Prolate spheroidal wave functions (PSWFs) provide a natural and effective tool for computing with bandlimited functions defined on an interval. As demonstrated by Slepian et. al., the so called generalized prolate spheroidal functions (GPSFs) extend this apparatus to higher dimensions. While the analytical and numerical apparatus in one dimension is fairly complete, the situation in higher dimensions is less satisfactory. This report attempts to improve the situation by providing analytical and numerical tools for GPSFs, including the efficient evaluation of eigenvalues, the construction of quadratures, interpolation formulae, etc. Our results are illustrated with several numerical examples. *This report is a draft; a complete version will be published at a later date.*

On generalized prolate spheroidal functions

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On Generalized Prolate Spheroidal Functions

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2.2 Zernike Polynomials

In this section we describe the properties of the Zernike polynomials, which are a sequence of orthogonal polynomials on the unit ball in $\mathbb{R}^{p+2}$. Zernike polynomials are defined via the formula

$$Z_{N,n}^{\ell}(x) = R_{N,n}(\|x\|) S_N^{\ell}(x/\|x\|), \tag{1}$$

for all $x \in \mathbb{R}^{p+2}$ such that $\|x\| \leq 1$, where N and n are nonnegative integers, S_N^{ℓ} are the orthonormal surface harmonics of degree N (see Section ??), and $R_{N,n}$ are polynomials of degree $2n + N$ defined via the formula

$$R_{N,n}(x) = x^N \sum_{m=0}^{n} (-1)^m \binom{n + N + \frac{p}{2}}{m} \binom{n}{m} (x^2)^{n-m} (1 - x^2)^m, \tag{2}$$

for all $0 \leq x \leq 1$. The polynomials $R_{N,n}$ satisfy the relation

$$R_{N,n}(1) = 1, \tag{3}$$

and are orthogonal with respect to the weight function $w(x) = x^{p+1}$, so that

$$\int_0^1 R_{N,n}(x) R_{N,m}(x) x^{p+1}\, dx = \frac{\delta_{n,m}}{2(2n + N + \frac{p}{2} + 1)}, \tag{4}$$

where

$$\delta_{n,m} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases} \tag{5}$$

We define the polynomials $\overline{R}_{N,n}$ via the formula

$$\overline{R}_{N,n}(x) = \sqrt{2(2n + N + p/2 + 1)}\, R_{N,n}(x), \tag{6}$$

so that

$$\int_0^1 \left(\overline{R}_{N,n}(x)\right)^2 x^{p+1}\, dx = 1, \tag{7}$$

where N and n are nonnegative integers. In an abuse of notation, we refer to both the polynomials $Z_{N,n}^{\ell}$ and the radial polynomials $R_{N,n}$ as Zernike polynomials where the meaning is obvious.

Remark 2.1 *When $p = -1$, the Zernike polynomials take the form*

$$Z^1_{0,n}(x) = R_{0,n}(|x|) = P_{2n}(x), \tag{8}$$

$$Z^2_{1,n}(x) = \mathrm{sgn}(x) \cdot R_{1,n}(|x|) = P_{2n+1}(x), \tag{9}$$

for $-1 \leq x \leq 1$ and nonnegative integer n, where P_n denotes the Legendre polynomial of degree n and

$$\mathrm{sgn}(x) = \begin{cases} 1 & \textit{if } x > 0, \\ 0 & \textit{if } x = 0, \\ -1 & \textit{if } x < 0, \end{cases} \tag{10}$$

for all real x.

Remark 2.2 *When $p = 0$, the Zernike polynomials take the form*

$$Z^1_{N,n}(x_1, x_2) = R_{N,n}(r)\cos(N\theta), \tag{11}$$

$$Z^2_{N,n}(x_1, x_2) = R_{N,n}(r)\sin(N\theta), \tag{12}$$

where $x_1 = r\cos(\theta)$, $x_2 = r\sin(\theta)$, and N and n are nonnegative integers.

2.2.1 Other Notational Conventions

Some authors index the Zernike polynomials by their degree. In the case $p = 0$, Born and Wolf [6] represent the Zernike polynomial

$$R_{N,n}(r) \tag{13}$$

by the notation

$$R^N_{2n+N}(r). \tag{14}$$

2.2.2 Special Values

The following formulas are valid for all nonnegative integers N and n, and for all $0 \leq x \leq 1$.

$$R_{N,0}(x) = x^N, \tag{15}$$

$$R_{N,1}(x) = x^N\big((N + p/2 + 2)x^2 - (N + p/2 + 1)\big), \tag{16}$$

$$R_{N,n}(1) = 1, \tag{17}$$

$$R^{(k)}_{N,n}(0) = 0 \quad \text{for } k = 0, 1, \ldots, N-1, \tag{18}$$

$$R^{(N)}_{N,n}(0) = (-1)^n N! \binom{n + N + \frac{p}{2}}{n}. \tag{19}$$

2.2.3 Hypergeometric Function

The polynomials $R_{N,n}$ are related to the hypergeometric function ${}_2F_1$ (see [3]) by the formula

$$R_{N,n}(x) = (-1)^n \binom{n+N+\frac{p}{2}}{n} x^N {}_2F_1\left(-n, n+N+\frac{p}{2}+1; N+\frac{p}{2}+1; x^2\right), \quad (20)$$

where $0 \leq x \leq 1$, and N and n are nonnegative integers.

2.2.4 Interrelations

The polynomials $R_{N,n}$ are related to the Jacobi polynomials via the formula

$$R_{N,n}(x) = (-1)^n x^N P_n^{(N+\frac{p}{2},0)}(1-2x^2), \quad (21)$$

where $0 \leq x \leq 1$, N and n are nonnegative integers, and $P_n^{(\alpha,\beta)}$, $\alpha, \beta > -1$, denotes the Jacobi polynomials of degree n (see [3]).

When $p = -1$, the polynomials $R_{N,n}$ are related to the Legendre polynomials via the formulas

$$R_{0,n}(x) = P_{2n}(x), \quad (22)$$

$$R_{1,n}(x) = P_{2n+1}(x), \quad (23)$$

where $0 \leq x \leq 1$, n is a nonnegative integer, and P_n denotes the Legendre polynomial of degree n (see [3]).

2.2.5 Limit Relations

The asymptotic behavior of the Zernike polynomials near 0 as the index n tends to infinity is described by the formula

$$\lim_{n\to\infty} \frac{(-1)^n R_{N,n}\left(\frac{x}{2n}\right)}{(2n)^{p/2}} = \frac{J_{N+p/2}(x)}{x^{p/2}}, \quad (24)$$

where $0 \leq x \leq 1$, N is a nonnegative integer, and J_ν denotes the Bessel functions of the first kind (see [3]).

2.2.6 Zeros

The asymptotic behavior of the zeros of the polynomials $R_{N,n}$ as n tends to infinity is described by the following relation. Let $x_{N,m}^{(n)}$ be the mth positive zero of $R_{N,n}$, so that $0 < x_{N,1}^{(n)} < x_{N,2}^{(n)} < \ldots$. Likewise, let $j_{\nu,m}$ be the mth positive zero of J_ν, so that $0 < j_{\nu,1} < j_{\nu,2} < \ldots$, where J_ν denotes the Bessel functions of the first kind (see [3]). Then

$$\lim_{n\to\infty} 2nx_{N,m}^{(n)} = j_{N+p/2,m}, \tag{25}$$

for any nonnegative integer N.

2.2.7 Inequalities

The inequality

$$|R_{N,n}(x)| \le \binom{n+N+\frac{p}{2}}{n} \tag{26}$$

holds for $0 \le x \le 1$ and nonnegative integer N and n.

2.2.8 Integrals

The polynomials $R_{N,n}$ satify the relation

$$\int_0^1 \frac{J_{N+p/2}(xy)}{(xy)^{p/2}} R_{N,n}(y) y^{p+1}\, dy = \frac{(-1)^n J_{N+p/2+2n+1}(x)}{x^{p/2+1}}, \tag{27}$$

where $x \ge 0$, N and n are nonnegative integers, and J_ν denotes the Bessel functions of the first kind.

2.2.9 Generating Function

The generating function associated with the polynomials $R_{N,n}$ is given by the formula

$$\frac{\left(1+z-\sqrt{1+2z(1-2x^2)+z^2}\right)^{N+p/2}}{(2zx)^{N+p/2}x^{p/2}\sqrt{1+2z(1-2x^2)+z^2}} = \sum_{n=0}^{\infty} R_{N,n}(x) z^n, \tag{28}$$

where $0 \le x \le 1$ is real, z is a complex number such that $|z| \le 1$, and N is a nonnegative integer.

2.2.10 Differential Equation

The polynomials $R_{N,n}$ satisfy the differential equation

$$(1-x^2)y''(x) - 2xy'(x) + \left(\chi_{N,n} + \frac{\frac{1}{4}-(N+\frac{p}{2})^2}{x^2}\right) y(x) = 0, \tag{29}$$

where

$$\chi_{N,n} = (N + \tfrac{p}{2} + 2n + \tfrac{1}{2})(N + \tfrac{p}{2} + 2n + \tfrac{3}{2}), \tag{30}$$

and

$$y(x) = x^{p/2+1} R_{N,n}(x), \tag{31}$$

for all $0 < x < 1$ and nonnegative integers N and n.

2.2.11 Recurrence Relations

The polynomials $R_{N,n}$ satisfy the recurrence relation

$$\begin{aligned} 2(n+1)(n+N+\tfrac{p}{2}+1)(2n+N+\tfrac{p}{2})R_{N,n+1}(x) \\ = -\big((2n+N+\tfrac{p}{2}+1)(N+\tfrac{p}{2})^2 + (2n+N+\tfrac{p}{2})_3(1-2x^2)\big)R_{N,n}(x) \\ - 2n(n+N+\tfrac{p}{2})(2n+N+\tfrac{p}{2}+2)R_{N,n-1}(x), \end{aligned} \tag{32}$$

where $0 \le x \le 1$, N is a nonnegative integer, n is a positive integer, and $(\cdot)_n$ is defined via the formula

$$(x)_n = x(x+1)(x+2)\dots(x+n-1), \tag{33}$$

for real x and nonnegative integer n. The polynomials $R_{N,n}$ also satisfy the recurrence relations

$$(2n+N+\tfrac{p}{2}+2)xR_{N+1,n}(x) = (n+N+\tfrac{p}{2}+1)R_{N,n}(x) + (n+1)R_{N,n+1}(x), \tag{34}$$

for nonnegative integers N and n, and

$$(2n+N+\tfrac{p}{2})xR_{N-1,n}(x) = (n+N+\tfrac{p}{2})R_{N,n}(x) + nR_{N,n-1}(x), \tag{35}$$

for integers $N \ge 1$ and $n \ge 0$, where $0 \le x \le 1$.

2.2.12 Differential Relations

The Zernike polynomials satisfy the differential relation given by the formula

$$\begin{aligned} (2n+N+\tfrac{p}{2})x(1-x^2)\frac{d}{dx}R_{N,n}(x) \\ = \big(N(2n+N+\tfrac{p}{2}) + 2n^2 - (2n+N)(2n+N+\tfrac{p}{2})x^2\big)R_{N,n}(x) \\ + 2n(n+N+\tfrac{p}{2})R_{N,n-1}(x), \end{aligned} \tag{36}$$

where $0 < x < 1$, N is a nonnegative integer, and n is a positive integer.

2.2.13 Spectral Differentiation

2.3 Generalized Prolate Spheroidal Functions

2.3.1 Basic Facts

In this section, we summarize several facts about generalized prolate spheroidal functions (GPSFs). Let B denote the closed unit ball in $\mathbb{R}^{p+2}$. Given a real number $c > 0$, we define the operator $F_c\colon L^2(B) \to L^2(B)$ via the formula

$$F_c[\psi](x) = \int_B \psi(t) e^{ic\langle x,t\rangle}\, dt, \tag{37}$$

for all $x \in B$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^{p+2}$. Clearly, F_c is compact. Obviously, F_c is also normal, but not self-adjoint. We denote the eigenvalues of F_c by $\lambda_0, \lambda_1, \dots, \lambda_n, \dots$, and assume that $|\lambda_j| \ge |\lambda_{j+1}|$ for each non-negative integer j. For each non-negative integer j, we denote by ψ_j the eigenfunction corresponding to λ_j, so that

$$\lambda_j \psi_j(x) = \int_B \psi_j(t) e^{ic\langle x,t\rangle}\, dt, \tag{38}$$

for all $x \in B$. We assume that $\|\psi_j\|_{L^2(B)} = 1$ for each j. The following theorem is proved in [1] and describes the eigenfunctions and eigenvalues of F_c.

Theorem 2.1 *Suppose that $c > 0$ is a real number and that F_c is defined by* (37). *Then the eigenfunctions $\psi_0, \psi_1, \dots, \psi_n, \dots$ of F_c are real, orthonormal, and complete in $L^2(B)$. For each j, the eigenfunction ψ_j is either even, in the sense that $\psi_j(-x) = \psi_j(x)$ for all $x \in B$, or odd, in the sense that $\psi_j(-x) = -\psi_j(x)$ for all $x \in B$. The eigenvalues corresponding to even eigenfunctions are real, and the eigenvalues corresponding to odd eigenfunctions are purely imaginary. The domain on which the eigenfunctions are defined can be extended from B to $\mathbb{R}^{p+2}$ by requiring that* (38) *hold for all $x \in \mathbb{R}^{p+2}$; the eigenfunctions will then be orthogonal on $\mathbb{R}^{p+2}$ and complete in the class of band-limited functions with bandlimit c.*

We define the self-adjoint operator $Q_c\colon L^2(B) \to L^2(B)$ via the formula

$$Q_c = \left(\frac{c}{2\pi}\right)^{p+2} F_c^* \cdot F_c. \tag{39}$$

Since F_c is normal, it follows that Q_c has the same eigenfunctions as F_c, and that the jth eigenvalue μ_j of Q_c is connected to λ_j via the formula

$$\mu_j = \left(\frac{c}{2\pi}\right)^{p+2} |\lambda_j|^2. \tag{40}$$

We also observe that

$$Q_c[\psi](x) = \left(\frac{c}{2\pi}\right)^{p/2+1} \int_B \frac{J_{p/2+1}\big(c\|x-t\|\big)}{\|x-t\|^{p/2+1}} \psi(t)\, dt, \tag{41}$$

for all $x \in \mathbb{R}^{p+2}$, where J_ν denotes the Bessel functions of the first kind and $\|\cdot\|$ denotes Euclidean distance in $\mathbb{R}^{p+2}$ (see Appendix A for a proof).

We observe that

$$Q_c[\psi](x) = \mathbb{1}_B(x) \cdot \mathcal{F}^{-1}\big[\mathbb{1}_{B(c)}(t) \cdot \mathcal{F}[\psi](t)\big](x), \tag{42}$$

where $\mathcal{F}\colon L^2(\mathbb{R}^{p+2}) \to L^2(\mathbb{R}^{p+2})$ is the $(p+2)$-dimensional Fourier transform, $B(c)$ denotes the set $\{\, x \in \mathbb{R}^{p+2} : \|x\| \le c \,\}$, and $\mathbb{1}_A$ is defined via the formula

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{43}$$

From (42) it follows that $\mu_j < 1$ for all j.

We observe further that Q_c is closely related to the operator $P_c\colon L^2(\mathbb{R}^{p+2}) \to L^2(\mathbb{R}^{p+2})$, defined via the formula

$$P_c[\psi](x) = \left(\frac{c}{2\pi}\right)^{p/2+1} \int_{\mathbb{R}^{p+2}} \frac{J_{p/2+1}\big(c\|x-t\|\big)}{\|x-t\|^{p/2+1}} \psi(t)\, dt, \tag{44}$$

which is the orthogonal projection onto the space of bandlimited functions on $\mathbb{R}^{p+2}$ with bandlimit $c > 0$.

2.3.2 Eigenfunctions and Eigenvalues of F_c

In this section we describe the eigenvectors and eigenvalues of the operator F_c, defined in (37). Suppose that ψ is some eigenfunction of the integral operator F_c, with corresponding complex eigenvalue λ, so that

$$\lambda\psi(x) = \int_B \psi(t) e^{ic\langle x,t\rangle}\, dt, \tag{45}$$

for all $x \in B$ (see Theorem 2.1).

Observation 2.3 *The operator F_c, defined by* (37)*, is spherically symmetric in the sense that, for any $(p+2) \times (p+2)$ orthogonal matrix U, F_c commutes with the operator $\hat{U}\colon L^2(B) \to L^2(B)$, defined via the formula*

$$\hat{U}[\psi](x) = \psi(Ux), \tag{46}$$

for all $x \in B$. Hence, the problem of finding the eigenfunctions and eigenvalues of F_c is amenable to the separation of variables.

Suppose that

$$\psi(x) = \Phi_N^\ell(\|x\|) S_N^\ell(x/\|x\|), \tag{47}$$

where S_N^ℓ, $\ell = 0, 1, \dots, h(N,p)$ denotes the spherical harmonics of degree N (see Section ??), and $\Phi_N^\ell(r)$ is a real-valued function defined on the interval $[0,1]$. We observe that

$$e^{ic\langle x,t\rangle} = \sum_{N=0}^{\infty} \sum_{\ell=1}^{h(N,p)} i^N (2\pi)^{p/2+1} \frac{J_{N+p/2}(c\|x\|\|t\|)}{(c\|x\|\|t\|)^{p/2}} S_N^\ell(x/\|x\|) S_N^\ell(t/\|t\|), \tag{48}$$

where $x, t \in B$, and where J_ν denotes the Bessel functions of the first kind (see Section VII of [1] for a proof). Substituting (47) and (48) into (45), we find that

$$\lambda \Phi_N^\ell(r) = i^N (2\pi)^{p/2+1} \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} \Phi_N^\ell(\rho) \rho^{p+1} \, d\rho, \tag{49}$$

for all $0 \le r \le 1$. We define the operator $H_{N,c} \colon L^2\big([0,1], \rho^{p+1}\, d\rho\big) \to L^2\big([0,1], \rho^{p+1}\, d\rho\big)$ via the formula

$$H_{N,c}[\Phi](r) = \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} \Phi(\rho) \rho^{p+1} \, d\rho, \tag{50}$$

where $0 \le r \le 1$, and observe that $H_{N,c}$ is clearly compact and self-adjoint, and does not depend on ℓ. Dropping the index ℓ, we denote by $\beta_{N,0}, \beta_{N,1}, \dots, \beta_{N,n}, \dots$ the eigenvalues of $H_{N,c}$, and assume that $|\beta_{N,n}| \ge |\beta_{N,n+1}|$ for each nonnegative integer n. For each nonnegative integer n, we let $\Phi_{N,n}$ denote the eigenvector corresponding to eigenvalue $\beta_{N,n}$, so that

$$\beta_{N,n} \Phi_{N,n}(r) = \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} \Phi_{N,n}(\rho) \rho^{p+1} \, d\rho, \tag{51}$$

for all $0 \le r \le 1$. Clearly, the eigenfunctions $\Phi_{N,n}$ are purely real. We assume that $\|\Phi_{N,n}\|_{L^2([0,1],\rho^{p+1}\, d\rho)} = 1$ and that $\Phi_{N,n}(1) > 0$ for each nonnegative integer N and n (see Theorem 9.6). It follows from (51) and (49) that the eigenvectors and eigenvalues of F_c are given by the formulas

$$\psi_{N,n}^\ell(x) = \Phi_{N,n}(\|x\|) S_N^\ell(x/\|x\|), \tag{52}$$

and

$$\lambda_{N,n}^\ell = i^N (2\pi)^{p/2+1} \beta_{N,n}, \tag{53}$$

respectively, where $x \in B$, N and n are nonnegative integers, and ℓ is an integer so that $1 \leq \ell \leq h(N, p)$ (see Section ??). We note in formula (53) the expected degeneracy of eigenvalues due to the spherical symmetry of the integral operator F_c (see Observation 2.3); we denote $\lambda^{\ell}_{N,n}$ by $\lambda_{N,n}$ where the meaning is clear. We also make the following observation.

Observation 2.4 *The domain on which the functions $\Phi_{N,n}$ are defined may be extended from the interval $[0, 1]$ to the complex plane $\mathbb{C}$ by requiring that (45) hold for all $r \in \mathbb{C}$. Moreover, the functions $\Phi_{N,n}$, extended in this way, are entire.*

2.3.3 The Dual Nature of GPSFs

In this section, we observe that the eigenfunctions $\Phi_{N,0}, \Phi_{N,1}, \dots, \Phi_{N,n}, \dots$ of the integral operator $H_{N,c}$, defined in (50), are also the eigenfunctions of a certain differential operator.

Let $\beta_{N,n}$ denote the eigenvalue corresponding to the eigenfunction $\Phi_{N,n}$, for all nonnegative integers N and n, so that

$$\beta_{N,n} \Phi_{N,n}(r) = \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} \Phi_{N,n}(\rho) \rho^{p+1} \, d\rho, \tag{54}$$

where $0 \leq r \leq 1$, N and n are nonnegative integers, and J_ν denotes the Bessel functions of the first kind (see (51)). Making the substitutions

$$\varphi_{N,n}(r) = r^{(p+1)/2} \Phi_{N,n}(r), \tag{55}$$

and

$$\gamma_{N,n} = c^{(p+1)/2} \beta_{N,n}, \tag{56}$$

we observe that

$$\gamma_{N,n} \varphi_{N,n}(r) = \int_0^1 J_{N+p/2}(cr\rho) \sqrt{cr\rho} \, \varphi_{N,n}(\rho) \, d\rho, \tag{57}$$

where $0 \leq r \leq 1$, and N and n are arbitrary nonnegative integers. We define the operator $M_{N,c} \colon L^2([0, 1]) \to L^2([0, 1])$ via the formula

$$M_{N,c}[\varphi](r) = \int_0^1 J_{N+p/2}(cr\rho) \sqrt{cr\rho} \, \varphi(\rho) \, d\rho, \tag{58}$$

where $0 \leq r \leq 1$, and N is an arbitrary nonnegative integer. Obviously, $M_{N,c}$ is compact and self-adjoint. Clearly, the eigenvalues of $M_{N,c}$ are $\gamma_{N,0}, \gamma_{N,1}, \dots, \gamma_{N,n}, \dots$, and $\varphi_{N,n}$ is the eigenfunction corresponding to eigenvalue $\gamma_{N,n}$, for each nonnegative integer n.

We define the differential operator $L_{N,c}$ via the formula

$$L_{N,c}[\varphi](x) = \frac{d}{dx}\left((1-x^2)\frac{d\varphi}{dx}(x)\right) + \left(\frac{\frac{1}{4}-(N+\frac{p}{2})^2}{x^2} - cx^2\right)\varphi(x), \tag{59}$$

where $0 < x < 1$, N is a nonnegative integer, and φ is twice continuously differentiable. Let C be the class of functions φ which are bounded and twice continuously differentiable on the interval $(0,1)$, such that $\varphi'(0) = 0$ if $p = -1$ and $N = 0$, and $\varphi(0) = 0$ otherwise. Then it is easy to show that, operating on function in class C, $L_{N,c}$ is self-adjoint. From Sturmian theory we obtain the following theorem (see [1]).

Theorem 2.2 *Suppose that $c > 0$, N is a nonnegative integer, and $L_{N,c}$ is defined via* (59)*. Then there exists a strictly increasing unbounded sequence of positive numbers $\chi_{N,0} < \chi_{N,1} < \ldots$ such that for each nonnegative integer n, the differential equation*

$$L_{N,c}[\varphi](x) + \chi_{N,n}\varphi(x) = 0 \tag{60}$$

has a solution which is bounded and twice continuously differentiable on the interval $(0,1)$, so that $\varphi'(0) = 0$ if $p = -1$ and $N = 0$, and $\varphi(0) = 0$ otherwise.

The following theorem is proved in [1].

Theorem 2.3 *Suppose that $c > 0$, N is a nonnegative integer, and the operators $M_{N,c}$ and $L_{N,c}$ are defined via* (58) *and* (59) *respectively. Suppose also that $\varphi\colon (0,1) \to \mathbb{R}$ is in $L^2([0,1])$, is twice differentiable, and that $\varphi'(0) = 0$ if $p = -1$ and $N = 0$, and $\varphi(0) = 0$ otherwise. Then*

$$L_{N,c}\big[M_{N,c}[\varphi]\big](x) = M_{N,c}\big[L_{N,c}[\varphi]\big](x), \tag{61}$$

for all $0 < x < 1$.

Since Theorem 2.2 shows that the eigenvalues of $L_{N,c}$ are not degenerate, Theorem 2.3 implies that $L_{N,c}$ and $M_{N,c}$ have the same eigenvectors.

2.3.4 Bandlimited Functions and GPSFs

2.3.5 Zernike Polynomials and GPSFs

In this section we describe the relationship between Zernike polynomials and GPSFs. We use $\varphi^c_{N,n}$, where $c > 0$ and N and n are arbitrary nonnegative integers, to denote the nth eigenfunction of $L_{N,c}$, defined in (59); we denote by $\chi_{N,n}(c)$ the eigenvalue corresponding to eigenfunction $\varphi^c_{N,n}$.

For $c = 0$, the eigenfunctions and eigenvalues of the differential operator $L_{N,c}$, defined in (59), are given by the formulas

$$T_{N,n}(x) = x^{(p+1)/2} R_{N,n}(x), \tag{62}$$

and

$$\chi_{N,n}(0) = (N + \tfrac{p}{2} + 2n + \tfrac{1}{2})(N + \tfrac{p}{2} + 2n + \tfrac{3}{2}), \tag{63}$$

respectively, where $0 \le x \le 1$, N and n are arbitrary nonnegative integers, and $R_{N,n}$ are Zernike polynomials defined by (2). We define the functions $\overline{T}_{N,n}$ via the formula

$$\overline{T}_{N,n}(x) = x^{(p+1)/2} \overline{R}_{N,n}(x), \tag{64}$$

where $0 \le x \le 1$, N and n are nonnegative integers, and $\overline{R}_{N,n}$ are the normalized Zernike polynomials defined by (6), so that

$$\int_0^1 \big(\overline{T}_{N,n}(x)\big)^2 \, dx = 1, \tag{65}$$

for all nonnegative integers N and n.

For small $c > 0$, the connection between Zernike polynomials and GPSFs is given by the formulas

$$\varphi^c_{N,n}(x) = \overline{T}_{N,n}(x) + o(c^2), \tag{66}$$

and

$$\chi_{N,n}(c) = \chi_{N,n}(0) + o(c^2), \tag{67}$$

as $c \to 0$, where $0 \le x \le 1$ and N and n are arbitrary nonnegative integers (see [1]).

For $c > 0$, the functions $T_{N,n}$ are also related to the integral operator $M_{N,c}$, defined in (58), via the formula

$$M_{N,c}\big[T_{N,n}\big](x) = \int_0^1 J_{N+p/2}(cxy)\sqrt{cxy}\, T_{N,n}(y)\, dy = \frac{(-1)^n J_{N+p/2+2n+1}(cx)}{\sqrt{cx}}, \tag{68}$$

where $x \ge 0$ and N and n are arbitrary nonnegative integers (see (27)).

2.4 Miscellaneous Analytical Facts

The following theorem is an identity involving the incomplete beta function.

Theorem 2.4 *Suppose that $a, b > 0$ are real numbers and n is a nonnegative integer. Then*

$$B_x(a+n,b) = \frac{\Gamma(a+n)}{\Gamma(a+b+n)}\left(\frac{\Gamma(a+b)}{\Gamma(a)}B_x(a,b) - (1-x)^b \sum_{k=1}^{n} \frac{\Gamma(a+b+k-1)}{\Gamma(a+k)} x^{a+k-1}\right) \tag{69}$$

for all $0 \leq x \leq 1$, where $B_x(a,b)$ denotes the incomplete beta function.

The following lemma is an identity involving the gamma function.

Lemma 2.5 *Suppose that n is a nonnegative integer. Then*

$$\sqrt{\pi} + \sum_{k=1}^{n} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} = \frac{2\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}. \tag{70}$$

The following two lemmas are identities involving the incomplete beta function.

Lemma 2.6 *Suppose that $0 \leq r \leq 1$. Then*

$$B_{1-r^2}(1, \tfrac{1}{2}) = 2(1-r). \tag{71}$$

Lemma 2.7 *Suppose that $0 \leq r \leq 1$. Then*

$$B_{1-r^2}(\tfrac{1}{2}, \tfrac{1}{2}) = 2\arccos(r). \tag{72}$$

2.4.1 The Area and Volume of a Hypersphere

The following theorem provides formulas for the volume and area of a $(p+2)$-dimensional hypersphere.

Theorem 2.8 *Suppose that $S^{p+2}(r) = \{x \in \mathbb{R}^{p+2} : \|x\| = r\}$ denotes the $(p+2)$-dimensional hypersphere of radius $r > 0$. Suppose further that $A_{p+2}(r)$ denotes the area of $S^{p+2}(r)$ and $V_{p+2}(r)$ denotes the volume enclosed by $S^{p+2}(r)$. Then*

$$A_{p+2}(r) = \frac{2\pi^{p/2+1}}{\Gamma(\frac{p}{2}+1)} r^{p+1}, \tag{73}$$

and

$$V_{p+2}(r) = \frac{\pi^{p/2+1}}{\Gamma(\frac{p}{2}+2)} r^{p+2}. \tag{74}$$

The following theorem provides a formula for the volume of the intersection of two $(p+2)$-dimensional hyperspheres (see, for example, [4]).

Theorem 2.9 *Suppose that $p \geq -1$ is an integer, let B denote the closed unit ball in $\mathbb{R}^{p+2}$, and let $B(c)$ denote the set $\{x \in \mathbb{R}^{p+2} : \|x\| \leq c\}$, where $c > 0$. Then*

$$\int_{\mathbb{R}^D} \mathbb{1}_B(u-t)\mathbb{1}_B(t)\,dt = V_{p+2}(1)\frac{B_{1-\|u\|^2/4}(\frac{p}{2}+\frac{3}{2},\frac{1}{2})}{B(\frac{p}{2}+\frac{3}{2},\frac{1}{2})}, \tag{75}$$

for all $u \in B(2)$, where $B(a,b)$ denotes the beta function, $B_x(a,b)$ denotes the incomplete beta function, V_{p+2} is defined by (74), and $\mathbb{1}_A$ is defined via the formula

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{76}$$

2.4.2 Bessel Functions

The primary analytical tool of this subsection is Theorem 2.15.

The following lemmas 2.10, 2.11, 2.12, 2.13, 2.14 describe the limiting behavior of certain integrals involving Bessel functions.

Lemma 2.10 *Suppose that $\nu > 0$. Then*

$$\int_0^1 (J_\nu(2cr))^2 \frac{1}{r}\,dr = \frac{1}{2\nu} + O\Big(\frac{1}{c}\Big), \tag{77}$$

as $c \to \infty$.

Lemma 2.11 *Suppose that $\nu > 0$. Then*

$$\int_0^1 (J_\nu(2cr))^2\,dr = \frac{1}{2\pi}\frac{\log(c)}{c} + o\Big(\frac{\log(c)}{c}\Big), \tag{78}$$

as $c \to \infty$.

Lemma 2.12 *Suppose that $\nu > 0$ is real and k is a positive integer. Then*

$$\int_0^1 (J_\nu(2cr))^2 r^k\,dr = O\Big(\frac{1}{c}\Big), \tag{79}$$

as $c \to \infty$.

Lemma 2.13 *Suppose that n is a positive integer. Then*

$$\int_0^1 \frac{(J_n(2cr))^2}{r} \arccos(r)\, dr = \frac{\pi}{4n} - \frac{1}{2\pi}\frac{\log(c)}{c} + o\Big(\frac{\log(c)}{c}\Big), \tag{80}$$

as $c \to \infty$.

Lemma 2.14 *Suppose that n and k are positive integers. Then*

$$\int_0^1 (J_n(2cr))^2 (1-r^2)^{k-\frac{1}{2}}\, dr = \frac{1}{2\pi}\frac{\log(c)}{c} + o\Big(\frac{\log(c)}{c}\Big), \tag{81}$$

as $c \to \infty$.

The following theorem describes the limiting behavior of a certain integral involving a Bessel function and the incomplete beta function.

Theorem 2.15 *Suppose that $p \geq -1$ is an integer. Then*

$$\int_0^1 \frac{(J_{p/2+1}(2cr))^2}{r} B_{1-r^2}(\tfrac{p}{2}+\tfrac{3}{2},\tfrac{1}{2})\, dr = \frac{\sqrt{\pi}\,\Gamma(\frac{p}{2}+\frac{3}{2})}{(p+2)\Gamma(\frac{p}{2}+2)} - \frac{1}{\pi}\frac{\log(c)}{c} + o\Big(\frac{\log(c)}{c}\Big) \tag{82}$$

as $c \to \infty$, where $B_x(a,b)$ denotes the incomplete beta function.

Proof. Suppose that $p \geq -1$ is an odd integer, and let $n = \frac{p}{2} + \frac{1}{2}$. Then

$$\int_0^1 \frac{\big(J_{p/2+1}(2cr)\big)^2}{r} B_{1-r^2}(\tfrac{p}{2}+\tfrac{3}{2},\tfrac{1}{2})\, dr = \int_0^1 \frac{\big(J_{n+1/2}(2cr)\big)^2}{r} B_{1-r^2}(1+n,\tfrac{1}{2})\, dr. \tag{83}$$

By Theorem 2.4 and Lemma 2.6, we observe that

$$\begin{aligned}
&\int_0^1 \frac{\big(J_{n+1/2}(2cr)\big)^2}{r} B_{1-r^2}(1+n,\tfrac{1}{2})\, dr \\
&\quad= \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \int_0^1 \frac{\big(J_{n+1/2}(2cr)\big)^2}{r} \Big(\frac{\sqrt{\pi}}{2} B_{1-r^2}(1,\tfrac{1}{2}) - r\sum_{k=1}^{n} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}(1-r^2)^k\Big)\, dr \\
&\quad= \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \int_0^1 \frac{\big(J_{n+1/2}(2cr)\big)^2}{r} \Big(\sqrt{\pi}(1-r) - r\sum_{k=1}^{n} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}(1-r^2)^k\Big)\, dr,
\end{aligned} \tag{84}$$

where $0 \leq r \leq 1$ and n is a nonnegative integer. By lemmas 2.10, 2.11, and 2.12, it follows that

$$\begin{aligned}
&\int_0^1 \frac{\big(J_{n+1/2}(2cr)\big)^2}{r} B_{1-r^2}(1+n,\tfrac{1}{2})\, dr \\
&\quad= \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\Big(\frac{\sqrt{\pi}}{2n+1} - \frac{1}{2\pi}\Big(\sqrt{\pi} + \sum_{k=1}^{n}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}\Big)\frac{\log(c)}{c}\Big) + o\Big(\frac{\log(c)}{c}\Big),
\end{aligned} \tag{85}$$

as $c \to \infty$, where $0 \le r \le 1$ and n is a nonnegative integer. Applying Lemma 2.5,

$$\begin{aligned}
&\int_0^1 (J_{n+1/2}(2cr))^2 B_{1-r^2}(1+n, \tfrac{1}{2})\, dr \\
&\quad = \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\Big(\frac{\sqrt{\pi}}{2n+1} - \frac{1}{\pi}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\frac{\log(c)}{c}\Big) + o\Big(\frac{\log(c)}{c}\Big) \\
&\quad = \frac{\sqrt{\pi}\,\Gamma(n+1)}{2(n+\frac{1}{2})\Gamma(n+\frac{3}{2})} - \frac{1}{\pi}\frac{\log(c)}{c} + o\Big(\frac{\log(c)}{c}\Big),
\end{aligned} \tag{86}$$

as $c \to \infty$, where $0 \le r \le 1$ and n is a nonnegative integer. Therefore,

$$\int_0^1 \frac{(J_{p/2+1}(2cr))^2}{r} B_{1-r^2}(\tfrac{p}{2}+\tfrac{3}{2}, \tfrac{1}{2})\, dr = \frac{\sqrt{\pi}\,\Gamma(\frac{p}{2}+\frac{3}{2})}{(p+2)\Gamma(\frac{p}{2}+2)} - \frac{1}{\pi}\frac{\log(c)}{c} + o\Big(\frac{\log(c)}{c}\Big), \tag{87}$$

as $c \to \infty$, for all $0 \le r \le 1$ and odd integers $p \ge -1$.

The proof in the case when $p \ge 0$ is an even integer is essentially identical.

■

3 Analytical Apparatus

3.1 Properties of GPSFs

The following theorem provides a formula for ratios of eigenvalues $\beta_{N,n}$ (see (51)), and finds use in the numerical evaluation of $\beta_{N,n}$.

Theorem 3.1 *Suppose that N is a nonnegative integer. Then*

$$\frac{\beta_{N,m}}{\beta_{N,n}} = \frac{\int_0^1 x\Phi'_{N,n}(x)\Phi_{N,m}(x)x^{p+1}\, dx}{\int_0^1 x\Phi'_{N,m}(x)\Phi_{N,n}(x)x^{p+1}\, dx}, \tag{88}$$

for each nonnegative integers n and m.

3.2 Decay of the Expansion Coefficients of GPSFs into Zernike Polynomials

Since the functions $\Phi_{N,n}$ are analytic on $\mathbb{C}$ for all nonnegative integers N and n (see Observation 2.4), and $\Phi^{(k)}_{N,n}(0) = 0$ for $k = 0, 1, \ldots, N-1$ (see Theorem 9.5), the functions $\Phi_{N,n}$ are representable by series of Zernike polynomials of the form

$$\Phi_{N,n}(r) = \sum_{k=0}^{\infty} b_k \overline{R}_{N,k}(r), \tag{89}$$

for all $0 \leq r \leq 1$, where $b_0, b_1, \ldots, b_k, \ldots$ are real numbers, and $\overline{R}_{N,n}$ is defined by (6). Theorem ?? in this section shows that the coefficients b_k decay exponentially, and establishes a bound for the decay rate.

4 Dimension of the Class of Bandlimited Functions

In this section, we investigate the properties of the eigenvalues $\mu_0, \mu_1, \ldots, \mu_j, \ldots$ of the operator Q_c, defined via formula (39). We denote by λ_j the eigenvalues of operator F_c, defined via formula (37), and let ψ_j denote the eigenfunctions corresponding to λ_j, for each nonnegative integer j.

The following two theorems evaluate the sums $\sum_{j=0}^{\infty} \mu_j$ and $\sum_{j=0}^{\infty} \mu_j^2$ respectively.

Theorem 4.1 *Suppose that $c > 0$. Then*

$$\sum_{j=0}^{\infty} \mu_j = \frac{c^{p+2}}{2^{p+2}\Gamma(\frac{p}{2}+2)^2}. \tag{90}$$

Proof. From (38), we observe the identity

$$\sum_{j=0}^{\infty} \lambda_j \psi_j(x) \psi_j(t) = e^{ic\langle x,t\rangle}, \tag{91}$$

for all $x, t \in B$, where B is the closed unit ball in $\mathbb{R}^{p+2}$, and the sum on the left hand side converges in the sense of $L^2(B) \otimes L^2(B)$. By taking the squared $L^2(B) \otimes L^2(B)$ norm of both sides and using (74), we obtain the formula

$$\sum_{j=0}^{\infty} |\lambda_j|^2 = \frac{\pi^{p+2}}{\Gamma(\frac{p}{2}+2)^2}. \tag{92}$$

Since

$$\mu_j = \left(\frac{c}{2\pi}\right)^{p+2} |\lambda_j|^2, \tag{93}$$

for all nonnegative integer j (see (38)), it follows that

$$\sum_{j=0}^{\infty} \mu_j = \frac{c^{p+2}}{2^{p+2}\Gamma(\frac{p}{2}+2)^2}. \tag{94}$$

■

Theorem 4.2 *Suppose that $c > 0$. Then*

$$\sum_{j=0}^{\infty} \mu_j^2 = \frac{c^{p+2}}{2^{p+2}\Gamma(\frac{p}{2}+2)^2} - \frac{c^{p+1}\log(c)}{\pi^2\Gamma(p+2)} + o\big(c^{p+1}\log(c)\big), \tag{95}$$

as $c \to \infty$.

Proof. By (41),

$$\sum_{j=0}^{\infty} \mu_j \psi_j(x)\psi_j(t) = \Big(\frac{c}{2\pi}\Big)^{p/2+1} \frac{J_{p/2+1}(c\|x-t\|)}{\|x-t\|^{p/2+1}}, \tag{96}$$

for all $x, t \in B$, where the sum on the left hand side converges in the sense of $L^2(B) \otimes L^2(B)$, and where J_ν denotes the Bessel functions of the first kind. Taking the squared $L^2(B) \otimes L^2(B)$ norm of both sides, we obtain the formula

$$\begin{aligned}
\sum_{j=0}^{\infty} \mu_j^2 &= \Big(\frac{c}{2\pi}\Big)^{p+2} \int_B \int_B \frac{\big(J_{p/2+1}(c\|x-t\|)\big)^2}{\|x-t\|^{p+2}}\,dx\,dt \\
&= \Big(\frac{c}{2\pi}\Big)^{p+2} \int_B \int_B \frac{\big(J_{p/2+1}(c\|x+t\|)\big)^2}{\|x+t\|^{p+2}}\,dx\,dt \\
&= \Big(\frac{c}{2\pi}\Big)^{p+2} \int_{\mathbb{R}^D} \int_{\mathbb{R}^D} \frac{\big(J_{p/2+1}(c\|x+t\|)\big)^2}{\|x+t\|^{p+2}} \mathbb{1}_B(x)\mathbb{1}_B(t)\,dx\,dt,
\end{aligned} \tag{97}$$

where $\mathbb{1}_A$ is defined via the formula

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{98}$$

Letting $u = x + t$, we observe that

$$\begin{aligned}
\sum_{j=0}^{\infty} \mu_j^2 &= \Big(\frac{c}{2\pi}\Big)^{p+2} \int_{\mathbb{R}^D} \int_{\mathbb{R}^D} \frac{\big(J_{p/2+1}(c\|u\|)\big)^2}{\|u\|^{p+2}} \mathbb{1}_B(u-t)\mathbb{1}_B(t)\,du\,dt \\
&= \Big(\frac{c}{2\pi}\Big)^{p+2} \int_{\mathbb{R}^D} \int_{\mathbb{R}^D} \frac{\big(J_{p/2+1}(c\|u\|)\big)^2}{\|u\|^{p+2}} \mathbb{1}_{B(2)}(u)\mathbb{1}_B(u-t)\mathbb{1}_B(t)\,du\,dt \\
&= \Big(\frac{c}{2\pi}\Big)^{p+2} \int_{B(2)} \frac{\big(J_{p/2+1}(c\|u\|)\big)^2}{\|u\|^{p+2}} \int_{\mathbb{R}^D} \mathbb{1}_B(u-t)\mathbb{1}_B(t)\,dt\,du.
\end{aligned} \tag{99}$$

Combining (99) and (75),

$$\sum_{j=0}^{\infty}\mu_j^2 = \Big(\frac{c}{2\pi}\Big)^{p+2}\int_{B(2)}\frac{\big(J_{p/2+1}(c\|u\|)\big)^2}{\|u\|^{p+2}}\cdot V_{p+2}(1)\frac{B_{1-\|u\|^2/4}(\frac{p}{2}+\frac{3}{2},\frac{1}{2})}{B(\frac{p}{2}+\frac{3}{2},\frac{1}{2})}\,du$$
$$= \Big(\frac{c}{2\pi}\Big)^{p+2}\frac{V_{p+2}(1)}{B(\frac{p}{2}+\frac{3}{2},\frac{1}{2})}\int_{B(2)}\frac{\big(J_{p/2+1}(c\|u\|)\big)^2}{\|u\|^{p+2}}B_{1-\|u\|^2/4}(\tfrac{p}{2}+\tfrac{3}{2},\tfrac{1}{2})\,du$$
$$= \Big(\frac{c}{2\pi}\Big)^{p+2}\frac{V_{p+2}(1)A_{p+2}(1)}{B(\frac{p}{2}+\frac{3}{2},\frac{1}{2})}\int_0^2\frac{\big(J_{p/2+1}(cr)\big)^2}{r}B_{1-r^2/4}(\tfrac{p}{2}+\tfrac{3}{2},\tfrac{1}{2})\,dr$$
$$= \Big(\frac{c}{2\pi}\Big)^{p+2}\frac{V_{p+2}(1)A_{p+2}(1)}{B(\frac{p}{2}+\frac{3}{2},\frac{1}{2})}\int_0^1\frac{\big(J_{p/2+1}(2cr)\big)^2}{r}B_{1-r^2}(\tfrac{p}{2}+\tfrac{3}{2},\tfrac{1}{2})\,dr, \quad (100)$$

where $V_{p+2}(1)$ denotes the volume of the unit ball in $\mathbb{R}^{p+2}$, $A_{p+2}(1)$ denotes the area of the unit sphere in $\mathbb{R}^{p+2}$, $B(a,b)$ denotes the beta function, and $B_x(a,b)$ denotes the incomplete beta function. Applying Theorem 2.8 to (100),

$$\sum_{j=0}^{\infty}\mu_j^2 = \frac{c^{p+2}}{2^{p+1}\sqrt{\pi}\Gamma(\frac{p}{2}+1)\Gamma(\frac{p}{2}+\frac{3}{2})}\int_0^1\frac{\big(J_{p/2+1}(2cr)\big)^2}{r}B_{1-r^2}(\tfrac{p}{2}+\tfrac{3}{2},\tfrac{1}{2})\,dr$$
$$= \frac{c^{p+2}}{\pi\Gamma(p+2)}\int_0^1\frac{\big(J_{p/2+1}(2cr)\big)^2}{r}B_{1-r^2}(\tfrac{p}{2}+\tfrac{3}{2},\tfrac{1}{2})\,dr. \quad (101)$$

Combining (101) and (82),

$$\sum_{j=0}^{\infty}\mu_j^2 = \frac{c^{p+2}}{\pi\Gamma(p+2)}\bigg(\frac{\sqrt{\pi}\,\Gamma(\frac{p}{2}+\frac{3}{2})}{(p+2)\Gamma(\frac{p}{2}+2)}-\frac{1}{\pi}\frac{\log(c)}{c}+o\Big(\frac{\log(c)}{c}\Big)\bigg)$$
$$= \frac{c^{p+2}}{2^{p+2}\Gamma(\frac{p}{2}+2)^2}-\frac{c^{p+1}\log(c)}{\pi^2\Gamma(p+2)}+o\big(c^{p+1}\log(c)\big), \quad (102)$$

as $c\to\infty$.

■

The following corollary follows immediately from theorems 4.1 and 4.2.

Corollary 4.3 *Suppose that $c>0$. Then*

$$\sum_{j=0}^{\infty}\mu_j(1-\mu_j) = \frac{c^{p+1}\log(c)}{\pi^2\Gamma(p+2)}+o\big(c^{p+1}\log(c)\big), \quad (103)$$

as $c\to\infty$.

From (90) and (103) we observe that the spectrum of Q_c consists of three parts:

$$\frac{c^{p+2}}{2^{p+2}\Gamma(\frac{p}{2}+2)^2} \tag{104}$$

eigenvalues close to 1;

$$\frac{c^{p+1}\log(c)}{\pi^2\Gamma(p+2)} \tag{105}$$

eigenvalues in the transition region; and the rest close to 0.

5 Numerical Evaluation of GPSFs

5.1 Numerical Evaluation of Eigenvalues λ_i

5.2 Numerical Evaluation of a Single Eigenvalue λ_i

6 Quadratures for Band-limited Functions

7 Interpolation via GPSFs

8 Numerical Results

9 Miscellaneous Properties of GPSFs

9.1 Growth and Oscillation Properties

9.2 Properties of the Derivatives of GPSFs

The following theorem follows immediately from (55) and (59).

Theorem 9.1 *Let $c > 0$. Then*

$$\begin{aligned}\frac{d}{dx}\Big((x^{p+1}-x^{p+3})\frac{d\Phi_{N,n}}{dx}(x)\Big)& \\ +\Big(\chi_{N,n}x^{p+1}-\frac{(p+1)(p+3)}{4}x^{p+1}-N(N+p)x^{p-1}-c^2x^{p+3}\Big)\Phi_{N,n}(x)&=0,\end{aligned} \tag{106}$$

where $0 < x < 1$ and N and n are arbitrary nonnegative integers.

Corollary 9.2 *Let $c > 0$. Then*

$$x^2(1-x^2)\Phi''_{N,n}(x) + \big((p+1)x - (p+3)x^3\big)\Phi'_{N,n}(x) \\ + \Big(\chi_{N,n}x^2 - \frac{(p+1)(p+3)}{4}x^2 - N(N+p) - c^2x^4\Big)\Phi_{N,n}(x) = 0, \qquad (107)$$

where $0 < x < 1$ and N and n are arbitrary nonnegative integers.

The following lemma connects the values of the $(k+2)$nd derivative of the function $\Phi_{N,n}$ with its derivatives of orders $k-4, k-3, \ldots, k+1$, and is obtained by repeated differentiation of (107).

Lemma 9.3 *Let $c > 0$. Then*

$$(x^2 - x^4)\Phi^{(k+2)}_{N,n}(x) + \big((2k+1+p)x - (4k+3+p)x^3\big)\Phi^{(k+1)}_{N,n}(x) \\ + \Big(k(k+p) - N(N+p) + \big[\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3) \\ - 3k(2k+1+p)\big]x^2 - c^2x^4\Big)\Phi^{(k)}_{N,n}(x) \\ + \Big(\big[2k\big(\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3)\big) - k(k-1)(4k+1+3p)\big]x - 4kc^2x^3\Big)\Phi^{(k-1)}_{N,n}(x) \\ + \Big(k(k-1)\big(\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3)\big) - k(k-1)(k-2)(k+p) - 6k(k-1)c^2x^2\Big)\Phi^{(k-2)}_{N,n}(x) \\ - 4k(k-1)(k-2)c^2x\Phi^{(k-3)}_{N,n}(x) - k(k-1)(k-2)(k-3)c^2\Phi^{(k-4)}_{N,n}(x) = 0, \qquad (108)$$

where $0 < x < 1$, N and n are arbitrary nonnegative integers, and k is an arbitrary integer so that $k \geq 4$. Also,

$$(x^2 - x^4)\Phi''_{N,n}(x) + \big((p+1)x - (p+3)x^3\big)\Phi'_{N,n}(x) \\ + \Big(-N(N+p) + \big[\chi_{N,n} - \tfrac{1}{4}(p+1)(p+2)\big]x^2 - c^2x^4\Big)\Phi_{N,n}(x) = 0, \quad (109)$$

and

$$(x^2 - x^4)\Phi^{(3)}_{N,n}(x) + \big((p+3)x - (p+7)x^3\big)\Phi''_{N,n}(x) \\ + \Big((p+1) - N(N+p) + \big[\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3) - 3(p+3)\big]x^2 - c^2x^4\Big)\Phi'_{N,n}(x) \\ + \Big(2\big[\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3)\big]x - 4c^2x^3\Big)\Phi_{N,n}(x) = 0, \quad (110)$$

and

$$
\begin{aligned}
&(x^2-x^4)\Phi^{(4)}_{N,n}(x)+\big((p+5)x-(p+11)x^3\big)\Phi^{(3)}_{N,n}(x)\\
&\quad+\Big(2(p+2)-N(N+p)+\big[\chi_{N,n}-\tfrac{1}{4}(p+1)(p+3)-6(p+5)\big]x^2-c^2x^4\Big)\Phi''_{N,n}(x)\\
&\quad+\Big(\big[4\big(\chi_{N,n}-\tfrac{1}{4}(p+1)(p+3)\big)-6(p+3)\big]x-8c^2x^3\Big)\Phi'_{N,n}(x)\\
&\quad+\Big(2\big(\chi_{N,n}-\tfrac{1}{4}(p+1)(p+3)\big)-12c^2x^2\Big)\Phi_{N,n}(x)=0, \quad (111)
\end{aligned}
$$

and

$$
\begin{aligned}
&(x^2-x^4)\Phi^{(5)}_{N,n}(x)+\big((p+7)x-(p+15)x^3\big)\Phi^{(4)}_{N,n}(x)\\
&\quad+\Big(3(p+3)-N(N+p)+\big[\chi_{N,n}-\tfrac{1}{4}(p+1)(p+3)-9(p+7)\big]x^2-c^2x^4\Big)\Phi^{(3)}_{N,n}(x)\\
&\quad+\Big(\big[6\big(\chi_{N,n}-\tfrac{1}{4}(p+1)(p+3)\big)-6(3p+13)\big]x-12c^2x^3\Big)\Phi''_{N,n}(x)\\
&\quad+\Big(6\big(\chi_{N,n}-\tfrac{1}{4}(p+1)(p+3)\big)-6(p+3)-36c^2x^2\Big)\Phi'_{N,n}(x)\\
&\quad-24c^2x\Phi_{N,n}(x)=0, \quad (112)
\end{aligned}
$$

where $0<x<1$ and N and n are arbitrary nonnegative integers.

The following corollary and theorem are obtained immediately from Lemma 9.3.

Corollary 9.4 *Let $c>0$. Then*

$$
\begin{aligned}
&\big(k(k+p)-N(N+p)\big)\Phi^{(k)}_{N,n}(0)\\
&\quad+\Big(k(k-1)\big(\chi_{N,n}-\tfrac{1}{4}(p+1)(p+3)\big)-k(k-1)(k-2)(k+p)\Big)\Phi^{(k-2)}_{N,n}(0)\\
&\quad-k(k-1)(k-2)(k-3)c^2\Phi^{(k-4)}_{N,n}(0)=0, \quad (113)
\end{aligned}
$$

where N and n are arbitrary nonnegative integers, and k is an arbitrary integer so that $k\ge 4$. Also,

$$N(N+p)\Phi_{N,n}(0)=0, \quad (114)$$

and

$$\big((p+1)-N(N+p)\big)\Phi'_{N,n}(0)=0, \quad (115)$$

and

$$\big(2(p+2)-N(N+p)\big)\Phi''_{N,n}(0)+2\big(\chi_{N,n}-\tfrac{1}{4}(p+1)(p+3)\big)\Phi_{N,n}(0)=0, \quad (116)$$

and

$$\big(3(p+3) - N(N+p)\big)\Phi^{(3)}_{N,n}(0) \\ + \Big(6\big(\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3)\big) - 6(p+3)\Big)\Phi'_{N,n}(0) = 0, \quad (117)$$

where N and n are arbitrary nonnegative integers.

Theorem 9.5 *If $N = 0$, then*

$$\Phi_{N,n}(0) \neq 0, \quad (118)$$

where n is an arbitrary nonnegative integer. If $N \geq 1$, then

$$\Phi^{(k)}_{N,n}(0) = 0 \quad \text{for } k = 0, 1, \ldots, N-1, \quad (119)$$

and

$$\Phi^{(N)}_{N,n}(0) \neq 0, \quad (120)$$

where n is an arbitrary nonnegative integer.

The following theorem follows directly from Theorem ??.

Theorem 9.6 *Suppose that N and n are nonnegative integers. Then*

$$\Phi_{N,n}(1) \neq 0. \quad (121)$$

9.3 Derivatives of GPSFs and Corresponding Eigenvalues With Respect to c

The following two theorems establish formulas for the derivatives of the eigenvalues $\mu_{N,n}$ (see (40)) and $\beta_{N,n}$ (see (51)) with respect to c.

Theorem 9.7 *Suppose that $c > 0$ is real and that N and n are nonnegative integers. Then*

$$\frac{\partial \beta_{N,n}}{\partial c} = \beta_{N,n}\frac{(\Phi_{N,n}(1))^2 - (p+2)}{2c}, \quad (122)$$

and

$$\frac{\partial \mu_{N,n}}{\partial c} = \frac{\mu_{N,n}}{c}((\Phi_{N,n}(1))^2 - (p+1)). \quad (123)$$

9.4 Integrals of Products of GPSFs and Their Derivatives

10 Appendix A

10.1 Derivation of the Integral Operator Q_c

In this section we derive an explicit formula for the integral operator Q_c, defined in (39).

Suppose that B denotes the closed unit ball in $\mathbb{R}^{p+2}$. From (39),

$$Q_c[\psi](x) = \left(\frac{c}{2\pi}\right)^{p+2} \int_B \int_B e^{ic\langle x-t,u\rangle} \psi(t)\, du\, dt, \tag{124}$$

for all $x \in B$. We observe that

$$e^{ic\langle v,u\rangle} = \sum_{N=0}^{\infty} \sum_{\ell=1}^{h(N,p)} i^N (2\pi)^{p/2+1} \frac{J_{N+p/2}(c\|u\|\|v\|)}{(c\|u\|\|v\|)^{p/2}} S_N^\ell(u/\|u\|) S_N^\ell(v/\|v\|), \tag{125}$$

for all $u, v \in B$, where S_N^ℓ denotes the spherical harmonics of degree N, and J_ν denotes Bessel functions of the first kind (see Section VII of [1]). Therefore,

$$\begin{aligned}
\int_B e^{ic\langle v,u\rangle}\, du &= (2\pi)^{p/2+1} \int_0^1 \frac{J_{p/2}(c\|v\|\rho)}{(c\|v\|\rho)^{p/2}} \rho^{p+1}\, d\rho \\
&= \frac{(2\pi)^{p/2+1}}{(c\|v\|)^{p/2}} \int_0^1 \rho^{p/2+1} J_{p/2}(c\|v\|\rho)\, d\rho \\
&= \left(\frac{2\pi}{c}\right)^{p/2+1} \frac{J_{p/2+1}(c\|v\|)}{\|v\|^{p/2+1}},
\end{aligned} \tag{126}$$

for all $v \in \mathbb{R}^{p+2}$, where the last equality follows from formula 6.561(5) in [5]. Combining (124) and (126),

$$Q_c[\psi](x) = \left(\frac{c}{2\pi}\right)^{p/2+1} \int_B \frac{J_{p/2+1}\big(c\|x-t\|\big)}{\|x-t\|^{p/2+1}} \psi(t)\, dt, \tag{127}$$

for all $x \in \mathbb{R}^{p+2}$.

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