The Laplace transform is frequently encountered in mathematics, physics, engineering and other areas. However, the spectral properties of the Laplace transform tend to complicate its numerical treatment; therefore, the closely related "truncated" Laplace transforms are often used in applications. In this dissertation, we construct efficient algorithms for the evaluation of the singular value decomposition (SVD) of such operators. The approach of this dissertation is somewhat similar to that introduced by Slepian et al. for the construction of prolate spheroidal wavefunctions in their classical study of the truncated Fourier transform. The resulting algorithms are applicable to all environments likely to be encountered in applications, including the evaluation of singular functions corresponding to extremely small singular values (e.g. 10^{-1000}).

On the Analytical and Numerical Properties of the Truncated Laplace Transform.

Roy R. Lederman†
Technical Report YALEU/DCS/TR-1490
May, 2014

† This author's research was supported in part by the ONR grants #N00014-11-1-0718 and #N00014-10-1-0570 and the NSF grant #1309858.

Keywords: *Truncated Laplace Transform, SVD.*

On the Analytical and Numerical Properties of the Truncated Laplace Transform

A Dissertation
Presented to the Faculty of the Graduate School
of
Yale University
in Candidacy for the Degree of
Doctor of Philosophy

by
Roy R. Lederman

Dissertation Director: Vladimir Rokhlin

May 2014

Contents

List of Figures

List of Tables

Acknowledgements

I would like to thank my advisor, Professor Vladimir Rokhlin, for his guidance in this dissertation. I would like to express my deep gratitude to Professor Ronald R. Coifman and Professor Vladimir Rokhlin for their superb mentorship and wise advice, for everything they taught me about mathematics and science, for many hours of discussions, and for their valued friendship.

I would like to thank Professor Peter Jones for his guidance and useful advice, and Dr. Andrei Osipov for many fruitful discussions. I would also like to thank Dr. Ronen Talmon for our productive brainstorming meetings, and Professor Gregory Beylkin for his help.

In addition, I would like to thank Karen Kavanaugh, Chris Hatchell and the Mathematics staff for their assistance. I would like to thank my professors, colleagues and classmates for making this journey not only possible, but also enjoyable.

I would also like to thank Laura and my dear friends for their support and understanding, and I would like to thank Moshe for his help.

Finally, I would like to thank Bella, Gil, Oren, Marcelle, Shaike, Manya and Shmuel for many years of love and support.

Chapter 1

Introduction

The Laplace transform $\tilde{\mathcal{L}}$ is a linear mapping $L^2(0,\infty) \to L^2(0,\infty)$; for a function $f \in L^2(0,\infty)$, it is defined by the formula:

$$\left(\tilde{\mathcal{L}}(f)\right)(\omega) = \int_0^\infty e^{-t\omega} f(t)\mathrm{d}t. \tag{1.1}$$

As is well-known, $\tilde{\mathcal{L}}$ has a continuous spectrum, and $\tilde{\mathcal{L}}^{-1}$ is not continuous (see, for example, [1]). These and related properties tend to complicate the numerical treatment of $\tilde{\mathcal{L}}$.

In addressing these problems, we find it useful to draw an analogy between the numerical treatment of the Laplace transform, and the numerical treatment of the Fourier transform $\tilde{\mathcal{F}}$; for a function $f \in L^1(\mathbb{R})$, the later is defined by the formula:

$$\left(\tilde{\mathcal{F}}(f)\right)(\omega) = \int_{-\infty}^\infty e^{-it\omega} f(t)\mathrm{d}t, \tag{1.2}$$

where $\omega \in \mathbb{R}$.

In various applications in mathematics and engineering, it is useful to define the "truncated" Fourier transform $\tilde{\mathcal{F}}_c : L^2(-1,1) \to L^2(-1,1)$; for a given $c > 0$, $\tilde{\mathcal{F}}_c$ of a function $f \in L^2(-1,1)$

is defined by the formula:

$$\left(\tilde{\mathcal{F}}_c(f)\right)(\omega) = \int_{-1}^{1} e^{-ict\omega} f(t)\mathrm{d}t. \tag{1.3}$$

The operator $\tilde{\mathcal{F}}_c$ has been analyzed extensively; one of the most notable discoveries, made by Slepian et al. in 1960, was that the integral operator $\tilde{\mathcal{F}}_c$ commutes with a second order differential operator (see [2]). This property of $\tilde{\mathcal{F}}_c$ was used in analytical and numerical investigation of the eigendecomposition of this operator, for example in [3] and [4].

For $0 < a < b < \infty$, the linear mapping $\mathcal{L}_{a,b} : L^2(a,b) \to L^2(0,\infty)$, defined by the formula

$$\left(\mathcal{L}_{a,b}(f)\right)(\omega) = \int_{a}^{b} e^{-t\omega} f(t)\mathrm{d}t, \tag{1.4}$$

will be referred to as the *truncated Laplace transform* of f; obviously, $\mathcal{L}_{a,b}$ is a bounded compact operator (see, for example, [1]) .

Bertero and Grünbaum discovered that each of the symmetric operators $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$ and $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*$ commutes with a differential operator (see [5]). These properties were used in the analysis of the truncated Laplace transform (see [5], [6]).

Despite the result in [5], more is known about the numerical and analytical properties of $\tilde{\mathcal{F}}_c$ than about the properties of $\mathcal{L}_{a,b}$.

In this dissertation, we introduce an algorithm for the efficient evaluation of the singular value decomposition (SVD) of $\mathcal{L}_{a,b}$, and analyze some of its properties. A more detailed analysis of the asymptotic properties of $\mathcal{L}_{a,b}$ will be presented in a separate paper.

The dissertation is organized as follows. Chapter 2 summarizes various standard mathematical facts and certain simple derivations that are used later in this dissertation. Chapter 2 also contains a definition of the SVD of the truncated Laplace transform and a summary of some known properties of the truncated Laplace transform. Chapter 3 contains the derivation of various properties of the truncated Laplace transform, which are used in the algorithms.

Chapter 4 describes the algorithms for the evaluation of the singular functions, singular values and associated eigenvalues. Chapter 5 contains numerical results obtained using the algorithms. Chapter 6 contains generalizations and conclusions.

Remark 1.1. Some authors define the truncated Laplace transform as in (1.4), but allow $a = 0$, or define the operator as a linear mapping $L^2(a, b) \to L^2(a, b)$. See, for example, [7].

Chapter 2

Mathematical preliminaries

In this chapter we introduce notation and summarize standard mathematical facts which we use in this dissertation. In addition, we present a brief derivation of some useful facts which we have failed to find in the literature.

2.1 Legendre Polynomials

Definition 2.1. *The Legendre polynomial P_k of degree $k \geq 0$, is defined by the formula*

$$P_k(x) = \frac{1}{2^k k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \left(x^2 - 1\right)^k . \tag{2.1}$$

As is well-known, the Legendre Polynomials of degrees $k = 0, 1....$ form an orthogonal basis in $L^2(-1, 1)$. The following well-known properties of the Legendre polynomials can be found inter alia in [8], [9]:

$$\int_{-1}^{1} (P_k(x))^2 \mathrm{d}x = \frac{2}{2k+1} \tag{2.2}$$

$$(k+1)P_{k+1}(x) = (2k+1)xP_k(x) - kP_{k-1}(x) \tag{2.3}$$

$$(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_k(x) = -kxP_k(x) + kP_{k-1}(x) \tag{2.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_k(x)\right) = -k(1+k)P_k(x) \tag{2.5}$$

$$(2k+1)P_k(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(P_{k+1}(x) - P_{k-1}(x)\right) \tag{2.6}$$

$$P_0(x) = 1 \tag{2.7}$$

$$P_1(x) = x \tag{2.8}$$

For all $k \geq 1$,

$$(k+1)P_{k+1}(x) = (2k+1)xP_k(x) - kP_{k-1}(x) \tag{2.9}$$

In this dissertation we will analyze functions in $L^2(0,1)$; it is therefore convenient to use the shifted Legendre polynomials, which are defined on the interval $(0, 1)$.

Definition 2.2. *The shifted Legendre polynomial of degree* $k \geq 0$*, which we will be denoting by* P_k^**, is defined via the Legendre polynomial* P_k *by the formula*

$$P_k^*(x) = P_k(2x - 1). \tag{2.10}$$

Clearly, the polynomials P_k^* form an orthogonal basis in $L^2(0, 1)$. The following properties of the shifted Legendre polynomials are easily derived from the properties of the Legendre polynomials by substituting (2.10) into (2.2-2.7).

$$\int_0^1 (P_k^*(x))^2 \mathrm{d}x = \frac{1}{2k+1} \tag{2.11}$$

$$xP_k^*(x) = \frac{1}{2}\left(\frac{kP_{k-1}^*(x)}{1+2k} + P_k^*(x) + \frac{(1+k)P_{k+1}^*(x)}{1+2k}\right) \tag{2.12}$$

$$x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}P_k^*(x) = \frac{k(1+k)}{2(1+2k)}\left(P_{k-1}^*(x) - P_{k+1}^*(x)\right) \tag{2.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}P_k^*(x)\right) = -k(1+k)P_k^*(x) \tag{2.14}$$

$$P_0^*(x) = 1 \tag{2.15}$$

As is evident from (2.2) and (2.11), neither the Legendre polynomials nor the shifted Legendre polynomials are normalized. In the discussion of the space of functions $L^2(0,1)$, we will find it convenient to use the orthonormal basis of the functions $\overline{P_k^*}(x)$.

Definition 2.3. *We define $\overline{P_k^*}(x)$ by the formula:*

$$\overline{P_k^*}(x) = P_k^*(x)\sqrt{2k+1}, \tag{2.16}$$

where $k = 0, 1, \ldots..$

Clearly, the polynomials $\overline{P_k^*}$ are an orthonormal basis in $L^2(0,1)$.

Observation 2.4. $\overline{P_0^*}$ is a constant

$$\overline{P_0^*}(x) = 1 \tag{2.17}$$

Observation 2.5. The derivative of $\overline{P_k^*}$ is a linear combination of $\overline{P_l^*}$, where $l < k$. The following expressions for the derivative are easily verified using (2.6), (2.16) and (2.10) :

$$\frac{\mathrm{d}}{\mathrm{d}x}\overline{P_{2j}^*}(x) = 2\sqrt{2(2j)+1}\sum_{l=0}^{j-1}\sqrt{2(2l+1)+1}\ \overline{P_{2l+1}^*}(x) \tag{2.18}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\overline{P_{2j+1}}(x) = 2\sqrt{2(2j+1)+1}\sum_{l=0}^{j-1}\sqrt{2(2l)+1}\ \overline{P_{2l}}(x) \tag{2.19}$$

2.2 Legendre Functions of the second kind

Definition 2.6. *The Legendre function of the second kind $Q_k(z)$ is defined by the formula*

$$Q_k(z) = \frac{1}{2}\int_{-1}^{1}(z-t)^{-1}P_k(t)\mathrm{d}t, \tag{2.20}$$

where $P_k(t)$ is defined in (2.1).

The following identities can be found, for example, in [8], [9]:

$$Q_k(z) = (-1)^{k+1} Q_k(-z), \tag{2.21}$$

$$Q_k(z) = \int_0^\infty \frac{\mathrm{d}\phi}{\left(z + \sqrt{z^2-1}\cosh(\phi)\right)^{k+1}}. \tag{2.22}$$

Having defined the shifted Legendre polynomials, we find it convenient to also define a shifted version of the Legendre function of the second kind.

Definition 2.7. *We define the shifted Legendre function of the second kind of degree k, which we will be denoting by Q_k^*, by the formula*

$$Q_k^*(z) = Q_k(2z-1) \tag{2.23}$$

By (2.16), (2.20), (2.21) and (2.23),

$$\int_0^1 (x+y)^{-1} \overline{P_k^*}(x)\mathrm{d}x = 2(-1)^k Q_k^*(y+1)\sqrt{2k+1} \quad y > 0 \tag{2.24}$$

and

$$Q^*{}_k(1+\delta/2) = Q^*{}_k(1+\delta) = \int_0^\infty \frac{\mathrm{d}\phi}{\left((1+\delta) + \sqrt{(1+\delta)^2-1}\cosh(\phi)\right)^{k+1}} \tag{2.25}$$

For a given $x > 1$, $Q_k^*(x)$ decays rapidly as k grows. The following lemma gives an upper bound for $|Q_k^*(z)|$, where $z \geq x$, as k grows.

Lemma 2.8. *Let $\delta > 0$. We introduce the notation $\tilde{\delta} = \sqrt{(1+\delta)^2 - 1}$. Then, for all $y \geq 0$,*

$$|Q_k^*(1 + \delta/2 + y)| < \frac{1}{\left(1+\tilde{\delta}\right)^{k+1}} \left(\log\left(2\frac{1+\tilde{\delta}}{\tilde{\delta}} \right) + 1 \right), \tag{2.26}$$

where Q_k^* is defined in (2.23).

Proof. By (2.25),

$$\begin{aligned} |{Q^*}_k(1+\delta/2+y)| =& |Q_k(1+\delta+2y)| = \\ &= \int_0^\infty \frac{\mathrm{d}\phi}{\left((1+\delta+y) + \sqrt{(1+\delta+y)^2-1}\cosh(\phi)\right)^{k+1}}. \end{aligned} \tag{2.27}$$

Since $(1+\delta+y) \geq (1+\delta)$,

$$\begin{aligned} |{Q^*}_k(1+\delta/2+y)| =& |Q_k(1+\delta+2y)| \leq \\ &\leq \int_0^\infty \frac{\mathrm{d}\phi}{\left((1+\delta) + \sqrt{(1+\delta)^2-1}\cosh(\phi)\right)^{k+1}}. \end{aligned} \tag{2.28}$$

Clearly, $\tilde{\delta} = \sqrt{(1+\delta)^2-1} > 0$, and by (2.22),

$$|{Q^*}_k(1+\delta/2+y)| < \int_0^\infty \frac{\mathrm{d}\phi}{\left(1+\tilde{\delta}\cosh(\phi)\right)^{k+1}}. \tag{2.29}$$

We define

$$\nu = \log\left(2\frac{1+\tilde{\delta}}{\tilde{\delta}}\right), \tag{2.30}$$

and break the integral in (2.29) into integrals on the two intervals $[0,\nu)$ and $[\nu,\infty)$:

$$|Q^*{}_k(1+\delta/2+y)| < \int_0^\nu \frac{\mathrm{d}\phi}{\left(1+\tilde{\delta}\cosh(\phi)\right)^{k+1}} + \int_\nu^\infty \frac{\mathrm{d}\phi}{\left(1+\tilde{\delta}\cosh(\phi)\right)^{k+1}}. \tag{2.31}$$

Clearly,

$$\frac{1}{\left(1+\tilde{\delta}\cosh(\phi)\right)^{k+1}} \leq \frac{1}{\left(1+\tilde{\delta}\right)^{k+1}}, \tag{2.32}$$

and

$$\frac{1}{\left(1+\tilde{\delta}\cosh(\phi)\right)^{k+1}} \leq \frac{1}{\left(\tilde{\delta}\exp(\phi)/2\right)^{k+1}}, \tag{2.33}$$

so that,

$$|Q^*{}_k(1+\delta/2+y)| < \frac{\nu}{\left(1+\tilde{\delta}\right)^{k+1}} + \int_\nu^\infty \frac{\mathrm{d}\phi}{\left(\tilde{\delta}\exp(\phi)/2\right)^{k+1}}. \tag{2.34}$$

Substituting (2.30) into the last inequality, we obtain

$$|Q^*_k(1+\delta/2+y)| < \frac{1}{\left(1+\tilde{\delta}\right)^{k+1}}\left(\log\left(2\frac{1+\tilde{\delta}}{\tilde{\delta}}\right) + \frac{1}{k+1}\right), \tag{2.35}$$

and from it, we obtain (2.26).

□

2.3 Laguerre functions

Definition 2.9. *The generalized Laguerre polynomial* $L_k^{(\alpha)}(x)$ *of order* $\alpha > -1$ *and degree* $k \geq 0$*, is defined by the formula*

$$L_k^{(\alpha)}(x) = \sum_{m=0}^{k} (-1)^m \binom{k+\alpha}{k-m} \frac{1}{m!} x^m \tag{2.36}$$

Definition 2.10. *The Laguerre polynomial* $L_k(x)$ *is the generalized Laguerre polynomial of order* 0*:*

$$L_k(x) = L_k^{(0)}(x) \tag{2.37}$$

As is well-known, the Laguerre polynomials are an orthonormal basis in the Hilbert space induced by the inner product

$$(f, g) = \int_0^\infty e^{-x} f(x) g(x) \mathrm{d}x \tag{2.38}$$

The following well-known properties of the generalized Laguerre polynomials can be found, inter alia, in [8]:

$$L_k^{\alpha-1}(x) = L_k^{\alpha}(x) - L_{k-1}^{\alpha}(x) \tag{2.39}$$

$$\frac{d}{\mathrm{d}x} L_k(x) = -L_{k-1}^{(1)} \tag{2.40}$$

$$xL_k(x) = -(k+1)L_{k+1}(x) + (2k+1)L_k(x) - kL_{k-1}(x) \tag{2.41}$$

$$\int_0^\infty e^{-xt} L_k(x)\mathrm{d}x = (t-1)^k t^{-k-1} \tag{2.42}$$

$$L_k(0) = 1 \tag{2.43}$$

$$L_0(x) = 1 \tag{2.44}$$

$$L_1(x) = 1 - x \tag{2.45}$$

For all $k \geq 1$,

$$(k+1)L_{k+1}(x) = (2k+1-x)L_k(x) - kL_{k-1}(x) \tag{2.46}$$

It is convenient to use functions which are orthonormal in the standard $L^2(0,\infty)$ sense. Therefore, we will use the Laguerre functions, as defined below, rather than the Laguerre polynomials.

Definition 2.11. *We define the Laguerre function, which we will be denoting by* Φ_k*, via the* *formula*

$$\Phi_k(x) = e^{-x/2} L_k(x). \tag{2.47}$$

Clearly, the Laguerre functions $\Phi_k(x)$ are an orthonormal basis in the standard $L^2(0, \infty)$ sense.

Observation 2.12. The derivative of a Laguerre function of degree k is a linear combination of Laguerre functions of degree k and lower. The following expression is easy to verify using (2.40) and (2.47):

$$\frac{d}{\mathrm{d}x}\Phi_k(x) = -\frac{1}{2}\Phi_k(x) - \sum_{l=0}^{k-1}\Phi_l(x). \tag{2.48}$$

2.4 The complete elliptic integral

Several slightly different definitions of the complete elliptic integral of the first kind can be found in the literature. In this dissertation, we will use the following definition.

Definition 2.13. *The complete elliptic integral of the first kind $K(m)$ is defined by the formula*

$$K(m) = \int_0^{\pi/2} \left(1 - m\sin^2(\theta)\right)^{-1/2} \mathrm{d}\theta. \tag{2.49}$$

2.5 Singular value decomposition (SVD) of integral operators

The SVD of integral operators and its key properties are summarized in the following theorem, which can be found in [10].

Theorem 2.14. *Suppose that the function $K : (c, d) \times (a, b) \to \mathbb{R}$ is square integrable, and let*

$T : L^2(a,b) \to L^2(c,d)$ be

$$(T(f))(x) = \int_a^b K(x,t)f(t)\mathrm{d}t. \tag{2.50}$$

Then, there exist two orthonormal sequences of functions $u_n : (a,b) \to \mathbb{R}$ and $v_n : (c,d) \to \mathbb{R}$ and a sequence $s_n \in \mathbb{R}$, for $n = 0, ...\infty$, such that

$$K(x,t) = \sum_{n=0}^{\infty} v_n(x)s_n u_n(t) \tag{2.51}$$

and that $s_0 \geq s_1 \geq ... \geq 0$. The sequence s_n is uniquely determined by K. Furthermore, the functions u_n are eigenfunctions of the operator $T^* \circ T$ and the values s_n are the square roots of the eigenvalues of $T^* \circ T$.

Observation 2.15. The function K can be approximated by discarding of small singular values (see [10]):

$$K(x,t) \simeq \sum_{n=0}^{p} v_n(x)s_n u_n(t) \tag{2.52}$$

2.6 Tridiagonal and five-diagonal matrices

In this section, we briefly describe a standard method for calculating eigenvectors and eigenvalues of symmetric tridiagonal and five-diagonal matrices.

2.6.1 Sturm sequence for tridiagonal and five-diagonal matrices

The Sturm sequence is a method for calculating the number of roots that a polynomial has in a given interval. In this dissertation, the Sturm sequence method for band matrices is used to calculate the number of negative eigenvalues of a matrix. The following theorems can be found, for example, in [11] and [12].

Theorem 2.16. Sturm sequence for tridiagonal matrices. *Let A be a symmetric $N \times N$ tridiagonal matrix, and let $A_{k,k} = a_1$ where $k = 1..N$, $A_{k,k+1} = A_{k+1,k} = b_{k+1}$ where $k = 1..N-1$. All other elements of A are 0.*

We define the sequences m_k and q_k as

$$
\begin{aligned}
m_0 =& 1 \\
m_1 =& a_1 \\
m_k =& a_1 m_{k-1} - b_k^2 m_{k-2} \quad , \quad k = 2, 3, ..., N
\end{aligned}
\tag{2.53}
$$

The number of sign changes in the sequence m_k is the number of eigenvalues of A that are smaller than 0.

Theorem 2.17. Sturm sequence for symmetric five-diagonal matrices. *Let A be a symmetric $N \times N$ five-diagonal matrix, and let $A_{k,k} = a_1$ where $k = 1..N$, $A_{k,k+1} = A_{k+1,k} = b_{k+1}$ where $k = 1..N-1$ and $A_{k,k+2} = A_{k+2,k} = c_{k+2}$ where $k = 1..N-2$.*

We define the sequences m_k and q_k as

$$
\begin{aligned}
q_k =& 0 \quad , \quad k \leq 0 \\
m_k =& 0 \quad , \quad k < 0 \\
m_0 =& 1 \\
q_{k-2} =& b_{k-1} m_{k-3} - c_{k-1} q_{k-3} \quad , \quad k = 3, 4, ..., N \\
m_k =& a_k m_{k-1} - b_k^2 m_{k-2} - c_k^2 (a_{k-1} m_{k-3} - c_{k-1}^2 m_{k-4}) + 2 b_k c_k q_{k-2} \quad , \\
& k = 1, 2, ..., N
\end{aligned}
\tag{2.54}
$$

The number of sign changes in the sequence m_k is the number of eigenvalues of A that are smaller than 0.

Remark 2.18. In implementations of this method, some scaling of the sequence is sometimes required in order to avoid overflows and underflows (see, for example, [13]).

Suppose that we wish to calculate λ_n, the n-th largest eigenvalue of the tridiagonal or five-diagonal matrix A. Let $\delta > 0$. We observe that the n-th largest eigenvalue of the matrix $(A - (\lambda_n + \delta)I)$ is negative. Therefore, the number of sign changes in the sequence m_k for the matrix $(A - (\lambda_n + \delta)I)$ is no smaller than n. Similarly, the n-th largest eigenvalue of the matrix $(A - (\lambda_n - \delta)I)$ is positive. Therefore, the number of sign changes in the sequence m_k for the matrix $(A - (\lambda_n - \delta)I)$ is strictly smaller than n.

We set a search range (α_1, α_2); we use the Sturm sequence to verify that λ_n is in the range, otherwise we extend the search range. We then use bisection to narrow the range (α_1, α_2) until $\alpha_2 - \alpha_1$ is smaller than the desired precision. λ_n is contained within the range, so $(\alpha_1 + \alpha_2)/2$ is a sufficient approximation for λ_n.

2.6.2 The inverse power method for tridiagonal and five-diagonal matrices

Let B be a symmetric matrix, and let $\lambda_n \neq 0$ be eigenvalue of B with the largest magnitude. Suppose that there is some $\delta > 0$ such that for any other eigenvalue λ_m of B, we have $|\lambda_n| > (1 + \delta)|\lambda_m|$. *The power method* is a well-known method for calculating the eigenvector v and the eigenvalue λ_n by iterative calculation of

$$v^{(k+1)} = Bv^{(k)}. \tag{2.55}$$

After a sufficient number of iterations,

$$Bv^{(k)} \approx \lambda_n v^{(k)}. \tag{2.56}$$

Let A be a symmetric tridiagonal or five-diagonal matrix, and let $\lambda_n \neq 0$ be an eigenvalue of A with multiplicity one. Then there exists $\delta > 0$ such that for any other eigenvalue λ_m of A, $|\lambda_n|(1 + \delta) < |\lambda_m|$. *The inverse power method* is a well-known method for calculating the eigenvector v and the eigenvalue λ_n, using the power method on $B = A^{-1}$. Instead of computing $B = A^{-1}$ explicitly, the power iteration $v^{(k+1)} = Bv^{(k)}$ is computed by solving

$$Av^{(k+1)} = v^{(k)}. \tag{2.57}$$

2.6.3 Calculating an eigenvector and an eigenvalue of a tridiagonal or five-diagonal matrix

Let A be a symmetric tridiagonal or five-diagonal matrix. Suppose that we would like to calculate the n-th largest eigenvalue λ_n and the corresponding eigenvector v of A, such that

$$Av = \lambda_n v \tag{2.58}$$

Assume that λ_n has multiplicity one.

First, we approximate the n-th eigenvalue λ_n using the Sturm sequence method described in theorems 2.16 and 2.17. We require the approximation $\tilde{\lambda}_n$ to be close to λ_n compared to the difference between λ_n and any other eigenvalue of A, but not equal to λ_n. In other words:

$$|\lambda_n - \tilde{\lambda}_n| \neq 0 \tag{2.59}$$

and

$$|\lambda_n - \tilde{\lambda}_n| \ll |\lambda_n - \lambda_m| \quad , \ \forall m \neq n \tag{2.60}$$

Next, we consider the matrix $(A - \tilde{\lambda}_n I)$. We observe that the eigenvector v that we wish to calculate is also an eigenvector of $(A - \tilde{\lambda}_n I)$, with the eigenvalue $\sigma_n = \lambda_n - \tilde{\lambda}_n \neq 0$. We observe that σ_n is smaller in magnitude than any other eigenvalue σ_m of $(A - \tilde{\lambda}_n I)$. We use the inverse power method to calculate v.

Finally, we obtain a better estimate for eigenvalue λ_n using (2.58).

2.7 The truncated Laplace transform

Definition 2.19. *For given* $0 < a < b < \infty$*, the truncated Laplace transform* $\mathcal{L}_{a,b}$ *is a linear mapping* $L^2(a,b) \to L^2(0,\infty)$*, defined by the formula*

$$(\mathcal{L}_{a,b}(f))(\omega) = \int_a^b e^{-t\omega} f(t)\mathrm{d}t, \tag{2.61}$$

where $0 \leq \omega < \infty$*.*

The adjoint operator of $\mathcal{L}_{a,b}$ is denoted by $(\mathcal{L}_{a,b})^*$. Obviously:

$$((\mathcal{L}_{a,b})^*(g))(t) = \int_0^\infty e^{-t\omega} g(\omega)\mathrm{d}\omega. \tag{2.62}$$

The operators $\mathcal{L}_{a,b}$ and $(\mathcal{L}_{a,b})^*$ are compact and injective, the range of $(\mathcal{L}_{a,b})^*$ is dense in $L^2(a,b)$ and the range of $\mathcal{L}_{a,b}$ is dense in $L^2(0,\infty)$ (see, for example, [1]).

2.8 The SVD of the truncated Laplace transform

In this section, we present the SVD of the truncated Laplace transform, which is the main tool we use to investigate the properties of this operator in this dissertation.

The kernel $K : (0,\infty) \times (a,b) \to \mathbb{R}$ of the integral operator $\mathcal{L}_{a,b}$ (defined in (2.61)) is defined by the formula

$$K(\omega, t) = e^{-\omega t}, \tag{2.63}$$

so that

$$(\mathcal{L}_{a,b}(f))(\omega) = \int_a^b K(\omega,t) f(t)\mathrm{d}t. \tag{2.64}$$

By theorem 2.14, there exist two orthonormal sequences of functions $u_n \in L^2(a,b)$ and $v_n \in L^2(0,\infty)$ such that

$$K(\omega,t) = \sum_{n=0}^{\infty} v_n(\omega) s_n u_n(t), \tag{2.65}$$

$$\mathcal{L}_{a,b}(u_n) = \alpha_n v_n, \tag{2.66}$$

and

$$(\mathcal{L}_{a,b})^*(v_n) = \alpha_n u_n. \tag{2.67}$$

We refer to the functions $u_n(t)$ as the *right singular functions*, and to the functions $v_n(\omega)$ as the *left singular functions*. We refer to $\alpha_n \geq 0$ as the *singular values*. The functions are numbered $n = 0, 1, ..$, and they are sorted according to the singular values, in descending order.

Observation 2.20. The multiplicity of α_n in this decomposition of $\mathcal{L}_{a,b}$ is one (see [5]).

Observation 2.21. A simple calculation shows that $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$ of a function $f \in L^2(a,b)$ is given by the formula

$$\left(\left((\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}\right)(f)\right)(t) = \int_a^b \frac{1}{t+s} f(s) \mathrm{d}s. \tag{2.68}$$

Clearly, $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$ is a symmetric positive semidefinite compact operator. By theorem 2.14, the right singular functions u_n of the operator $\mathcal{L}_{a,b}$ are also the eigenfunctions of the operator $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$, and the singular values α_n are the square roots of the eigenvalues of

$(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$. In other words,

$$(((\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b})(u_n))(t) = \int_a^b \frac{1}{t+s} u_n(s)\mathrm{d}s = \alpha_n^2 u_n(t). \tag{2.69}$$

Observation 2.22. Similarly, $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*$ of a function $g \in L^2(0,\infty)$ is given by the formula

$$(((\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*)(g))(\omega) = \int_0^\infty \frac{e^{-a(\omega+\rho)} + e^{-b(\omega+\rho)}}{\omega+\rho} g(\rho)\mathrm{d}\rho. \tag{2.70}$$

By theorem 2.14, the left singular functions v_n of $\mathcal{L}_{a,b}$ are the eigenfunctions of $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*$ and the singular values α_n are the square roots of the eigenvalues of $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*$. In other words,

$$((\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*)(v_n))(\omega) = \int_0^\infty \frac{e^{-a(\omega+\rho)} + e^{-b(\omega+\rho)}}{\omega+\rho} v_n(\rho)\mathrm{d}\rho = \alpha_n^2 v_n(\omega). \tag{2.71}$$

2.9 A differential operator related to the right singular functions u_n

It has been observed in [5] that the integral operator $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$ (defined in (2.68)) commutes with a differential operator.

Theorem 2.23. *The differential operator $\tilde{D}_t$, defined by the formula*

$$\left(\tilde{D}_t(f)\right)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left((t^2-a^2)(b^2-t^2)\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right) - 2(t^2-a^2)f(t), \tag{2.72}$$

commutes with the integral operator $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$ (defined in (2.68)) in $L^2(a,b)$.

It has also been shown in [5] that the eigenvalues of the operators $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$ and $\tilde{D}_t$ have a multiplicity of one. It follows from theorem 2.23, and the multiplicity of the eigenvalues, that the eigenfunctions of the integral operator $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$ are the regular eigenfunctions of the differential operator $\tilde{D}_t$. By (2.69), these functions are the right singular functions u_n of $\mathcal{L}_{a,b}$.

Furthermore, it has been shown that if the eigenfunctions of $\tilde{D}_t$ are sorted according to the eigenvalues of $\tilde{D}_t$, in descending order, the n-th eigenfunction of $\tilde{D}_t$ is the n-th singular function of $\mathcal{L}_{a,b}$. Therefore, u_n is both the $n+1$-th right singular function of $\mathcal{L}_{a,b}$, the $n+1$-th eigenfunction of $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$, and the $n+1$-th eigenfunction of $\tilde{D}_t$.

We denote the eigenvalues of the differential operator $\tilde{D}_t$ by $\tilde{\chi}_n$. By theorem 2.23, u_n is the solution to the differential equation

$$\left(\tilde{D}_t(u_n)\right)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left((t^2-a^2)(b^2-t^2)\frac{\mathrm{d}}{\mathrm{d}t}u_n(t)\right) - 2(t^2-a^2)u_n(t) = \tilde{\chi}_n u_n(t). \tag{2.73}$$

2.10 The function ψ_n associated with the right singular function u_n

The right singular functions u_n of $\mathcal{L}_{a,b}$ (the operator defined in (2.61)) are defined on the interval (a,b). It is convenient to scale and shift the interval (a,b) to $(0,1)$.

We define the variable $x \in (0,1)$ by the formula

$$x = \frac{t-a}{b-a}, \qquad t = a + (b-a)x. \tag{2.74}$$

The functions ψ_k are defined using the change of variables (2.74), as follows.

Definition 2.24. *The function $\psi_n(x)$ is defined via the corresponding right singular function u_n, by the formula*

$$\psi_n(x) = \sqrt{b-a}\; u_n(a+(b-a)x). \tag{2.75}$$

Observation 2.25. Since the function u_n is normalized on (a,b), it is clear from (2.75) that ψ_n is normalized on $(0,1)$

$$\int_0^1 (\psi_n(x))^2 \,\mathrm{d}x = 1, \tag{2.76}$$

and that the sequence of functions ψ_n forms an orthonormal basis in $L^2(0,1)$.

By (2.68) and (2.74), the functions ψ_n are the eigenfunctions of the integral operator $T^* \circ T$, where $T^* \circ T$ of a function f is defined by the formula

$$\left((T^* \circ T)\, \tilde{f} \right)(x) = \int_0^1 \frac{1}{x+y+\beta} \tilde{f}(y)\mathrm{d}y, \tag{2.77}$$

and where β is defined by the the formula:

$$\beta = \frac{2a}{b-a}. \tag{2.78}$$

Clearly, $T^* \circ T$ has the same eigenvalues as $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$:

$$\left((T^* \circ T)\,(\psi_n) \right)(x) = \int_0^1 \frac{1}{x+y+\beta} \psi_n(y)\mathrm{d}y = \alpha_n^2 \psi_n(x). \tag{2.79}$$

Similarly, by (2.72) and (2.74), ψ_n are the eigenfunctions of the differential operator D_x, which is defined by the formula

$$\left(D_x(f) \right)(x) = \frac{\mathrm{d}}{\mathrm{d}x} \left(x(1-x)(\beta+x)(\beta+1+x) \frac{\mathrm{d}}{\mathrm{d}x} f(x) \right) - 2x(x+\beta)f(x). \tag{2.80}$$

In other words,

$$\begin{gathered}
\left(D_x(\psi_n)\right)(x) = \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1-x)(\beta+x)(\beta+1+x)\frac{\mathrm{d}}{\mathrm{d}x}\psi_n(x)\right) - 2x(x+\beta)\psi_n(x) = \\
\chi_n\psi_n(x),
\end{gathered} \tag{2.81}$$

where χ_n are the eigenvalues of D_x.

2.11 A differential operator related to the left singular functions v_n

It has been observed in [5] that the integral operator $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*$ (defined in (2.70)) commutes with a differential operator.

Theorem 2.26. *The differential operator $\hat{D}_\omega$, defined by the formula*

$$\begin{gathered}
\left(\hat{D}_\omega(f)\right)(\omega) = \left(\left(\mathcal{L}_{a,b} \circ \tilde{D} \circ (\mathcal{L}_{a,b})^{-1}\right)(f)\right)(\omega) = \\
-\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left(\omega^2\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}f(\omega)\right) + (a^2+b^2)\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\omega^2\frac{\mathrm{d}}{\mathrm{d}\omega}f(\omega)\right) + \left(-a^2b^2\omega^2 + 2a^2\right)f(\omega),
\end{gathered} \tag{2.82}$$

commutes with the integral operator $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*$ (defined in (2.70)). The left singular functions v_n are the eigenfunctions of $\hat{D}_\omega$.

We denote the eigenvalues of $\hat{D}_\omega$ by χ_k^*. By theorem 2.26, the function v_n is the solution of the differential equation

$$\begin{aligned}
&\left(\hat{D}_\omega(v_k)\right)(\omega) = \\
&= -\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left(\omega^2 \frac{\mathrm{d}^2}{\mathrm{d}\omega^2} v_k(\omega)\right) + (a^2+b^2)\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\omega^2 \frac{\mathrm{d}}{\mathrm{d}\omega} v_k(\omega)\right) + (-a^2b^2\omega^2 + 2a^2)v_k(\omega) = \\
&= \chi_k^* v_k(\omega).
\end{aligned} \tag{2.83}$$

Observation 2.27. The eigenvalues of $\hat{D}_\omega$ are equal to the eigenvalues of $\tilde{D}_t$:

$$\tilde{\chi}_n = \chi_n^* \tag{2.84}$$

2.12 The functions $(\mathcal{L}_{a,b})^*(\Phi_k)$

Having introduced the operator $\mathcal{L}_{a,b}$ in (2.61) and its adjoint $(\mathcal{L}_{a,b})^*$ in (2.62), we now discuss the properties of the function generated by applying $(\mathcal{L}_{a,b})^*$ to the Laguerre function Φ_k (defined in (2.47)). By (2.42), (2.47) and (2.61),

$$\begin{aligned}
\left((\mathcal{L}_{a,b})^*(\Phi_k)\right)(t) &= \int_0^\infty e^{-\omega t}\Phi_k(\omega)\mathrm{d}\omega = \int_0^\infty e^{-\omega(t+1/2)} L_k(\omega)\mathrm{d}\omega = \\
&= \left(t - \frac{1}{2}\right)^k \left(t + \frac{1}{2}\right)^{-k-1}.
\end{aligned} \tag{2.85}$$

In particular, at $t = 1/2$, (2.85) becomes

$$\left((\mathcal{L}_{a,b})^*(\Phi_k)\right)(1/2) = \int_0^\infty e^{-q/2}\Phi_k(q)\mathrm{d}q = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.86}$$

Differentiating (2.85), we obtain

$$\left((\mathcal{L}_{a,b})^*(\Phi_k)\right)'(t) = (8k - 8t + 4)(2t-1)^{k-1}(2t+1)^{-k-2}, \tag{2.87}$$

which, at $t = 1/2$, becomes

$$\left(\left(\mathcal{L}_{a,b}\right)^{*}\left(\Phi_{k}\right)\right)'(1/2) = \begin{cases} -1 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.88}$$

Chapter 3

Analytical apparatus

In this part of the dissertation, we discuss certain useful properties of the truncated Laplace transform; we begin with a brief discussion of the scaling properties of the truncated Laplace transform, and with a definition of a standard form of the truncated Laplace transform. We proceed to define the transform C_γ and discuss various symmetry properties associated with it. We then discuss the expansions of u_n and v_n in orthonormal bases, and show that the calculations of u_n and v_n can be phrased as benign eigensystem calculations. This chapter is concluded with brief discussions of several miscellaneous useful properties.

3.1 On the scaling properties of the truncated Laplace transform

The truncated Laplace transform (as defined in (2.61)) can be generalized to the form

$$\left(\mathcal{L}_{a,b,c}(f)\right)(\omega) = \int_a^b e^{-ct\omega} f(t)\mathrm{d}t. \tag{3.1}$$

with arbitrary $0 < c < \infty, 0 < a < b < \infty$.

Observation 3.1. The properties of the truncated Laplace transform are determined by the ratio

$$\gamma = b/a > 1 \tag{3.2}$$

(see, for example, [1]).

Observation 3.2. The particular choice

$$\begin{aligned} a =& \frac{1}{2\sqrt{\gamma}}, \\ b =& \frac{\sqrt{\gamma}}{2}, \\ c =& 1, \end{aligned} \tag{3.3}$$

yields several useful properties, which we will discuss in this dissertation.

Due to observations 3.2 and 3.1, in the remainder of this dissertation we will be assuming without loss of generality that the values of a, b and c are as defined in (3.3). In other words, we will restrict our attention to the following form of the truncated Laplace transform:

Definition 3.3. *For a given $1 < \gamma < \infty$, we will denote by $\mathcal{L}_\gamma : L^2(\frac{1}{2\sqrt{\gamma}}, \frac{\sqrt{\gamma}}{2}) \to L^2(0, \infty)$ the operator defined by*

$$\mathcal{L}_\gamma = \mathcal{L}_{\frac{1}{2\sqrt{\gamma}}, \frac{\sqrt{\gamma}}{2}} = \mathcal{L}_{\frac{1}{2\sqrt{\gamma}}, \frac{\sqrt{\gamma}}{2}, 1}. \tag{3.4}$$

The operator $\mathcal{L}_\gamma$ will be referred to as the "standard form" of the truncated Laplace transform.

Obviously, $\mathcal{L}_\gamma$ of a function $f \in L^2(\frac{1}{2\sqrt{\gamma}}, \frac{\sqrt{\gamma}}{2})$ is defined by the formula

$$\left(\mathcal{L}_\gamma(f)\right)(\omega) = (\mathcal{L}_{\frac{1}{2\sqrt{\gamma}}, \frac{\sqrt{\gamma}}{2}, 1}(f))(\omega) = \int_{\frac{1}{2\sqrt{\gamma}}}^{\frac{\sqrt{\gamma}}{2}} e^{-t\omega} f(t) \mathrm{d}t. \tag{3.5}$$

Where there is no danger of confusion, we write $\mathcal{L}$ instead of $\mathcal{L}_\gamma, \mathcal{L}_{a,b}$ and $\mathcal{L}_{a,b,c}$, and we denote the adjoint of $\mathcal{L}$ by $\mathcal{L}^*$.

Remark 3.4. Combining (3.2) with (2.78), we observe that the quantity β (defined in (2.78)) is related to γ by the formula

$$\beta = \frac{2a}{b-a} = \frac{2}{\gamma - 1}. \tag{3.6}$$

Remark 3.5. Let $\tilde{u}_n$, $\tilde{v}_n$ and $\tilde{\alpha}_n$ be the $n+1$-th right singular function, left singular function and singular value of $\mathcal{L}_{\tilde{a},\tilde{b},\tilde{c}}$, such that

$$\mathcal{L}_{\tilde{a},\tilde{b},\tilde{c}}(\tilde{u}_n) = \tilde{\alpha}_n \tilde{v}_n. \tag{3.7}$$

Let $\gamma = \tilde{b}/\tilde{a}$ and let u_n, v_n and α_n be the $n+1$-th singular functions and singular value of $\mathcal{L}_\gamma$. Then, the SVD of $\mathcal{L}_{\tilde{a},\tilde{b},\tilde{c}}$ is related to the SVD of the standard form $\mathcal{L}_\gamma$ by:

$$\tilde{u}_n(t) = \sqrt{\frac{a}{\tilde{a}}}\, u_n(ta/\tilde{a}), \tag{3.8}$$

$$\tilde{v}_n(\omega) = \sqrt{\frac{\tilde{a}c}{a}}\, v_n(\omega c\tilde{a}/a), \tag{3.9}$$

and

$$\tilde{\alpha}_n = \alpha_n/\sqrt{c}. \tag{3.10}$$

3.2 The transform C_γ

In this section we define the transform C_γ which is useful in the discussion of certain symmetry properties.

Definition 3.6. *We define the new variable $s \in \mathbb{R}$ via*

$$s = 2\log(2t)/\log(\gamma). \tag{3.11}$$

For $\gamma > 1$, we define the transform C_γ of a function f by the the formula

$$\left(C_\gamma(f)\right)(s) = \gamma^{s/4}\ f\left(\gamma^{s/2}/2\right). \tag{3.12}$$

Observation 3.7. A simple calculation shows that

$$\int_{s(t_1)}^{s(t_2)} \left(C_\gamma(f)\right)(s)\left(C_\gamma(g)\right)(s)\mathrm{d}s = \frac{4}{\log\gamma}\int_{t_1}^{t_2} f(t)g(t)\mathrm{d}t. \tag{3.13}$$

We are particularly interested in the case where a, b are as defined in (3.3). In this case, C_γ becomes a mapping $L^2\left(\frac{1}{2\sqrt{\gamma}}, \frac{\sqrt{\gamma}}{2}\right) \to L^2(-1,1)$.

3.2.1 The functions $\left(C_\gamma \circ \mathcal{L}^*\right)(\Phi_k)$

In this subsection, we discuss certain properties of the Laguerre functions Φ_k (defined in (2.47)), related to the operator C_γ (defined in (3.12)).

A simple calculation shows that C_γ of the function $\mathcal{L}^*(\Phi_k)$ (see (2.85)) is given by

$$\left(\left(C_\gamma \circ \mathcal{L}^*\right)(\Phi_k)\right)(s) = \gamma^{s/4}\left(\gamma^{s/2}-1\right)^k\left(\gamma^{s/2}+1\right)^{-k-1}. \tag{3.14}$$

Clearly,

$$|((C_\gamma \circ \mathcal{L}^*)(\Phi_k))(s)| = \frac{\gamma^{s/4}}{\gamma^{s/2}+1}\left|\frac{\gamma^{s/2}-1}{\gamma^{s/2}+1}\right|^k, \tag{3.15}$$

from which it immediately follows that

$$|((C_\gamma \circ \mathcal{L}^*)(\Phi_k))(s)| \leq \frac{1}{2}\left|\frac{\gamma^{s/2}-1}{\gamma^{s/2}+1}\right|^k. \tag{3.16}$$

Observation 3.8. For all $1 < \gamma < \infty$ and $s \in \mathbb{R}$, we have $\left|\frac{\gamma^{s/2}-1}{\gamma^{s/2}+1}\right| < 1$; it is therefore obvious from (3.15) that $|((C_\gamma \circ \mathcal{L}^*)(\Phi_k))(s)|$ decays exponentially as k grows.

Observation 3.9. By (3.14),

$$((C_\gamma \circ \mathcal{L}^*)(\Phi_k))(s) = (-1)^k ((C_\gamma \circ \mathcal{L}^*)(\Phi_k))(-s). \tag{3.17}$$

In other words, for an even k, the function $(C_\gamma \circ \mathcal{L}^*(\Phi_k))(s)$ is even; and for an odd k, it is odd.

Observation 3.10. By (3.14), at the point $s = 0$,

$$((C_\gamma \circ \mathcal{L}^*)(\Phi_k))(0) = \begin{cases} 1/2 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.18}$$

Observation 3.11. By differentiating (3.14) and setting $s = 0$ we obtain:

$$((C_\gamma \circ \mathcal{L}^*)(\Phi_k))'(0) = \begin{cases} \log(\gamma)/4 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.19}$$

3.2.2 C_γ of the right singular function u_n

Definition 3.12. *We introduce the function U_n, which we define by the formula*

$$U_n(s) = \left(C_\gamma(u_n)\right)(s), \tag{3.20}$$

where u_n is a right singular function of the operator $\mathcal{L}_{a,b}$, and C_γ is defined in (3.12).

By (3.13), and since u_n is normalized on (a, b), the norm of U_n on $\left(2\frac{\log(2a)}{\log(\gamma)}, 2\frac{\log(2b)}{\log(\gamma)}\right)$ is:

$$\int_{2\frac{\log(2a)}{\log(\gamma)}}^{2\frac{\log(2b)}{\log(\gamma)}} (U_n(s))^2 \,\mathrm{d}s = \frac{4}{\log\gamma} \tag{3.21}$$

Equation (3.21) holds for an arbitrary choice of a and b such that $b/a = \gamma$. In this dissertation we assume $a = \frac{1}{2\sqrt{\gamma}}, b = \frac{\sqrt{\gamma}}{2}$ (as defined in (3.3)). By substituting (3.3) into (3.21), the interval $s \in \left(2\frac{\log(2a)}{\log(\gamma)}, 2\frac{\log(2b)}{\log(\gamma)}\right)$ becomes $s \in (-1, 1)$.

In the case of $\mathcal{L}_\gamma$ (the standard form of the truncated Laplace transform, defined in (3.5)), by (2.72) and (3.20), the functions U_n are the eigenfunctions of the differential operator $\tilde{\tilde{D}}_s$, defined by the formula

$$\begin{aligned}\left(\tilde{\tilde{D}}_s(f)\right)(s) = &\left(\log\left(\sqrt{\gamma}\right)\right)^{-2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\gamma^2 + 1 - 2\gamma\cosh\left(2s\log\left(\sqrt{\gamma}\right)\right)\right)\frac{\mathrm{d}}{\mathrm{d}s}f(s)\\ &-\left(\frac{3}{2}\gamma\cosh\left(2s\log\left(\sqrt{\gamma}\right)\right) + \frac{1}{4}\gamma^2 - \frac{7}{4}\right)f(s).\end{aligned} \tag{3.22}$$

A simple calculation shows that the eigenvalues of $\tilde{\tilde{D}}_s$, which we denote by μ_n, are related to the eigenvalues $\tilde{\chi}_n$ (defined in (2.72)) by the formula:

$$\mu_n = 4\gamma\tilde{\chi}_n. \tag{3.23}$$

3.3 The symmetry property of u_n and U_n

By [5], the right singular functions u_n of $\mathcal{L}_{a,b}$ (the operator defined in (2.61)) satisfy a form of symmetry around the point $\sqrt{ab}$. In the case of the standard form $\mathcal{L}_\gamma$, defined in (3.5), we have $\sqrt{ab} = 1/2$, and the symmetry relation is:

$$u_n\left(\frac{1}{4t}\right) = (-1)^n 2t u_n(t) \tag{3.24}$$

Observation 3.13. In the case of standard form $\mathcal{L}_\gamma$, it follows from (3.20), that the functions U_n (defined in (3.20)) are even and odd functions in the regular sense:

$$U_n(s) = \left(C_\gamma(u_n)\right)(s) = (-1)^n U_n(-s). \tag{3.25}$$

In particular, at the point $s = 0$, we have:

$$U_{2j+1}(0) = \left(C_\gamma(u_{2j+1})\right)(0) = 0, \tag{3.26}$$

and

$$U'_{2j}(0) = \left(C_\gamma(u_{2j})\right)'(0) = 0. \tag{3.27}$$

Remark 3.14. The functions U_n are even and odd functions around the point $s = 0$ in the case of the standard form $\mathcal{L}_\gamma$ (as defined in (3.5)). Similar symmetry exists for C_γ of the right singular functions of $\mathcal{L}_{a,b}$ (as defined in (2.61)), however the center of symmetry is not necessarily $s = 0$.

3.4 The differential operator D_x and the expansion of ψ_n in the basis of $\overline{P_k^*}$

In this section we consider the expansion of functions $f \in L^2(0,1)$ in the orthonormal basis of the polynomials $\overline{P_k^*}$ (defined in (2.16)):

$$f(x) = \sum_{k=0}^{\infty} h_k \overline{P_k^*}(x). \tag{3.28}$$

Lemma 3.15 describes the operation of D_x (defined in (2.80)) on a basis function $\overline{P_k^*}$. The result is used to express the functions ψ_n (defined in (2.75)) via a five-terms recurrence relation or a solution to a benign eigensystem, specified in theorem 3.16.

Lemma 3.15. *Applying the differential operator D_x to the polynomial $\overline{P_k^*}$ yields a linear combination of $\overline{P_{k-2}^*}$,$\overline{P_{k-1}^*}$,$\overline{P_k^*}$,$\overline{P_{k+1}^*}$ and $\overline{P_{k+2}^*}$:*

$$\begin{aligned}
&\left(D_x(\overline{P_k^*})\right)(x) = \\
&= -\tfrac{(k-1)^2k^2}{4\sqrt{2k-3}(2k-1)\sqrt{2k+1}}\overline{P_{k-2}^*}(x) \\
&- \tfrac{k^3(1+\beta)}{\sqrt{2k-1}\sqrt{2k+1}}\overline{P_{k-1}^*}(x) \\
&- \tfrac{\left(-4-6\beta-2k\beta(2+3\beta)+k^2\left(7+12\beta+2\beta^2\right)+\left(2k^3+k^4\right)\left(7+16\beta+8\beta^2\right)\right)}{2(2k-1)(2k+3)}\overline{P_k^*}(x) \\
&- \tfrac{(k+1)^3(1+\beta)}{\sqrt{2k+1}\sqrt{2k+3}}\overline{P_{k+1}^*}(x) \\
&- \tfrac{(k+1)^2(k+2)^2}{4\sqrt{2k+1}(2k+3)\sqrt{2k+5}}\overline{P_{k+2}^*}(x),
\end{aligned} \tag{3.29}$$

where $\beta = \frac{2a}{b-a} = \frac{2}{\gamma-1}$ (as defined in 3.6).

Proof. By the definition of D_x (in (2.80)),

$$\begin{aligned}
&\left(D_x(P_k^*)\right)(x) = \\
&= \frac{d}{\mathrm{dx}}\left((\beta+x)(\beta+1+x)x(1-x)\frac{d}{\mathrm{dx}}P_k^*(x)\right) - 2x(x+\beta)P_k^*(x).
\end{aligned} \tag{3.30}$$

Using the chain rule,

$$
\begin{aligned}
&\left(D_x(P_k^*)\right)(x) = \\
=&\left(\frac{d}{\mathrm{dx}}(\beta+x)(\beta+1+x)\right)\left(x(1-x)\frac{d}{\mathrm{dx}}P_k^*(x)\right) \\
&+(\beta+x)(\beta+1+x)\frac{d}{\mathrm{dx}}\left(x(1-x)\frac{d}{\mathrm{dx}}P_k^*(x)\right)-2x(x+\beta)P_k^*(x)= \\
=&(1+2x+2\beta)\left(x(1-x)\frac{d}{\mathrm{dx}}P_k^*(x)\right) \\
&+\left(x^2+x(1+2\beta)+\beta+\beta^2\right)\frac{d}{\mathrm{dx}}\left(x(1-x)\frac{d}{\mathrm{dx}}P_k^*(x)\right) \\
&-2x(x+\beta)P_k^*(x)
\end{aligned}
\tag{3.31}
$$

Using identities (2.12), (2.13) and (2.14),

$$
\begin{aligned}
&\left(D_x(P_k^*)\right)(x) = \\
&-\frac{(-1+k)^2k^2P_{k-2}^*(x)}{4(-1+2k)(1+2k)} \\
&-\frac{k^3(1+\beta)P_{k-1}^*(x)}{1+2k} \\
&-\frac{\left(-4+7k^2+14k^3+7k^4-6\beta-4k\beta+12k^2\beta+32k^3\beta+16k^4\beta-6k\beta^2+2k^2\beta^2+16k^3\beta^2+8k^4\beta^2\right)}{2(-1+2k)(3+2k)}P_k^*(x) \\
&-\frac{(1+k)^3(1+\beta)P_{k+1}^*(x)}{1+2k} \\
&-\frac{(1+k)^2(2+k)^2P_{k+2}^*(x)}{4(1+2k)(3+2k)}.
\end{aligned}
\tag{3.32}
$$

Finally, substituting (2.11) into (3.32) gives (3.29). □

Theorem 3.16. *Let the function $\psi_n(x)$ be as defined in (2.75). Let $h^n = (h_0^n, h_1^n, ...)^\top$ be the* *vector of coefficients in the expansion of $\psi_n(x)$ in the basis of the polynomials $\overline{P_k^*}$:*

$$\psi_n(x) = \sum_{k=0}^{\infty} h_k^n \overline{P_k^*}(x) \tag{3.33}$$

Then, h^n is the $n+1$-th eigenvector of M :

$$M h^n = \chi_n h^n, \tag{3.34}$$

where M is the five-diagonal matrix

$$\begin{aligned}
M_{k-2,k} &= -\tfrac{(k-1)^2 k^2}{4\sqrt{2k-3}(2k-1)\sqrt{2k+1}} \\
M_{k-1,k} &= -\tfrac{k^3(1+\beta)}{\sqrt{2k-1}\sqrt{2k+1}} \\
M_{k,k} &= -\tfrac{\left(-4-6\beta-2k\beta(2+3\beta)+k^2\left(7+12\beta+2\beta^2\right)+\left(2k^3+k^4\right)\left(7+16\beta+8\beta^2\right)\right)}{2(2k-1)(2k+3)} \\
M_{k+1,k} &= -\tfrac{(k+1)^3(1+\beta)}{\sqrt{2k+1}\sqrt{2k+3}} \\
M_{k+2,k} &= -\tfrac{(k+1)^2(k+2)^2}{4\sqrt{2k+1}(2k+3)\sqrt{2k+5}},
\end{aligned} \tag{3.35}$$

and where χ_n are the eigenvalues of the differential operator D_x, and $k = 0, 1, 2....$

Proof. By (2.81), $\psi_n(x)$ is an eigenfunction of D_x, with the eigenvalue χ_n. Since the differential operator is linear,

$$\left(D_x(\psi_n)\right)(x) = \sum_{k=0}^{\infty} h_k^n \left(D_x(\overline{P_k^*})\right)(x) = \chi_n \sum_{k=0}^{\infty} h_k^n \overline{P_k^*}(x). \tag{3.36}$$

Using lemma 3.15, (3.36) becomes (3.34). □

Observation 3.17. Clearly, h_k^n is the inner product of ψ_n and $\overline{P_k^*}$:

$$h_k^n = \int_0^1 \overline{P_k^*}(x)\psi_n(x)\mathrm{d}x. \tag{3.37}$$

3.4.1 The decay of the coefficients in the expansion of ψ_n

Since the functions ψ_n are smooth regular solutions of a differential operator, they can be efficiently expressed using an orthogonal basis of polynomials. In other words, we expect the coefficients in the expansion of ψ_n in terms of the polynomials $\overline{P_k^*}$ to decay rapidly. In this subsection, we obtain a bound for this decay.

Lemma 3.18. *Let* $0 < \beta < \infty$ *and* $0 \leq y \leq 1$*. We introduce the notation*

$$\tilde{\beta} = \sqrt{(1 + (2\beta)^2) - 1} = \sqrt{4\beta(1+\beta)}. \tag{3.38}$$

Then,

$$\int_0^1 \left(\int_0^1 \frac{1}{x+y+\beta} \overline{P_k^*}(x) \mathrm{d}x \right)^2 \mathrm{d}y \leq \left(\frac{2\sqrt{2k+1}}{\left(1+\tilde{\beta}\right)^{k+1}} \left(\log\left(2\frac{1+\tilde{\beta}}{\tilde{\beta}} \right) + 1 \right) \right)^2, \tag{3.39}$$

where $\overline{P_k^*}$ *is defined in (2.16).*

Proof. We recall from (2.24) that

$$\left| \int_0^1 (x+y+\beta)^{-1} \overline{P_k^*}(x) \mathrm{d}x \right| = 2Q_k^*(y+\beta+1)\sqrt{2n+1}, \tag{3.40}$$

where Q_k^* is defined in (2.23). So, by lemma 2.8,

$$\left| \int_0^1 (x+y+\beta)^{-1} \overline{P_k^*}(x) \mathrm{d}x \right| < \frac{2\sqrt{2k+1}}{\left(1+\tilde{\beta}\right)^{k+1}} \left(\log\left(2\frac{1+\tilde{\beta}}{\tilde{\beta}} \right) + 1 \right). \tag{3.41}$$

By squaring (3.41) and integrating over y, we obtain (3.39). □

Lemma 3.19. *Let* h_k^n *be the* $k+1$*-th coefficient in the expansion defined in (3.33), of the* *function ψ_n (defined in (2.75)) in the basis of the polynomials $\overline{P_k^*}$ (defined in (2.16)). Then,*

$$|h_k^n| \leq \alpha_n^{-2} \frac{2\sqrt{2k+1}}{\left(1+\tilde{\beta}\right)^{k+1}} \left(\log \left(2 \frac{1+\tilde{\beta}}{\tilde{\beta}} \right) + 1 \right) \tag{3.42}$$

Where

$$\tilde{\beta} = \sqrt{(1+(2\beta)^2)-1} = \sqrt{4\beta(1+\beta)} \tag{3.43}$$

and β is as defined in (3.6).

Proof. We substitute (2.79) into (3.37) and change the order of integration:

$$\begin{aligned} h_k^n &= \alpha_n^{-2} \int_0^1 \int_0^1 \frac{1}{x+y+\beta} \overline{P_k^*}(x) \psi_n(y) \mathrm{d}x \mathrm{d}y = \\ &= \alpha_n^{-2} \int_0^1 \psi_n(y) \left(\int_0^1 \frac{1}{x+y+\beta} \overline{P_k^*}(x) \mathrm{d}x \right) \mathrm{d}y. \end{aligned} \tag{3.44}$$

By the Cauchy-Schwarz inequality,

$$|h_k^n| \leq \alpha_n^{-2} \sqrt{\int_0^1 (\psi_n(y))^2 \,\mathrm{d}y} \sqrt{\int_0^1 \left(\int_0^1 \frac{1}{x+y+\beta} \overline{P_k^*}(x) \mathrm{d}x \right)^2 \mathrm{d}y}. \tag{3.45}$$

By (2.76) and (3.39),

$$|h_k^n| \leq \alpha_n^{-2} \sqrt{1} \left(\frac{2\sqrt{2k+1}}{\left(1+\tilde{\beta}\right)^{k+1}} \left(\log \left(2 \frac{1+\tilde{\beta}}{\tilde{\beta}} \right) + 1 \right) \right). \tag{3.46}$$

□

3.5 The differential operator $\hat{D}_\omega$ and the expansion of v_n in the basis of Φ_k

In this section we consider the expansion of functions $g \in L^2(0, \infty)$ in the basis of the Laguerre functions Φ_k (the functions defined in (2.47)):

$$g(\omega) = \sum_{k=0}^{\infty} \eta_k \Phi_k(\omega). \tag{3.47}$$

Lemma 3.21 describes the operation of the differential operator $\hat{D}_\omega$ (define in (2.83)) on Φ_k. This relation is used to express the expansion of the left singular function v_n of the operator $\mathcal{L}_{a,b}$ (the operator defined in (2.61)) via a five-terms recurrence relation, or as a solution to a benign eigensystem described in theorem 3.22.

Remark 3.20. Lemma 3.21, theorem 3.22 and the discussion in section 3.5.5 apply to the operators associated with $\mathcal{L}_{a,b}$ (defined in (2.61)) with an arbitrary choice of $0 < a < b < \infty$. Subsections 3.5.1 and 3.5.2 apply to the special cases of $\mathcal{L}_{1/2,\gamma/2}$ and $\mathcal{L}_{1/(2\gamma),1/2}$. Subsections 3.5.3 and 3.5.4 treat to the standard form of the truncated Laplace transform $\mathcal{L}_\gamma$, as defined in (3.5).

Lemma 3.21. *Applying the differential operator $\hat{D}_\omega$ (defined in (2.83)) to the Laguerre function Φ_k (defined in (2.47)) yields a linear combination of the Laguerre functions Φ_{k-2}, Φ_{k-1},*

Φ_k, Φ_{k+1} and Φ_{k+2}:

$$
\begin{aligned}
&\left(\hat{D}_\omega(\Phi_k)\right)(\omega) = \\
&-\frac{1}{16}\left(4a^2-1\right)\left(4b^2-1\right)(k-1)k\Phi_{k-2}(\omega) \\
&+\frac{1}{4}k^2\left(16a^2b^2-1\right)\Phi_{k-1}(\omega) \\
&+\frac{1}{8}\left(k(k+1)\left(-48a^2b^2-4a^2-4b^2-3\right)+\left(-16a^2b^2+12a^2-4b^2-1\right)\right)\Phi_k(\omega) \\
&+\frac{1}{4}(k+1)^2\left(16a^2b^2-1\right)\Phi_{k+1}(\omega) \\
&-\frac{1}{16}\left(4a^2-1\right)\left(4b^2-1\right)(k+2)(k+1)\Phi_{k+2}(\omega).
\end{aligned}
\tag{3.48}
$$

Proof. Applying $\hat{D}_\omega$ to a Laguerre function Φ_k yields

$$
\begin{aligned}
&\left(\hat{D}_\omega(\Phi_k)\right)(x) = \\
&= -\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\omega^2\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\Phi_k(\omega)+(a^2+b^2)\frac{\mathrm{d}}{\mathrm{d}\omega}\omega^2\frac{\mathrm{d}}{\mathrm{d}\omega}\Phi_k(x)+(-a^2b^2\omega^2+2a^2)\Phi_k(\omega)
\end{aligned}
\tag{3.49}
$$

A somewhat tedious derivation from (3.49), using identities (2.39), (2.40) and (2.41), yields (3.48). ☐

Theorem 3.22. *Let $v_n(\omega)$ be the $n+1$-th left singular function of the truncated Laplace transform. Let $\eta^n = (\eta_0^n, \eta_1^n, ...)^\top$ be the vector of coefficients in the expansion of $v_n(\omega)$ in the basis of Laguerre functions Φ_k (the functions defined in (2.47)), such that*

$$
v_n(\omega) = \sum_{k=0}^{\infty}\eta_k^n\Phi_k(\omega). \tag{3.50}
$$

Then, η^n is the $n+1$-th eigenvector of $\hat{M}$:

$$\hat{M}\eta^n = \chi_n^* \eta^n \tag{3.51}$$

where $\hat{M}$ is the symmetric five-diagonal matrix

$$\begin{aligned}
\hat{M}_{k-2,k} &= -\frac{1}{16}\left(4a^2-1\right)\left(4b^2-1\right)(k-1)k \\
\hat{M}_{k-1,k} &= +\frac{1}{4}k^2\left(16a^2b^2-1\right) \\
\hat{M}_{k,k} &= +\frac{1}{8}\left(k(k+1)\left(-48a^2b^2-4a^2-4b^2-3\right)+\left(-16a^2b^2+12a^2-4b^2-1\right)\right) \\
\hat{M}_{k+1,k} &= +\frac{1}{4}(k+1)^2\left(16a^2b^2-1\right) \\
\hat{M}_{k+2,k} &= -\frac{1}{16}\left(4a^2-1\right)\left(4b^2-1\right)(k+2)(k+1),
\end{aligned} \tag{3.52}$$

and where χ_n^* are the eigenvalues of $\hat{D}_\omega$ (defined in (2.83)), and $k = 0, 1, 2 \ldots$.

Proof. By (2.83), the left singular function v_n is an eigenfunction of the differential operator $\hat{D}_\omega$, and therefore

$$\left(\hat{D}_\omega(v_n)\right)(\omega) = \chi^* v_n(\omega). \tag{3.53}$$

Substituting (3.50) into (3.53) and using the linearity of the differential operator, we obtain:

$$\sum_{k=0}^{\infty} \eta_k^n \left(\hat{D}_\omega(\Phi_k)\right)(\omega) = \chi^* \sum_{k=0}^{\infty} \eta_k^n \Phi_k(\omega). \tag{3.54}$$

Using lemma 3.21 and (3.54), we obtain (3.51). □

Observation 3.23. Clearly,

$$\eta_k^n = \int_0^\infty v_n(\omega)\Phi_k(\omega)\mathrm{d}\omega. \tag{3.55}$$

Remark 3.24. Expressing the left singular functions v_n in a similar way, using Hermite polynomials or parabolic cylinder functions, yields a similar framework, with a seven-diagonal matrix $\hat{M}$.

3.5.1 A special case of theorem 3.22: $a = 1/2$

We observe that there are two special choices of a and b for which the matrix $\hat{M}$ (defined in (3.52)) becomes tridiagonal. We briefly describe these two cases in this subsection and in the next subsection.

The substitution of $a = 1/2, b = \gamma/2$ into (3.52) yields the first tridiagonal case of $\hat{M}$:

$$\begin{aligned} \hat{M}_{k-1,k} &= +\frac{1}{4}\left(\gamma^2-1\right)k^2 \\ \hat{M}_{k,k} &= +\frac{1}{4}\left(-\gamma^2-2\left(\gamma^2+1\right)k^2-2\left(\gamma^2+1\right)k+1\right) \\ \hat{M}_{k+1,k} &= +\frac{1}{4}\left(\gamma^2-1\right)(k+1)^2 \end{aligned} \tag{3.56}$$

3.5.2 A special case of theorem 3.22: $b = 1/2$

A substituting of $a = \frac{1}{2\gamma}, b = 1/2$ into (3.52) yields the second tridiagonal case of $\hat{M}$:

$$\begin{aligned} \hat{M}_{k-1,k} &= -\frac{\left(\gamma^2-1\right)k^2}{4\gamma^2} \\ \hat{M}_{k,k} &= -\frac{\left(2\left(\gamma^2+1\right)k^2+2\left(\gamma^2+1\right)k+\gamma^2-1\right)}{4\gamma^2} \\ \hat{M}_{k+1,k} &= -\frac{\left(\gamma^2-1\right)(k+1)^2}{4\gamma^2} \end{aligned} \tag{3.57}$$

3.5.3 A special case of theorem 3.22: the standard form of the truncated Laplace transform, as defined in 3.5

We now consider theorem 3.22 in the case of the standard form $\mathcal{L}_\gamma$ (defined in (3.5)); in other words, we set $a = \frac{1}{2\sqrt{\gamma}}$ and $b = \frac{\sqrt{\gamma}}{2}$ (as defined in (2.61)). We will show that in this case, the even-numbered left singular functions v_{2j} are expanded using only the even-numbered Laguerre functions Φ_{2m}, and that the odd-numbered left singular functions v_{2j+1} are expanded using only the odd-numbered Laguerre functions Φ_{2m+1}. Furthermore, we will show that the expansions of v_{2j} and v_{2j+1} can be obtained from two benign tridiagonal eigensystems.

Observation 3.25. We substitute $a = \frac{1}{2\sqrt{\gamma}}, b = \frac{\sqrt{\gamma}}{2}$ (as specified in (3.3)) into (3.50). We observe that the first off diagonal of $\hat{M}$ vanishes, but the second off diagonal does not vanish:

$$\begin{aligned} \hat{M}_{k-2,k} &= +\frac{(\gamma-1)^2(k-1)k}{16\gamma} \\ \hat{M}_{k,k} &= +\frac{\left(\left(-\gamma^2-6\gamma-1\right)k(k+1)-\gamma^2-2\gamma+3\right)}{8\gamma} \\ \hat{M}_{k+2,k} &= +\frac{(\gamma-1)^2(k+1)(k+2)}{16\gamma} \end{aligned} \tag{3.58}$$

Let $\hat{M}_{i,j}$ be an entry of $\hat{M}$ that does not vanish. Then, we observe that both i and j must be even or both must be odd. In other words, the non-zero elements can be found only in even-numbered columns of even-numbered rows, and in odd-numbered columns of odd-numbered rows of $\hat{M}$.

We split the matrix $\hat{M}$ into two matrices; one of the matrices contains all the even rows of all the even columns, and the other matrix contains all the odd rows of all the odd columns.

These are the two tridiagonal matrices $\hat{M}^{even}$ and $\hat{M}^{even}$, specified by the formulas:

$$\begin{aligned}\hat{M}^{even}_{j-1,j} =& \frac{(\gamma-1)^2(2j-1)2j}{16\gamma} \\ \hat{M}^{even}_{j,j} =& \frac{\left(\left(-\gamma^2-6\gamma-1\right)2j(2j+1)-\gamma^2-2\gamma+3\right)}{8\gamma} \\ \hat{M}^{even}_{j+1,j} =& \frac{(\gamma-1)^2(2j+1)(2j+2)}{16\gamma}\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}\hat{M}^{odd}_{j-1,j} =& \frac{(\gamma-1)^2(2j)(2j+1)}{16\gamma} \\ \hat{M}^{odd}_{j,j} =& \frac{\left(\left(-\gamma^2-6\gamma-1\right)(2j+1)(2j+2)-\gamma^2-2\gamma+3\right)}{8\gamma} \\ \hat{M}^{odd}_{j+1,j} =& \frac{(\gamma-1)^2(2j+2)(2j+3)}{16\gamma}.\end{aligned} \tag{3.60}$$

We introduce the notation $\eta^{even,j}$ and $\chi^{*,even}_j$ for the $j+1$-th eigenvector and eigenvalue of $\hat{M}^{even}$, and $\eta^{odd,j}$ and $\chi^{*,odd}_j$ for the $j+1$-th eigenvector and eigenvalue of $\hat{M}^{odd}$;

$$\hat{M}^{even}\eta^{even,j} = \chi^{*,even}_j\eta^{even,j}, \tag{3.61}$$

and

$$\hat{M}^{odd}\eta^{odd,j} = \chi^{*,odd}_j\eta^{odd,j}. \tag{3.62}$$

Observation 3.26. Let χ^* be an eigenvalue of $\hat{M}$. Then, χ^* is either an eigenvalue of $\hat{M}^{even}$, or an eigenvalue of $\hat{M}^{odd}$. Any eigenvalue of $\hat{M}^{even}$ or $\hat{M}^{odd}$ is an eigenvalue of $\hat{M}$.

Observation 3.27. The vector $\left(\eta^{even,j}_0, 0, \eta^{even,j}_1, 0, ...\right)^\top$ is an eigenvector of $\hat{M}$ with the eigenvalue $\chi^{*,even}_j$.

Observation 3.28. The vector $\left(0, \eta^{odd,j}_0, 0, \eta^{odd,j}_1, 0, ...\right)^\top$ is an eigenvector of $\hat{M}$ with the eigenvalue $\chi_j^{*,odd}$.

We introduce the notation $v_j^{even}(\omega)$, $v_j^{odd}(\omega)$:

$$v_j^{even}(\omega) = \sum_{l=0}^{\infty} \eta_l^{even,j} \Phi_{2l}(\omega) \tag{3.63}$$

$$v_j^{odd}(\omega) = \sum_{l=0}^{\infty} \eta_l^{odd,j} \Phi_{2l+1}(\omega). \tag{3.64}$$

Observation 3.29. Each function in the sequences v_j^{even} and v_j^{odd} is a left singular function. Each left singular function is either in the sequence of functions v_j^{even}, or in the sequence v_j^{odd}.

It remains to be shown which function, in which of the two sequences v_j^{even} and v_j^{odd}, corresponds to the $n+1$-th left singular function v_n.

Lemma 3.30. *Let $\eta^{even,j}$ be the $j+1$-th eigenvector of $\hat{M}^{even}$ and let v_j^{even} be as defined in (3.63).*

Let $\eta^{odd,j}$ be the $j+1$-th eigenvector of $\hat{M}^{odd}$ (defined in (3.60)) and let v_j^{odd} be as defined in (3.64).

Then,

$$\begin{aligned} v_{2j}(\omega) &= v_j^{even}(\omega) \\ v_{2j+1}(\omega) &= v_j^{odd}(\omega). \end{aligned} \tag{3.65}$$

Proof. By observation 3.29, v_j^{even} is a left singular function. Let u_m be the corresponding right singular function of the truncated Laplace transform $\mathcal{L}$ (the operator defined in (3.5)). By (3.63) and (2.67),

$$u_m = \alpha_m^{-1} \sum_{l=0}^{\infty} \eta_l^{even,j} (\mathcal{L}_\gamma)^*(\Phi_{2l}). \tag{3.66}$$

We multiply both sides of (3.66) by the operator C_γ (as defined in (3.12)), and use the definition of U_n in (3.20), to obtain

$$U_m = \alpha_m^{-1} \sum_{l=0}^{\infty} \eta_l^{even,j} \left(C_\gamma \circ (\mathcal{L}_\gamma)^*\right)(\Phi_{2l}). \tag{3.67}$$

By (3.17), the functions $\left(\left(C_\gamma \circ (\mathcal{L}_\gamma)^*\right)(\Phi_{2l})\right)(s)$ are even functions, so U_m is an even function; therefore, by (3.25), m must be an even number. In other words, v_j^{even} is the even-numbered left singular function v_m.

By a similar argument, v_j^{odd} is an odd-numbered left singular function. In other words, the sequence of functions v_j^{even} is the sequence of even-numbered left singular functions v_n and the sequence of functions v_j^{odd} is the sequence of odd-numbered left singular functions v_n. Based on these facts, it is a matter of simple bookkeeping to obtain (3.65) using observation 3.26.

□

3.5.4 Additional properties of v_n in the case of the standard form of the truncated Laplace transform

η_0^n and η_1^n, the first two coefficients in the expansion (3.50) of v_n, are related to $U_n(0)$ (the function defined in (3.20), at the $s = 0$) and to the value of the derivative $U_n'(0)$.

Lemma 3.31. *Let u_n and v_n be the $n+1$-th right and left singular function of $\mathcal{L}_\gamma$ (defined in (3.5)). Let η_0^n and η_1^n be the first and second coefficients in the expansion defined in (3.50), of v_n in the basis of Laguerre functions Φ_k (the functions defined in (2.47)). Let U_n be $C_\gamma u_n$, as defined in (3.20). Then:*

$$U_n(0) = \alpha_n^{-1} \eta_0^n / 2, \tag{3.68}$$

and

$$U_n'(0) = \alpha_n^{-1}\eta_1^n \log(\gamma)/4. \tag{3.69}$$

Proof. By (3.50) and (2.67),

$$u_n = \alpha_n^{-1}\sum_{k=0}^{\infty}\eta_k^n(\mathcal{L}_\gamma)^*(\Phi_k). \tag{3.70}$$

We apply the operator C_γ (defined in (3.12)) to (3.70), and use (3.20) to obtain

$$U_n(s) = \alpha_n^{-1}\sum_{k=0}^{\infty}\eta_k^n\left(\left(C_\gamma\circ(\mathcal{L}_\gamma)^*\right)(\Phi_k)\right)(s). \tag{3.71}$$

In particular, at the point $s = 0$,

$$U_n(0) = \alpha_n^{-1}\sum_{k=0}^{\infty}\eta_k^n\left(\left(C_\gamma\circ(\mathcal{L}_\gamma)^*\right)(\Phi_k)\right)(0) \tag{3.72}$$

We then use (3.18) to obtain (3.68).

We differentiate (3.71) and set $s = 0$;

$$U_n'(0) = \alpha_n^{-1}\sum_{k=0}^{\infty}\eta_k^n\left(\left(C_\gamma\circ(\mathcal{L}_\gamma)^*\right)(\Phi_k)\right)'(0). \tag{3.73}$$

We use (3.19) to obtain (3.69). □

Remark 3.32. Similar relations for the value of the right singular function $u_n(1/2)$ of $\mathcal{L}_\gamma$ (3.5) at $t = 1/2$ and for the derivative $u_n'(1/2)$ are easy to obtain from lemma 3.31 or by a similar construction.

Remark 3.33. In the other spacial cases of $\mathcal{L}_{a,b}$, where $a = 1/2$ or $b = 1/2$, similar relations exist between the value of the function u_n at the ends of the interval $[a, b]$ and the first coefficients in the expansion.

3.5.5 Decay of the coefficients in the expansion of v_n in the basis of Φ_k

The left singular functions v_n are smooth functions, and they are therefore efficiently expressed using Laguerre functions Φ_k (the functions defined in (2.47)). In other words, we expect the coefficients in the expansion (3.50) to decay rapidly. In this section we derive a bound for the rate of decay of these coefficients.

Lemma 3.34. *Given an arbitrary choice of $0 < a < b < \infty$, we consider the SVD of the operator $\mathcal{L}_{a,b}$ (defined in 2.61). Let η_k^n be the $k+1$-th coefficient in the expansion of the $n+1$-th left singular function v_n in the basis of Laguerre functions (the functions Φ_k, defined in (2.47)).*
We define $\gamma = b/a$ and introduce the notation

$$s_{max} = \max\left(\left|2\frac{\log 2a}{\log\gamma}\right|, \left|2\frac{\log 2b}{\log\gamma}\right|\right). \tag{3.74}$$

Then,

$$s_{max} \geq 1 \tag{3.75}$$

and

$$|\eta_k^n| \leq \alpha_n^{-1}\sqrt{\frac{2}{\log\gamma}}\left|\frac{\gamma^{s_{max}/2}-1}{\gamma^{s_{max}/2}+1}\right|^k. \tag{3.76}$$

In particular, in the case $\mathcal{L}_\gamma = \mathcal{L}_{\frac{1}{2\sqrt{\gamma}},\frac{\sqrt{\gamma}}{2}}$ (as defined in (3.5)),

$$|\eta_k^n| \leq \alpha_n^{-1}\sqrt{\frac{2}{\log\gamma}}\left|1-\frac{2}{1+\sqrt{\gamma}}\right|^k. \tag{3.77}$$

Proof. By (3.55) and (2.66),

$$\eta_k^n = \alpha_n^{-1} \int_0^\infty \left(\int_a^b e^{-\omega t} u_n(t) \mathrm{d}t \right) \Phi_k(\omega) \mathrm{d}\omega. \tag{3.78}$$

Changing the order of integration and using (2.62),

$$\eta_k^n = \alpha_n^{-1} \int_a^b u_n(t) \left((\mathcal{L}_{a,b})^* (\Phi_k) \right) (t) \mathrm{d}t. \tag{3.79}$$

A simple calculation using (3.11), (3.12) and (3.20) shows that

$$\eta_k^n = \alpha_n^{-1} \int_{2\frac{\log 2a}{\log \gamma}}^{2\frac{\log 2b}{\log \gamma}} U_n(s) \left(\left(C_\gamma \circ (\mathcal{L}_{a,b})^* \right) (\Phi_k) \right) (s) \mathrm{d}s. \tag{3.80}$$

By the Cauchy-Schwarz inequality,

$$|\eta_k^n| \le \alpha_n^{-1} \sqrt{\int_{2\frac{\log 2a}{\log \gamma}}^{2\frac{\log 2b}{\log \gamma}} (U_n(s))^2 \,\mathrm{d}s} \sqrt{\int_{2\frac{\log 2a}{\log \gamma}}^{2\frac{\log 2b}{\log \gamma}} \left(\left(C_\gamma \circ (\mathcal{L}_{a,b})^* \right) (\Phi_k) \right)^2 (s) \mathrm{d}s}, \tag{3.81}$$

and by (3.21),

$$|\eta_k^n| \le \alpha_n^{-1} \frac{2}{\sqrt{\log \gamma}} \sqrt{\int_{2\frac{\log 2a}{\log \gamma}}^{2\frac{\log 2b}{\log \gamma}} \left(\left(C_\gamma \circ (\mathcal{L}_{a,b})^* \right) (\Phi_k) \right)^2 (s) \mathrm{d}s}. \tag{3.82}$$

We observe that s_{max} is the supremum of $|s|$, where $s \in (2\frac{\log 2a}{\log \gamma}, 2\frac{\log 2b}{\log \gamma})$. In other words, s_{max} is the largest magnitude of the variable s in the integration (3.82). It is easy to observe that s_{max} is no smaller than 1. By (3.16),

$$\left| \left(\left(C_\gamma \circ \left((\mathcal{L}_{a,b})^* \right) (\Phi_k) \right) \right) (s) \right| \le \frac{1}{2} \left| \frac{\gamma^{s_{max}/2} - 1}{\gamma^{s_{max}/2} + 1} \right|^k. \tag{3.83}$$

For a given ratio $\gamma = b/a$, it is easy to observe that the length of the interval $(2\frac{\log 2a}{\log \gamma}, 2\frac{\log 2b}{\log \gamma})$

in the integral (3.82) is 2. So, by (3.82) and (3.83),

$$|\eta_k^n| \leq \alpha_n^{-1} \frac{\sqrt{2}}{\sqrt{\log \gamma}} \left| \frac{\gamma^{s_{max}/2} - 1}{\gamma^{s_{max}/2} + 1} \right|^k. \tag{3.84}$$

In the case of the standard form of the truncated Laplace transform $\mathcal{L}_\gamma$, where $a = \frac{1}{2\sqrt{\gamma}}, b = \frac{\sqrt{\gamma}}{2}$, this interval becomes $(-1, 1)$, and $s_{max} = 1$. So, for the standard form of the truncated Laplace transform,

$$|\eta_k^n| \leq \alpha_n^{-1} \frac{\sqrt{2}}{\sqrt{\log \gamma}} \left| \frac{\gamma^{1/2} - 1}{\gamma^{1/2} + 1} \right|^k = \alpha_n^{-1} \frac{\sqrt{2}}{\sqrt{\log \gamma}} \left| 1 - \frac{2}{1 + \sqrt{\gamma}} \right|^k. \tag{3.85}$$

□

Observation 3.35. Let $\tilde{v}_n$ be the $n+1$-th left singular function of $\mathcal{L}_{a,b}$ (defined in (2.61)). Let v_n be the $n+1$-th left singular function of $\mathcal{L}_\gamma$ (the operator in the standard form, as defined in (3.5)), where $b/a = \gamma$. Let the vectors η^n and $\tilde{\eta}^n$ represent the expansions, defined in (3.50), of v_n and $\tilde{v}_n$.

In the case of $\mathcal{L}_\gamma$, we have $s_{max} = 1$, and it is easy to observe that the bound (3.77) for $|\eta_k^n|$ decays faster than the bound (3.76) for the general $|\tilde{\eta}_k^n|$.

3.6 A remark about the limit $\gamma \to 1$

Throughout this dissertation we have assumed that the parameter $\gamma = b/a$ in (3.5) is strictly larger than 0. In this section, we describe some properties of the differential operator $\hat{D}_\omega$ (defined in (2.82)) and its eigenfunctions v_n at the limit $\gamma \to 1$. Other aspects of this limit are discussed in [6].

By substituting $b = a$ into (2.82),

$$\left(\hat{D}_\omega(f)\right)(\omega) = -\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\omega^2\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}f(\omega) + 2a^2\frac{\mathrm{d}}{\mathrm{d}\omega}\omega^2\frac{\mathrm{d}}{\mathrm{d}\omega}f(\omega) + \left(-a^4\omega^2 + 2a^2\right)f(\omega). \tag{3.86}$$

In particular, substituting $a = b = 1/2$ into (2.82) yields

$$\left(\hat{D}_\omega(f)\right)(\omega) = -\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\omega^2\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}f(\omega) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\omega}\omega^2\frac{\mathrm{d}}{\mathrm{d}\omega}f(\omega) + \left(-\frac{1}{16}\omega^2 + \frac{1}{2}\right)f(\omega). \tag{3.87}$$

Theorem 3.22 provides a relation between the operator $\hat{D}_\omega$ and the matrix $\hat{M}$ (see (3.52)). By substituting $a = b = 1/2$ into (3.52), we obtain a diagonal matrix:

$$\hat{M}_{k,k} = -\,k(k+1). \tag{3.88}$$

Clearly, the eigenvalues of this matrix are

$$\chi^* = -k(k+1), \tag{3.89}$$

and the eigenvectors are simply $(0, .., 0, 1, 0, ...)^\top$. By theorem 3.22, this means that for $\gamma = 1$, the $n+1$-th eigenfunction v_n of the differential operator $\hat{D}_\omega$, is the Laguerre function Φ_n (the function defined in (2.47)), and the $n+1$-th eigenvalue of $\hat{D}_\omega$ is $\chi_n^* = -n(n+1)$. In other words, the Laguerre function Φ_n is the solution of the differential equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\omega^2\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\Phi_n + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\omega}\omega^2\frac{\mathrm{d}}{\mathrm{d}\omega}\Phi_n + \left(-\frac{1}{16}\omega^2 + \frac{1}{2} + n(n+1)\right)\Phi_n = 0. \tag{3.90}$$

3.7 A relation between the $n+1$-th and $m+1$-th singular functions, and the ratio α_n/α_m

Lemma 3.36. *Let u_n and u_m be right singular functions, and let α_n and α_m be the corresponding singular values of $\mathcal{L}_{a,b}$ (defined in (2.61). Let ψ_n and ψ_m be the corresponding functions defined in (2.75). Then:*

$$\frac{\alpha_m^2}{\alpha_n^2} = \frac{\int_0^1 \psi_n'(x)\psi_m(x)\mathrm{d}x}{\int_0^1 \psi_n(x)\psi_m'(x)\mathrm{d}x} \tag{3.91}$$

and

$$\frac{\alpha_m^2}{\alpha_n^2} = \frac{\int_a^b u_n'(t)u_m(t)\mathrm{d}t}{\int_a^b u_m'(t)u_n(t)\mathrm{d}t}, \tag{3.92}$$

if the integrals are not 0.

Proof. We recall from (2.67) that

$$u_n(t) = \frac{1}{\alpha_n}\left(\mathcal{L}^*(v_n)\right)(t) = \frac{1}{\alpha_n}\int_0^\infty e^{-\omega t}v_n(\omega)\mathrm{d}\omega. \tag{3.93}$$

Therefore, the derivative of $u_n(t)$ is

$$u_n'(t) = \frac{1}{\alpha_n}\int_0^\infty (-\omega)e^{-\omega t}v_n(\omega)\mathrm{d}\omega. \tag{3.94}$$

We multiply both sides of the expression by $u_m(t)$, integrate both sides, and change the order of integration:

$$\int_a^b u_n'(t)u_m(t)\mathrm{d}t = \frac{1}{\alpha_n}\int_a^b \left(\int_0^\infty (-\omega)e^{-\omega t}v_n(\omega)\mathrm{d}\omega\right)u_m(t)\mathrm{d}t. \tag{3.95}$$

By rearranging the result, we obtain

$$\int_a^b u_n'(t)u_m(t)\mathrm{d}t = \frac{\alpha_m}{\alpha_n}\int_0^\infty (-\omega)v_n(\omega)v_m(\omega)\mathrm{d}\omega. \tag{3.96}$$

m and n are clearly interchangeable, so that

$$\int_0^\infty (-\omega)v_n(\omega)v_m(\omega)\mathrm{d}\omega = \frac{\alpha_m}{\alpha_n}\int_a^b u_m'(t)u_n(t)\mathrm{d}t. \tag{3.97}$$

By substituting (3.97) into (3.96), we obtain (3.92). The identity (2.75) is used to obtain (3.91).

□

A similar relation exists for the left singular functions and their derivatives:

Lemma 3.37. *Let v_n and v_m be left singular functions and let α_n and α_m be the corresponding singular values. Then:*

$$\frac{\alpha_m^2}{\alpha_n^2} = \frac{\int_0^\infty v_n'(\omega)v_m(\omega)\mathrm{d}\omega}{\int_0^\infty v_n(\omega)v_m'(\omega)\mathrm{d}\omega}, \tag{3.98}$$

if the integrals are not equal to 0.

The proof is similar to the proof of lemma 3.36.

3.8 A relation between $v_n(0)$, h_0^n and the singular value α_n

The following lemma provides the relation between h_0^n (the first coefficient in the expansion (3.33) of u_n), $v_n(0)$, and the corresponding singular value α_n.

Lemma 3.38. *Let $v_n(\omega)$ be a left singular functions of $\mathcal{L}_\gamma$ (the operator defined in (3.5)). Let u_k be the corresponding right singular function, and let α_n be the corresponding singular value.*

Let ψ_n be as defined in (2.75) and let h^n be the vector of coefficients defined in (3.33). Then,

$$\alpha_n = \sqrt{\frac{\gamma - 1}{2\sqrt{\gamma}} \frac{h_0^n}{v_n(0)}} \tag{3.99}$$

Proof. By the definition of the SVD (2.66),

$$\left(\mathcal{L}(u_n)\right)(\omega) = \alpha_n v_n(\omega). \tag{3.100}$$

In particular, at $\omega = 0$:

$$\alpha_n v_n(0) = \left(\mathcal{L}(u_n)\right)(0) = \int_a^b u_n(t)\mathrm{d}t \tag{3.101}$$

Using the change of variables (2.74), and substituting (2.79) into the last expression, we obtain:

$$\alpha_n v_n(0) = (b-a)\int_0^1 u_n(a + (b-a)x)\mathrm{d}x = \sqrt{(b-a)}\int_0^1 \psi_n(x)\mathrm{d}x \tag{3.102}$$

Expressing ψ_n using the expansion defined in (3.33):

$$\alpha_n v_n(0) = \sqrt{(b-a)}\int_0^1 \left(\sum_{m=0}^{\infty} h_m^n \overline{P_m^*}(x)\right)\mathrm{d}x \tag{3.103}$$

By (2.17), $\overline{P_0^*}(x) \equiv 1$, and since all the other polynomials $\overline{P_k^*}$ are orthogonal to it,

$$\alpha_n v_n(0) = \sqrt{(b-a)}h_0^n \tag{3.104}$$

Substituting the values of a and b, defined in (3.3) into (3.104), we obtain (3.99).

□

3.9 A closed form approximation of the eigenvalues $\tilde{\chi}_n, \chi_n^*, \chi_n$ and singular values α_n

The eigenvalues of differential operators $\tilde{D}_t$, D_x and $\hat{D}$, as the operators are defined in (2.72), (2.80) and (2.82), and as they appear in equations (2.73) , (2.81) and (2.83), have closed form asymptotic expressions; in the case of the standard form $\mathcal{L}_\gamma$ (the operator defined in (3.5)), the eigenvalues of the differential operators are:

$$\chi_n \frac{4\gamma}{(\gamma-1)^2} = \tilde{\chi}_n = \chi_n^* = -\frac{2\gamma^2 + 4\gamma + \frac{\pi^2(\gamma+1)^2\left(n+\frac{1}{2}\right)^2}{K\left(\frac{(\gamma-1)^2}{(\gamma+1)^2}\right)^2} - 6}{16\gamma}\left(1 + O(n^{-2})\right), \tag{3.105}$$

where $K(m)$ is the complete elliptic integral of the first kind (as defined in (2.49)) . These eigenvalues are negative and roughly proportional to $-n(n+1)$. The proof for this asymptotic expression is involved, and it will be provided at a later date.

The singular values α_n also have a closed form asymptotic expression; in the case of the standard form $\mathcal{L}_\gamma$, the singular values are:

$$\alpha_n = \sqrt{2\pi}\exp\left(-\frac{\sqrt{3-\gamma(\gamma+8\chi_n^*+2)}K\left(\frac{4\gamma}{(\gamma+1)^2}\right)}{\sqrt{2}(\gamma+1)}\right)\left(1 + O(n^{-1})\right), \tag{3.106}$$

where $K(m)$ is the complete elliptic integral of the first kind (as defined in (2.49)). The proof for this asymptotic expression is involved, and it will be provided at a later date.

Chapter 4

Algorithms

4.1 Evaluation of the right singular functions u_n

In this section we introduce an algorithm for the numerical evaluation of $u_n(t)$, the $n+1$-th right singular function of $\mathcal{L}_\gamma$ (the operator defined in (3.5)).

We recall that $u_n(t)$ can be easily calculated from the function $\psi_n(x)$ using (2.75). We also recall that $\psi_n(x)$ is efficiently represented in the basis of $\overline{P^*}_k$ (the polynomials defined in (2.16)) and that the expansion of $\psi_n(x)$ in $\overline{P^*}_k$ is related to the $n+1$-th eigenvector of the 5-diagonal matrix specified in theorem 3.16.

The algorithm for obtaining the right singular function $u_n(t)$ is therefore:

- Compute h^n, the $n+1$-th eigenvector of the matrix M, defined in (3.35).
- Compute the function $\psi_n(x)$ from h^n, using the expansion specified in (3.33).
- Obtain $u_n(t)$ from $\psi_n(x)$ using (2.75).

The calculation of the eigenvalues and eigenvectors is done using the Sturm sequence method and the inverse power method, as described in section 2.6.

4.2 Evaluation of the left singular functions v_n

In this section we introduce an algorithm for the numerical evaluation of v_n, the left singular functions of $\mathcal{L}_\gamma$ (the operator defined in (3.5)).

We recall that $v_n(\omega)$, is efficiently expressed in the basis of Laguerre functions as specified in (3.50). We recall that the coefficients of even-numbered left singular functions of $\mathcal{L}_\gamma$ are given by the eigenvectors of the matrix $\hat{M}^{even}$, specified in (3.61). Therefore, the algorithm for computing an even-numbered left singular function v_n where $n = 2j$ is:

- Compute $\eta^{even,j}$, the $j+1$-th eigenvector of the matrix $\hat{M}^{even}$ specified in (3.59).
- Compute the function $v_j^{even}(\omega)$ from $\eta^{even,j}$, using the expansion specified in (3.63).
- By (3.65), $v_n(\omega) = v_j^{even}(\omega)$.

Similarly, the algorithm for computing an odd-numbered left singular function v_n where $n = 2j+1$ is:

- Compute $\eta^{odd,j}$, the $j+1$-th eigenvector of the matrix $\hat{M}^{odd}$ specified in (3.60).
- Compute the function $v_j^{odd}(\omega)$ from $\eta^{odd,j}$, using the expansion specified in (3.64).
- By (3.65), $v_n(\omega) = v_j^{odd}(\omega)$.

The calculation of the eigenvalues and eigenvectors is done using the Sturm sequence method and the inverse power method, as described in section 2.6.

Remark 4.1. Clearly, the left singular functions of $\mathcal{L}_{a,b}$ (the operator defined in (2.61)) can be computed directly, using the matrix $\hat{M}$ described in theorem 3.22.

4.3 Evaluation of the singular values α_n

In this section, we introduce two algorithms for computing the singular value α_n of $\mathcal{L}_\gamma$ (the operator defined in (3.5)).

4.3.1 Calculating the singular value α_n from α_m, via lemma 3.36 or lemma 3.37

Lemma 3.36 provides a way to calculate α_{n+1} from a known α_n using the right singular functions. Suppose that we have the first singular value α_0. Then, we calculate the functions ψ_0 and ψ_1 (the functions as defined in (2.75)) and use (3.91) to calculate α_1 from α_0. To obtain the other singular values, we calculate every α_{n+1} from the previous α_n, using ψ_{n+1} and ψ_n.

There are several obvious methods for evaluating α_0 via numerical integration; for example:

$$\alpha_0 = \sqrt{\frac{\left(\left(\mathcal{L}^* \circ \mathcal{L}\right)(u_0)\right)(t)}{u_0(t)}} \tag{4.1}$$

and

$$\alpha_0 = \frac{\left(\mathcal{L}(u_0)\right)(\omega)}{v_0(\omega)}. \tag{4.2}$$

We use the relation provided in lemma 3.38 to evaluate α_0; in section 4.3.2 we use lemma 3.38 to calculate arbitrary α_n directly.

Remark 4.2. A similar algorithm, based on the left singular functions v_n rather than the right singular functions, is easy to construct using lemma 3.37.

Remark 4.3. Clearly, if $\left(\left(\mathcal{L}^* \circ \mathcal{L}\right)(u_n)\right)(t)$ and $u_n(t)$ are available at sufficient precision, relations like (4.1) can be used to evaluate arbitrary singular values. However, in general, the condition number of the problem does not allow high precision calculation of small singular values using direct integration; and since α_n decays exponentially (see (3.106)), only very few singular values can be calculated by direct numerical integration. The method in section 4.3.2 provides an alternative way of evaluating the singular value α_n.

4.3.2 Calculating the singular value α_n via lemma (3.38)

Let $v_n(\omega)$ be the $n+1$-th left singular function of $\mathcal{L}_\gamma$ (the operator defined in (3.5)). Let α_n be the $n+1$-th singular value. Let h^n be the $n+1$-th eigenvector of the matrix M (the matrix defined in (3.35)). By lemma 3.38,

$$\alpha_n = \sqrt{\frac{\gamma - 1}{2\sqrt{\gamma}} \frac{h_0^n}{v_n(0)}}. \tag{4.3}$$

The values of h_0^n and $v_n(0)$ are obtained through the evaluation of the right and left singular functions, as described in sections 4.1 and 4.2.

Remark 4.4. By (3.63), (3.64), (2.43) and (2.47), $v_{2j}(0)$ is simply the sum of the entries in the eigenvector $\eta^{even,j}$ of $\hat{M}^{even}$ (the matrix defined in (3.59)) and $v_{2j+1}(0)$ is simply the sum of the entries in the eigenvector $\eta^{odd,j}$ of $\hat{M}^{odd}$ (the matrix defined in (3.60));

$$v_{2j}(0) = \sum_{k=0}^{\infty} \eta_k^{even,j}, \tag{4.4}$$

$$v_{2j+1}(0) = \sum_{k=0}^{\infty} \eta_k^{odd,j}. \tag{4.5}$$

Remark 4.5. It has been shown in [14] that in some band matrices, such as the matrix M in (3.35), the first element of the vector h^n can be computed to relative precision, and not just to absolute precision. The analysis is somewhat involved, and it will be reported at a later date.

Chapter 5

Implementation and numerical results

Algorithms for the evaluation of the right singular functions u_n, left singular functions v_n and singular values α_n of $\mathcal{L}_\gamma$ were implemented in FORTRAN 77. In this section, we present examples of numerical experiments. The gfortran compiler, and double precision arithmetic were used in all the experiments, except for the last experiment, where the Fujitsu compiler and quadruple precision were used.

In figure 5.1 we present examples of right singular functions u_n and left singular functions v_n of $\mathcal{L}_\gamma$ (the operator defined in (3.5)), with the parameter $\gamma = 1.1$. The right singular functions are plotted on the interval $\left(\frac{1}{2\sqrt{\gamma}}, \frac{\sqrt{\gamma}}{2}\right)$. The left singular functions are plotted on a subset of the interval $(0, \infty)$.

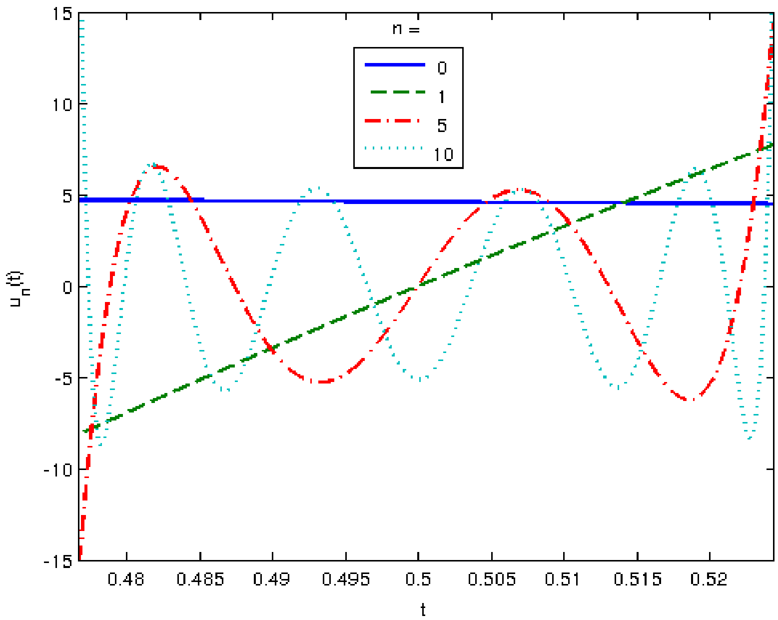
Figure 5.2 is the same as figure 5.1, but with the parameter $\gamma = 10$. Figure 5.3 is the same as figure 5.1, with $\gamma = 10^5$, and a different selection of n.

The singular values α_n of $\mathcal{L}_\gamma$, over a range of n and a range of γ, are presented in table 5.1 and figure 5.4.

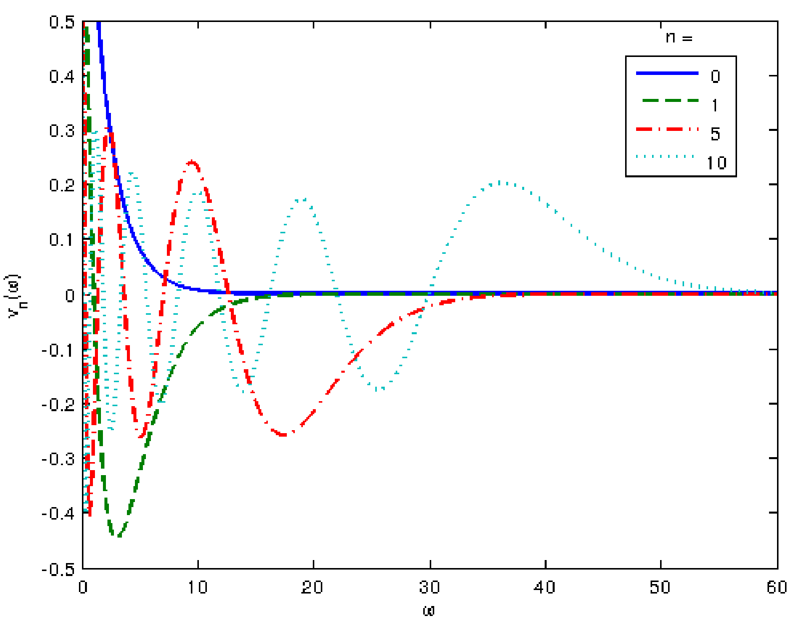
The eigenvalues χ^* of the differential operator $\hat{D}_\omega$ (as defined in (2.82)) are presented in table 5.2 and figure 5.5.

In figure 5.6 we plot of $u_n(a)$; the right singular function of $\mathcal{L}_\gamma$, evaluated at the point $a = \frac{1}{2\sqrt{\gamma}}$. In figure 5.7 we plot $v_n(0)$; the left singular function, evaluated at the point $\omega = 0$. An analysis of the properties of u_n and v_n at these endpoints will be presented at a later date.

In table 5.3 we present several singular values smaller than 10^{-1000}.

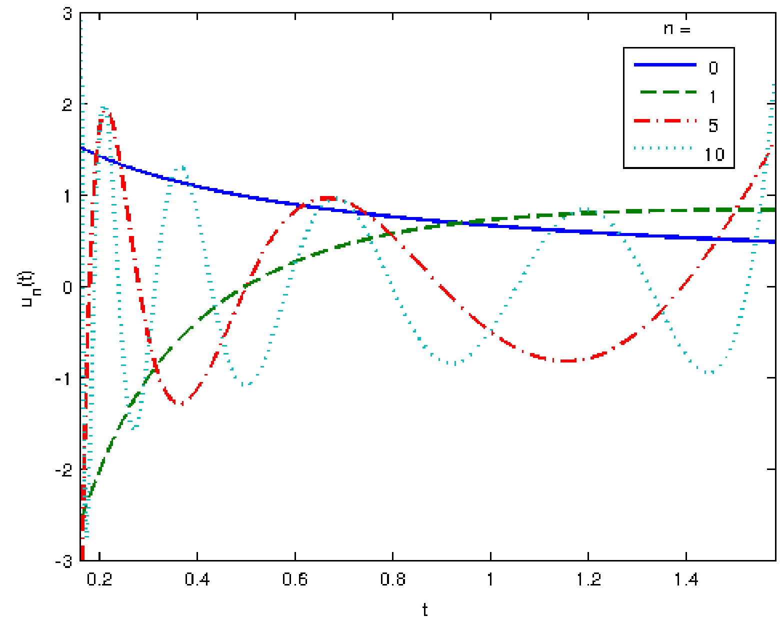


(a) Right singular functions u_n.

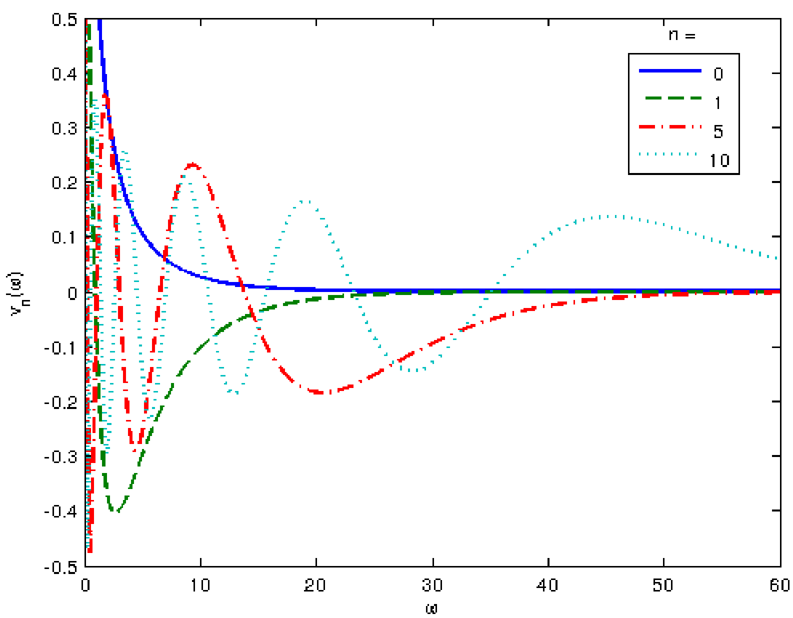


(b) Left singular functions v_n.

Figure 5.1: Singular functions of $\mathcal{L}_\gamma$, where $\gamma = 1.1$.

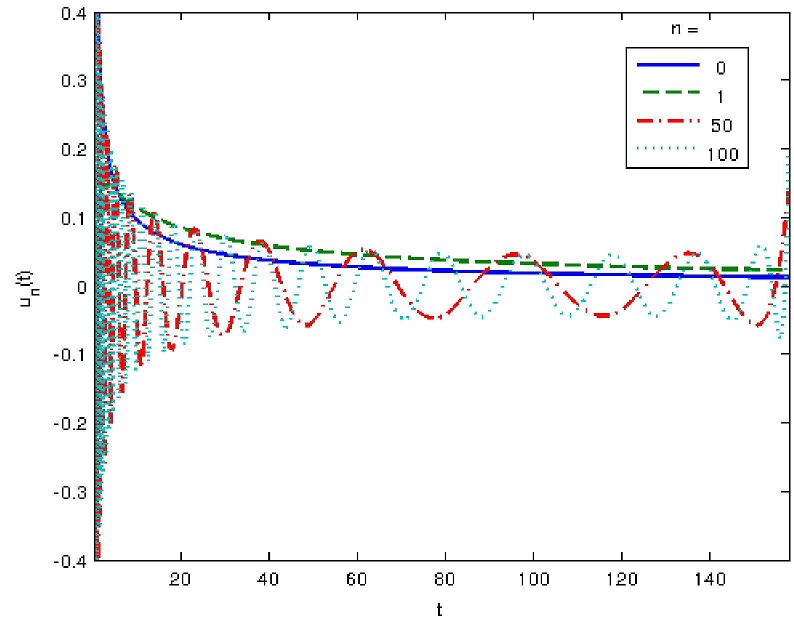


(a) Right singular functions u_n.

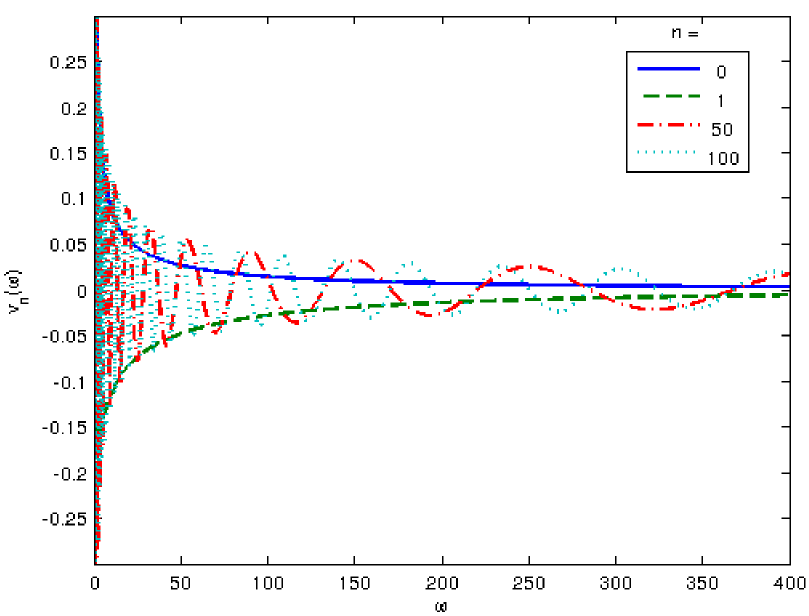


(b) Left singular functions v_n.

Figure 5.2: Singular functions of $\mathcal{L}_\gamma$, where $\gamma = 10$.



(a) Right singular functions u_n.



(b) Left singular functions v_n.

Figure 5.3: Singular functions of $\mathcal{L}_\gamma$, where $\gamma = 10^5$.

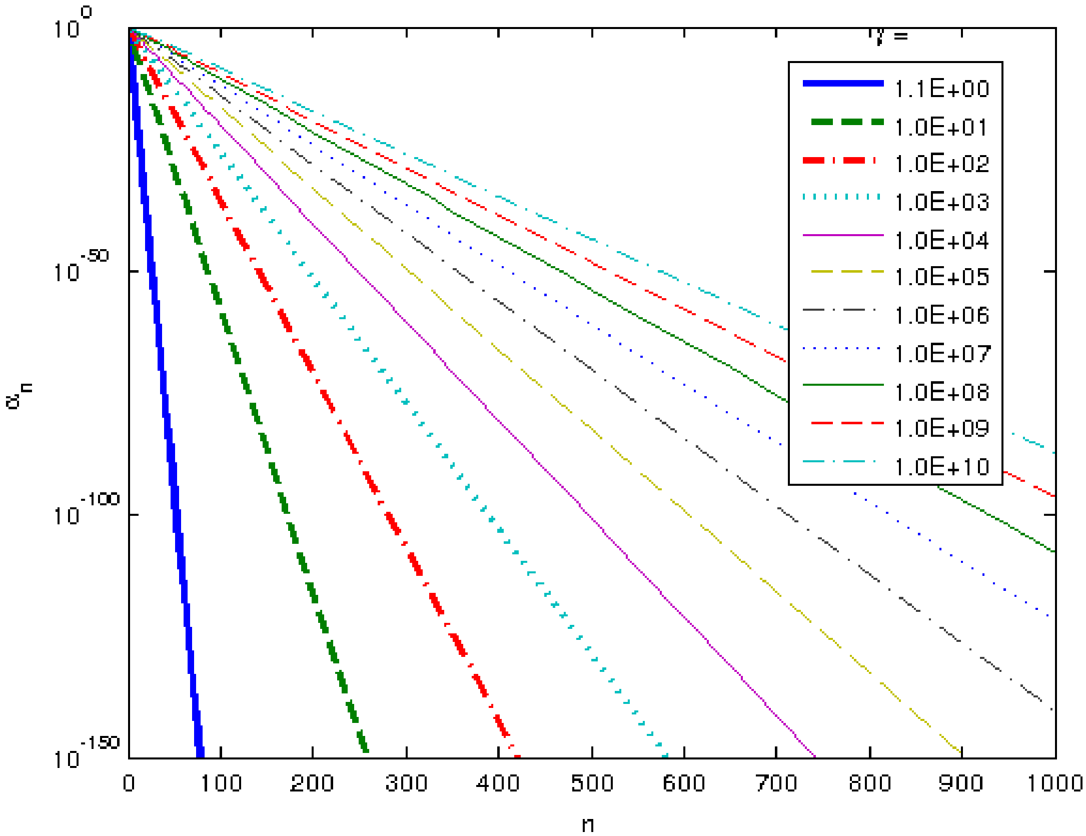


Figure 5.4: Singular values α_n of $\mathcal{L}_\gamma$.

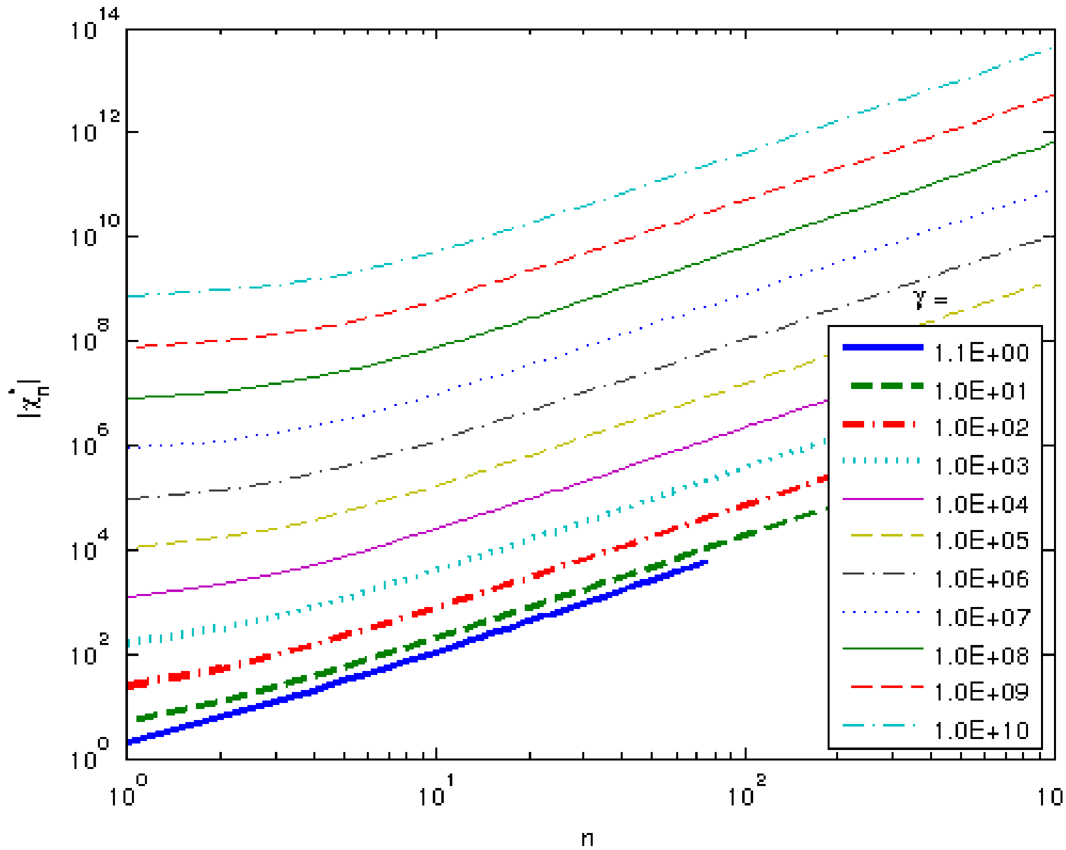


Figure 5.5: The magnitude of the eigenvalues χ_n^* of the differential operator $\hat{D}_\omega$ (defined in (2.82)).

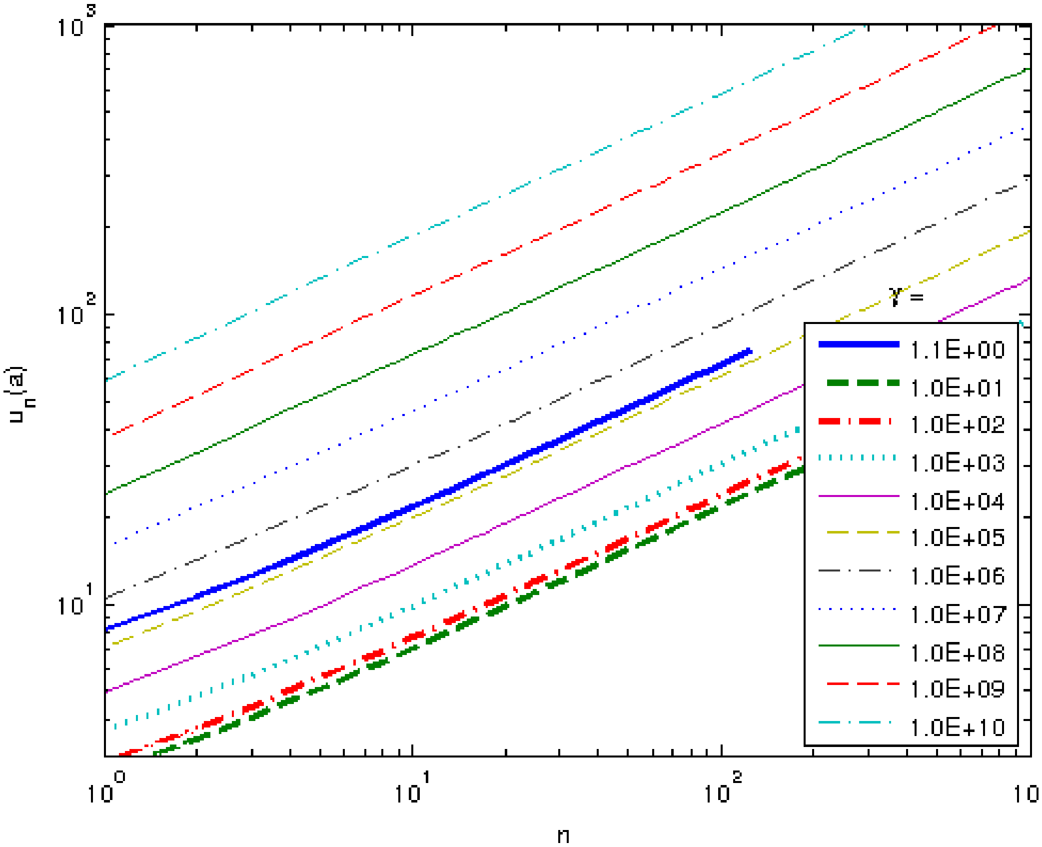


Figure 5.6: $u_n\left(1/(2\sqrt{\gamma})\right)$. The right singular functions, evaluated at $t = a = 1/(2\sqrt{\gamma})$.

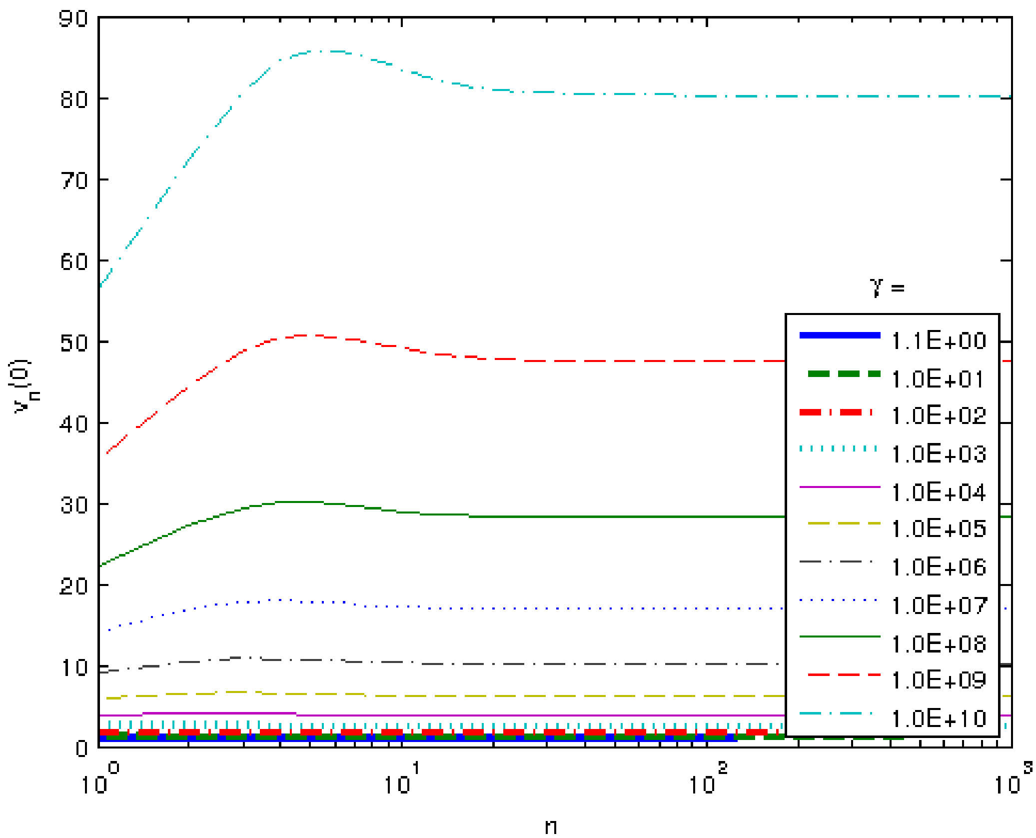


Figure 5.7: $v_n(0)$. The left singular functions, evaluated at $\omega = 0$.

Table 5.1: Singular values α_n of $\mathcal{L}_\gamma$

n	γ =1.1E+00	γ =1.0E+01	γ =1.0E+02	γ =1.0E+04	γ =1.0E+06	γ =1.0E+08	γ =1.0E+10
0	2.18280E-01	1.02356E+00	1.31941E+00	1.55687E+00	1.64778E+00	1.69163E+00	1.71595E+00
1	3.00227E-03	3.09878E-01	6.68211E-01	1.12288E+00	1.35702E+00	1.48763E+00	1.56644E+00
2	3.69344E-05	8.39567E-02	3.04070E-01	7.39927E-01	1.04024E+00	1.23673E+00	1.36792E+00
3	4.46186E-07	2.23263E-02	1.35394E-01	4.73173E-01	7.71417E-01	9.95863E-01	1.16064E+00
4	5.35677E-09	5.90020E-03	5.98904E-02	2.99697E-01	5.64351E-01	7.89136E-01	9.68344E-01
10	1.55445E-20	1.94760E-06	4.34546E-04	1.86336E-02	8.19585E-02	1.81020E-01	2.96456E-01
20	8.99018E-40	3.00805E-12	1.15751E-07	1.77967E-04	3.20877E-03	1.50761E-02	3.95113E-02
30	5.17974E-59	4.62827E-18	3.07157E-11	1.69324E-06	1.25143E-04	1.25067E-03	5.24436E-03
40	2.98139E-78	7.11415E-24	8.14269E-15	1.60942E-08	4.87574E-06	1.03648E-04	6.95389E-04
50	1.71536E-97	1.09308E-29	2.15775E-18	1.52914E-10	1.89890E-07	8.58629E-06	9.21696E-05
60	9.86744E-117	1.67918E-35	5.71671E-22	1.45257E-12	7.39392E-09	7.11151E-07	1.22140E-05
70	5.67549E-136	2.57922E-41	1.51440E-25	1.37967E-14	2.87871E-10	5.88936E-08	1.61838E-06
80		3.96141E-47	4.01148E-29	1.31033E-16	1.12070E-11	4.87688E-09	2.14423E-07
90		6.08399E-53	1.06254E-32	1.24442E-18	4.36274E-13	4.03827E-10	2.84079E-08
100		9.34359E-59	2.81432E-36	1.18179E-20	1.69830E-14	3.34375E-11	3.76350E-09
150		7.98028E-88	3.66768E-54	9.12588E-31	1.51765E-21	1.30108E-16	1.53547E-13
200		6.81449E-117	4.77880E-72	7.04566E-41	1.35593E-28	5.06159E-22	6.26325E-18
250		5.81852E-146	6.22602E-90	5.43916E-51	1.21135E-35	1.96895E-27	2.55460E-22
300			8.11118E-108	4.19880E-61	1.08214E-42	7.65882E-33	1.04190E-26
350			1.05669E-125	3.24121E-71	9.66687E-50	2.97907E-38	4.24935E-31
400			1.37659E-143	2.50198E-81	8.63540E-57	1.15876E-43	1.73305E-35
450				1.93132E-91	7.71391E-64	4.50712E-49	7.06798E-40
500				1.49081E-101	6.89071E-71	1.75309E-54	2.88254E-44
550				1.15077E-111	6.15533E-78	6.81876E-60	1.17559E-48
600				8.88291E-122	5.49840E-85	2.65220E-65	4.79437E-53
650				6.85676E-132	4.91158E-92	1.03159E-70	1.95528E-57
700				5.29275E-142	4.38737E-99	4.01240E-76	7.97413E-62
750					3.91910E-106	1.56063E-81	3.25205E-66
800					3.50081E-113	6.07013E-87	1.32627E-70
850					3.12716E-120	2.36099E-92	5.40884E-75
900					2.79338E-127	9.18312E-98	2.20585E-79
950					2.49523E-134	3.57179E-103	8.99599E-84
1000					2.22890E-141	1.38925E-108	3.66878E-88

Table 5.2: Eigenvalues χ_n^* of the differential operator $\hat{D}_\omega$ (defined in 2.82).

n	γ =1.1E+00	γ =1.0E+01	γ =1.0E+02	γ =1.0E+04	γ =1.0E+06	γ =1.0E+08	γ =1.0E+10
0	$-4.65907E-02$	$-1.37081E+00$	$-9.86948E+00$	$-7.68147E+02$	$-7.02559E+04$	$-6.73781E+06$	$-6.58542E+08$
1	$-2.04886E+00$	$-4.99310E+00$	$-2.38874E+01$	$-1.24392E+03$	$-9.47054E+04$	$-8.24168E+06$	$-7.60836E+08$
2	$-6.05341E+00$	$-1.22170E+01$	$-5.13245E+01$	$-2.12394E+03$	$-1.38128E+05$	$-1.08500E+07$	$-9.35829E+08$
3	$-1.20602E+01$	$-2.30561E+01$	$-9.25437E+01$	$-3.43924E+03$	$-2.02129E+05$	$-1.46429E+07$	$-1.18769E+09$
4	$-2.00693E+01$	$-3.75087E+01$	$-1.47522E+02$	$-5.19520E+03$	$-2.87361E+05$	$-1.96677E+07$	$-1.51947E+09$
10	$-1.10172E+02$	$-2.00102E+02$	$-7.66098E+02$	$-2.49694E+04$	$-1.24793E+06$	$-7.62302E+07$	$-5.24213E+09$
20	$-4.20524E+02$	$-7.60147E+02$	$-2.89677E+03$	$-9.30877E+04$	$-4.55775E+06$	$-2.71192E+08$	$-1.80759E+10$
30	$-9.31103E+02$	$-1.68151E+03$	$-6.40208E+03$	$-2.05154E+05$	$-1.00030E+07$	$-5.91946E+08$	$-3.91911E+10$
40	$-1.64191E+03$	$-2.96419E+03$	$-1.12820E+04$	$-3.61167E+05$	$-1.75837E+07$	$-1.03849E+09$	$-6.85869E+10$
50	$-2.55294E+03$	$-4.60820E+03$	$-1.75366E+04$	$-5.61128E+05$	$-2.72998E+07$	$-1.61081E+09$	$-1.06263E+11$
60	$-3.66420E+03$	$-6.61352E+03$	$-2.51658E+04$	$-8.05036E+05$	$-3.91512E+07$	$-2.30893E+09$	$-1.52220E+11$
70	$-4.97569E+03$	$-8.98016E+03$	$-3.41696E+04$	$-1.09289E+06$	$-5.31381E+07$	$-3.13283E+09$	$-2.06458E+11$
80		$-1.17081E+04$	$-4.45480E+04$	$-1.42470E+06$	$-6.92604E+07$	$-4.08252E+09$	$-2.68976E+11$
90		$-1.47974E+04$	$-5.63011E+04$	$-1.80045E+06$	$-8.75181E+07$	$-5.15799E+09$	$-3.39774E+11$
100		$-1.82480E+04$	$-6.94288E+04$	$-2.22014E+06$	$-1.07911E+08$	$-6.35925E+09$	$-4.18853E+11$
150		$-4.09208E+04$	$-1.55687E+05$	$-4.97785E+06$	$-2.41908E+08$	$-1.42523E+10$	$-9.38456E+11$
200		$-7.26265E+04$	$-2.76311E+05$	$-8.83424E+06$	$-4.29289E+08$	$-2.52901E+10$	$-1.66507E+12$
250		$-1.13365E+05$	$-4.31300E+05$	$-1.37893E+07$	$-6.70055E+08$	$-3.94725E+10$	$-2.59870E+12$
300			$-6.20655E+05$	$-1.98431E+07$	$-9.64207E+08$	$-5.67996E+10$	$-3.73934E+12$
350			$-8.44376E+05$	$-2.69955E+07$	$-1.31174E+09$	$-7.72713E+10$	$-5.08700E+12$
400			$-1.10246E+06$	$-3.52467E+07$	$-1.71266E+09$	$-1.00888E+11$	$-6.64167E+12$
450				$-4.45965E+07$	$-2.16697E+09$	$-1.27649E+11$	$-8.40335E+12$
500				$-5.50450E+07$	$-2.67466E+09$	$-1.57554E+11$	$-1.03720E+13$
550				$-6.65922E+07$	$-3.23574E+09$	$-1.90605E+11$	$-1.25478E+13$
600				$-7.92381E+07$	$-3.85020E+09$	$-2.26800E+11$	$-1.49305E+13$
650				$-9.29826E+07$	$-4.51805E+09$	$-2.66139E+11$	$-1.75202E+13$
700				$-1.07826E+08$	$-5.23928E+09$	$-3.08624E+11$	$-2.03170E+13$
750					$-6.01390E+09$	$-3.54253E+11$	$-2.33207E+13$
800					$-6.84190E+09$	$-4.03026E+11$	$-2.65315E+13$
850					$-7.72328E+09$	$-4.54945E+11$	$-2.99493E+13$
900					$-8.65806E+09$	$-5.10008E+11$	$-3.35741E+13$
950					$-9.64621E+09$	$-5.68215E+11$	$-3.74059E+13$
1000					$-1.06878E+10$	$-6.29568E+11$	$-4.14447E+13$

Table 5.3: Examples of singular values α_n smaller than 10^{-1000}

γ	n	α_n
$1.1E+0$	520	$8.70727E-1002$
$1.0E+1$	1721	$3.66934E-1001$
$1.0E+2$	2797	$5.29961E-1001$
$1.0E+3$	3872	$5.71146E-1001$
$1.0E+4$	4946	$9.44191E-1001$
$1.0E+5$	6021	$8.89748E-1001$

Chapter 6

Conclusions and generalizations

In this dissertation we have introduced efficient algorithms for the evaluation of the singular functions and singular values of the truncated Laplace transform.

Among the obvious generalizations of this work, is the Laplace transform in higher dimensions. Another closely related object is the *two-sided* band-limited Laplace transform, $\tilde{\tilde{\mathcal{L}}}_c$; for a given $c \in \mathbb{C}$ and a function $f \in L^2(-1,1)$, the later is defined by the formula

$$\left(\tilde{\tilde{\mathcal{L}}}_c(f)\right)(\omega) = \int_{-1}^{1} e^{-ct\omega} f(t)\mathrm{d}t. \tag{6.1}$$

As we will report in more detail at a later date, much of the analysis of $\tilde{\mathcal{F}}_c$ (the operator defined in (1.3)) has a natural extension to $\tilde{\tilde{\mathcal{L}}}_c$.

One of the results of this work will be the construction of interpolation formulas in the span of right or left singular functions, as well as associated quadrature formulas.

In a future paper we will discuss asymptotic properties of the truncated Laplace transform and of the associated differential operators, and the relations between all these operators.

Bibliography

[1] M. Bertero, P. Boccacci, and E. R. Pike, "On the recovery and resolution of exponential relaxation rates from experimental data: A singular-value analysis of the laplace transform inversion in the presence of noise," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 383, pp. 15–29, Sept. 1982.

[2] D. Slepian and H. Pollak, "Prolate spheroidal wave functions, fourier analysis, and uncertainty i," *Bell Syst. Tech. J*, Jan. 1961.

[3] H. Xiao, V. Rokhlin, and N. Yarvin, "Prolate spheroidal wavefunctions, quadrature and interpolation," *Inverse Problems*, vol. 17, pp. 805–838, Aug. 2001.

[4] A. Osipov, V. Rokhlin, and H. Xiao, *Prolate spheroidal wave functions of order zero: mathematical tools for bandlimited approximation.* New York: Springer, 2013.

[5] M. Bertero and F. A. Grunbaum, "Commuting differential operators for the finite laplace transform," *Inverse Problems*, vol. 1, pp. 181–192, Aug. 1985.

[6] M. Bertero, F. A. Grunbaum, and L. Rebolia, "Spectral properties of a differential operator related to the inversion of the finite laplace transform," *Inverse Problems*, vol. 2, pp. 131–139, May 1986.

[7] A. Ben and T. Moumni, "On the spectrum of the finite laplace transform with some applications," *Applicable Analysis and Discrete Mathematics*, vol. 6, no. 2, pp. 304–316, 2012.

[8] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions: with formulas, graphs and mathematical tables.* New York: National Bureau of Standards, 1966.

[9] I. S. Gradshtein, *Table of integrals, series, and products.* San Diego: Academic Press, 6th ed ed., 2000.

[10] N. Yarvin and V. Rokhlin, "Generalized gaussian quadratures and singular value decompositions of integral operators," *SIAM Journal on Scientific Computing*, vol. 20, pp. 699–718, Jan. 1998.

[11] J. H. Wilkinson, "Calculation of the eigenvalues of a symmetric tridiagonal matrix by the method of bisection," *Numerische Mathematik*, vol. 4, pp. 362–367, Dec. 1962.

[12] J. R. Kuttler, "Upper and lower bounds for eigenvalues of torsion and bending problems by finite difference methods," *Zeitschrift fr angewandte Mathematik und Physik ZAMP*, vol. 21, pp. 326–342, May 1970.

[13] W. Barth, R. S. Martin, and J. H. Wilkinson, "Calculation of the eigenvalues of a symmetric tridiagonal matrix by the method of bisection," *Numerische Mathematik*, vol. 9, pp. 386–393, Apr. 1967.

[14] A. Osipov, "Evaluation of small elements of the eigenvectors of certain symmetric tridiagonal matrices with high relative accuracy," *arXiv*, vol. 1208.4906, Aug. 2012.