Fourier Methods with Extended Stability Intervals for the Korteweg-de Vries Equation.

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Abstract

A full Leap Frog Fourier method for integrating the Korteweg-de Vries (KdV) equation $u_t + uu_x + \epsilon u_{xxx} = 0$ has an $O(N^{-3})$ stability constraint on the time step, where N is the number of Fourier modes used[7]. In this paper, we propose two new Fourier methods which have better stability properties. One method treats the linear dispersive u_{xxx} term implicitly without solving linear systems by integrating in time in the Fourier space and the nonlinear uu_x term by Leap Frog. The second method uses basis functions which solve the linear part of the KdV equation and Leap Frog for time integration. We carry out a linearized stability analysis of the proposed schemes and prove that a version of the first scheme possesses a certain kind of unconditionally stability and that the second scheme has an $O(N^{-1})$ stability limit. In addition, we analyze a linearization of a nonlinear finite element scheme proposed by Winther that treats the u_{xxx} term by Crank-Nicholson. Numerical experiments on soliton solutions show that the linearized stability analysis gives accurate predictions for all the nonlinear schemes and that the Fourier methods are more accurate than the finite element method.

1. Introduction

Fourier methods have been shown to perform well on suitable time dependent problems. Among the problems which have been considered are: equations with variable coefficients [4, 5, 14, 15, 2], the Navier-Stokes equations [5,6,12,13], metereological equations [9] and the Korteweg-de Vries equation [5,6,7]. The theory of these methods is discussed among others by Orszag in [12,13,14] or Gottlieb & Orszag in [8].

We shall consider the solution of the periodic initial value problem for the Korteweg-de Vries equation on $I = [0, L]$ by three-level spectral and pseudospectral methods. The solution $u(x,t)$ is thus defined by

$$\begin{aligned} &u_t + uu_x + \epsilon u_{xxx} = 0 && (x,t) \in I \times [0,\infty), \\ &u(x,0) = u_0(x) && x \in I, \\ &u(x,t) = u(x+L,t) && (x,t) \in R \times [0,\infty). \end{aligned}$$

We show that one can take advantage of specific characteristics of the equation so as to improve the performance of these methods.

It has been observed in [5,6] that because of the linear dispersive term u_{xxx}, a Leap Frog Fourier scheme for solving the Korteweg-de Vries equation gives rise to a very restrictive stability condition (the timestep k must satisfy $k < O(N^{-3})$, where N is the number of points in space). However, we can obtain a significantly larger stability limit for the timestep k by treating the u_{xxx} term implicitly. Finite element methods in which a Crank Nicholson scheme is used for the u_{xxx} term and a Leap Frog scheme for the uu_x term, have been proposed by Winther in [19]. He analyzed these schemes for the solution of the full nonlinear Korteweg-de Vries equation and proved that they are unconditionally stable in a certain sense. Here we examine a Fourier scheme for solving the Korteweg-de Vries equation in which the Leap Frog discretization is applied to the uu_x term and u_{xxx} is approximated by

$$\theta D^3 V^- + (1-\theta) D^3 V^+ \qquad 0 \le \theta \le 1,$$

where D represents differentiation, V^+ denotes the solution V to the discretized problem at the next timestep and V^- at the preceding timestep.

Analyzing the proposed scheme for a linearization of the Korteweg-de Vries equation we prove that there exists a type of unconditional stability (to be defined in Section 4), in the case that $\theta = \frac{1}{2}$ (i.e. Crank Nicholson) but that for other values of θ the method is unstable except for special cases. For the Crank Nicholson scheme our analysis shows that the timestep k has to be bounded by a maximal timestep k_{max} *which is independent of N.*

We examine how the unconditional stability of the partial Crank Nicholson scheme is obtained only together with an unavoidable inaccuracy because of large numerical dispersion.

It is not necessary to treat the linear u_{xxx} term implicitly to extend the stability limit beyond the $O(N^{-3})$ limit. We propose an alternative scheme in which we use basis functions which solve the linear part of the Korteweg-de Vries exactly. As a result the u_{xxx} term no longer affects the stability properties of the method and we obtain a Leap Frog like limit for the uu_x term only, thus giving an $O(N^{-1})$ stability limit for k. The stability limit for this scheme is therefore not quite as attractive as the one for the scheme in which the linear term is treated implicitly. Although the alternative scheme is theoretically attractive and probably can be extended to more complicated equations, it does not perform as well as the one in which Crank Nicholson is used on the dispersive term in cases where the nonlinear uu_x term dominates.

In § **2** we introduce our notation and definitions. In § **3** we consider the implementation of implicit Fourier methods and propose a scheme for the solution of the Korteweg-de Vries equation. In § **4** we analyze the stability properties of the proposed scheme for a linearization of the Korteweg-de Vries equation. In § **5** we propose a scheme which uses modified basis functions for limiting numerical dispersion and thus improving accuracy. In § **6** we apply the linearized stability analysis to a finite element scheme which was proposed by Winther in [19] and obtain stability results which are comparable to those for the Fourier methods. In § **7** we discuss the results of a number of numerical experiments. These indicate that the linearized stability analysis gives relatively accurate predictions for stability intervals of the nonlinear problem and show the superiority of Fourier methods over finite difference methods for sufficiently well-behaved problems. In § **8** we summarize and comment on the accuracy of the theoretical analysis in predicting the numerical results.

2. Notation and Definitions.

Let the solution u be approximated by the finite Fourier series $V(x,t) = \Sigma_j \tilde{v}_j(t)\phi_j(x)$, where the ϕ_j are defined by $\phi_j(x) = e^{i\xi_j x}$ with $\xi_j = \frac{2\pi j}{L}$ and $j \in \{-N, -(N-1), \cdots, -1, 0, 1, \cdots, N\}$. Define the points $x_p = \frac{pL}{2N+1}$ in the domain I with $p \in \{0, 1, \cdots, 2N-1, 2N\}$. Let $\mathbf{v}$ be a vector in C^{2N+1}. (The ϕ_j are orthonormal with respect to the inner product $\langle f, g\rangle = \frac{1}{L}\int_0^L f(x)g^*(x)dx$ and eigenfunctions of differentiation with respect to x in physical space. Partial dervatives $\frac{\partial}{\partial x}$ of V can therefore be taken by multiplying each Fourier basis function $\phi_j(x)$ for frequency ξ_j by its eigenvalue $i\xi_j$.)

The discrete Fourier transform $\tilde{\mathbf{v}} = F(\mathbf{v})$ of the vector $\mathbf{v}$ is given by

$$\tilde{v}_j = \sum_{p=0}^{2N} e^{-i\xi_j x_p} v_p.$$

The inverse discrete Fourier transform $\mathbf{v} = F^{-1}(\tilde{\mathbf{v}})$ of the vector $\tilde{\mathbf{v}}$ is thus defined as

$$v_p = \frac{1}{2N+1} \sum_{j=-N}^{N} e^{i\xi_j x_p} \tilde{v}_j.$$

In the spectral or Galerkin approximation we obtain differential-difference equations for the approximate solution vector by substituting the approximating series $V(x,t) = \Sigma_j \tilde{v}_j(t)\phi_j(x)$ for u in the Korteweg-de Vries equation and imposing the Galerkin conditions

$$\frac{d}{dt}\langle V_t, \phi_j\rangle + \langle VV_x + \epsilon V_{xxx}, \phi_j\rangle = 0 \quad \forall j \in \{-N, \cdots, N\}.$$

The expressions $\langle VV_x, \phi_j\rangle$, which we shall write as $P_{G_j}(VV_x)$ (i.e. the j-th component of the Galerkin projection of VV_x), result in convolutions of Fourier coefficients.

In the pseudospectral or collocation approximation we obtain the semidiscrete equations for the approximate solution vector by substituting the approximating series $V(x,t) = \Sigma_j \tilde{v}_j(t)\phi_j(x)$ for u in the Korteweg-de Vries equation and requiring

$$V_t(x_p,t) + VV_x(x_p,t) + \epsilon V_{xxx}(x_p,t) = 0$$

to be satisfied at the $2N+1$ points x_p. The collocation projection P_{col} of the VV_x term is thus formed by interpolation with a linear combination of the ϕ_j's at the $2N+1$ points x_p. Calculation of the $2N+1$ components of $P_{col}(VV_x)$ requires convolutions again, although not spectral convolutions.

The deviations from convolutions which arise as a result of the collocation projection are explained in terms of "aliasing" by Orszag in [12,13,14].

Because we use three-level time stepping schemes we need two initial approximate solution vectors. These can be obtained from initial functions by collocation and a one-step method.

We can implement an efficient, time $O(N\log(N))$ calculation of the convolutions for the discrete VV_x term with the Fast Fourier Transform (FFT) [1,13] by the convolution theorem. Then both $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{v_x}}$ have to be Fourier transformed, however. If we rewrite VV_x as $(\frac{1}{2}V^2)_x$ on the other hand, only $\tilde{\mathbf{v}}$ need be transformed, saving one FFT per time-step. We thus only need one transformation from $\tilde{\mathbf{v}}$ to $F^{-1}(\tilde{\mathbf{v}}) = \mathbf{v}$ in $O(N\log(N))$ time, form $\mathbf{v}.\mathbf{v}$ (multiply corresponding elements of vectors) in $O(N)$ time by just doing local products, and transform back in $O(N\log(N))$ time. Finally we differentiate by multiplying the N spectral coefficients by the eigenvalues of the corresponding spectral basis functions.

Therefore we will henceforth use $\frac{\partial}{\partial x}P(\frac{1}{2}V^2)$ rather than $P(VV_x)$, where P is either P_G or P_{col}, although $\frac{\partial}{\partial x}P_{col}(\frac{1}{2}V^2) \neq P_{col}(VV_x)$. We choose the collocation which requires the smallest amount of work because there is no reason to assume that either collocation projection is a more accurate approximation than the other one.

3. Implicit Fourier methods

In most articles on Fourier methods only explicit time integration schemes have been considered. However, implicit time stepping can be realized straightforwardly and simply with these methods by time stepping in Fourier space rather than in physical space, because implementing an implicit scheme for a linear pde requires only simple divisions in Fourier space and does not require solution of linear systems. Implicit Fourier schemes can therefore be implemented efficiently by integrating in time in Fourier space rather than in physical space.

Consider the linear part of the Korteweg-de Vries equation

$$u_t + \epsilon u_{xxx} = 0 \quad \text{on} \qquad I = [0, L].$$

Suppose we decide to solve this equation using a spectral method and the Crank Nicholson time stepping scheme with a timestep k

$$\frac{\tilde{v}_\xi^+ - \tilde{v}_\xi^-}{2k} - i\epsilon\xi^3 \frac{\tilde{v}_\xi^+ + \tilde{v}_\xi^-}{2} = 0$$

obtaining

$$\tilde{v}_\xi^+ = \frac{1 + ik\epsilon\xi^3}{1 - ik\epsilon\xi^3} \tilde{v}_\xi^-,$$

where $\tilde{v}^+$ indicates a Fourier component of the approximate solution V at the next timestep and $\tilde{v}^-$ at the preceding timestep.

From this equation we immediately see that

$$|\tilde{v}_\xi^+| = \left| \frac{1 + ik\epsilon\xi^3}{1 - ik\epsilon\xi^3} \right| |\tilde{v}_\xi^-|,$$

so that $|\tilde{v}_\xi^+| = |\tilde{v}_\xi^-|$ for all frequencies ξ. We therefore see that with a spectral method Crank Nicholson is unconditionally stable and conservative, just like for a finite difference method.

We can use the above observations about implicit Fourier methods for implementing an implicit time stepping scheme for the Korteweg-de Vries equation. Because the nonlinear term generates convolutions of Fourier coefficients however, it is preferable to treat the nonlinear term explicitly so that we can use the convolution theorem for its evaluation. Thus we discretize the nonlinear uu_x term with the Leap Frog scheme and use an implicit scheme for the u_{xxx} term. The resulting Fourier-finite difference scheme is

$$\frac{\tilde{v}_j^+ - \tilde{v}_j^-}{2k} + i\xi_j P_j(\frac{1}{2}V^2) - i\epsilon\xi_j^3(\theta\tilde{v}_j^- + (1-\theta)\tilde{v}_j^+) = 0, \tag{1}$$

where P can be either P_G or P_{col}.

4. Linearized stability analysis of the proposed scheme.

In this section we analyze the stability of Scheme (1) applied to the linearized Korteweg-de Vries equation

$$u_t + \alpha u_x + \epsilon u_{xxx} = 0 \quad \text{on} \quad I = [0, L].$$

Analyzing this equation we are able to apply the usual stability analysis for discrete approximations to hyperbolic pde's with constant coefficients with all its implications [16]. The stability properties of the scheme are determined by the location of the roots of its characteristic polynomial [16]. Stability limits on the size of k follow from limits on the size of the roots of the polynomial.

For stability analysis we introduce the following two classes of polynomials.

Definition 1: A polynomial $\phi(z)$ which has only roots z with $|z| < 1$ is called a Schur polynomial.

Definition 2: A polynomial $\phi(z)$ which has no roots z with $|z| > 1$ and only simple roots with $|z| = 1$ is called a simple von Neumann polynomial.

We define stability by the following

Definition 3: A numerical scheme is stable if and only if its characteristic polynomial $\phi(z)$ is a simple von Neumann polynomial.

It is possible to examine if a given polynomial $\phi(z)$ is a simple von Neumann polynomial by reducing it to a polynomial of lower degree using the theory which originated from Schur [17, 18, 3, 10]. This theory, which is exposed by Miller in [11] and Chan in [3], simplifies the algebra needed to determine the conditions under which the characteristic polynomial $\phi(z)$ is simple von Neumann considerably.

Given the polynomial $\phi(z) = \sum_{j=0}^{N} a_j z^j$ of degree N ($a_N \neq 0$) with $a_0 \neq 0$ we can obtain a polynomial $\phi_1(z)$ of degree N-1 by introducing $\phi^*(z) = \sum_{j=0}^{N} a^*_{N-j} z^j$ and defining

$$\phi_1(z) = \frac{\phi^*(0)\phi(z) - \phi(0)\phi^*(z)}{z}.$$

Now we use the following two theorems:

Theorem 1: $\phi(z)$ is a Schur polynomial if and only if $|\phi^*(0)| > |\phi(0)|$ and $\phi_1(z)$ is a Schur polynomial.

Theorem 2: $\phi(z)$ is a simple von Neumann polynomial if and only if either $|\phi^*(0)| > |\phi(0)|$ and $\phi_1(z)$ is a simple von Neumann polynomial or $\phi_1(z) \equiv 0$ and $\frac{d}{dz}\phi(z)$ is a Schur polynomial.

We apply the above theory to the linearized Scheme (1):

$$\frac{\tilde{v}^+_\xi - \tilde{v}^-_\xi}{2k} + i\alpha\xi\tilde{v}_\xi - i\epsilon\xi^3(\theta\tilde{v}^-_\xi + (1-\theta)\tilde{v}^+_\xi) = 0. \tag{2}$$

The stability polynomial $\phi(z)$ [16] corresponding to the above scheme is given by

$$\phi(z) = (1 - i2(1-\theta)k\epsilon\xi^3)z^2 + 2ik\alpha\xi z - (1 + i2\theta k\epsilon\xi^3).$$

Because $|\phi^*(0)| < |\phi(0)|$ if $\theta > \frac{1}{2}$ the characteristic polynomial of the scheme has at least one root with size larger than 1 if we make the method too explicit.

Next $|\phi^*(0)| > |\phi(0)|$ if $\theta < \frac{1}{2}$ so that $\phi(z)$ is a simple von Neumann polynomial if and only if $\phi_1(z)$ is. But

$$\phi_1(z) = (2\theta - 1)4k^2\epsilon^2\xi^6 z - (2\theta - 1)4k^2\alpha\epsilon\xi^4,$$

and we therefore need

$$\frac{(1-2\theta)4k^2|\alpha|\epsilon\xi^4}{(1-2\theta)4k^2\epsilon^2\xi^6} < 1,$$

or

$$\frac{|\alpha|}{\epsilon} < \xi^2.$$

Whether this condition is met for all frequencies ξ_j in the Fourier scheme is independent of the timestep k. Apart from special cases in which α and ϵ are such that this condition happens to be satisfied for all used frequencies ξ_j, the method will use a number of frequencies ξ_j which are smaller than this limit so that the method will be unstable at those frequencies.

In the case that $\theta = \frac{1}{2}$ however, $|\phi^*(0)| = |\phi(0)|$ and $\phi_1(z) \equiv 0$, so that we should consider $\frac{d}{dz}\phi(z)$:

$$2(1 - ik\epsilon\xi^3)z + 2ik\alpha\epsilon\xi.$$

The stability requirement that this polynomial be a Shur polynomial demands that

$$1 + k^2\epsilon^2\xi^6 > k^2\alpha^2\xi^2,$$

which depends only on the absolute value of ξ. Therefore it follows from the symmetry of the frequency-spectrum with respect to the origin and from the inequality trivially being fulfilled for $\xi = 0$, that we only need to consider the inequality for ξ_1 to ξ_N. Rewriting the criterion as

$$1 + k^2\xi^2(\epsilon^2\xi^4 - |\alpha|^2) > 0$$

shows moreover that no instabilities occur for all frequencies ξ for which $\epsilon^2\xi^4 - |\alpha|^2 \geq 0$. Thus no stability requirements result from the frequencies ξ in the spectrum of the method for which $\xi^2 \geq \frac{|\alpha|}{\epsilon}$. Hence the method is unconditionally stable if $\xi_1^2 \geq \frac{|\alpha|}{\epsilon}$, or

$$\left(\frac{2\pi}{L}\right)^2 \geq \frac{|\alpha|}{\epsilon} \quad \text{on } I.$$

If $\xi_1^2 < \frac{|\alpha|}{\epsilon}$ however, stability restrictions for k follow from the frequencies below the limit $\sqrt{\frac{|\alpha|}{\epsilon}}$:

$$k^2 < \frac{1}{\xi^2(|\alpha|^2 - \epsilon^2\xi^4)}$$

for the $\xi_j \leq \sqrt{\frac{|\alpha|}{\epsilon}}$.

But $\frac{1}{\xi^2(|\alpha|^2-\epsilon^2\xi^4)}$ approaches $+\infty$ at $\xi^2 = \frac{|\alpha|}{\epsilon}$, assumes its minimum $\frac{3\sqrt{3}}{2}\frac{\epsilon}{|\alpha|^3}$ when $(\xi^*)^2 = \frac{|\alpha|}{\sqrt{3}\epsilon}$ and approaches $+\infty$ again at 0. *This means that* $\sqrt{\frac{3\sqrt{3}}{2}\frac{\epsilon}{|\alpha|^3}}$ *is the absolute minimum for the stability limit of Scheme (2). Whenever k is below this limit the linearized Scheme (2) is stable.*

If $\sqrt{\frac{|\alpha|}{\sqrt{3}\epsilon}} \leq \xi_1 \leq \sqrt{\frac{|\alpha|}{\epsilon}}$ the strongest restriction for k therefore follows for ξ_1, so that

$$k^2 < \frac{1}{\xi_1^2(|\alpha|^2 - \epsilon^2\xi_1^4)}.$$

With $\xi_1 \leq \sqrt{\frac{|\alpha|}{\sqrt{3}\epsilon}} \leq \xi_N$ the interval of frequencies contains the most restrictive frequency $\xi^* = \sqrt{\frac{|\alpha|}{\sqrt{3}\epsilon}}$ so that we obtain only one stability condition

$$k^2 < \frac{3\sqrt{3}}{2}\frac{\epsilon}{|\alpha|^3}.$$

When finally $\sqrt{\frac{|\alpha|}{\sqrt{3}\epsilon}} \geq \xi_N$ the most stringent limit on k is obtained for ξ_N and demands

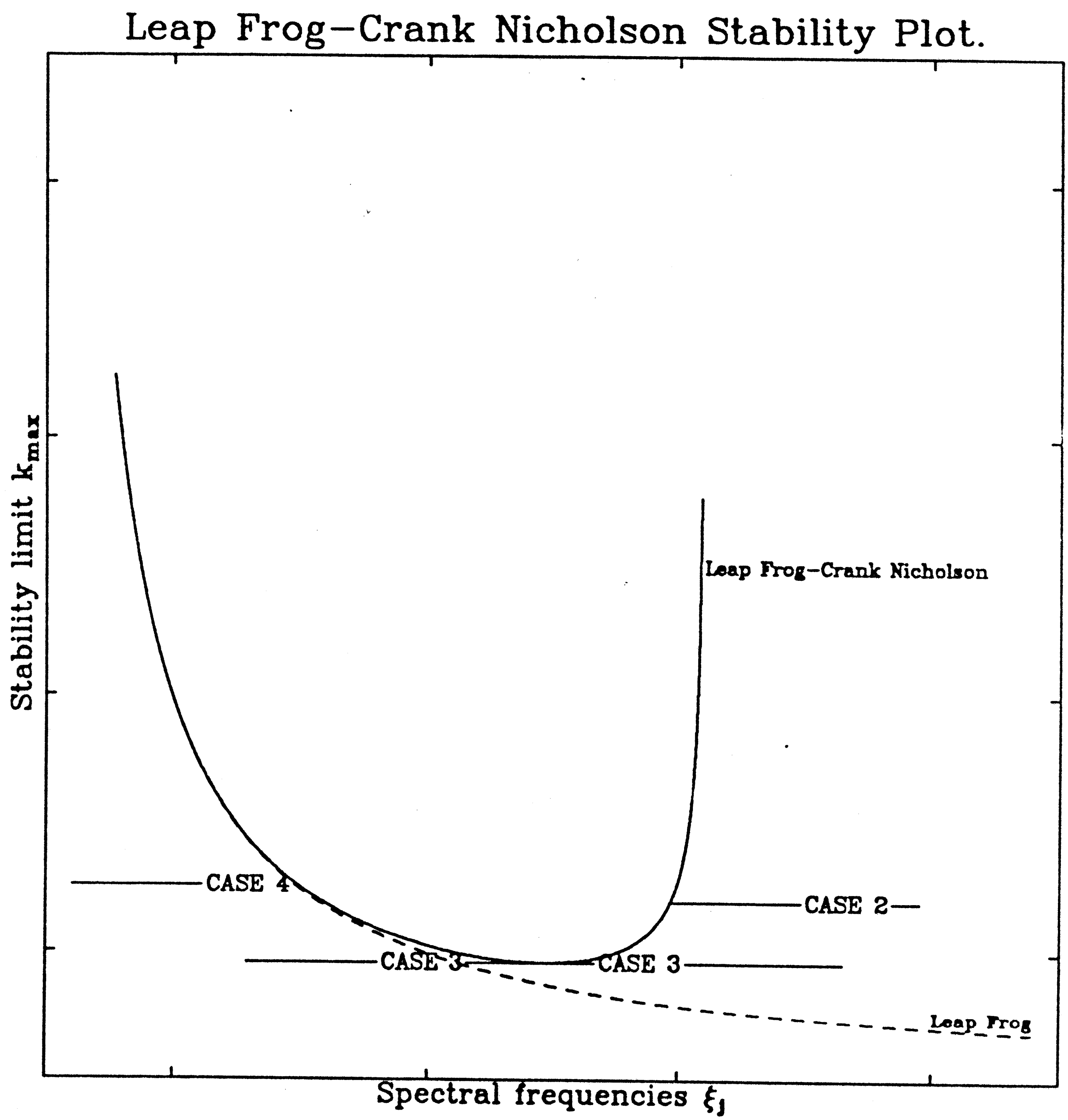
$$k^2 < \frac{1}{\xi_N^2(|\alpha|^2 - \epsilon^2\xi_N^4)}.$$

The stability analysis for the linearized version of the Korteweg-de Vries equation thus shows that there exists an interval $[0, k_0]$, where $k_0 = \sqrt{\frac{3\sqrt{3}}{2}\frac{\epsilon}{|\alpha|^3}}$, such that for k in this interval unconditional stability ensues. There is no limitation on the mesh-size of the space discretization whatsoever. This is attractive using spectral methods because of the high accuracy in space for these schemes.

Moreover the combined Leap Frog-Crank Nicholson scheme inherits being conservative from Leap Frog and Crank Nicholson as can be checked easily from the expression for the roots $z_{1,2}$ of $\phi(z)$ in the case $\theta = \frac{1}{2}$

$$z_{1,2} = \frac{\pm\sqrt{1 - k^2\xi^2|\alpha|^2 + k^2\epsilon^2\xi^6} - ik\alpha\xi}{1 - ik\epsilon\xi^3}$$

We summarize the results of this Section in the following


Leap Frog–Crank Nicholson Stability Plot.
Stability limit k_{max}
Spectral frequencies ξ_j
Leap Frog-Crank Nicholson
Leap Frog
CASE 4
CASE 3
CASE 3
CASE 2

Theorem 3: For the linear periodic initial value problem

$$\begin{aligned} &u_t + \alpha u_x + \epsilon u_{xxx} = 0 \quad (x,t) \in I \times [0,\infty), \\ &u(x,0) = u_0(x) \qquad x \in I, \\ &u(x,t) = u(x+L,t) \quad (x,t) \in R \times [0,\infty). \end{aligned}$$

where $\alpha, \epsilon \in R$, the Fourier-finite difference scheme

$$\frac{\tilde{v}_\xi^+ - \tilde{v}_\xi^-}{2k} + i\alpha\xi\tilde{v}_\xi - i\epsilon\xi^3(\theta\tilde{v}_\xi^- + (1-\theta)\tilde{v}_\xi^+) = 0 \qquad (0 \le \theta \le 1)$$

is unconditionally unstable if $\theta > \frac{1}{2}$,
is unconditionally unstable if $\theta < \frac{1}{2}$ except if α and ϵ satisfy $\sqrt{\frac{|\alpha|}{\epsilon}} < \frac{2\pi}{L}$
in which case the scheme is unconditionally stable,
is conservative and unconditionally stable if $\theta = \frac{1}{2}$ in the sense that

(1) if $\sqrt{\frac{|\alpha|}{\epsilon}} \le \frac{2\pi}{L}$ then there are no restrictions on k.

(2) if $\sqrt{\frac{|\alpha|}{\sqrt{3}\epsilon}} \le \frac{2\pi}{L} \le \sqrt{\frac{|\alpha|}{\epsilon}}$ then $k^2 < \frac{1}{(\frac{2\pi}{L})^2(|\alpha|^2 - \epsilon^2(\frac{2\pi}{L})^4)}$.

(3) if $\frac{2\pi}{L} \le \sqrt{\frac{|\alpha|}{\sqrt{3}\epsilon}} \le \frac{2\pi N}{L}$ then $k^2 < k_0^2 = \frac{3\sqrt{3}}{2}\frac{\epsilon}{|\alpha|^3}$.

(4) if $\frac{2\pi N}{L} \le \sqrt{\frac{|\alpha|}{\sqrt{3}\epsilon}}$ then $k^2 < \frac{1}{(\frac{2\pi N}{L})^2(|\alpha|^2 - \epsilon^2(\frac{2\pi N}{L})^4)}$.

The dependence of the stability conditions on the spectrum of the frequencies used is clarified in the picture. This shows that the Crank Nicholson stability result is valid not only if the nonlinear Leap Frog term is absent, but also if it is sufficiently dominated by the Crank Nicolson part of the equation (Case (1)). The Leap Frog part of the equation affects stability first at the smallest positive frequency reflecting that for this frequency Leap Frog dominates Crank Nicholson least (Case (2)). Once $\xi_1 \le \sqrt{\frac{|\alpha|}{\sqrt{3}\epsilon}}$ and ξ_N decreases towards $\sqrt{\frac{|\alpha|}{\sqrt{3}\epsilon}}$ the Leap Frog stability limit gradually takes over (Case (3)). Finally the Leap Frog limit dominates and the stability limit for the scheme is obtained for ξ_N. The Leap Frog limit increases as ξ_N decreases (Case (4)).

As an overall result of the interaction of the Leap Frog and Crank Nicholson schemes we have obtained a scheme which is unconditionally stable because the unconditional stability of the Crank Nicholson part of the finite difference scheme dominates the Leap Frog limit exactly at those frequencies where the latter is most restrictive, namely the high frequencies. (See picture)

5. Modified Basis Functions

For the linear part of the Korteweg-de Vries equation $u_t + \epsilon u_{xxx} = 0$, we obtain the following dispersion relation

$$\omega = -\epsilon\xi^3.$$

For the Crank Nicholson discretization in time however, the dependence of $\tilde{\omega}$ on ξ is given by

$$\tilde{\omega} = -\frac{1}{k}\arcsin(k\epsilon\xi^3\cos(\xi^3 h)).$$

Thus numerical dispersion will be small only if $k\epsilon\xi_N^3 \ll 1$ so that we obtain an accuracy limit on the size of k. Although the solutions to the discretized equation can be bounded for larger k because of an extensive stability interval, they might be immensely inaccurate.

The combined Leap Frog-Crank Nicholson scheme which we proposed acquires its unconditional stability because Crank Nicholson dominates Leap Frog exactly at those frequencies at which the latter has its lowest stability limit. But at those high frequencies Crank Nicholson is less accurate.

Fornberg and Whitham [7] used the solutions

$$f(x,t) = e^{i(\xi x + \xi^3 t)}$$

to the linear part of the Korteweg-de Vries equation, to modify the coefficients in their full Leap Frog scheme for increasing their $O(N^{-3})$ stability limit. We consider another technique for discretising the Korteweg-de Vries equation using these above solutions which results in an $O(N^{-1})$ stability limit.

The approach is to use these solutions to the linear dispersive part of the Korteweg-de Vries equation as basis functions for a spectral method. In other words we try to approximate the solution to $u_t + uu_x + \epsilon u_{xxx} = 0$ by

$$W(x,t) = \sum_{k=-N}^{N} \tilde{w}_k(t) e^{i(\xi_k x + \xi_k^3 t)}$$

With this scheme our basis functions are running waves which travel at the correct speed for the linear part of our pde. Because for high frequencies ξ the u_{xxx} term dominates the non-linear uu_x term this means that we may expect to be able to avoid the inaccuracy of the Crank Nicholson scheme for high-frequency modes.

Substituting this last expression for $u(x,t)$ in the pde we can discretize using either the Galerkin or the collocation projection and obtain

$$\frac{d}{dt}\tilde{w}_k(t) + \frac{\partial}{\partial x}P\Big(\frac{1}{2}\sum_{q=-N}^{N}\sum_{p=-N}^{N}\tilde{w}_q(t)\tilde{w}_p(t)e^{i(\xi_q^3+\xi_p^3-\xi_k^3)t}e^{i\xi_q x}e^{i\xi_p x}\Big) = 0.$$

Apart from the correction factors $e^{i(\xi_q^3+\xi_p^3-\xi_k^3)t}$ the nonlinear term gives rise to convolutions which were already present in the Leap Frog-Crank Nicholson scheme.

For the linearized scheme we therefore obtain a Leap Frog like limit under all circumstances:

$$k < \frac{1}{|\alpha|\xi_N} = \frac{L}{|\alpha|2\pi N}.$$

The stability limit derived in Section 3 is always less restrictive than the Leap Frog limit for the nonlinear part. Which method is more attractive depends therefore on how accuracy requirements and stability requirements balance out. (See picture) This is examined in detail in the next section.

For this method in which we use modified basis functions the stability properties do depend on the length of the domain. The stability limit for a problem on $[0, L]$ is $\frac{L}{|\alpha|2\pi N}$. At the ξ^* which is most restrictive for the Leap Frog-Crank Nicholson scheme the Leap Frog scheme with extended eigenfunctions results in the limit $\frac{L}{|\alpha|\xi^*} = \sqrt{\sqrt{3}\frac{\epsilon}{|\alpha|^3}}$. The difference between the two stability limits at ξ^* is therefore only a factor $\sqrt{\frac{3}{2}}$.

6. A Finite Element Scheme

Finite element methods for the Korteweg-de Vries equation have been analyzed by Winther in [19], where he proved convergence for a class of these schemes. We implemented one of the schemes he proposed:

$$\frac{1}{2}\left(\frac{u_{j+1}^{n+1} - u_{j+1}^{n-1}}{2k} + \frac{u_j^{n+1} - u_j^{n-1}}{2k}\right) + \frac{1}{2h}((u_{j+1}^n)^2 - (u_j^n)^2) +$$
$$\frac{\epsilon}{h^3}\left(\frac{u_{j+2}^{n+1} + u_{j+2}^{n-1}}{2} - 3\frac{u_{j+1}^{n+1} + u_{j+1}^{n-1}}{2} + 3\frac{u_j^{n+1} + u_j^{n-1}}{2} - \frac{u_{j-1}^{n+1} + u_{j-1}^{n-1}}{2}\right) = 0$$

Linearizing and analyzing the stability of the linearized finite-difference scheme like we did for the Fourier methods we obtain the following stability polynomial:

$$\phi(z) = \left[\cos(\xi\tfrac{h}{2}) + \epsilon\frac{k}{h^3}(2i)^3\sin^3(\xi\tfrac{h}{2})\right]z^2 + 2\alpha\frac{k}{h}(2i)\sin(\xi\tfrac{h}{2})z - \left[\cos(\xi\tfrac{h}{2}) - \epsilon\frac{k}{h^3}(2i)^3\sin^3(\xi\tfrac{h}{2})\right]$$

The structure of this polynomial is essentially identical to that of the one we obtained for the Fourier method. Necessary and sufficient for stability is therefore that $\frac{d}{dz}\phi(z)$ be a Schur polynomial:

$$1 - \sin^2(\xi\tfrac{h}{2}) + k^2\epsilon^2(\tfrac{h}{2})^{-6}\sin^2(\xi\tfrac{h}{2})\{\sin^2(\xi\tfrac{h}{2}) + \frac{\alpha}{\epsilon}(\frac{h}{2})^2\}\{\sin^2(\xi\tfrac{h}{2}) - \frac{\alpha}{\epsilon}(\frac{h}{2})^2\} > 0.$$

For stability we obtain the necessary condition

$$(\frac{h}{2})^2 \le \frac{\epsilon}{\alpha}$$

from the above requirement while $\sin^2(\xi\frac{h}{2}) = 1$.

	Leap Frog-Crank Nicholson spectral	Leap Frog-Crank Nicholson pseudospectral	Modified basis functions spectral	Modified basis functions pseudospectral	Finite elements $N = 500$
k_{th}	$9.19 * 10^{-2}$	$9.19 * 10^{-2}$	$1.52 * 10^{-2}$	$1.52 * 10^{-2}$	$9.19 * 10^{-2}$
k_{obs}	$9.9 * 10^{-2}$	$9.9 * 10^{-2}$	$8 * 10^{-3}$	$8 * 10^{-3}$	$1 * 10^{-1}$
e_{max}	$9.33 * 10^{-5}$	$9.33 * 10^{-5}$	$3.28 * 10^{-4}$	$3.28 * 10^{-4}$	$7.24 * 10^{-3}$
e_{max}	$3.53 * 10^{-5}$	$3.53 * 10^{-5}$	$8.51 * 10^{-5}$	$8.51 * 10^{-5}$	$7.17 * 10^{-3}$

Table 1: Comparison of the performance of the discussed schemes on the propagation of a Korteweg-de Vries soliton with $\beta = 1.5$ and $\epsilon = 1$ on the interval $[0, 20]$. We use 64 basis functions or discrete points, $T_{max} = 10$. Shown are: The theoretical stability limit k_{th}, The observed stability limit k_{obs} (approximate), The maximal error e_{max} when $k = 1 * 10^{-3}$, The maximal error e_{max} when $k = 5 * 10^{-4}$. (Errors are scaled by $\frac{1}{\sqrt{N}}$ so that they are independent of the number of points.)

Under the assumption that $h^2 \ll \frac{4\epsilon}{\alpha}$ (accuracy demands require a very small h anyway) we obtain

$$k^2 < \frac{3\sqrt{3}}{2}\frac{\epsilon}{|\alpha|^3}(1 - \frac{1}{4\sqrt{3}}|\alpha|h^2)$$

as the second stability condition.

7. Numerical Results

We compared the performance of the three methods presented before and verified the linear stability analysis by numerical experiments. Although the linearized analysis does not distinguish between the Galerkin and the pseudospectral approach, the resulting data seem to indicate that neither method performs better than the other one. The results for Winther's scheme are shown together with those for the Fourier methods in Table 1.

Because of the much lower accuracy of finite difference methods we had to use a much finer space grid with 500 points to achieve comparable results in a numerical sense. Needless to say it took considerably more time to run the finite element method.

We compared the performance of the different schemes for the solution of the Korteweg-de Vries equation by examining the accuracy and stability with which they propagate a Korteweg-de Vries soliton

$$u(x,t) = 3\frac{\beta^2}{\cosh^2(\frac{\beta x - \beta^3 t}{2})}.$$

(The errors were computed in the L_2-norm.) All schemes perform very well on non-trivial problems like colliding solitons too. All runs were done on a FPS 164 attached processor (53-bit mantissa) with a Vax-11/780 host.

Comparing the errors for the two different k's, we see that for the Fourier methods we do have $e_{max} = O(k^2)$ approximately, as we should for a centered finite difference approximation in time.

That we obtained hardly any smaller error by decreasing k for the finite element method reflects that the error is still dominated by the space discretization and one actually should take a still much finer mesh for limiting the spatial discretization error.

Overall, the finite element method compared unfavorably with the spectral methods on the problem considered here. Korteweg-de Vries solitons being analytical functions, this was to be expected.

Experiments with the θ scheme where $\theta \neq \frac{1}{2}$ indicated that those cases are unstable indeed.

8. Conclusion.

We have proposed two new Fourier methods for solving the KdV equation that have improved stability properties over conventional explicit methods. The key features through which this is achieved are the use of implicit Fourier methods and modified basis functions. Both ideas can be extended to higher dimensional problems and other types of equations in a straightforward manner.

In the linearized stability analysis the $\theta = \frac{1}{2}$, Crank Nicholson scheme is the only three-level Fourier scheme in the class of schemes considered here for which some kind of unconditional stability is obtained.

Numerical experiments show that the Fourier methods are considerably more accurate and substantially faster than finite element methods if the problems are well suited.

Although a linearized stability analysis is not sufficient for proving stability and convergence of the corresponding non-linear schemes, obtaining estimates of stability intervals and accuracy requirements is often desirable in practice. Our experiments show that this analysis does give accurate predictions about the properties of the nonlinear methods for the problems considered here. The Korteweg-de Vries equation is an example of a time-dependent problem with a linear-dispersive term and a nonlinear term. The kind of analysis given here can be extended with minor modifications to other equations of this type: e.g. modifications of the Korteweg-de Vries equation

$$u_t + (p+1)u^p u_x + \epsilon u_{xxx} = 0, \quad p \in N$$

and the Schrödinger equation

$$iu_t + u_{xx} - V(x)u = 0.$$

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