domain decomposition method. The exact eigen-decomposition of this matrix in the simple case considered in this paper allows us to compare the performance of various preconditioners. It also illustrates somewhat more clearly the origin of the pseudo differential operator used in Dryja's original preconditioner and the way in which the Golub-Mayers preconditioner is an improvement. Our analysis reveals for the first time the dependence of the performance of these preconditioners on the aspect ratios of the subdomains. Based on the results here, it is straightforward to derive a preconditioner for irregular domains which takes into account the aspect ratios of the subdomains. Furthermore, the knowledge of the eigen-decomposition of C makes it possible to construct a direct domain-decomposed fast Poisson solver on a rectangle divided into many strips and boxes. We referred the interested readers to [4].

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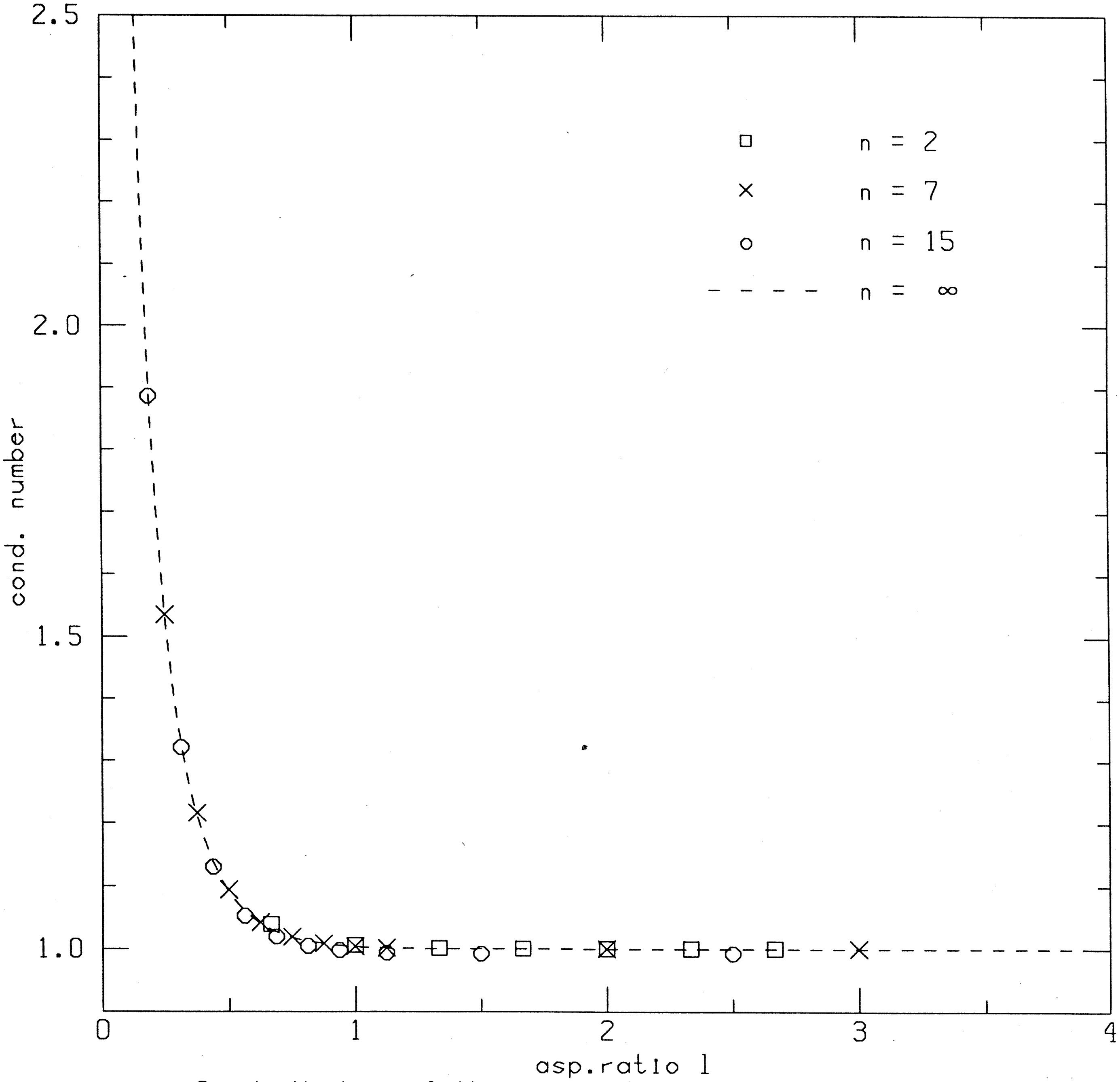
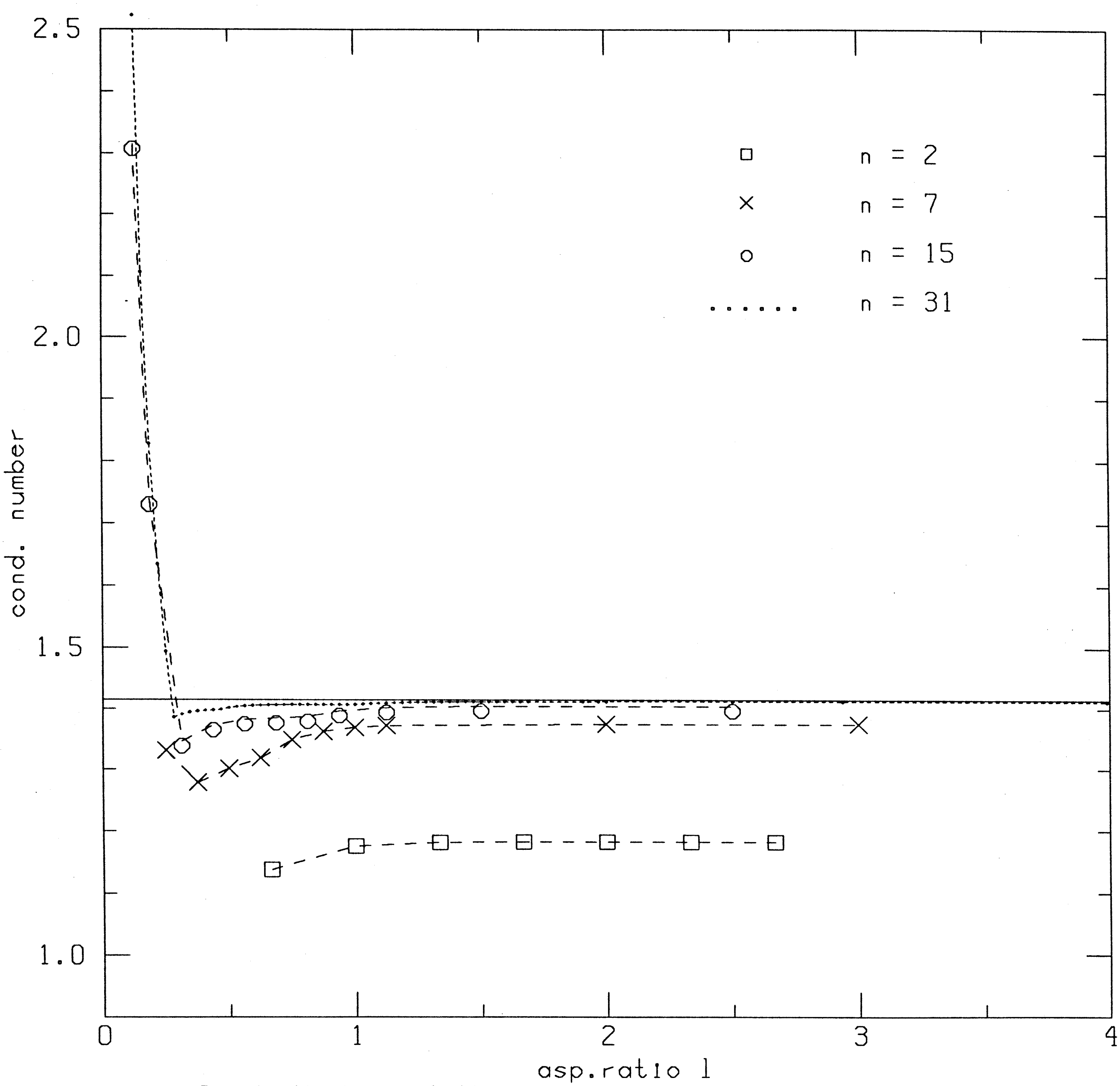
Figure 1: Dependence of $K(M_G^{-1}C)$ on h and aspect ratio l



Cond. Number of the preconditioned capacitance matrix
Golub and Mayers preconditoner

Figure 2: Dependence of $K(M_D^{-1}C)$ on h and aspect ratio l



Cond. Number of the preconditioned capacitance matrix
Dryja preconditoner

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