Prolate spheroidal wave functions (PSWFs) play an important role in various areas, from physics (e.g. wave phenomena, fluid dynamics) to engineering (e.g. signal processing, filter design). One of the principal reasons for the importance of PSWFs is that they are a natural and efficient tool for computing with bandlimited functions, that frequently occur in the abovementioned areas. This is due to the fact that PSWFs are the eigenfunctions of the integral operator, that represents timelimiting followed by lowpassing. Needless to say, the behavior of this operator is governed by the decay rate of its eigenvalues. Therefore, investigation of this decay rate plays a crucial role in the related theory and applications - for example, in construction of quadratures, interpolation, filter design, etc.
The significance of PSWFs and, in particular, of the decay rate of the eigenvalues of the associated integral operator, was realized at least half a century ago. Nevertheless, perhaps surprisingly, despite vast numerical experience and existence of several asymptotic expansions, a non-trivial explicit upper bound on the magnitude of the eigenvalues has been missing for decades.
The principal goal of this paper is to close this gap in the theory of PSWFs. We analyze the integral operator associated with PSWFs, to derive fairly tight non-asymptotic upper bounds on the magnitude of its eigenvalues. Our results are illustrated via several numerical experiments.

Explicit upper bounds on the eigenvalues associated with prolate spheroidal wave functions

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Contents

1 Introduction

A function $f : \mathbb{R} \to \mathbb{R}$ is bandlimited of band limit $c > 0$, if there exists a function $\sigma \in L^2[-1, 1]$ such that

$$f(x) = \int_{-1}^{1} \sigma(t) e^{icxt} \, dt. \tag{1}$$

In other words, the Fourier transform of a bandlimited function is compactly supported. While (1) defines f for all real x, one is often interested in bandlimited functions, whose argument is confined to an interval, e.g. $-1 \leq x \leq 1$. Such functions are encountered in physics (wave phenomena, fluid dynamics), engineering (signal processing), etc. (see e.g. [13], [19], [20]).

About 50 years ago it was observed that the eigenfunctions of the integral operator $F_c : L^2[-1, 1] \to L^2[-1, 1]$, defined via the formula

$$F_c[\varphi](x) = \int_{-1}^{1} \varphi(t) e^{icxt} \, dt, \tag{2}$$

provide a natural tool for dealing with bandlimited functions, defined on the interval $[-1, 1]$. Moreover, it was observed (see [8], [9], [11]) that the eigenfunctions of F_c are precisely the prolate spheroidal wave functions (PSWFs), well known from the mathematical physics

[16], [19]. The PSWFs are the eigenfunctions of the differential operator L_c, defined via the formula

$$L_c[\varphi](x) = -\frac{d}{dx}\left((1-x^2)\cdot\frac{d\varphi}{dx}(x)\right) + c^2x^2. \tag{3}$$

In other words, the integral operator F_c commutes with the differential operator L_c [8], [18]. This property, being remarkable by itself, also plays an important role in both the analysis of PSWFs and the associated numerical algorithms [2], [3].

Obviously, the behavior of the operator F_c is governed by the decay rate of its eigenvalues. Over the last half a century, several related asymptotic expansions, as well as results of numerous numerical experiments, have been published; moreover, implications of the decay rate of the eigenvalues to both theory and applications have been extensively covered in the literature - see, for example, [1], [3], [4]. [5], [6], [8], [9], [10], [11], [12], [14], [15], [17]. It is perhaps surprising, however, that a non-trivial explicit upper bound on the magnitude of the eigenvalues of F_c has been missing for decades. This paper closes this gap in the theory of PSWFs.

This paper is mostly devoted to the analysis of the integral operator F_c, defined via (2). More specifically, several explicit upper bounds for the magnitude of the eigenvalues of F_c are derived. These bounds turn out to be fairly tight. The analysis is illustrated through several numerical experiments.

Some of the results of this paper are based on the recent analysis of the differential operator L_c, defined via (3), that appear in [22]. Nevertheless, the techniques used in this paper are quite different from those of [22]. The implications of the recent analysis of both L_c and F_c to numerical algorithms involving PSWFs are being currently investigated.

This paper is organized as follows. In Section 2, we summarize a number of well known mathematical facts to be used in the rest of this paper. In Section 3, we provide a summary of the principal results of this paper, and discuss several consequences of these results. In Section 4, we introduce the necessary analytical apparatus and carry out the analysis. In Section 5, we illustrate the analysis via several numerical examples.

2 Mathematical and Numerical Preliminaries

In this section, we introduce notation and summarize several facts to be used in the rest of the paper.

2.1 Prolate Spheroidal Wave Functions

In this subsection, we summarize several facts about the PSWFs. Unless stated otherwise, all these facts can be found in [3], [4], [6], [8], [9], [22].

Given a real number $c > 0$, we define the operator $F_c : L^2[-1,1] \to L^2[-1,1]$ via the formula

$$F_c[\varphi](x) = \int_{-1}^{1} \varphi(t) e^{icxt}\, dt. \tag{4}$$

Obviously, F_c is compact. We denote its eigenvalues by $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ and assume that they are ordered such that $|\lambda_n| \geq |\lambda_{n+1}|$ for all natural $n \geq 0$. We denote by ψ_n the eigenfunction corresponding to λ_n. In other words, the following identity holds for all integer $n \geq 0$ and all real $-1 \leq x \leq 1$:

$$\lambda_n \psi_n(x) = \int_{-1}^{1} \psi_n(t) e^{icxt}\, dt. \tag{5}$$

We adopt the convention[1] that $\|\psi_n\|_{L^2[-1,1]} = 1$. The following theorem describes the eigenvalues and eigenfunctions of F_c.

Theorem 1. *Suppose that $c > 0$ is a real number, and that the operator F_c is defined via (4) above. Then, the eigenfunctions $\psi_0, \psi_1, \ldots$ of F_c are purely real, are orthonormal and are complete in $L^2[-1,1]$. The even-numbered functions are even, the odd-numbered ones are odd. Each function ψ_n has exactly n simple roots in $(-1,1)$. All eigenvalues λ_n of F_c are non-zero and simple; the even-numbered ones are purely real and the odd-numbered ones are purely imaginary; in particular, $\lambda_n = i^n |\lambda_n|$.*

We define the self-adjoint operator $Q_c : L^2[-1,1] \to L^2[-1,1]$ via the formula

$$Q_c[\varphi](x) = \frac{1}{\pi} \int_{-1}^{1} \frac{\sin(c(x-t))}{x-t}\, \varphi(t)\, dt. \tag{6}$$

Clearly, if we denote by $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the unitary Fourier transform, then

$$Q_c[\varphi](x) = \chi_{[-1,1]}(x) \cdot \mathcal{F}^{-1}\left[\chi_{[-c,c]}(\xi) \cdot \mathcal{F}[\varphi](\xi)\right](x), \tag{7}$$

i.e. Q_c represents low-passing followed by time-limiting. Q_c relates to F_c, defined via (4), by

$$Q_c = \frac{c}{2\pi} \cdot F_c^* \cdot F_c, \tag{8}$$

and the eigenvalues μ_n of Q_n satisfy the identity

$$\mu_n = \frac{c}{2\pi} \cdot |\lambda_n|^2, \tag{9}$$

for all integer $n \geq 0$. Moreover, Q_c has the same eigenfunctions ψ_n as F_c. In other words,

$$\mu_n \psi_n(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{\sin(c(x-t))}{x-t}\, \psi_n(t)\, dt, \tag{10}$$

for all integer $n \geq 0$ and all $-1 \leq x \leq 1$. Also, Q_c is closely related to the operator $P_c : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, defined via the formula

$$P_c[\varphi](x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(c(x-t))}{x-t}\, \varphi(t)\, dt, \tag{11}$$

which is a widely known orthogonal projection onto the space of functions of band limit $c > 0$ on the real line $\mathbb{R}$.

The following theorem about the eigenvalues μ_n of the operator Q_c, defined via (6), can be traced back to [6]:

[1] This convention agrees with that of [3], [4] and differs from that of [8].

Theorem 2. *Suppose that $c > 0$ and $0 < \alpha < 1$ are positive real numbers, and that the operator $Q_c : L^2[-1,1] \to L^2[-1,1]$ is defined via (6) above. Suppose also that the integer $N(c,\alpha)$ is the number of the eigenvalues μ_n of Q_c that are greater than α. In other words,*

$$N(c,\alpha) = \max\left\{k = 1, 2, \dots \; : \; \mu_{k-1} > 0\right\}. \tag{12}$$

Then,

$$N(c,\alpha) = \frac{2}{\pi} c + \left(\frac{1}{\pi^2} \log \frac{1-\alpha}{\alpha}\right) \log c + O\left(\log c\right). \tag{13}$$

According to (13), there are about $2c/\pi$ eigenvalues whose absolute value is close to one, order of $\log c$ eigenvalues that decay exponentially, and the rest of them are very close to zero.

The eigenfunctions ψ_n of Q_c turn out to be the PSWFs, well known from classical mathematical physics [16]. The following theorem, proved in a more general form in [11], formalizes this statement.

Theorem 3. *For any $c > 0$, there exists a strictly increasing unbounded sequence of positive numbers $\chi_0 < \chi_1 < \dots$ such that, for each integer $n \geq 0$, the differential equation*

$$\left(1 - x^2\right) \psi''(x) - 2x \cdot \psi'(x) + \left(\chi_n - c^2 x^2\right) \psi(x) = 0 \tag{14}$$

has a solution that is continuous on $[-1,1]$. Moreover, all such solutions are constant multiples of the eigenfunction ψ_n of F_c, defined via (4) above.

In the following theorem, that appears in [4], an upper bound on $|\lambda_n|$ in terms of n and c is described.

Theorem 4. *Suppose that $c > 0$ is a real number, and $n \geq 0$ is a non-negative integer. Suppose also that λ_n is the nth eigenvalue of the operator F_c, defined via (4). Suppose furthermore that the real number $\nu(n,c)$ is defined via the formula*

$$\nu(n,c) = \frac{\sqrt{\pi} \cdot c^n \left(n!\right)^2}{(2n)! \cdot \Gamma(n + 3/2)}, \tag{15}$$

where Γ denotes the gamma function. Then,

$$|\lambda_n| \leq \nu(n,c). \tag{16}$$

Moreover,

$$\lambda_n(c) = i^n \nu(n,c) \cdot e^{R(n,c)}, \tag{17}$$

where the real number $R(n,c)$ is defined via the formula

$$R(n,c) = \int_0^c \left(\frac{2\left(\psi_n^\tau(1)\right)^2 - 1}{2\tau} - \frac{n}{\tau}\right) d\tau. \tag{18}$$

The function ψ_n^τ in (18) is the nth PSWF corresponding to the band limit τ.

The following approximation formula for $|\lambda_n|$ appears in Theorem 18 of [4], without proof (though the authors do illustrate its accuracy via several numerical examples).

Theorem 5. *Suppose that $c \geq 1$ is a real number, and that $n \geq c$ is a positive integer. Suppose also that the real number $p_0(n,c)$ is defined via the formula*

$$p_0(n,c) = \sqrt{\frac{2\pi}{c}} \cdot \exp\left[-\sqrt{\chi_n} \cdot \left(F\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right) - E\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right)\right)\right], \tag{19}$$

where F, E are the complete elliptic integrals, defined, respectively, via (38), (39) in Section 2.3. Then,

$$\left|\frac{|\lambda_n|}{p_0(n,c)} - 1\right| = O\left(\frac{1}{\sqrt{cn}}\right). \tag{20}$$

Remark 1. *Obviously, (20) cannot be used in rigorous analysis, due to the lack of both error estimates and proof. In addition, the assumption $n \geq c$ turns out to be rather restrictive. Nevertheless, in Section 4 we establish several upper bounds on $|\lambda_n|$, whose form is somewhat similar to that of $p_0(n,c)$. The approximate formula (20) will only be used in the discussion of the accuracy of these bounds, in Section 3.2.*

The following four theorems contain relatively recent results. All of them appear in [22].

Many properties of the PSWF ψ_n depend on whether the eigenvalue χ_n of the ODE (14) is greater than or less than c^2. In the following theorem, that appears in [22], we describe a simple relationship between c, n and χ_n.

Theorem 6. *Suppose that $n \geq 2$ is a non-negative integer.*

- *If $n \leq (2c/\pi) - 1$, then $\chi_n < c^2$.*
- *If $n \geq (2c/\pi)$, then $\chi_n > c^2$.*
- *If $(2c/\pi) - 1 < n < (2c/\pi)$, then either inequality is possible.*

In the following theorem, that appears in [22], we describe upper and lower bounds on χ_n in terms of n and c.

Theorem 7. *Suppose that $n \geq 2$ is a positive integer, and that $\chi_n > c^2$. Then,*

$$\begin{aligned} n < \frac{2}{\pi}\int_0^1 \sqrt{\frac{\chi_n - c^2 t^2}{1-t^2}}\, dt = \\ \frac{2}{\pi}\sqrt{\chi_n} \cdot E\left(\frac{c}{\sqrt{\chi_n}}\right) < n+3, \end{aligned} \tag{21}$$

where the function $E : [0,1] \to \mathbb{R}$ is defined via (39) in Section 2.3.

In the following theorem, we provide another upper bound on χ_n in terms of n.

Theorem 8. *Suppose that $n \geq 2$ is a positive integer, and that $\chi_n > c^2$. Then,*

$$\chi_n < \left(\frac{\pi}{2}(n+1)\right)^2. \tag{22}$$

In the following theorem, we describe an upper bound on the reciprocal of $|\psi_n(0)|$ for even n.

Theorem 9. *Suppose that $n > 0$ is an even integer, and that $\chi_n > c^2$. Then,*

$$\frac{1}{|\psi_n(0)|} \leq 4 \cdot \sqrt{n \cdot \frac{\chi_n}{c^2}}. \tag{23}$$

Remark 2. *Detailed numerical experiments, conducted by the author, seem to indicate that, in fact,*

$$\frac{1}{|\psi_n(0)|} = O(1). \tag{24}$$

In other words, the inequality (23) is rather crude; on the other hands, it has been rigorously proved, and is sufficient for our purposes.

2.2 Legendre Polynomials and PSWFs

In this subsection, we list several well known facts about Legendre polynomials and the relationship between Legendre polynomials and PSWFs. All of these facts can be found, for example, in [7], [3] [21].

The Legendre polynomials $P_0, P_1, P_2, \dots$ are defined via the formulae

$$\begin{aligned} P_0(t) &= 1, \\ P_1(t) &= t, \end{aligned} \tag{25}$$

and the recurrence relation

$$(k+1)\,P_{k+1}(t) = (2k+1)\,tP_k(t) - kP_{k-1}(t), \tag{26}$$

for $k = 1, 2, \dots$. The Legendre polynomials $\{P_k\}_{k=0}^{\infty}$ constitute a complete orthogonal system in $L^2\,[-1,1]$. The normalized Legendre polynomials are defined via the formula

$$\overline{P_k}(t) = P_k(t) \cdot \sqrt{k + 1/2}, \tag{27}$$

for $k = 0, 1, 2, \dots$. The $L^2\,[-1,1]$-norm of each normalized Legendre polynomial equals to one, i.e.

$$\int_{-1}^{1} \left(\overline{P_k}(t)\right)^2 \, dt = 1. \tag{28}$$

Therefore, the normalized Legendre polynomials constitute an orthonormal basis for $L^2[-1,1]$. In particular, for every real $c > 0$ and every integer $n \geq 0$, the prolate spheroidal wave function ψ_n, corresponding to the band limit c, can be expanded into the series

$$\psi_n(x) = \sum_{k=0}^{\infty} \beta_k^{(n,c)} \cdot \overline{P_k}(x), \tag{29}$$

for $-1 \leq x \leq 1$, where $\beta_0^{(n,c)}, \beta_1^{(n,c)}, \dots$ are defined via the formula

$$\beta_k^{(n,c)} = \int_{-1}^{1} \psi_n(x) \cdot \overline{P_k}(x)\, dx, \tag{30}$$

for all $k = 0, 1, 2, \dots$. The sequence $\beta_0^{(n,c)}, \beta_1^{(n,c)}, \dots$ satisfies the recurrence relation

$$\begin{aligned}
A_{0,0} \cdot \beta_0^{(n,c)} + A_{0,2} \cdot \beta_2^{(n,c)} &= \chi_n \cdot \beta_0^{(n,c)}, \\
A_{1,1} \cdot \beta_1^{(n,c)} + A_{1,3} \cdot \beta_3^{(n,c)} &= \chi_n \cdot \beta_1^{(n,c)}, \\
A_{k,k-2} \cdot \beta_{k-2}^{(n,c)} + A_{k,k} \cdot \beta_k^{(n,c)} + A_{k,k+2} \cdot \beta_{k+2}^{(n,c)} &= \chi_n \cdot \beta_k^{(n,c)},
\end{aligned} \tag{31}$$

for all $k = 2, 3, \dots$, where $A_{k,k}$, $A_{k+2,k}$, $A_{k,k+2}$ are defined via the formulae

$$\begin{aligned}
A_{k,k} &= k(k+1) + \frac{2k(k+1)-1}{(2k+3)(2k-1)} \cdot c^2, \\
A_{k,k+2} = A_{k+2,k} &= \frac{(k+2)(k+1)}{(2k+3)\sqrt{(2k+1)(2k+5)}} \cdot c^2,
\end{aligned} \tag{32}$$

for all $k = 0, 1, 2, \dots$. In other words, the infinite vector $\beta = \left\{\beta_k^{(n,c)}\right\}_{k=0}^{\infty}$ satisfies the identity

$$(A - \chi_n I) \cdot \beta = 0, \tag{33}$$

where the non-zero entries of the infinite symmetric matrix A are given via (32).

2.3 Elliptic Integrals

In this subsection, we summarize several facts about elliptic integrals. These facts can be found, for example, in section 8.1 in [7], and in [21].

The incomplete elliptic integrals of the first and second kind are given, respectively, by the formulae

$$F(y,k) = \int_0^y \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}, \tag{34}$$

$$E(y,k) = \int_0^y \sqrt{1 - k^2 \sin^2 t}\, dt, \tag{35}$$

where $0 \le y \le \pi/2$ and $0 \le k \le 1$. By performing the substitution $x = \sin t$, we can write (34) and (35) as

$$F(y,k) = \int_0^{\sin(y)} \frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}}, \tag{36}$$

$$E(y,k) = \int_0^{\sin(y)} \sqrt{\frac{1-k^2x^2}{1-x^2}}\, dx. \tag{37}$$

The complete elliptic integrals of the first and second kind are given, respectively, by the formulae

$$F(k) = F\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}, \tag{38}$$

$$E(k) = E\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2 t}\, dt, \tag{39}$$

where $0 \le k \le 1$.

3 Summary and Discussion

In this section, we summarize some of the properties of prolate spheroidal wave functions and the associated eigenvalues, proved in Section 4. In particular, we present several upper bounds on $|\lambda_n|$ and discuss their accuracy. The PSWFs and related notions were introduced in Section 2.1. Throughout this section, the band limit $c > 0$ is assumed to be a fixed positive real number.

3.1 Summary of Analysis

In the following two propositions, we provide some upper bounds on the eigenvalues χ_n of the ODE (14). They are proved in Theorem 25, 26, 31 in Section 4.3.

Proposition 1. *Suppose that n is a positive integer, and that*

$$n > \frac{2c}{\pi} + \frac{2}{\pi^2} \cdot \delta \cdot \log\left(\frac{4e\pi c}{\delta}\right), \tag{40}$$

for some

$$0 < \delta < \frac{5\pi}{4} \cdot c. \tag{41}$$

Then,

$$\chi_n > c^2 + \frac{4}{\pi} \cdot \delta \cdot c. \tag{42}$$

Proposition 2. *Suppose that n is a positive integer, and that*

$$\frac{2c}{\pi} + 3 \leq n \leq \frac{2c}{\pi} + \frac{2}{\pi^2} \cdot \delta \cdot \log\left(\frac{4e\pi c}{\delta}\right), \tag{43}$$

for some

$$3 < \delta < \frac{5\pi}{4} \cdot c. \tag{44}$$

Then,

$$\chi_n < c^2 + 16 \cdot \delta \cdot c. \tag{45}$$

The following is the principal result of this paper. It is proved in Theorem 23 in Section 4.2 (see also Remark 5), and is illustrated in Experiments 2, 3 in Section 5.

Proposition 3. *Suppose that $n > 0$ is an even integer number, and that λ_n is the nth eigenvalue of the integral operator F_c, defined via* (4), (5) *in Section 2.1. Suppose also that*

$$n > \frac{2c}{\pi} + \sqrt{42}. \tag{46}$$

Suppose furthermore that the real number $\zeta(n, c)$ is defined via the formula

$$\begin{aligned} \zeta(n,c) = {} & \frac{7}{2\,|\psi_n(0)|} \cdot \frac{\left(4 \cdot \chi_n / c^2 - 2\right)^4}{3 \cdot \chi_n / c^2 - 1} \cdot \left(\chi_n - c^2\right)^{\frac{1}{4}} \cdot \\ & \exp\left[-\sqrt{\chi_n} \cdot \left(F\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right) - E\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right)\right)\right], \end{aligned} \tag{47}$$

where χ_n is the nth eigenvalue of the differential operator L_c, defined via (4) *in Section 2.1, and F, E are the complete elliptic integrals, defined, respectively, via* (38), (39) *in Section 2.3. Then,*

$$|\lambda_n| < \zeta(n, c). \tag{48}$$

Remark 3. *It follows from the combination of Remark 2 in Section 2.1 and Proposition 2 above that*

$$\begin{aligned} & \zeta(n,c) = \\ & O((\delta c)^{1/4}) \cdot \exp\left[-\sqrt{\chi_n} \cdot \left(F\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right) - E\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right)\right)\right], \end{aligned} \tag{49}$$

where n, δ are as in (43), (44).

In the following proposition, we describe another upper bound on $|\lambda_n|$, which is weaker than the one presented in Proposition 3, but has a simpler form. It is proved in Theorem 24 in Section 4.3.

Proposition 4. *Suppose that $n > 0$ is an even integer number, and that λ_n is the nth eigenvalue of the integral operator F_c, defined via (4), (5) in Section 2.1. Suppose also that*

$$n > \frac{2c}{\pi} + \sqrt{42}. \tag{50}$$

Suppose furthermore that the real number $\eta(n,c)$ is defined via the formula

$$\eta(n,c) = 18 \cdot (n+1) \cdot \left(\frac{\pi \cdot (n+1)}{c}\right)^7 \cdot \\ \exp\left[-\sqrt{\chi_n} \cdot \left(F\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right) - E\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right)\right)\right], \tag{51}$$

where χ_n is the nth eigenvalue of the differential operator L_c, defined via (4) in Section 2.1, and F, E are the complete elliptic integrals, defined, respectively, via (38), (39) in Section 2.3. Then,

$$|\lambda_n| < \eta(n,c). \tag{52}$$

Remark 4. *According to Proposition 4,*

$$\eta(n,c) = O(c) \cdot \exp\left[-\sqrt{\chi_n} \cdot \left(F\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right) - E\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right)\right)\right], \tag{53}$$

as long as n is proportional to c.

Both $\zeta(n,c)$ and $\eta(n,c)$, defined, respectively, via (47) in Proposition 3 and (51) in Proposition 24, depend on χ_n, which somewhat obscures their behavior. In the following proposition, we eliminate this inconvenience by providing yet another upper bound on $|\lambda_n|$. It is proved in Theorem 33 in Section 4.3 and is illustrated via Experiment 3 in Section 5.

Proposition 5. *Suppose that $\delta > 0$ is a real number, and that*

$$3 < \delta < \frac{c}{16}. \tag{54}$$

Suppose also that n is a positive integer, and that

$$n > \frac{2}{\pi}c + \frac{2}{\pi^2} \cdot \delta \cdot \log\left(\frac{4e\pi c}{\delta}\right). \tag{55}$$

Suppose furthermore that the real number $\xi(n,c)$ is defined via the formula

$$\xi(n,c) = 7056 \cdot c \cdot \exp\left[-\delta\left(1 - \frac{\delta}{2\pi c}\right)\right]. \tag{56}$$

Then,

$$|\lambda_n| < \xi(n,c). \tag{57}$$

3.2 Accuracy of Upper Bounds on $|\lambda_n|$

In this subsection, we discuss the accuracy of the upper bounds on $|\lambda_n|$, presented in Propositions 3, 4, 5. In this discussion, we use the analysis of Section 4; previously reported results; and numerous numerical experiments, some of which are described in Section 5. Throughout this subsection, we suppose that n is a positive integer in the range

$$\frac{2c}{\pi} < n < \frac{2c}{\pi} + O(\log(c)). \tag{58}$$

According to the combination of Theorem 5 in Section 2.1 and Remark 3,

$$\frac{\zeta(n,c)}{|\lambda_n|} = O(c^{3/4}), \tag{59}$$

where $\zeta(n,c)$ is that of Proposition 3. On the other hand, both $|\lambda_n|$ and $\zeta(n,c)$ decay with n roughly exponentially, at the same rate. Therefore, the inequality (48) in Proposition 3 is reasonably tight (see also Experiment 2, Experiment 3 in Section 5).

The factor $O(c^{3/4})$ in (59) is an artifact of the analysis in Section 4.1. The first source of inaccuracy is the inequality (76) in the proof of Theorem 11. In this inequality, $\left|a_k^{(n,c)}\right|$ bounded from above by 1, while numerical experiments indicate that

$$\left|a_k^{(n,c)}\right| < O(c^{-1/2}), \tag{60}$$

for all integer $k > 0$. This contributes to the factor of order $c^{1/2}$ in (59). The second source of inaccuracy is Theorem 14, which gives rise to the factor

$$\frac{\left(4 \cdot \chi_n/c^2 - 2\right)^4}{3 \cdot \chi_n/c^2 - 1} \cdot \left(\chi_n - c^2\right)^{\frac{1}{4}} = O(c^{1/4}) \tag{61}$$

in (47) (see also Proposition 2). This contributes to another factor of order $c^{1/4}$ in (59).

In Propositions 4, 5 we introduce two additional upper bounds on $|\lambda_n|$; these bounds are weaker than $\zeta(n,c)$. More specifically,

$$\frac{\eta(n,c)}{\zeta(n,c)} = O(c^{3/4}) = \frac{\xi(n,c)}{\zeta(n,c)}, \tag{62}$$

due to Remarks 3, 4 and Proposition 5. Both $\eta(n,c)$ and $\xi(n,c)$ are derived from $\zeta(n,c)$ in Theorems 24, 33, respectively. There are two sources of the discrepancy (62). First, in the proofs of Theorems 24, 33, the term $\left(\chi_n - c^2\right)^{1/4}$ is bounded from above by $O(c^{1/2})$, while, in fact, it is of order $c^{1/4}$ (see (61) above). Additional factor of order $c^{1/2}$ in (62) is due to Theorem 9 and Remark 2 in Section 2.1. See also results of numerical experiments, reported in Section 5.

Finally, we observe that the upper bound $\nu(n,c)$ on $|\lambda_n|$, introduced in Theorem 4 in Section 2.1, is useless for n as in (58), due to the combination of Theorem 34 and Remark 11 in Section 4.3. On the other hand, $\nu(n,c)$ can be used to understand the behavior of $|\lambda_n|$ as $n \to \infty$, for a fixed $c > 0$.

4 Analytical Apparatus

The purpose of this section is to provide the analytical apparatus to be used in the rest of the paper. This principal results of this section are Theorems 23, 24.

4.1 Legendre Expansion

In this subsection, we analyze the Legendre expansion of PSWFs, introduced in Section 2.2. This analysis will be subsequently used in Section 4.2 to prove the principal result of this paper.

The following theorem is a direct consequence of the results outlined in Section 2.1 and Section 2.2.

Theorem 10. *Suppose that $c > 0$ is a real number, and $n > 0$ is an even positive integer. Suppose also that the numbers $a_1^{(n,c)}, a_2^{(n,c)}, \dots$ are defined via the formula*

$$a_k^{(n,c)} = \int_{-1}^{1} \psi_n(t) \cdot \overline{P_{2k-2}}(t)\, dt, \quad k = 1, 2, \dots, \tag{63}$$

where $\psi_n(t)$ is the nth PSWF corresponding to band limit c, and $\overline{P_k}(t)$ is the kth normalized Legendre polynomial. Then, the sequence $\left\{a_k^{(n,c)}\right\}$ satisfies the recurrence relation

$$\begin{aligned} & c_1 \cdot a_2^{(n,c)} + b_1 \cdot a_2^{(n,c)} = 0, \\ & c_{k+1} \cdot a_{k+2}^{(n,c)} + b_{k+1} \cdot a_{k+1}^{(n,c)} + c_k \cdot a_k^{(n,c)} = 0, \quad k \geq 1, \end{aligned} \tag{64}$$

where the numbers $c_1, c_2, \dots$ are defined via the formula

$$c_k = \frac{2k \cdot (2k-1)}{(4k-1) \cdot \sqrt{(4k-3) \cdot (4k+1)}} \cdot c^2, \quad k \geq 1, \tag{65}$$

and the numbers $b_1, b_2, \dots$ are defined via the formula

$$b_k = 2 \cdot (k-1) \cdot (2k-1) + \frac{2 \cdot (2k-1) \cdot (2k-2) - 1}{(4k-1) \cdot (4k-5)} \cdot c^2 - \chi_n, \quad k \geq 1. \tag{66}$$

Here χ_n is the nth eigenvalue of the prolate differential equation (14)*. Moreover,*

$$\psi_n(t) = \sum_{k=1}^{\infty} a_k^{(n,c)} \cdot \overline{P_{2k-2}}(t), \tag{67}$$

and

$$\sum_{k=1}^{\infty} \left(a_k^{(n,c)}\right)^2 = 1. \tag{68}$$

Proof. To establish (64) and (67), we combine (29), (32), (33) in Section 2.2 with Theorem 1 in Section 2.1. The identity (68) follows from the fact that the normalized Legendre polynomials constitute an orthonormal basis for $L^2[-1,1]$. ■

In the rest of the section, $c > 0$ is a fixed real number, and $n > 0$ is an even positive integer.

The following theorem provides an upper bound on $\left|a_1^{(n,c)}\right|$ in terms of the elements of another sequence.

Theorem 11. *Suppose that the sequence $\alpha_1, \alpha_2, \ldots$ is defined via the formula*

$$\alpha_k = \frac{a_k^{(n,c)}}{a_1^{(n,c)}}, \quad k \geq 1, \tag{69}$$

where $a_1^{(n,c)}, a_2^{(n,c)}, \ldots$ are defined via (63) in Theorem 10. Then, the sequence $\alpha_1, \alpha_2, \ldots$ satisfies the recurrence relation

$$\begin{aligned} &\alpha_1 = 1, \\ &\alpha_2 = B_0, \\ &\alpha_{k+2} = B_k \cdot \alpha_{k+1} - A_k \cdot \alpha_k, \quad k \geq 1, \end{aligned} \tag{70}$$

where the sequence $A_1, A_2, \ldots$ is defined via the formula

$$A_k = \frac{k \cdot (2k-1) \cdot (4k+3)}{(k+1) \cdot (2k+1) \cdot (4k-1)} \cdot \sqrt{\frac{4k+5}{4k-3}}, \quad k \geq 1, \tag{71}$$

and the sequence $B_0, B_1, \ldots$ is defined via the formula

$$\begin{aligned} B_k = &\left(\frac{\chi_n - 2k \cdot (2k+1)}{c^2} \right) \cdot \frac{(4k+3) \cdot \sqrt{(4k+1) \cdot (4k+5)}}{(2k+1) \cdot (2k+2)} - \\ &\frac{(4k \cdot (2k+1) - 1) \cdot \sqrt{(4k+1) \cdot (4k+5)}}{(4k-1) \cdot (2k+1) \cdot (2k+2)}, \quad k \geq 0. \end{aligned} \tag{72}$$

Moreover, for all $k = 1, 2, \ldots$,

$$\left| a_1^{(n,c)} \right| \leq \frac{1}{|\alpha_k|}. \tag{73}$$

Proof. Due to (64) in Theorem 10, the recurrence relation (70) holds with A_k, B_k's defined via the formulae

$$A_k = \frac{c_k}{c_{k+1}}, \quad B_k = -\frac{b_{k+1}}{c_{k+1}}, \tag{74}$$

where c_k, b_k's are defined, respectively, via (65) and (66). We observe that

$$\frac{1}{c_{k+1}} = \frac{(4k+3) \cdot \sqrt{(4k+1) \cdot (4k+5)}}{(2k+1) \cdot (2k+2)} \cdot \frac{1}{c^2} \tag{75}$$

and readily obtain both (71) and (72). Next, due to (68) and (69),

$$1 \geq \left| a_k^{(n,c)} \right| = \left| \frac{a_k^{(n,c)}}{a_1^{(n,c)}} \right| \cdot \left| a_1^{(n,c)} \right| = |\alpha_k| \cdot \left| a_1^{(n,c)} \right|, \tag{76}$$

for all $k = 1, 2, \ldots$, which implies (73). $\blacksquare$

It is somewhat easier to analyze a rescaled version of the sequence $\{\alpha_k\}$ defined via (69) in Theorem 11. This observation is reflected in the following theorem.

Theorem 12. *Suppose that the sequence $\beta_1, \beta_2, \dots$ is defined via the formula*

$$\beta_k = \alpha_k \cdot \sqrt{\frac{2}{4k-3}}, \quad k \geq 1, \tag{77}$$

where $\alpha_1, \alpha_2, \dots$ are defined via (69) in Theorem 11 above. Suppose also that the sequence $B_0^\chi, B_1^\chi, \dots$ is defined via the formula

$$B_k^\chi = \frac{(4k+1)\cdot(4k+3)}{(2k+1)\cdot(2k+2)} \cdot \left[\frac{\chi_n - c^2 - 2k\cdot(2k+1)}{c^2} \right], \quad k \geq 0. \tag{78}$$

Then, the sequence $\beta_1, \beta_2, \dots$ satisfies the recurrence relation

$$\begin{aligned} &\beta_1 = \sqrt{2}, \\ &\beta_2 = \tilde{B}_0 \cdot \sqrt{2}, \\ &\beta_{k+2} = \tilde{B}_k \cdot \beta_{k+1} - \tilde{A}_k \cdot \beta_k, \quad k \geq 1, \end{aligned} \tag{79}$$

where $\tilde{A}_0, \tilde{A}_1, \dots$ are defined via the formula

$$\tilde{A}_k = \frac{k\cdot(2k-1)\cdot(4k+3)}{(k+1)\cdot(2k+1)\cdot(4k-1)}, \quad k \geq 0, \tag{80}$$

and $\tilde{B}_0, \tilde{B}_1, \dots$ are defined via the formula

$$\tilde{B}_k = B_k^\chi + 1 + \tilde{A}_k, \quad k \geq 0. \tag{81}$$

Proof. Due to (70) and (77), we have for $k = 1, 2, \dots$

$$\begin{aligned} \beta_{k+2} &= \sqrt{\frac{2}{4k+5}} \cdot \alpha_{k+2} = \sqrt{\frac{2}{4k+5}} \cdot B_k \cdot \alpha_{k+1} - \sqrt{\frac{2}{4k+5}} \cdot A_k \cdot \alpha_k \\ &= \sqrt{\frac{4k+1}{4k+5}} \cdot B_k \cdot \sqrt{\frac{2}{4k+1}} \cdot \alpha_{k+1} - \sqrt{\frac{4k-3}{4k+5}} \cdot A_k \cdot \sqrt{\frac{2}{4k-3}} \cdot \alpha_k, \end{aligned} \tag{82}$$

and hence the recurrence relation (79) holds with

$$\tilde{A}_k = \sqrt{\frac{4k-3}{4k+5}} \cdot A_k, \quad \tilde{B}_k = \sqrt{\frac{4k+1}{4k+5}} \cdot B_k. \tag{83}$$

It remains to compute $\tilde{A}_k$'s and $\tilde{B}_k$'s. First, we observe that (80) follows immediately from the combination of (71) with (83). Second, we combine (72) with (83) to conclude that, for

$k = 1, 2, \ldots,$

$$\begin{aligned}
\tilde{B}_k &= \left[\frac{\chi_n - 2k\cdot(2k+1)}{c^2}\right]\cdot\frac{(4k+3)\cdot(4k+1)}{(2k+1)\cdot(2k+2)} - \frac{(8k^2+4k-1)\cdot(4k+1)}{(4k-1)\cdot(2k+1)\cdot(2k+2)} \\
&= \frac{(4k+1)\cdot(4k+3)}{(2k+1)\cdot(2k+2)}\cdot\left[\frac{\chi_n - c^2 - 2k\cdot(2k+1)}{c^2}\right] + \\
&\quad \frac{(4k+3)\cdot(4k+1)\cdot(4k-1)-(4k+1)\cdot(8k^2+4k-1)}{(4k-1)\cdot(2k+1)\cdot(2k+2)} \\
&= \frac{(4k+1)\cdot(4k+3)}{(2k+1)\cdot(2k+2)}\cdot\left[\frac{\chi_n - c^2 - 2k\cdot(2k+1)}{c^2}\right] + 1 + \\
&\quad \frac{(4k+3)\cdot(4k+1)\cdot(4k-1)-(4k+1)\cdot(8k^2+4k-1)-(4k-1)\cdot(2k+1)\cdot(2k+2)}{(4k-1)\cdot(2k+1)\cdot(2k+2)} \\
&= \frac{(4k+1)\cdot(4k+3)}{(2k+1)\cdot(2k+2)}\cdot\left[\frac{\chi_n - c^2 - 2k\cdot(2k+1)}{c^2}\right] + 1 + \tilde{A}_k, \qquad (84)
\end{aligned}$$

which completes the proof. ■

The following theorem, in which we establish the monotonicity of both $\{\alpha_k\}$ and $\{\beta_k\}$ up to a certain value of k, is a consequence of Theorem 12.

Theorem 13. *Suppose that $\chi_n > c^2$, and that $\beta_1, \beta_2, \ldots$ are defined via (77) in Theorem 12. Suppose also that the integer k_0 is defined via the formula*

$$\begin{aligned}
k_0 &= \max_k \left\{k = 1, 2, \ldots \;:\; 2k\cdot(2k+1) < \chi_n - c^2\right\} \\
&= \max_k \left\{k = 1, 2, \ldots \;:\; k \leq \frac{1}{2}\cdot\sqrt{\chi_n - c^2 + \frac{1}{4}} - \frac{1}{4}\right\}. \qquad (85)
\end{aligned}$$

Then,

$$\sqrt{2} = \beta_1 < \beta_2 < \cdots < \beta_{k_0} < \beta_{k_0+1} < \beta_{k_0+2}, \qquad (86)$$

and also,

$$1 = \alpha_1 < \alpha_2 < \cdots < \alpha_{k_0} < \alpha_{k_0+1} < \alpha_{k_0+2}, \qquad (87)$$

where the sequences $\{\alpha_k\}$ and $\{\beta_k\}$ are defined via (69) *and* (77)*, respectively.*

Proof. Due to (81) in Theorem 12 and the assumption that $\chi_n > c^2$,

$$\tilde{B}_0 = \frac{3}{2}\cdot\frac{\chi_n - c^2}{c^2} + 1 > 1. \qquad (88)$$

Therefore, due to (79) in Theorem 12,

$$\beta_2 = \tilde{B}_0 \cdot \beta_1 > \beta_1. \qquad (89)$$

By induction, suppose that $1 \le k \le k_0$ and assume that $\beta_k < \beta_{k+1}$. We combine (79), (80), (81) and (85) to conclude that

$$\beta_{k+2} = \beta_{k+1} + \tilde{B}_k \cdot \beta_{k+1} + \tilde{A}_k \cdot (\beta_{k+1} - \beta_k) > \beta_{k+1}, \tag{90}$$

since $\tilde{A}_k, \tilde{B}_k > 0$, which implies (86). To establish (87), we use (77) and observe that

$$\frac{\alpha_{k+1}}{\alpha_k} = \sqrt{\frac{4k+1}{4k-3}} \cdot \frac{\beta_{k+1}}{\beta_k} > \sqrt{\frac{4k+1}{4k-3}} > 1, \tag{91}$$

for $1 \le k \le k_0 + 1$. ■

In the following theorem, we bound the sequence $\beta_1, \beta_2, \dots,$ defined via (77) in Theorem 12, by another sequence from below.

Theorem 14. *Suppose that $\chi_n > c^2$, and that the sequence $\rho_1, \rho_2, \dots,$ is defined via the formula*

$$\rho_k = \frac{(4k-6) \cdot (4k-4) \cdot (4k+7)}{(4k-2) \cdot (4k) \cdot (4k+3)}, \tag{92}$$

for $k = 1, 2, \dots$. Suppose also that the sequence $A_1^{new}, A_2^{new}, \dots$ is defined via the formula

$$A_k^{new} = \tilde{A}_k \cdot \rho_k, \tag{93}$$

for $k = 1, 2, \dots,$ where $\tilde{A}_k$ is defined via (80) in Theorem 12. Suppose furthermore that the sequence $\beta_1^{new}, \beta_2^{new}, \dots$ is defined via the formulae

$$\begin{aligned} \beta_1^{new} &= \beta_1, \\ \beta_2^{new} &= \beta_2, \\ \beta_3^{new} &= \beta_3, , \\ \beta_{k+2}^{new} &= (B_k^\chi + 1) \cdot \beta_{k+1}^{new} + A_k^{new} \cdot (\beta_{k+1}^{new} - \beta_k^{new}), \quad k \ge 2, \end{aligned} \tag{94}$$

where $\beta_1, \beta_2, \dots$ are defined via (77), and B_k^χ is defined via (78) in Theorem 12. Then,

$$A_k^{new} = \frac{4k-4}{4k+4} \cdot \frac{4k-6}{4k+2} \cdot \frac{4k+7}{4k-1}, \tag{95}$$

for $k = 0, 1, \dots,$ and also

$$0 = A_1^{new} < A_2^{new} < A_3^{new} < \dots < A_k^{new} < \dots < 1. \tag{96}$$

Moreover,

$$\sqrt{2} = \beta_1^{new} < \beta_2^{new} < \dots < \beta_{k_0}^{new} < \beta_{k_0+1}^{new} < \beta_{k_0+2}^{new}, \tag{97}$$

where k_0 is defined via (85) in Theorem 13. In addition,

$$\beta_1^{new} \le \beta_1, \quad \beta_2^{new} \le \beta_2, \quad \dots, \quad \beta_{k_0+1}^{new} \le \beta_{k_0+1}, \quad \beta_{k_0+2}^{new} \le \beta_{k_0+2}. \tag{98}$$

Proof. The identity (95) follows immediately from the combination of (80) and (92). The monotonicity of $\{A_k^{new}\}$ follows from the fact that

$$\frac{dA_k}{dk} = \frac{(((3+k)\cdot 8k-19)\cdot 2k-51)\cdot 8k+2}{(4k-1)^2\cdot(k+1)^2\cdot(2k+1)^2}, \tag{99}$$

which is positive for all $k \geq 2$; combining this observation with the fact that A_k^{new} tends to 1 as $k \to \infty$, we conclude (96).

It follows from (94) by induction that $\beta_{j+2}^{new} > \beta_{j+1}^{new}$ as long as $B_j^{\chi} > 0$, which holds for $j \leq k_0$ due to (78) and (85). This observation implies (97).

It remains to prove (98). We observe that, due to (92), the sequence $0 = \rho_1, \rho_2, \dots$ grows monotonically and is bounded from above by 1. Combined with (93), this implies that

$$A_k^{new} < \tilde{A}_k, \quad k = 1, 2, \dots. \tag{100}$$

Eventually, we show by induction that

$$\beta_{k+1}^{new} - \beta_k^{new} \leq \beta_{k+1} - \beta_k \quad \text{and} \quad \beta_{k+1}^{new} \leq \beta_{k+1}, \tag{101}$$

for $k = 1, 2, \dots, k_0 + 1$, with k_0 defined via (85). For $k = 1, 2$, the inequalities (101) hold due to (94). We assume that they hold for some $k \leq k_0$. First, we combine (78), (77), (85), (94), (100) and the induction hypothesis to conclude that

$$\begin{aligned} \beta_{k+2}^{new} - \beta_{k+1}^{new} &= B_k^{\chi} \cdot \beta_{k+1}^{new} + A_k^{new} \cdot (\beta_{k+1}^{new} - \beta_k^{new}) \\ &\leq B_k^{\chi} \cdot \beta_{k+1} + \tilde{A}_k \cdot (\beta_{k+1} - \beta_k). \end{aligned} \tag{102}$$

Then, we combine (78), (77), (85), (94), (100) and the induction hypothesis to conclude that

$$\begin{aligned} &\beta_{k+2} - \beta_{k+2}^{new} = \\ &(B_k^{\chi} + 1) \cdot (\beta_{k+1} - \beta_{k+1}^{new}) + \tilde{A}_k \cdot (\beta_{k+1} - \beta_k) - A_k^{new} \cdot (\beta_{k+1}^{new} - \beta_k^{new}) > \\ &\beta_{k+1} - \beta_{k+1}^{new} > 0, \end{aligned} \tag{103}$$

which finishes the proof. ■

Theorem 14 allows us to find a lower bound on β_k by finding a lower bound on β_k^{new}, for $k \leq k_0 + 2$. In the following theorem, we simplify the recurrence relation (94) by rescaling $\{\beta_k^{new}\}$.

Theorem 15. *Suppose that $\chi_n > c^2 + 6$, and that the sequence $\beta_1^{new}, \beta_2^{new}, \dots$ is defined via (94) in Theorem 14. Suppose also that the sequence $f_1, f_2, \dots$ is defined via the formula*

$$f_k = \frac{(4k-4)\cdot(4k-6)}{4k-1}, \tag{104}$$

for $k = 1, 2, \dots$, and the sequence $\gamma_1, \gamma_2, \dots$ is defined via the formulae

$$\begin{aligned} &\gamma_1 = \beta_1^{new}, \\ &\gamma_k = f_k \cdot \beta_k^{new}, \quad k \geq 2. \end{aligned} \tag{105}$$

Then, the sequence $\gamma_1, \gamma_2, \ldots$ satisfies the formulae

$$\gamma_1 = \sqrt{2}, \tag{106}$$

$$\gamma_2 = \frac{8}{7\sqrt{2}} \cdot \left(2 + 3 \cdot \frac{\chi_n - c^2}{c^2}\right), \tag{107}$$

$$\gamma_3 = \frac{16\sqrt{2}}{11} \cdot \left(3 + 15 \cdot \frac{\chi_n - c^2}{c^2} + \frac{105}{8} \cdot \frac{\chi_n - c^2}{c^2} \cdot \frac{\chi_n - c^2 - 6}{c^2} - \frac{105}{2c^2}\right), \tag{108}$$

$$\gamma_{k+2} = \left(B_k^I + B_k^{II}\right) \cdot \gamma_{k+1} - \gamma_k, \quad k \geq 2, \tag{109}$$

where the sequences $\left\{B_k^I\right\}$ and $\left\{B_k^{II}\right\}$ are defined via the formulae

$$B_k^I = \frac{4 \cdot (4k+1) \cdot (4k+3)^2}{4k \cdot (4k-2) \cdot (4k+7)} \cdot \left[\frac{\chi_n - c^2 - 2k \cdot (2k+1)}{c^2}\right], \tag{110}$$

for $k = 1, 2, \ldots,$ and

$$B_k^{II} = 2 + \frac{60}{32k^4 + 32k^3 - 38k^2 + 7k}, \tag{111}$$

for $k = 1, 2, \ldots,$ respectively. Moreover,

$$\frac{245}{22} \cdot \frac{\chi_n - c^2 - 6}{c^2} = B_1^I > B_2^I > \cdots > B_{k_0}^I > 0, \tag{112}$$

where k_0 is defined via (85)*, and*

$$\frac{42}{11} = B_1^{II} > B_2^{II} > \cdots > B_k^{II} > \cdots > 2. \tag{113}$$

Proof. The identity (106) follows immediately from (94) and (105). Then, it follows from (71), (72), that

$$A_1 = \frac{7}{6}, \quad B_0 = \frac{\sqrt{5}}{2} \cdot \left(\frac{3\chi_n}{c^2} - 1\right) = \frac{\sqrt{5}}{2} \cdot \left(2 + 3 \cdot \frac{\chi_n - c^2}{c^2}\right), \tag{114}$$

moreover,

$$\begin{aligned} B_1 &= \frac{7\sqrt{5}}{4} \cdot \frac{\chi_n - 6}{c^2} - \frac{11\sqrt{5}}{12} = \frac{7\sqrt{5}}{4} \cdot \frac{\chi_n - c^2 - 6}{c^2} + \frac{7\sqrt{5}}{4} - \frac{11\sqrt{5}}{12} \\ &= \frac{\sqrt{5}}{12} \cdot \left(10 + 21 \cdot \frac{\chi_n - c^2 - 6}{c^2}\right). \end{aligned} \tag{115}$$

We combine (114) with (70), (77), (94), (104), (105) to conclude that

$$\gamma_2 = \frac{8}{7} \cdot \beta_2 = \frac{8}{7} \cdot \sqrt{\frac{2}{5}} \cdot \alpha_2 = \frac{8}{7} \cdot \sqrt{\frac{2}{5}} \cdot B_0, \tag{116}$$

from which (107) follows. Then we combine (114), (115) with (70), (77), (94), (104), (105) to conclude that

$$\begin{aligned}\gamma_3 &= \frac{48}{11}\cdot\beta_3 = \frac{48}{11}\cdot\frac{\sqrt{2}}{3}\cdot\alpha_3 = \frac{48\sqrt{2}}{33}\cdot(B_1\alpha_2 - A_1\alpha_1) = \frac{48\sqrt{2}}{33}\cdot(B_1B_0 - A_1)\\ &= \frac{16\sqrt{2}}{11}\cdot\left(\frac{5}{24}\cdot\left(2+3\cdot\frac{\chi_n - c^2}{c^2}\right)\cdot\left(10+21\cdot\frac{\chi_n-c^2-6}{c^2}\right)-\frac{7}{6}\right),\end{aligned} \tag{117}$$

which simplifies to yield (108). The relation (109) is established by using (78), (94), (93), (104), (105) to expand, for $k \geq 2$,

$$\begin{aligned}\gamma_{k+2} &= f_{k+2}\cdot\beta_{k+2}^{new} = f_{k+2}\cdot(B_k^{\chi}+1+A_k^{new})\cdot\beta_{k+1}^{new} - f_{k+2}\cdot A_k^{new}\cdot\beta_k^{new}\\ &= \frac{f_{k+2}}{f_{k+1}}\cdot(B_k^{\chi}+1+A_k^{new})\cdot\gamma_{k+1} - \frac{f_{k+2}}{f_k}\cdot A_k^{new}\cdot\gamma_k.\end{aligned} \tag{118}$$

Since, due to (93), (104), we have

$$\begin{aligned}&\frac{f_{k+2}}{f_k}\cdot A_k^{new} =\\ &\frac{(4n+4)\cdot(4n+2)}{4n+7}\cdot\frac{4n-1}{(4n-4)\cdot(4n-6)}\cdot\frac{(4n-4)\cdot(4n-6)\cdot(4n+7)}{(4n+4)\cdot(4n+2)\cdot(4n-1)} = 1,\end{aligned} \tag{119}$$

the identity (109) readily follows from (118), (119), with

$$B_k^I = \frac{f_{k+2}}{f_{k+1}}\cdot B_k^{\chi} \tag{120}$$

and

$$B_k^{II} = \frac{f_{k+2}}{f_{k+1}}\cdot\left(A_k^{new}+1\right). \tag{121}$$

We substitute (78), (104) into (120) to obtain (110). Next,

$$\begin{aligned}&\frac{d}{dk}\left[\frac{4\cdot(4k+1)\cdot(4k+3)^2}{4k\cdot(4k-2)\cdot(4k+7)}\right] = \frac{9}{14k^2}+\frac{512}{21\cdot(7+4k)^2}-\frac{50}{3\cdot(2k-1)^2} <\\ &\frac{1}{(k-1/2)^2}\cdot\left(\frac{9}{14}+\frac{512}{21\cdot 16}-\frac{50}{12}\right) = -\frac{2}{(k-1/2)^2} < 0,\end{aligned} \tag{122}$$

for $k \geq 1$. Due to (85), the term inside the square brackets of (110) is positive for $k \geq k_0$ and monotonically decreases as k grows, which, combined with (122), implies (112). Eventually, we substitute (93), (104) into (121) and use (119) to obtain, for $k \geq 1$,

$$B_k^{II} = \frac{f_{k+2}+f_k}{f_{k+1}}, \tag{123}$$

which yields (111) through straightforward algebraic manipulations. The monotonicity relation (113) follows immediately from (111). ■

We analyze the sequence $\{\gamma_k\}$ from Theorem 15 by considering the ratios of its consecutive elements. The latter are bounded from below by the largest eigenvalue of the characteristic equation of the recurrence relation (109). In the following two theorems, we elaborate on these ideas.

Theorem 16. *Suppose that $\chi_n > c^2$, and that the sequence $r_1, r_2, \dots$ is defined via the formula*

$$r_k = \frac{\gamma_{k+1}}{\gamma_k}, \tag{124}$$

for $k = 1, 2, \dots$, where the sequence $\gamma_1, \gamma_2, \dots$ is defined via (105) in Theorem 15. Suppose also that the sequence $\sigma_1, \sigma_2, \dots$ is defined via the formula

$$\sigma_k = \frac{B_k^I + B_k^{II}}{2} + \sqrt{\left(\frac{B_k^I + B_k^{II}}{2}\right)^2 - 1}, \tag{125}$$

for $k = 1, 2, \dots$, where B_k^I, B_k^{II} are defined via (110),(111) in Theorem 15, respectively. Then,

$$r_2 > B_2^I + B_2^{II}. \tag{126}$$

Moreover, if $B_2^I + B_2^{II} > 2$, then $\sigma_2 > 0$, and

$$r_2 > \sigma_2. \tag{127}$$

Proof. We use (110), (111) to obtain

$$B_2^I + B_2^{II} = \frac{44}{21} + \frac{121}{20} \cdot \frac{\chi_n - c^2 - 20}{c^2}. \tag{128}$$

Next, we plug (107),(108) into (124) to obtain

$$\begin{aligned} r_2 = \frac{28}{11} \cdot & \left(3 + 15 \cdot \frac{\chi_n - c^2}{c^2} + \frac{105}{8} \cdot \frac{\chi_n - c^2}{c^2} \cdot \frac{\chi_n - c^2 - 6}{c^2} - \frac{105}{2c^2}\right) \cdot \\ & \left(2 + 3 \cdot \frac{\chi_n - c^2}{c^2}\right)^{-1}. \end{aligned} \tag{129}$$

We subtract (128) from (129) to obtain, by performing elementary algebraic manipulations,

$$\begin{aligned} & r_2 - (B_2^I + B_2^{II}) = \\ & \frac{247}{77} + \frac{1119}{220} \cdot \frac{\chi_n - c^2}{c^2} - \frac{98}{33} \cdot \left(2 + 3 \cdot \frac{\chi_n - c^2}{c^2}\right)^{-1} + \frac{596}{11c^2} > \\ & \frac{247}{77} - \frac{98}{66} = \frac{398}{231} > 0, \end{aligned} \tag{130}$$

which implies (126). Due to (125), σ_2 is positive if and only if $B_2^I + B_2^{II} > 2$; in that case,

$$B_2^I + B_2^{II} > \sigma_2, \tag{131}$$

which, combined with (126), implies (127). ■

The following theorem extends Theorem 16.

Theorem 17. *Suppose that $\chi_n > c^2$, and that $k_0 > 2$, where k_0 is defined via* (85) *in Theorem 13. Suppose also that the sequences $r_1, r_2, \dots$ and $\sigma_1, \sigma_2, \dots$ are defined, respectively, via* (124), (125) *in Theorem 16. Then,*

$$\sigma_1 > \sigma_2 > \sigma_3 > \dots > \sigma_{k_0} > 1. \tag{132}$$

In addition,

$$r_2 > r_3 > \dots > r_{k_0} > 1. \tag{133}$$

Moreover,

$$r_2 > \sigma_2 > 1, \quad r_3 > \sigma_3 > 1, \quad \dots, \quad r_{k_0} > \sigma_{k_0} > 1. \tag{134}$$

Proof. We combine (110), (111), (112), (113) in Theorem 15 with (125) in Theorem 16 to conclude that, for $k = 1, 2, \dots, k_0$,

$$\sigma_k > \frac{B_k^I + B_k^{II}}{2} > \frac{B_k^{II}}{2} > 1. \tag{135}$$

We use this in combination with (112) and (113) to conclude that (132) holds. Then, we use (135) and Theorem 16 to conclude that

$$r_2 > \sigma_2 > 1. \tag{136}$$

Next, we prove (134) by induction on $k \leq k_0$. The case $k = 2$ is handled by (136). Suppose that $2 < k < k_0$, and (134) is true for k, i.e.

$$r_k > \sigma_k > 1. \tag{137}$$

We consider the quadratic equation

$$x^2 - (B_k^I + B_k^{II}) \cdot x + 1 = 0, \tag{138}$$

in the unknown x. Due to (125) and (135), σ_k is the largest root of the quadratic equation (138), and, moreover, $\sigma_k^{-1} < 1$ is its second (smallest) root. Thus, the left hand side of (138) is negative if and only if $x \in (\sigma_k^{-1}, \sigma_k)$. We combine this observation with (137) to conclude that

$$r_k^2 - (B_k^I + B_k^{II}) \cdot r_k + 1 > 0, \tag{139}$$

and, consequently,

$$r_k > (B_k^I + B_k^{II}) - \frac{1}{r_k}. \tag{140}$$

Then, we substitute (124) into (109) to obtain

$$r_{k+1} = \frac{\gamma_{k+2}}{\gamma_{k+1}} = \frac{(B_k^I + B_k^{II}) \cdot \gamma_{k+1} - \gamma_k}{\gamma_{k+1}} = (B_k^I + B_k^{II}) - \frac{1}{r_k}. \tag{141}$$

By combining (140) with (141) we conclude that

$$r_k > r_{k+1}. \tag{142}$$

Moreover, we combine (137) with (141) and use the fact that σ_k is a root of (138) to obtain the inequality

$$r_{k+1} = (B_k^I + B_k^{II}) - \frac{1}{r_k} > (B_k^I + B_k^{II}) - \frac{1}{\sigma_k} = \sigma_k. \tag{143}$$

However, combined with the already proved (132) and the fact that $k < k_0$, the inequality (143) implies that

$$r_{k+1} > \sigma_{k+1}. \tag{144}$$

This completes the proof of (134). The relation (133) follows from the inequality (142) above. ∎

In the following theorem, we bound the product of several σ_k's by a definite integral.

Theorem 18. *Suppose that $\chi_n > c^2$, and that $k_0 > 2$, where k_0 is defined via (85) in Theorem 13. Suppose also that the real valued function g_n is defined via the formula*

$$g_n(x) = \\ 1 + 2 \cdot \left(\frac{\chi_n - c^2}{c^2} - \left(\frac{2x}{c}\right)^2 \right) + \sqrt{\left[1 + 2 \cdot \left(\frac{\chi_n - c^2}{c^2} - \left(\frac{2x}{c}\right)^2 \right) \right]^2 - 1}, \tag{145}$$

for the real values of x satisfying the inequality $4x^2 \leq \chi_n - c^2$. Suppose furthermore that the sequence $\sigma_1, \sigma_2, \ldots$ is defined via the formula (125) in Theorem 16. Then,

$$\sigma_2 \cdot \sigma_3 \cdot \dots \cdot \sigma_{k_0 - 1} > (g_n(0))^{-4} \cdot \exp \int_0^{\left(\sqrt{\chi_n - c^2}\right)/2} \log\left(g_n(x)\right)\, dx. \tag{146}$$

Proof. We observe that, for $k = 1, 2, \ldots$, we have the inequality

$$4 \cdot k^2 < 2k \cdot (2k+1) < 4 \cdot (k+1)^2 < 2(k+1) \cdot (2(k+1)+1). \tag{147}$$

In combination with (85), this implies that, for $k = 1, \ldots, k_0$,

$$\chi_n - c^2 - 4 \cdot k^2 > 0. \tag{148}$$

Moreover, due to (110), (111) in Theorem 15, the inequality

$$2 < 2 + 4 \cdot \left(\frac{\chi_n - c^2}{c^2} - \left(\frac{2 \cdot (k+1)}{c} \right)^2 \right) < B_k^I + B_k^{II} \tag{149}$$

holds for $k = 1, \dots, k_0 - 1$, where B_k^I, B_k^{II} are defined via (110), (111), respectively. We combine (149) with (125) in Theorem 16 and (145) above to obtain the inequality

$$\sigma_k > g_n(k+1), \tag{150}$$

which holds for $k = 1, \dots, k_0 - 1$. Consequently, using the monotonicity of g_n,

$$\begin{aligned} & \sigma_2 \cdot \sigma_3 \cdot \dots \cdot \sigma_{k_0 - 1} > \\ & g_n(3) \cdot g_n(4) \cdot \dots \cdot g_n(k_0) = \frac{g_n(0) \cdot g_n(1) \cdot \dots \cdot g_n(k_0 - 1) \cdot g_n(k_0)^2}{g_n(0) \cdot g_n(1) \cdot g_n(2) \cdot g_n(k_0)} > \\ & g_n(0)^{-4} \cdot \exp\left(\log(g_n(0)) + \dots + \log(g_n(k_0+1)) + 2 \cdot \log(g_n(k_0)) \right). \end{aligned} \tag{151}$$

Obviously, due to (148), the inequality

$$\log(g_n(k)) > \int_k^{k+1} \log(g_n(x)) \, dx \tag{152}$$

holds for $k = 0, \dots, k_0 - 1$. Next, due to (85) and (147), we have

$$k_0 < \frac{1}{2} \sqrt{\chi_n - c^2} < k_0 + 2. \tag{153}$$

Therefore,

$$\begin{aligned} 2 \cdot \log(g_n(k_0)) &> \left(\frac{1}{2} \sqrt{\chi_n - c^2} - k_0 \right) \cdot \log(g_n(k_0)) \\ &> \int_{k_0}^{\left(\sqrt{\chi_n - c^2}\right)/2} g_n(x) \, dx. \end{aligned} \tag{154}$$

Thus, the inequality (146) follows from the combination of (151), (152) and (154). ■

4.2 Principal Result

In this subsection, we use the tools developed in Section 4.1 to derive an upper bound on $|\lambda_n|$. Theorem 23 is the principal result of this subsection.

In the following theorem, we simplify the integral in (146) by expressing it in terms of elliptic functions.

Theorem 19. *Suppose that $\chi_n > c^2$, and that the real-valued function g_n is defined via the formula* (145) *in Theorem 18. Then,*

$$\int_0^{\left(\sqrt{\chi_n - c^2}\right)/2} \log\left(g_n(x)\right) \, dx = \frac{\chi_n - c^2}{c} \cdot \int_0^{\pi/2} \frac{\sin^2(\theta) \, d\theta}{\sqrt{1 + \frac{\chi_n - c^2}{c^2} \cdot \cos^2(\theta)}}. \tag{155}$$

Moreover,

$$\int_0^{\left(\sqrt{\chi_n - c^2}\right)/2} \log\left(g_n(x)\right)\, dx = \sqrt{\chi_n} \cdot \left[F\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}} \right) - E\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}} \right) \right], \tag{156}$$

where F, E are the elliptic integrals defined, respectively, via the formula (38), (39) *in Section 2.3.*

Proof. We use (145) and perform the change of variable

$$s = \frac{2x}{\sqrt{\chi_n - c^2}} \tag{157}$$

in the left-hand side of (155) to obtain

$$\begin{aligned}
&\int_0^{\left(\sqrt{\chi_n - c^2}\right)/2} \log\left(g_n(x)\right)\, dx = \\
&\frac{\sqrt{\chi_n - c^2}}{2} \cdot \int_0^1 \log\left(g_n\left(\frac{s\sqrt{\chi_n - c^2}}{2} \right)\right)\, ds = \\
&\frac{V \cdot c}{2} \cdot \int_0^1 \log\left(1 + 2V^2(1-s^2) + \sqrt{(1+2V^2(1-s^2))^2 - 1} \right)\, ds = \\
&\frac{V \cdot c}{2} \cdot \int_0^1 \log(h(s))\, ds,
\end{aligned} \tag{158}$$

where V is defined via the formula

$$V = \sqrt{\frac{\chi_n - c^2}{c^2}}, \tag{159}$$

and the function $h : [0,1] \to \mathbb{R}$ is defined via the formula

$$h(s) = 1 + 2V^2(1-s^2) + \sqrt{(1+2V^2(1-s^2))^2 - 1}. \tag{160}$$

We observe that $\log(h(1)) = 0$ and $h(0)$ is finite, hence

$$\int_0^1 \log(h(s))\, ds = \left[s \cdot \log(h(s)) \right]_0^1 - \int_0^1 \frac{s \cdot h'(s)}{h(s)}\, ds = -\int_0^1 \frac{s \cdot h'(s)}{h(s)}\, ds. \tag{161}$$

Then, we differentiate $h(s)$, defined via (160), with respect to s to obtain

$$\begin{aligned}
h'(s) &= -2V^2 \cdot 2s + \frac{2 \cdot (1 + 2V^2(1-s^2)) \cdot (-2V^2 \cdot 2s)}{2\sqrt{(1+2V^2(1-s^2))^2 - 1}} \\
&= -4V^2 s \cdot \left(1 + \frac{1 + 2V^2(1-s^2)}{\sqrt{(1+2V^2(1-s^2))^2 - 1}} \right) \\
&= -\frac{4V^2 s \cdot h(s)}{\sqrt{(1+2V^2(1-s^2))^2 - 1}}.
\end{aligned} \tag{162}$$

We substitute (162) into (161) to obtain

$$\begin{aligned}
&\int_0^1 \log(h(s))\, ds = \\
&\int_0^1 \frac{4V^2 s^2}{\sqrt{(1+2V^2(1-s^2))^2 - 1}}\, ds = \\
&\int_0^1 \frac{4V^2 s^2}{\sqrt{4V^4(1-s^2)^2 + 4V^2(1-s^2)}}\, ds = \\
&2V \cdot \int_0^1 \frac{s^2}{\sqrt{(1-s^2)\cdot(1+V^2(1-s^2))}}\, ds. \qquad (163)
\end{aligned}$$

We perform the change of variable

$$s = \sin(\theta), \quad ds = \cos(\theta)\cdot d\theta, \qquad (164)$$

to transform (163) into

$$\int_0^1 \log(h(s))\, ds = 2V \cdot \int_0^{\pi/2} \frac{\sin^2(\theta)\, d\theta}{\sqrt{1 + V^2 \cdot \cos^2(\theta)}}. \qquad (165)$$

We combine (158), (159) and (165) to obtain the formula (155). Next, we express (155) in terms of the elliptic integrals $F(k)$ and $E(k)$, defined, respectively, via (38),(39) in Section 2.3. We note that

$$\begin{aligned}
F(k) - E(k) &= \int_0^{\pi/2} \frac{k^2 \sin^2 t\, dt}{\sqrt{1 - k^2 \sin^2 t}} \\
&= \frac{k^2}{\sqrt{1-k^2}} \cdot \int_0^{\pi/2} \frac{\sin^2 t\, dt}{\sqrt{1 + \frac{k^2}{1-k^2}\cdot \cos^2 t}}. \qquad (166)
\end{aligned}$$

Motivated by (155) and (166), we solve the equation

$$\frac{k^2}{1-k^2} = \frac{\chi_n - c^2}{c^2} \qquad (167)$$

in the unknown k, to obtain the solution

$$k = \sqrt{\frac{\chi_n - c^2}{\chi_n}}. \qquad (168)$$

We plug (168) into (166) to conclude that

$$\begin{aligned}
&F\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right) - E\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right) = \\
&\frac{\chi_n - c^2}{c\sqrt{\chi_n}} \cdot \int_0^{\pi/2} \frac{\sin^2(\theta)\, d\theta}{\sqrt{1 + \frac{\chi_n - c^2}{c^2}\cdot \cos^2(\theta)}}. \qquad (169)
\end{aligned}$$

We combine (155) with (169) to obtain (156). ■

In the following theorem, we establish a relationship between the eigenvalue λ_n of the integral operator F_c defined via (4) in Section 2.1, and the value of $a_1^{(n,c)}$ defined via (63) above.

Theorem 20. *Suppose that $n > 0$ is an even integer number, and that λ_n is the nth eigenvalue of the integral operator F_c defined via (4) in Section 2.1. In other words, λ_n satisfies the identity (5) in Section 2.1. Suppose also, that the sequence $a_1^{(n,c)}, a_2^{(n,c)}, \dots$ is defined via the formula (63) above. Then,*

$$\lambda_n = \frac{\sqrt{2}}{\psi_n(0)} \cdot a_1^{(n,c)}, \tag{170}$$

where ψ_n is the nth prolate spheroidal wave function defined in Section 2.1.

Proof. Due to (5) in Section 2.1, (25), (27) in Section 2.2, and (63) above,

$$\lambda_n \cdot \psi_n(0) = \int_{-1}^{1} \psi_n(t)\, dt = \sqrt{2} \cdot \int_{-1}^{1} \psi_n(t) \cdot \overline{P_0}(t)\, dt = \sqrt{2} \cdot a_1^{(n,c)}, \tag{171}$$

from which (170) readily follows. ■

In the following theorem, we provide an upper bound on $|\lambda_n|$ in terms of the elements of the sequence $\{\gamma_k\}$, defined via (105) in Theorem 15 above.

Theorem 21. *Suppose that $n > 0$ is an even integer number, and that λ_n is the nth eigenvalue of the integral operator F_c, defined via (4), (5) in Section 2.1. Suppose also that $\chi_n > c^2$, and that $k_0 > 2$, where k_0 is defined via (85) in Theorem 13. Suppose furthermore, that the sequence $\gamma_1, \gamma_2, \dots$ is defined via (105) in Theorem 15. Then,*

$$|\lambda_n| < \frac{2}{|\psi_n(0)|} \cdot \frac{(4k_0-4)\cdot(4k_0-6)}{(4k_0-1)\cdot\sqrt{4k_0-3}} \cdot \frac{1}{\gamma_{k_0}}. \tag{172}$$

Proof. We combine the inequality (73) in Theorem 11 with the identity (170) in Theorem 20, to conclude that

$$|\lambda_n| = \frac{\sqrt{2}}{|\psi_n(0)|} \cdot |a_1^{(n,c)}| < \frac{\sqrt{2}}{|\psi_n(0)|} \cdot \frac{1}{\alpha_{k_0}} = \frac{2}{|\psi_n(0)|} \cdot \frac{1}{\sqrt{4k_0-3}} \cdot \frac{1}{\beta_{k_0}}, \tag{173}$$

where β_{k_0} is defined via (77) in Theorem 12. Next, we combine (94), (98) in Theorem 14, (104),(105) in Theorem 15, and (173) to obtain the inequality

$$\begin{aligned} |\lambda_n| \; &< \frac{2}{|\psi_n(0)|} \cdot \frac{1}{\sqrt{4k_0-3}} \cdot \frac{1}{\beta_{k_0}} \le \frac{2}{|\psi_n(0)|} \cdot \frac{1}{\sqrt{4k_0-3}} \cdot \frac{1}{\beta_{k_0}^{new}} \\ &= \frac{2}{|\psi_n(0)|} \cdot \frac{(4k_0-4)\cdot(4k_0-6)}{(4k_0-1)\cdot\sqrt{4k_0-3}} \cdot \frac{1}{\gamma_{k_0}}, \end{aligned} \tag{174}$$

which is precisely (172). ■

The following theorem is a direct consequence of Theorems 6, 7 in Section 2.1.

Theorem 22. *Suppose that $n > 0$ is a positive integer. Suppose also that $n > (2c/\pi) + \sqrt{42}$. Then,*

$$\chi_n > c^2 + 42, \tag{175}$$

and also,

$$k_0 > 2, \tag{176}$$

where k_0 is defined via (85) in Theorem 13.

Proof. Suppose that $c^2 < \chi_n \geq c^2 + 2$. Then, due to Theorem 6,

$$\begin{aligned} n &< \frac{2}{\pi} \int_0^1 \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}}\, dt \leq \frac{2}{\pi} \int_0^1 \sqrt{c^2 + \frac{42}{1 - t^2}}\, dt \\ &< \frac{2c}{\pi} + \frac{2\sqrt{42}}{\pi} \cdot \int_0^1 \frac{dt}{\sqrt{1 - t^2}} = \frac{2c}{\pi} + \sqrt{42}. \end{aligned} \tag{177}$$

We combine (177) with Theorem 6 to conclude (175). Then, we combine (175) with (85) in Theorem 13 to conclude (176). ∎

The following theorem is the principal result of this paper.

Theorem 23. *Suppose that $n > 0$ is an even integer number, and that λ_n is the nth eigenvalue of the integral operator F_c, defined via (4), (5) in Section 2.1. Suppose also that $\chi_n > c^2 + 42$. Suppose furthermore that the real number $\zeta(n, c)$ is defined via the formula*

$$\begin{aligned} \zeta(n, c) = \frac{7}{2\,|\psi_n(0)|} \cdot \frac{\left(4 \cdot \chi_n / c^2 - 2\right)^4}{3 \cdot \chi_n / c^2 - 1} \cdot \left(\chi_n - c^2\right)^{\frac{1}{4}} \cdot \\ \exp\left[-\sqrt{\chi_n} \cdot \left(F\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right) - E\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right)\right)\right], \end{aligned} \tag{178}$$

where F, E are the complete elliptic integrals, defined, respectively, via (38), (39) in Section 2.3. Then,

$$|\lambda_n| < \zeta(n, c). \tag{179}$$

Proof. We start with observing that, due to (85) in Theorem 13 and (153) in Theorem 18, the inequality $\chi_n > c^2 + 42$ implies that $k_0 > 2$. We combine (105) in Theorem 15, (124), (125) in Theorem 16 and (134) in Theorem 17, to obtain the inequality

$$\begin{aligned} \gamma_{k_0} &= \gamma_2 \cdot \frac{\gamma_3}{\gamma_2} \cdot \dots \cdot \frac{\gamma_{k_0-1}}{\gamma_{k_0-2}} \cdot \frac{\gamma_{k_0}}{\gamma_{k_0-1}} \\ &= \gamma_2 \cdot r_2 \cdot \dots \cdot r_{k_0-2} \cdot r_{k_0-1} \\ &> \gamma_2 \cdot \left(\sigma_2 \cdot \dots \cdot \sigma_{k_0-1}\right). \end{aligned} \tag{180}$$

Next, we substitute (145), (146) in Theorem 18 into (180) to obtain the inequality

$$\begin{aligned}\gamma_{k_0} &> \gamma_2 \cdot (g_n(0))^{-4} \cdot \exp \int_0^{\left(\sqrt{\chi_n - c^2}\right)/2} \log\left(g_n(x)\right)\, dx \\ &> \gamma_2 \cdot \left(2 + 4 \cdot \frac{\chi_n - c^2}{c^2}\right)^{-4} \cdot \exp \int_0^{\left(\sqrt{\chi_n - c^2}\right)/2} \log\left(g_n(x)\right)\, dx, \end{aligned} \tag{181}$$

where the function g_n is defined via (145). Then, we plug the identity (155) from Theorem 19 into (181) to obtain the inequality

$$\begin{aligned}\frac{1}{\gamma_{k_0}} < \frac{1}{\gamma_2} \cdot \left(2 + 4 \cdot \frac{\chi_n - c^2}{c^2}\right)^4 \cdot & \\ \exp\left[-\frac{\chi_n - c^2}{c} \cdot \int_0^{\pi/2} \frac{\sin^2(\theta)\, d\theta}{\sqrt{1 + \frac{\chi_n - c^2}{c^2} \cdot \cos^2(\theta)}}\right]&. \end{aligned} \tag{182}$$

We use (85) in Theorem 13 and (153) in Theorem 18 to conclude that

$$\frac{(4k_0 - 4) \cdot (4k_0 - 6)}{(4k_0 - 1) \cdot \sqrt{4k_0 - 3}} < \sqrt{4k_0} < \sqrt{2} \cdot \left(\chi_n - c^2\right)^{\frac{1}{4}}. \tag{183}$$

We substitute (183) into (172) in Theorem 21 to obtain

$$|\lambda_n| < \frac{2}{|\psi_n(0)|} \cdot \sqrt{2} \cdot \left(\chi_n - c^2\right)^{\frac{1}{4}} \cdot \frac{1}{\gamma_{k_0}}. \tag{184}$$

Finally, we combine (107) in Theorem 15 with (182), (184) to obtain

$$\begin{aligned}|\lambda_n| < \frac{7}{2\,|\psi_n(0)|} \cdot \left(\chi_n - c^2\right)^{\frac{1}{4}} \cdot \left(2 + 3 \cdot \frac{\chi_n - c^2}{c^2}\right)^{-1} \cdot \left(2 + 4 \cdot \frac{\chi_n - c^2}{c^2}\right)^4 \cdot & \\ \exp\left[-\frac{\chi_n - c^2}{c} \cdot \int_0^{\pi/2} \frac{\sin^2(\theta)\, d\theta}{\sqrt{1 + \frac{\chi_n - c^2}{c^2} \cdot \cos^2(\theta)}}\right]&. \end{aligned} \tag{185}$$

Eventually, we combine (156) in Theorem 19 with (185) to conclude (179). ■

Remark 5. *The assumptions of Theorem 23 are satisfied if n is an even integer such that*

$$n > \frac{2c}{\pi} + \sqrt{42}, \tag{186}$$

since, in this case, $\chi_n > c^2 + 42$ due to Theorem 22.

4.3 Weaker But Simpler Bounds

In this subsection, we use Theorem 23 in Section 4.2 to derive several upper bounds on $|\lambda_n|$. While these bounds are weaker than $\zeta(n, c)$ defined via (178), they have a simpler form, and contribute to a better understanding of the decay of $|\lambda_n|$. The principal results of this subsection are Theorems 24, 33.

In the following theorem, we simplify the inequality (179). The resulting upper bound on $|\lambda_n|$ is weaker than (179) in Theorem 23, but has a simpler form.

Theorem 24. *Suppose that $n > 0$ is an even integer number, and that λ_n is the nth eigenvalue of the integral operator F_c, defined via (4), (5) in Section 2.1. Suppose also that $\chi_n > c^2 + 42$. Suppose furthermore that the real number $\eta(n,c)$ is defined via the formula*

$$\eta(n,c) = 18 \cdot (n+1) \cdot \left(\frac{\pi \cdot (n+1)}{c} \right)^7 \cdot \exp\left[-\sqrt{\chi_n} \cdot \left(F\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}} \right) - E\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}} \right) \right) \right], \tag{187}$$

where F, E are the complete elliptic integrals, defined, respectively, via (38), (39) in Section 2.3. Then,

$$|\lambda_n| < \eta(n,c). \tag{188}$$

Proof. We use (22) in Theorem 8 in Section 2.1 to conclude that

$$\left(\chi_n - c^2\right)^{1/4} < (\chi_n)^{1/4} < \left(\frac{\pi}{2} \cdot (n+1) \right)^{1/2}. \tag{189}$$

Next,

$$\left(2 + 3 \cdot \frac{\chi_n - c^2}{c^2} \right)^{-1} \cdot \left(2 + 4 \cdot \frac{\chi_n - c^2}{c^2} \right)^4 < 2^7 \cdot \left(\frac{\chi_n}{c^2} \right)^3. \tag{190}$$

We combine Theorems 8, 9 in Section 2.1 with (189), (190) to conclude that

$$\begin{aligned}
&\frac{1}{|\psi_n(0)|} \cdot \frac{\left(4 \cdot \chi_n / c^2 - 2\right)^4}{3 \cdot \chi_n / c^2 - 1} \cdot \left(\chi_n - c^2\right)^{\frac{1}{4}} < \\
&4 \cdot \sqrt{n \cdot \frac{\chi_n}{c^2}} \cdot \frac{\left(4 \cdot \chi_n / c^2 - 2\right)^4}{3 \cdot \chi_n / c^2 - 1} \cdot \left(\chi_n - c^2\right)^{\frac{1}{4}} < \\
&4 \cdot (n+1)^{1/2} \cdot 2^7 \cdot \left(\frac{\chi_n}{c^2} \right)^{7/2} \cdot \left(\frac{\pi}{2} \cdot (n+1) \right)^{1/2} < \\
&4 \cdot \sqrt{\frac{\pi}{2}} \cdot 2^7 \cdot (n+1) \cdot \left(\frac{\pi \cdot (n+1)}{2c} \right)^7 = \sqrt{\frac{\pi}{2}} \cdot (n+1) \cdot \left(\frac{\pi \cdot (n+1)}{c} \right)^7.
\end{aligned} \tag{191}$$

We conclude by combining the inequality (179) in Theorem 23 above with the inequality (191). ■

Both $\zeta(n,c)$ and $\eta(n,c)$, defined, respectively, via (178) in Theorem 23 and (187) in Theorem 24, contain an exponential term (of the form $\exp[\ldots]$). This term depends on band limit c and prolate index n through χ_n, which somewhat obscures its behavior. The following theorem eliminates this inconvenience.

Theorem 25. *Suppose that n is a positive integer such that $n > 2c/\pi$, and that the function $f : [0, \infty) \to \mathbb{R}$ is defined via the formula*

$$f(x) = -1 + \int_0^{\pi/2} \sqrt{x + \cos^2(\theta)}\, d\theta. \tag{192}$$

Suppose also that the function $H : [0, \infty) \to \mathbb{R}$ is the inverse of f, in other words,

$$y = f(H(y)) = -1 + \int_0^{\pi/2} \sqrt{H(y) + \cos^2(\theta)}\, d\theta, \tag{193}$$

for all $y \geq 0$. Suppose furthermore that the function $G : [0, \infty) \to \mathbb{R}$ is defined via the formula

$$G(x) = \int_0^{\pi/2} \frac{\sin^2(\theta)\, d\theta}{\sqrt{1 + x \cdot \cos^2(\theta)}}, \tag{194}$$

for $x \geq 0$. Then,

$$H\left(\frac{n\pi}{2c} - 1\right) < \frac{\chi_n - c^2}{c^2} < H\left(\frac{n\pi}{2c} - 1 + \frac{3\pi}{2c}\right). \tag{195}$$

Moreover,

$$\begin{aligned} & c \cdot H\left(\frac{n\pi}{2c} - 1\right) \cdot G\left(H\left(\frac{n\pi}{2c} - 1\right)\right) < \\ & \sqrt{\chi_n} \cdot \left(F\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right) - E\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right)\right), \end{aligned} \tag{196}$$

where F, E are the complete elliptic integrals, defined, respectively, via (38), (39) *in Section 2.3.*

Proof. Obviously, the function f, defined via (192), is monotonically increasing. Moreover, $f(0) = 0$, and

$$\lim_{x \to \infty} f(x) = \infty. \tag{197}$$

Therefore, $H(y)$ is well defined for all $y \geq 0$, and, moreover, the function H is monotonically increasing. This observation, combined with Theorems 6, 7 in Section 2.1, implies the inequality (195).

Next, the right hand side of (196) increases with χ_n, due to the combination of (38), (39) in Section 2.3. This observation, combined with (169) in the proof of Theorem 19, (194) and (195), implies (196). ■

Remark 6. *The functions H, G, defined, respectively, via* (193), (194) *above, do not depend on either of n, c, χ_n. Therefore, while the right-hand side of* (196) *does depend on χ_n, its left-hand side depends solely on c and n.*

In the following theorem, we provide simple lower and upper bounds on H, defined via (193) in Theorem 25.

Theorem 26. *Suppose that the function $H : [0, \infty) \to \mathbb{R}$ is defined via* (193) *in Theorem 25. Then,*

$$s \le H\left(\frac{s}{4} \cdot \log \frac{16e}{s}\right) \le s + \frac{s^2}{5}, \tag{198}$$

for all real $0 \le s \le 5$.

Proof. The proof of (198) is straightforward and elementary, and will be omitted. It is based on well known properties of the elliptic integral E defined via (39) in Section 2.3. The correctness of Theorem 26 has been validated numerically. ■

Remark 7. *The relative error of the lower bound in* (198) *is below 0.07 for all $0 \le s \le 5$; moreover, this error grows roughly linearly with s to ≈ 0.085 for $0 \le s \le 0.1$. The relative error of the upper bound in* (198) *grows roughly linearly with s to 1, for $0 \le s \le 5$.*

In the following theorem, we provide simple lower and upper bound on G, defined via (194) in Theorem 25.

Theorem 27. *Suppose that the function $G : [0, \infty) \to \mathbb{R}$ is defined via* (194) *in Theorem 25. Then,*

$$\frac{\pi}{4} \cdot \left(1 - \frac{x}{8}\right) \le G(x) \le \frac{\pi}{4}, \tag{199}$$

for all real $0 \le x \le 5$.

Proof. The proof of (199) is elementary and will be omitted. It is based on Taylor expansion of G about zero, for $x > 0$. The correctness of Theorem 27 has been validated numerically. ■

Remark 8. *The relative errors of both lower and upper bounds in* (199) *are below 0.6 for all $0 \le x \le 5$; moreover, these errors are below 0.01 for all $0 \le x \le 0.1$, and grow roughly linearly with x in this interval.*

The following theorem is in the spirit of Theorems 26, 27.

Theorem 28. *Suppose that the functions $H, G : [0, \infty) \to \mathbb{R}$ are defined, respectively, via* (193), (194) *in Theorem 25. Then,*

$$\frac{\pi}{4} \cdot s \cdot \left(1 - \frac{s}{8}\right) \le H\left(\frac{s}{4} \cdot \log \frac{16e}{s}\right) \cdot G\left(H\left(\frac{s}{4} \cdot \log \frac{16e}{s}\right)\right) \le \frac{\pi}{4} \cdot s, \tag{200}$$

for all real $0 \le s \le 5$. Moreover, the function $x \to H(x) \cdot G(H(x))$ is monotonically increasing.

Proof. The proof uses Theorems 26, 27, is elementary, and will be omitted. The correctness of Theorem 28 has been validated numerically. ■

Remark 9. *The relative errors of both lower and upper bounds in* (200) *are below 0.5 for all $0 \le s \le 5$. Moreover, these errors are below 0.01 for all $0 \le s \le 0.1$, and grow roughly linearly with s in this interval.*

The following theorem is a consequence of Theorems 25 - 28.

Theorem 29. *Suppose that $\delta > 0$ is a real number, such that*

$$0 < \delta < \frac{5\pi}{4} \cdot c. \tag{201}$$

Suppose also that n is a positive integer, such that

$$n > \frac{2}{\pi} c + \frac{2}{\pi^2} \cdot \delta \cdot \log\left(\frac{4e\pi c}{\delta}\right). \tag{202}$$

Then,

$$\delta \cdot \left(1 - \frac{\delta}{2\pi c}\right) < \sqrt{\chi_n} \cdot \left(F\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right) - E\left(\sqrt{\frac{\chi_n - c^2}{\chi_n}}\right)\right), \tag{203}$$

where F, E are the complete elliptic integrals, defined, respectively, via (38), (39) *in Section 2.3.*

Proof. It follows from (202) that

$$\frac{\pi n}{2c} - 1 > \frac{1}{\pi} \cdot \frac{\delta}{c} \cdot \log\left(\frac{4e\pi c}{\delta}\right). \tag{204}$$

We defined $s > 0$ via the formula

$$s = \frac{4\delta}{\pi c}, \tag{205}$$

and observe that $0 < s < 5$ due to (201). We combine (204), (205) and Theorem 28 to obtain

$$\begin{aligned}
& H\left(\frac{n\pi}{2c} - 1\right) \cdot G\left(H\left(\frac{n\pi}{2c} - 1\right)\right) > \\
& H\left(\frac{1}{\pi} \cdot \frac{\delta}{c} \cdot \log\left(\frac{4e\pi c}{\delta}\right)\right) \cdot G\left(H\left(\frac{1}{\pi} \cdot \frac{\delta}{c} \cdot \log\left(\frac{4e\pi c}{\delta}\right)\right)\right) = \\
& H\left(\frac{s}{4} \cdot \log \frac{16e}{s}\right) \cdot G\left(H\left(\frac{s}{4} \cdot \log \frac{16e}{s}\right)\right) \geq \\
& \frac{\pi}{4} \cdot s \cdot \left(1 - \frac{s}{8}\right) = \frac{\delta}{c} \cdot \left(1 - \frac{\delta}{2\pi c}\right).
\end{aligned} \tag{206}$$

We substitute (206) into the inequality (196) in Theorem 25 to obtain (203). ■

In the following theorem, we derive an upper bound on χ_n in terms of χ_{n-3}.

Theorem 30. *Suppose that $n > (2c)/\pi + 3$ is a positive integer. Then,*

$$\chi_n < \chi_{n-3} + 6 \cdot \sqrt{\chi_{n-3}}. \tag{207}$$

Proof. Due to Theorems 6, 7 i n Section 2.1,

$$
\begin{aligned}
\frac{2}{\pi} \int_0^{\pi/2} \sqrt{\chi_n - c^2 \sin^2(s)}\, ds < n + 3 = (n-3) + 6 \\
< \frac{2}{\pi} \int_0^{\pi/2} \sqrt{\chi_{n-3} - c^2 \sin^2(s)}\, ds.
\end{aligned}
\tag{208}
$$

It follows from (208) that

$$
\begin{aligned}
3\pi \; & > \int_0^{\pi/2} \left(\sqrt{\chi_n - c^2 \sin^2(s)} - \sqrt{\chi_{n-3} - c^2 \sin^2(s)} \right) \, ds \\
& = \int_0^{\pi/2} \frac{\chi_n - \chi_{n-3}}{\sqrt{\chi_n - c^2 \sin^2(s)} + \sqrt{\chi_{n-3} - c^2 \sin^2(s)}} \, ds \\
& > \int_0^{\pi/2} \frac{\chi_n - \chi_{n-3}}{\sqrt{\chi_{n-3}}} \, ds = \frac{\pi}{2} \cdot \frac{\chi_n - \chi_{n-3}}{\sqrt{\chi_{n-3}}},
\end{aligned}
$$

which implies (207). ∎

In the following theorem, we derive an upper bound on χ_n in terms of n.

Theorem 31. *Suppose that n is a positive integer, and that*

$$
\frac{2c}{\pi} + 3 < n \le \frac{2}{\pi} c + \frac{2}{\pi^2} \cdot \delta \cdot \log\left(\frac{4e\pi c}{\delta} \right),
\tag{209}
$$

for some

$$
3 < \delta < \frac{5\pi}{4} \cdot c.
\tag{210}
$$

Then,

$$
\frac{\chi_{n-3} - c^2}{c^2} < \frac{8}{\pi} \cdot \frac{\delta}{c},
\tag{211}
$$

and, moreover,

$$
\frac{\chi_n - c^2}{c^2} < 16 \cdot \frac{\delta}{c}.
\tag{212}
$$

Proof. We combine (209), (210), (195) in Theorem 25 and (198) in Theorem 26 to obtain

$$
\begin{aligned}
\frac{\chi_{n-3} - c^2}{c^2} \; & < H\left(\frac{n\pi}{2c} - 1 \right) < H\left(\frac{1}{\pi} \cdot \frac{\delta}{c} \cdot \log\left(\frac{4e\pi c}{\delta} \right) \right) \\
& < \frac{4\delta}{\pi c} \cdot \left(1 + \frac{4}{5\pi} \cdot \frac{\delta}{c} \right),
\end{aligned}
\tag{213}
$$

which implies (211). We substitute (209), (210) into (207) in Theorem 30 to obtain

$$\begin{aligned} \chi_n &< c^2 + \frac{8\delta c}{\pi} + 6 \cdot \sqrt{c^2 + \frac{8\delta c}{\pi}} \\ &< c^2 + \frac{8\delta c}{\pi} + 6c \cdot \left(1 + \frac{4\delta}{\pi c}\right) < c^2 + c \cdot \left(\frac{8\delta}{\pi} + 36\right), \end{aligned} \tag{214}$$

which implies (212). ■

In the following theorem, we derive an upper bound on the non-exponential term of $\zeta(n, c)$, defined via (178) in Theorem 23.

Theorem 32. *Suppose that n is an even positive integer, and that*

$$\frac{2c}{\pi} + 3 < n \le \frac{2}{\pi} c + \frac{2}{\pi^2} \cdot \delta \cdot \log\left(\frac{4e\pi c}{\delta}\right), \tag{215}$$

for some

$$3 < \delta < \frac{5\pi}{4} \cdot c. \tag{216}$$

Then,

$$\begin{aligned} &\frac{7}{2\,|\psi_n(0)|} \cdot \frac{\left(4 \cdot \chi_n/c^2 - 2\right)^4}{3 \cdot \chi_n/c^2 - 1} \cdot \left(\chi_n - c^2\right)^{\frac{1}{4}} < \\ &\frac{896}{3} \cdot \delta^{1/4} \cdot c^{3/4} \cdot \left(1 + \frac{12\delta}{c}\right) \cdot \left(1 + \frac{32\delta}{c}\right)^3. \end{aligned} \tag{217}$$

Proof. We use (212) to obtain

$$\begin{aligned} \frac{\left(4 \cdot \chi_n/c^2 - 2\right)^4}{3 \cdot \chi_n/c^2 - 1} &= \frac{4}{3} \cdot \frac{\left(4 \cdot (\chi_n - c^2)/c^2 + 2\right)^4}{4 \cdot (\chi_n - c^2)/c^2 + 8/3} < \frac{32}{3} \cdot \left(1 + 2 \cdot \frac{\chi_n - c^2}{c^2}\right)^3 \\ &< \frac{32}{3} \cdot \left(1 + \frac{32\delta}{c}\right)^3. \end{aligned} \tag{218}$$

Then, we use (212) to obtain

$$\left(\chi_n - c^2\right)^{\frac{1}{4}} < (16\delta c)^{\frac{1}{4}} = 2 \cdot (\delta c)^{\frac{1}{4}}. \tag{219}$$

Next, we combine Theorems 7, 9 in Section 2.1 with Theorem 31 to obtain

$$\begin{aligned} \frac{1}{|\psi_n(0)|} &< 4 \cdot \sqrt{n} \cdot \sqrt{\frac{\chi_n}{c^2}} < \frac{4}{c} \cdot (\chi_n)^{\frac{3}{4}} < \frac{4}{c} \cdot c^{\frac{3}{2}} \cdot \left(1 + \frac{16\delta}{c}\right)^{\frac{3}{4}} \\ &< 4 \cdot c^{\frac{1}{2}} \cdot \left(1 + \frac{12\delta}{c}\right). \end{aligned} \tag{220}$$

We combine (218), (219), (220) to obtain (217). ■

The following theorem is one of the principal results of this subsection.

Theorem 33. *Suppose that $\delta > 0$ is a real number, and that*

$$3 < \delta < \frac{c}{16}. \tag{221}$$

Suppose also that n is a positive integer, and that

$$n > \frac{2}{\pi}c + \frac{2}{\pi^2} \cdot \delta \cdot \log\left(\frac{4e\pi c}{\delta}\right). \tag{222}$$

Suppose furthermore that the real number $\xi(n,c)$ is defined via the formula

$$\xi(n,c) = 7056 \cdot c \cdot \exp\left[-\delta\left(1 - \frac{\delta}{2\pi c}\right)\right]. \tag{223}$$

Then,

$$|\lambda_n| < \xi(n,c). \tag{224}$$

Proof. Suppose first that n is an even positive integer of the form

$$n = \frac{2}{\pi}c + \frac{2}{\pi^2} \cdot \hat{\delta} \cdot \log\left(\frac{4e\pi c}{\hat{\delta}}\right), \tag{225}$$

for some $3 < \hat{\delta} < c/16$. Due to Theorem 32,

$$\begin{aligned}
&\frac{7}{2\,|\psi_n(0)|} \cdot \frac{\left(4 \cdot \chi_n/c^2 - 2\right)^4}{3 \cdot \chi_n/c^2 - 1} \cdot \left(\chi_n - c^2\right)^{\frac{1}{4}} < \\
&\frac{896}{3} \cdot \left(\frac{c}{16}\right)^{1/4} \cdot c^{3/4} \cdot \left(1 + \frac{12}{16}\right) \cdot \left(1 + \frac{32}{16}\right)^3 = 7056 \cdot c.
\end{aligned} \tag{226}$$

We observe that the right-hand side of (226) is independent of $\hat{\delta}$. We combine this observation with (226), (179) in Theorem 23, (203) in Theorem 29, and the fact that $|\lambda_n|$ decrease monotonically with n, to obtain (224). ∎

Definition 1 ($\delta(n)$)**.** *Suppose that n is a positive integer, and that*

$$\frac{2c}{\pi} < n < \frac{10c}{\pi}. \tag{227}$$

We define the real number $\delta(n)$ to be the solution of the equation

$$n = \frac{2}{\pi}c + \frac{2}{\pi^2} \cdot X \cdot \log\left(\frac{4e\pi c}{X}\right), \tag{228}$$

in the unknown X in the interval $0 < X < 4\pi c$.

Remark 10. *We observe that the right-hand side of* (228) *is an increasing function of* δ *in the range* $0 < \delta < 4\pi c$. *Therefore,* $\delta(n)$ *is well defined.*

We conclude this subsection with the following theorem, that describes the behavior of the upper bound $\nu(n, c)$ on $|\lambda_n|$ (see (15), (16) in Theorem 4 in Section 2.1).

Theorem 34. *Suppose that* n *is a positive integer, and that*

$$\frac{2}{\pi} \cdot c \leq n < \left(\frac{2}{\pi} + \frac{1}{25}\right) \cdot c. \tag{229}$$

Then,

$$\nu(n, c) \geq \frac{1}{10}, \tag{230}$$

where $\nu(n, c)$ *is defined via* (15) *in Theorem 4 in Section 2.1.*

Proof. We carry out elementary calculations, involving the well known Stirling's approximation formula for the gamma function, to obtain the inequality

$$\nu(n, c) \geq \frac{\sqrt{2\pi n}}{2n+1} \cdot \left(\frac{ce}{4n}\right)^n, \tag{231}$$

for all n in the range (229). We use (231) to obtain the inequality

$$\begin{aligned}
\log(\nu(n, c)) &> \log \frac{1}{\sqrt{n}} + n \cdot \log\left(\frac{ce}{4n}\right) \\
&> -\frac{1}{2} \cdot \log(c) + \left(\frac{2}{\pi} + \frac{1}{25}\right) \cdot c \cdot \log\left(\frac{e/4}{2/\pi + 1/25}\right) \\
&> -\frac{1}{2} \cdot \log(c) + \frac{c}{500} \geq \frac{1}{2} \cdot (1 - \log(250)) > -2.27.
\end{aligned} \tag{232}$$

The inequality (230) follows directly from (232). ■

Remark 11. *According to Theorem 34, the inequality* (16) *of Theorem 4 in Section 2.1 is trivial for* $n < (2/\pi + 1/25) \cdot c$. *In particular, for such* n *this inequality is useless.*

5 Numerical Results

In this section, we illustrate the results of Section 4 via several numerical experiments. All the calculations were implemented in FORTRAN (the Lahey 95 LINUX version) and were carried out in either double or quadruple precision. The algorithms for the evaluation of PSWFs and the associated eigenvalues were based on [3].

c	n	$(\pi n)/(2c)$	$\|\lambda_n\|$	$\mu_n = (c/2\pi) \cdot \|\lambda_n\|^2$
10	0	0.00000E+00	0.79267E+00	0.10000E+01
10	3	0.47124E+00	0.79183E+00	0.99790E+00
10	6	0.94248E+00	0.52588E+00	0.44015E+00
100	0	0.00000E+00	0.25066E+00	0.10000E+01
100	31	0.48695E+00	0.25066E+00	0.10000E+01
100	63	0.98960E+00	0.18589E+00	0.54997E+00
1000	0	0.00000E+00	0.79267E-01	0.10000E+01
1000	318	0.49951E+00	0.79267E-01	0.10000E+01
1000	636	0.99903E+00	0.57640E-01	0.52877E+00
10000	0	0.00000E+00	0.25066E-01	0.10000E+01
10000	3183	0.49998E+00	0.25066E-01	0.10000E+01
10000	6366	0.99997E+00	0.16644E-01	0.44088E+00
100000	0	0.00000E+00	0.79267E-02	0.10000E+01
100000	31830	0.49998E+00	0.79267E-02	0.10000E+01
100000	63661	0.99998E+00	0.60295E-02	0.57861E+00

Table 1: *Behavior of* $|\lambda_n|$ *for* $0 \leq n \leq 2c/\pi$. *Corresponds to Experiment 1 in Section 5.*

5.1 Experiment 1

In this experiment, we demonstrate the behavior of $|\lambda_n|$ with $0 \leq n \leq 2c/\pi$, for several values of band limit $c > 0$.

For each of five different values of $c = 10, 10^2, 10^3, 10^4, 10^5$, we do the following. First, we evaluate $|\lambda_n|$ numerically, for $n = 0$, $n \approx c/\pi$ and $n \approx 2c/\pi$. For each such n, we also compute $\mu_n = (c/2\pi) \cdot |\lambda_n|$. Here λ_n is the nth eigenvalue of the integral operator F_c, and μ_n is the nth eigenvalue of the integral operator Q_c (see (4), (5), (6), (9) in Section 2.1).

The results of Experiment 1 are shown in Table 1. This table has the following structure. The first two columns contain the band limit c and the prolate index n, respectively. The third column contains the ratio of n to $2c/\pi$. The fourth column contains $|\lambda_n|$. The last column contains the eigenvalue μ_n of the integral operator Q_c (see (6), (9) in Section 2.1).

Several observations can be made from Table 1.

1. For all five values of band limit c, the eigenvalue μ_n decreases from ≈ 1 to $\approx 1/2$, as n increases from 0 to $(c/2\pi)$. In other words, the first $2c/\pi$ eigenvalues λ_n have roughly the same magnitude $\approx \sqrt{2\pi/c}$. This observation confirms Theorem 2 in Section 2.1.

2. Due to Theorem 6 in Section 2.1, the bounds on the decay of $|\lambda_n|$, established in Section 4, hold for n greater than $2c/\pi$ only (see also Remark 5). Thus, Table 1 indicates that this assumption on n is, in fact, not restrictive, since the first $2c/\pi$ eigenvalues have roughly constant magnitude.

5.2 Experiment 2

In this experiment, we illustrate Theorem 23. As opposed to Experiment 1, we demonstrate the behavior of $|\lambda_n|$ for $n > 2c/\pi$.

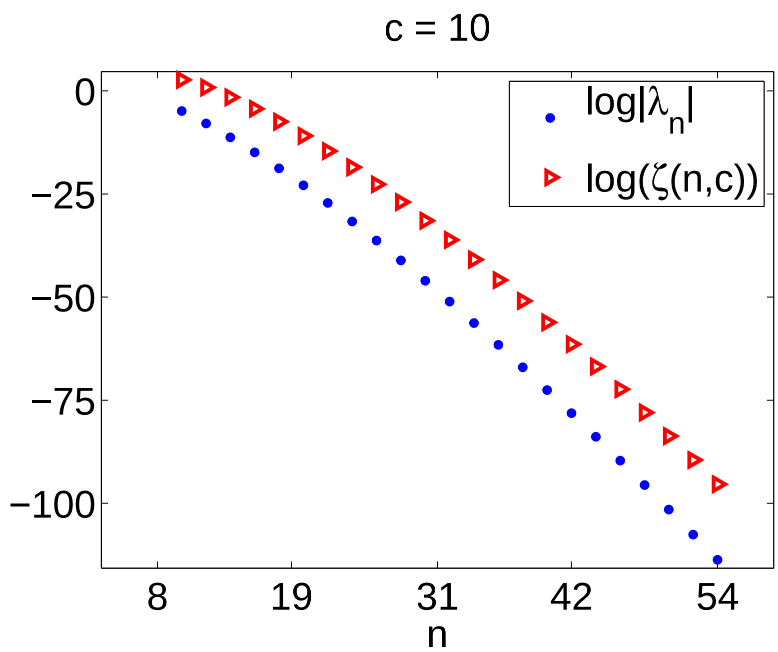

Figure 1: *Illustration of Theorem 23 with* $c = 10$*. Corresponds to Experiment 2 in Section 5.*

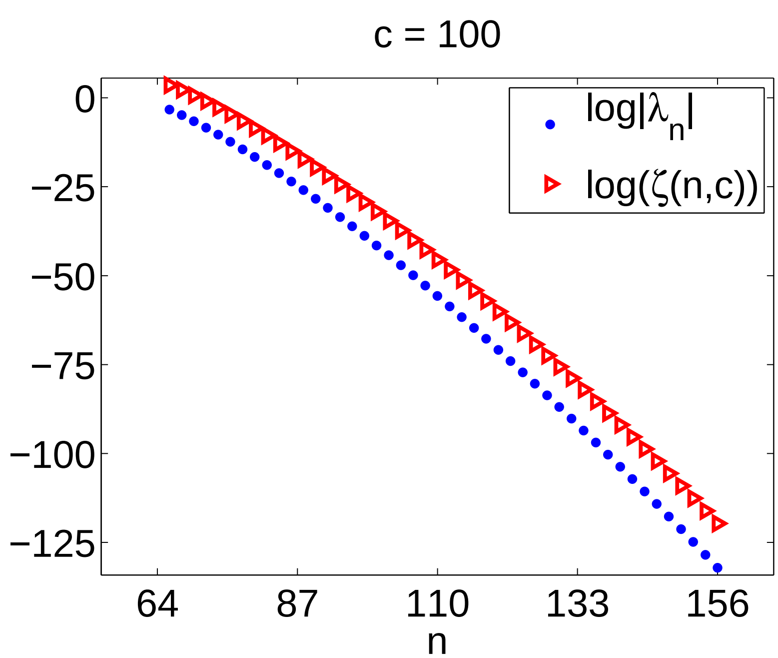

Figure 2: *Illustration of Theorem 23 with* $c = 100$*. Corresponds to Experiment 2 in Section 5.*

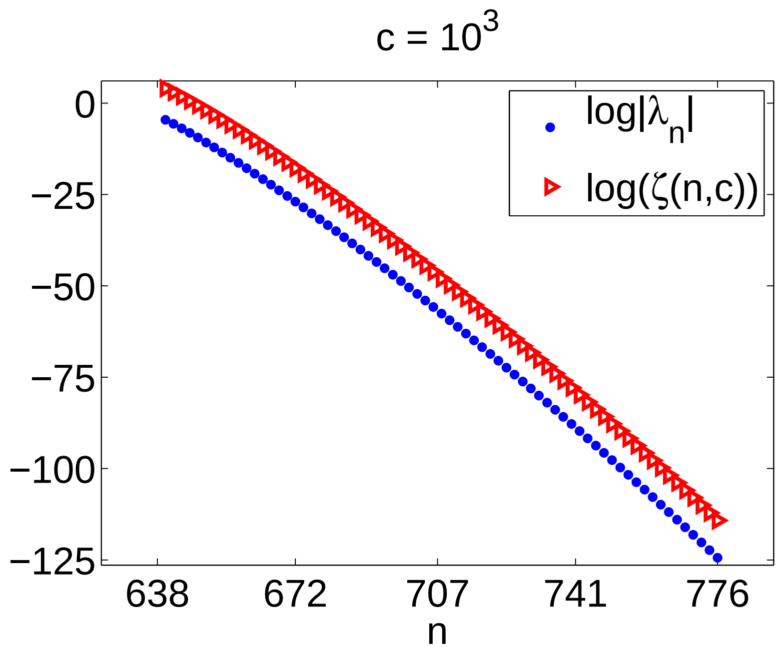

Figure 3: *Illustration of Theorem 23 with $c = 1,000$. Corresponds to Experiment 2 in Section 5.*

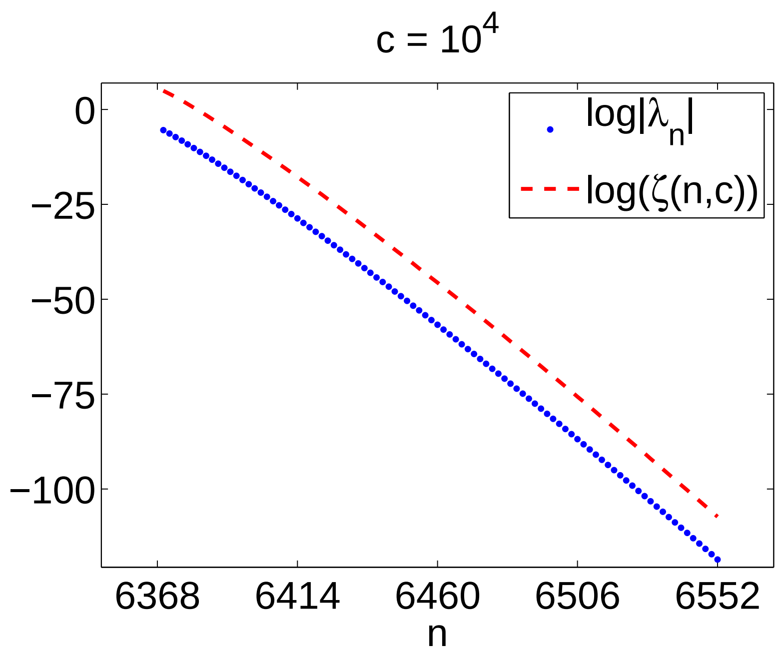

Figure 4: *Illustration of Theorem 23 with $c = 10,000$. Corresponds to Experiment 2 in Section 5.*

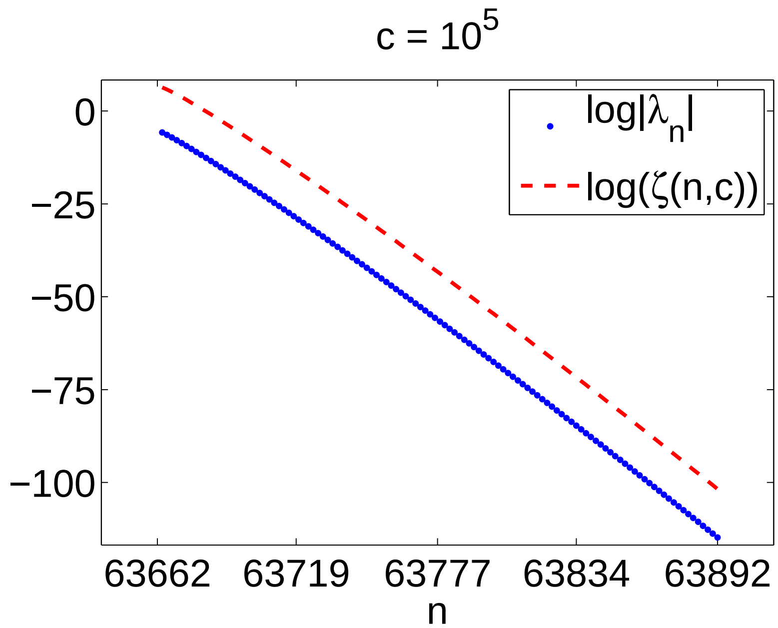

Figure 5: *Illustration of Theorem 23 with* $c = 100,000$. *Corresponds to Experiment 2 in Section 5.*

In this experiment, we proceed as follows. First, we pick band limit $c > 0$ (more or less arbitrarily). Then, for each even integer n in the range

$$\frac{2c}{\pi} < n < \frac{2c}{\pi} + 20 \cdot \log(c), \tag{233}$$

we evaluate numerically $|\lambda_n|$ and $\zeta(n,c)$, where the latter is defined via (178) in Theorem 23.

The results of Experiment 2 are shown in Figures 1 - 5 and in Table 2. In Figures 1 - 5, we plot both $\log(|\lambda_n|)$ and $\log(\zeta(n,c))$ as functions of n. Each of Figures 1 - 5 corresponds to a certain value of band limit ($c = 10, 10^2, 10^3, 10^4, 10^5$, respectively).

Table 2 has the following structure. The first column contains precision $\varepsilon = e^{-50}, e^{-100}$. The second column contains band limit c. The third column contains the integer $n_1(\varepsilon)$, defined via the formula

$$n_1(\varepsilon) = \min_k \left\{ k > 2c/\pi \ : \ |\lambda_k| < \varepsilon \right\}. \tag{234}$$

In other words, $n_1(\varepsilon)$ is the integer satisfying the inequality

$$|\lambda_{n_1(\varepsilon)-1}| > \varepsilon > |\lambda_{n_1(\varepsilon)}|. \tag{235}$$

The fourth column contains $\Delta_1(\varepsilon)$, defined to be the difference between $n_1(\varepsilon)$ and $2c/\pi$, scaled by $\log(c)$. In other words,

$$\Delta_1(\varepsilon) = \frac{n_1(\varepsilon) - 2c/\pi}{\log(c)}. \tag{236}$$

ε	c	$n_1(\varepsilon)$	$\Delta_1(\varepsilon)$	$n_2(\varepsilon)$	$\Delta_2(\varepsilon)$	$n_2(\varepsilon)-n_1(\varepsilon)$
e^{-50}	10	32	0.11133E+02	38	0.13738E+02	6
e^{-50}	10^2	107	0.94107E+01	114	0.10931E+02	7
e^{-50}	10^3	700	0.91752E+01	712	0.10912E+02	12
e^{-50}	10^4	6450	0.90987E+01	6468	0.11053E+02	18
e^{-50}	10^5	63765	0.89484E+01	63792	0.11294E+02	27
e^{-100}	10	50	0.18950E+02	56	0.21556E+02	6
e^{-100}	10^2	138	0.16142E+02	146	0.17879E+02	8
e^{-100}	10^3	753	0.16848E+02	764	0.18440E+02	11
e^{-100}	10^4	6526	0.17350E+02	6542	0.19087E+02	16
e^{-100}	10^5	63864	0.17547E+02	63890	0.19806E+02	26

Table 2: *Illustration of Theorem 23. Corresponds to Experiment 2 in Section 5.*

The fifth column contains the even integer $n_2(\varepsilon)$, defined via the formula

$$n_2(\varepsilon) = \min_k \{k > 2c/\pi \ : \ k \text{ is even}, \ |\zeta(k,c)| < \varepsilon\}. \tag{237}$$

In other words, $n_2(\varepsilon)$ is the even integer satisfying the inequality

$$|\zeta(n_2(\varepsilon)-2,c)| > \varepsilon > |\zeta(n_2(\varepsilon),c)|. \tag{238}$$

The sixth column contains $\Delta_2(\varepsilon)$, defined to be the difference between $n_2(\varepsilon)$ and $2c/\pi$, scaled by $\log(c)$. In other words,

$$\Delta_2(\varepsilon) = \frac{n_2(\varepsilon) - 2c/\pi}{\log(c)}. \tag{239}$$

The last column contains the difference between $n_2(\varepsilon)$ and $n_1(\varepsilon)$.

Several observations can be made from Figures 1 - 5 and Table 2.

1. In all figures, $|\lambda_n| < \zeta(n,c)$, as expected, which confirms Theorem 23.
2. For each c, both $|\lambda_n|$ and $\zeta(n,c)$ decay roughly exponentially fast with n.
3. For each c, both $|\lambda_n|$ and $\zeta(n,c)$ decrease to roughly e^{-125}, as n increases from $2c/\pi$ to $2c/\pi + 20 \cdot \log(c)$. In particular,
$$\left|\lambda_{2c/\pi+20\cdot\log(c)}\right| \approx e^{-125}, \tag{240}$$
for $c = 10, 10^2, 10^3, 10^4, 10^5$. The fact that the right-hand side of (240) is the same for all c is somewhat surprising. However, this is not coincidental, as will be illustrated in Experiment 3 below.
4. For $c = 10^2, 10^3, 10^4, 10^5$, it suffices to take $n \approx 2c/\pi + 9 \cdot \log(c)$ to ensure that $|\lambda_n| \approx e^{-50}$ (see third column in Table 2). In addition, it suffices to take $n \approx 2c/\pi + 17 \cdot \log(c)$ to ensure that $|\lambda_n| \approx e^{-100}$. In other words,
$$n_1(\varepsilon) \approx \frac{2c}{\pi} + 0.17 \cdot \log\left(\frac{1}{\varepsilon}\right) \cdot \log(c), \tag{241}$$
where $n_1(\varepsilon)$ is defined via (234) above (see also (240)).

5. For $c = 10^2, 10^3, 10^4, 10^5$, it suffices to take $n \approx 2c/\pi + 11 \cdot \log(c)$ to ensure that $\zeta(n, c) \approx e^{-50}$ (see fifth column in Table 2). In addition, it suffices to take $n \approx 2c/\pi + 19 \cdot \log(c)$ to ensure that $\zeta(n, c) \approx e^{-100}$. In other words,

$$n_2(\varepsilon) \approx \frac{2c}{\pi} + 0.2 \cdot \log\left(\frac{1}{\varepsilon}\right) \cdot \log(c), \tag{242}$$

where $n_2(\varepsilon)$ is defined via (237) above (see also (240), (241)).

6. The difference $n_2(\varepsilon) - n_1(\varepsilon)$ is roughly independent of ε, and grows only slowly as c increases (see last column of Table 2). In other words, suppose that one needs to determine n such that $|\lambda_k| < e^{-50}$ for all $k \geq n$. Due to (234), $n_1(e^{-50})$ would be the minimal such n. On the other hand, $n = n_2(e^{-50})$ is only larger by 6 for $c = 10$ and by 27 for $c = 10^5$.

5.3 Experiment 3

In this experiment, we illustrate Theorem 33. We proceed as follows. First, we pick band limit $c > 0$ (more or less arbitrarily). Then, we define the positive integer $n_{\max}$ to be the minimal even integer such that

$$n_{\max} > \frac{2}{\pi}c + \frac{2}{\pi^2} \cdot 150 \cdot \log\left(\frac{4e\pi c}{150}\right) \approx \frac{2}{\pi}c + 30.4 \cdot \log(0.23 \cdot c). \tag{243}$$

Then, for each positive even integer n in the range

$$\frac{2c}{\pi} < n < n_{\max}, \tag{244}$$

we evaluate the following quantities:

- the eigenvalue λ_n of the operator F_c (see (4), (5) in Section 2.1);
- $\delta(n)$ of Definition 1 in Section 4.3;
- $\zeta(n, c)$, defined via (178) in Theorem 23 in Section 4.2;
- $\xi(n, c)$, defined via (223) in Theorem 33 in Section 4.3.

The results of Experiment 3 are shown in Figures 6, 7, that correspond, respectively, to band limit $c = 10^4$ and $c = 10^5$. In each of Figures 6, 7, we plot $\log(|\lambda_n|)$, $-\delta(n)$, $\log(\zeta(n, c))$ and $\log(\xi(n, c))$ as functions of n.

Several observations can be made from Figures 6, 7, and from more detailed experiments by the author.

1. In both figures,

$$\log(|\lambda_n|) < -\delta(n) < \log(\zeta(n, c)) < \log(\xi(n, c)), \tag{245}$$

for all n. This observation confirms both Theorem 23 of Section 4.2 and Theorem 33 of Section 4.3. Also, $\xi(n, c)$ is weaker than $\zeta(n, c)$ as an upper bound on $|\lambda_n|$, as expected.

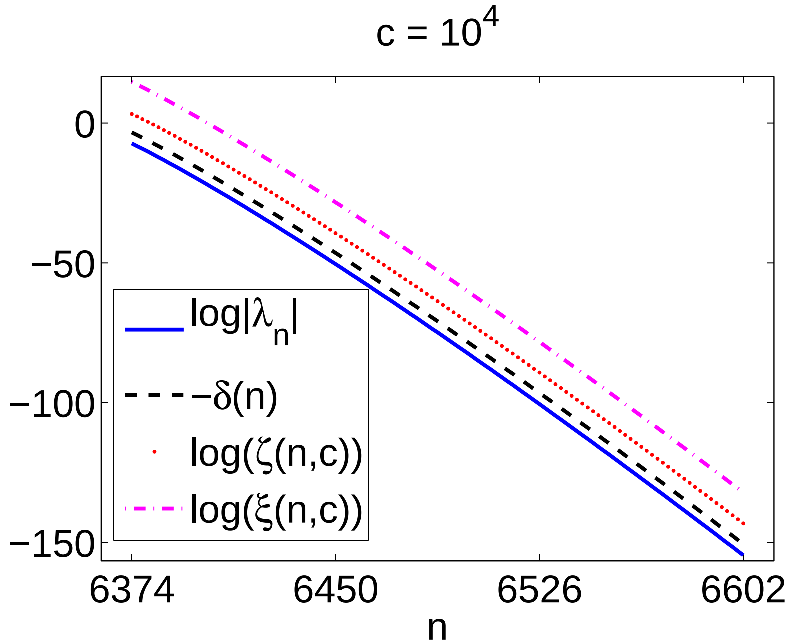

Figure 6: *Illustration of Theorem 33 with* $c = 10,000$. *Corresponds to Experiment 3 in Section 5.*

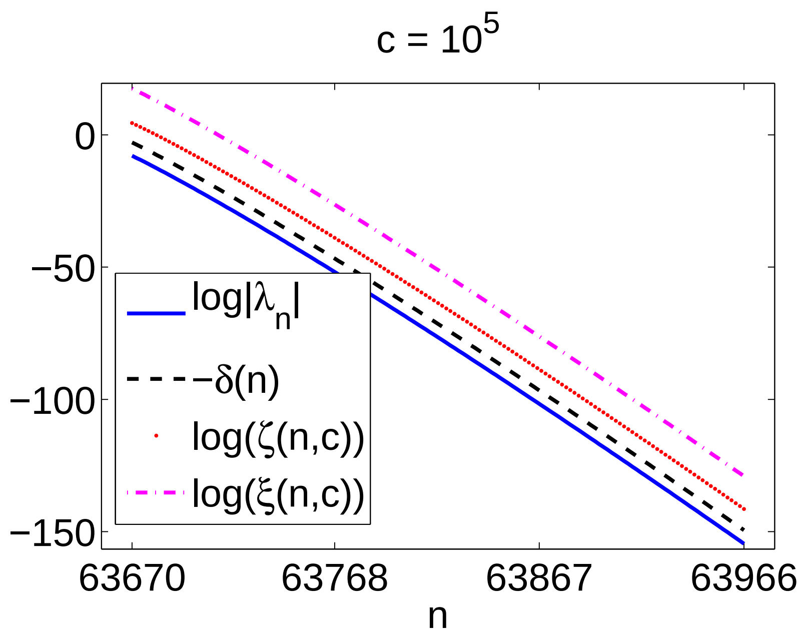

Figure 7: *Illustration of Theorem 33 with* $c = 100,000$. *Corresponds to Experiment 3 in Section 5.*

2. All the four functions, plotted in Figures 6, 7, decay roughly exponentially with n. Moreover,

$$\log(|\lambda_n|) \approx \log \sqrt{\frac{2\pi}{c}} - \delta(n), \tag{246}$$

in correspondence with Theorem 5 in Section 2.1. In particular, even the weakest bound $\xi(n,c)$ correctly captures the exponential decay of $|\lambda_n|$. On the other hand, $\xi(n,c)$ overestimates $|\lambda_n|$ by a roughly constant factor of order $c^{3/2}$ (see also Section 3.2).

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