In the present paper we describe an algorithm for the evaluation of Bessel functions $J_\nu(x)$, $Y_\nu(x)$ and $H_\nu^{(j)}(x)$ $(j = 1, 2)$ of arbitrary positive orders and arguments at a constant CPU time. The algorithm employs Taylor series, the Debye asymptotic expansions and numerical evaluation of the Sommerfeld integral, and is based on the following two observations.
1) The Debye asymptotic expansions, contrary to what appears to be a popular belief, are not expansions in inverse powers of (large) parameter ν but turn out to be uniform expansions in inverse powers of (large) parameter $g_1 = (x - \nu)/x^{1/3}$ for $x > \nu$ and (large) parameter $g_2 = (\nu - x)/\nu^{1/3}$ for $x < \nu$.
2) For x and ν such that both Taylor and Debye expansions do not provide a specified accuracy Bessel functions can be computed at a constant CPU time via (numerical) evaluation of the Sommerfeld integral along contours of steepest descents.
In addition, in Appendix B we obtain certain new estimates concerning decay of the functions $J_\nu(x)$ and $-1/Y_\nu(x)$ of fixed x and large ν, and in Appendix C we show that functions $J_\nu(x)$ of integer ν provide the solution for a certain system of coupled harmonic oscillators.

On the Evaluation of Bessel Functions

Gregory Matviyenko
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1. Introduction

Bessel functions of argument x and order ν of the first kind $J_\nu(x)$, second kind $Y_\nu(x)$ and third kind $H_\nu^{(1)}(x)$, $H_\nu^{(2)}(x)$ (Hankel functions), play an important role in physics, mathematics and engineering. Applications of Bessel functions usually require an algorithm for the rapid evaluation of these functions with sufficiently high accuracy.

For arguments $x \sim 1$ and arbitrary ν Bessel functions can be computed via their Taylor expansions (see Subsection 2.2 below). If $x \gg 1$ and $\nu < x^{\frac{1}{2}}$ these functions can be evaluated by means of the Hankel asymptotic expansion (see Subsection 2.3 below). However, there exists a wide region of values of x and ν where both Taylor and Hankel expansions do not provide any reasonable numerical approximation.

Most of the existing algorithms for the evaluation of Bessel functions in this region are based on the recurrence relation (see, for example [1])

$$f_{\nu-1}(x) + f_{\nu+1}(x) = \frac{2\nu}{x}\, f_\nu(x), \tag{1}$$

where $f_\nu(x)$ denotes one of the functions $J_\nu(x)$, $Y_\nu(x)$ or $H_\nu^{(j)}(x)$ (j=1,2). The asymptotic estimate of the complexity of these algorithms is of order $O(x)$ for functions of the first kind and of order $O(\nu)$ for functions of the second and third kind (see, for example, [1]).

In this paper we present an algorithm for the evaluation of an individual Bessel function of an arbitrary nonnegative order and argument at a constant CPU time. The method is based on the following two observations.

1) The Debye asymptotic expansions [3], proposed in 1909 and since that time considered expansions for large orders (see, for example [1], [4], [5], [6]), are found to have a much wider range of validity. Namely, we show that for $x > \nu$ this expansion for function $H_\nu^{(1)}(x)$ is a *uniform* asymptotic expansion in inverse powers of (large) parameter $g_1 = (x - \nu)/x^{\frac{1}{3}}$ (see Theorem 3.1 and Observation 3.1 below). Moreover, it turns out that for $\nu = 0$ the Debye asymptotic expansion coincides with the Hankel asymptotic expansion (see Theorem 3.2 below). For $x < \nu$ the Debye expansions for functions $J_\nu(x)$ and $Y_\nu(x)$ are proved to be *uniform* asymptotic expansions in inverse powers of (large) parameter $g_2 = (\nu - x)/\nu^{\frac{1}{3}}$ (see Theorem 3.3 and Observation 3.2 below).

2) For the values of x and ν for which both Taylor and Debye expansions do not provide a specified accuracy, Bessel functions can be computed (at a constant CPU time) by means of numerical evaluation of the Sommerfeld integral taken along Debye contours (the definitions of the Sommerfeld integral and Debye contours are presented in Subsection 2.4 below). It is worth noting that Debye contours were extensively investigated in connection with the derivations of various asymptotic expansions for Bessel functions (see, for example, [3], Ch. 8 of [4], [7]). However, the possibility of using them in numerical computations seems to have been overlooked.

The plan of the paper is as follows. In Section 2 we summarize certain mathematical facts to be used in the rest of the paper. In Section 3 we analyze the error terms of the Debye asymptotic expansions. Numerical evaluation of the Sommerfeld integral is discussed in Section 4. In Section 5 we briefly discuss the implementation of our numerical scheme. In Section 6 we present a formal description of the algorithm. In Appendix A we discuss round-off errors. In addition, in Appendix B we obtain asymptotic solutions (with respect to ν) of equations $J_\nu(x) = \epsilon$ and $-1/Y_\nu(x) = \epsilon$ for large fixed positive $x < \nu$ and sufficiently small $\epsilon > 0$. Finally, in Appendix C we show that functions $J_\nu(x)$ of integer ν describe displacements of coupled harmonic oscillators on a line.

2. Relevant mathematical facts

In this section we present a number of well known formulae to be used in the rest of the paper.

2.1 Connections between the three kinds of Bessel functions

All the formulae presented in this subsection can be found, for example, in [1].

Functions $H_\nu^{(j)}(x)$ $(j = 1, 2)$ are expressed through $J_\nu(x)$ and $Y_\nu(x)$ as

$$H_\nu^{(1)}(x) = J_\nu(x) + i\, Y_\nu(x), \tag{2}$$

$$H_\nu^{(2)}(x) = J_\nu(x) - i\, Y_\nu(x). \tag{3}$$

For nonnegative x and ν, both $J_\nu(x)$ and $Y_\nu(x)$ are real. Thus

$$J_\nu(x) = Re\ H_\nu^{(1)}(x), \tag{4}$$

$$Y_\nu(x) = Im\ H_\nu^{(1)}(x), \tag{5}$$

$$H_\nu^{(2)}(x) = H_\nu^{(1)}(x) - 2iIm\ H_\nu^{(1)}(x). \tag{6}$$

2.2 Taylor expansions

For small arguments, function $J_\nu(x)$ is normally evaluated via the formula

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \left(-\frac{x^2}{4}\right)^k \frac{1}{k!\ \Gamma(\nu+k+1)}. \tag{7}$$

If ν is not an integer, function $Y_\nu(x)$ is computed as

$$Y_\nu(x) = \frac{J_\nu(x)\ \cos(\pi\ \nu) - J_{-\nu}(x)}{\sin(\pi\ \nu)}. \tag{8}$$

For integer ν the formula (8) can not be used and is replaced by

$$\begin{aligned} Y_\nu(x) &= -\frac{1}{\pi} \left(\frac{x}{2}\right)^{-\nu} \sum_{k=0}^{\nu-1} \left(\frac{x^2}{4}\right)^k \frac{(\nu-k-1)!}{k!} + \frac{2}{\pi} \ln\left(\frac{x}{2}\right) J_\nu(x) \\ &\quad -\frac{1}{\pi} \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \left(-\frac{x^2}{4}\right)^k \frac{1}{(\nu+k)!\ k!} (\psi(k+1) + \psi(\nu+k+1)), \end{aligned} \tag{9}$$

where $\psi(z) = \frac{d}{dz}\ln(\Gamma(z))$. Formulae (7)-(9) can be found, for example, in [1].

2.3 The Hankel asymptotic expansion

The Hankel asymptotic expansion has the form (see, for example, [1], Ch. 7 of [4] , Ch. 7 of [5])

$$H_\nu^{(1)}(x) = \left(\frac{2}{\pi\ x}\right)^{\frac{1}{2}} \exp(i\chi(x,\nu)) \left(\sum_{n=0}^{N} b_n(\nu) \left(-\frac{i}{x}\right)^n + \theta_{N+1,h}(\nu,x)\right), \tag{10}$$

where

$$\chi(x,\nu) = x - \frac{\pi}{4} - \frac{1}{2}\pi\nu, \tag{11}$$

and $\theta_{N+1,h}(\nu,x)$ is the error term.

Coefficients $b_n(\nu)$ satisfy the recurrence relation

$$b_0(\nu) = 1, \tag{12}$$

$$b_{n+1}(\nu) = \frac{(2n+1)^2 - 4\,\nu^2}{8\,(n+1)}\, b_n(\nu), \quad (n = 0, 1, \cdots), \tag{13}$$

whereas the error term $\theta_{N+1,h}(\nu, x)$ is bounded by (see, for example, Ch. 7 of [5])

$$|\theta_{N+1,h}(\nu, x)| \leq 2\, \frac{b_{N+1}(\nu)}{x^{N+1}} \cdot \exp\left(\frac{|\nu^2 - \frac{1}{4}|}{x}\right). \tag{14}$$

2.4 The Sommerfeld integral

All the formulae, presented in this subsection can be found, for example, in Ch. 8 of [4].

For the values of x and ν for which both Taylor and Debye expansions do not provide a specified accuracy we computed function $H_\nu^{(1)}(x)$ by means of numerical evaluation of the so called Sommerfeld integral:

$$H_\nu^{(1)}(x) = \frac{1}{\pi i} \int_{-\infty}^{\infty + i\pi} \exp(x \sinh(w) - \nu w)\, dw. \tag{15}$$

Following [4] we will write

$$w = u + iv, \tag{16}$$

where both u and v are real. Integration in (15) is performed along an arbitrary contour that has the following asymptotes:

$$\lim_{u \to -\infty} v = 0, \tag{17}$$

$$\lim_{u \to \infty} v = \pi. \tag{18}$$

Observation 2.1

As paths of integration in (15) it is natural to choose the so called Debye contours on which the integrand of (15) does not oscillate (see, for example, Ch. 8 of [4]). We note in passing that the Debye contours are a particular example of contours of steepest descents that are widely used for the evaluation of the asymptotic expansions of certain contour integrals (see, for example, Ch. 8 of [4], [5], [6]).

The Debye contours $u = u(v)$ are curves on the complex w-plane (16) that (generally speaking) are implicitly defined via equations

$$p(u, v) = 0. \tag{19}$$

For $x > \nu$,

$$p(u, v) = \cosh(u) - \frac{\sin(\beta) + (v - \beta)\ \cos(\beta)}{\sin(v)}, \tag{20}$$

where

$$\cos(\beta) = \frac{\nu}{x}. \tag{21}$$

It immediately follows from (21) that for nonnegative x and ν,

$$0 \le \beta \le \frac{\pi}{2}. \tag{22}$$

For $x < \nu$,

$$p(u, v) = v, \quad \text{if } u \le \alpha, \tag{23}$$

and

$$p(u, v) = \cosh(u) - \cosh(\alpha)\ \frac{v}{\sin(v)}, \quad \text{if } u > \alpha, \tag{24}$$

where

$$\cosh(\alpha) = \frac{\nu}{x}. \tag{25}$$

For $x = \nu$,

$$p(u, v) = v, \quad \text{if } u \le 0, \tag{26}$$

and

$$p(u, v) = \cosh(u) - \frac{v}{\sin(v)}, \quad \text{if } u > 0. \tag{27}$$

Finally we note that all the Debye contours, associated with function $H_\nu^{(1)}(x)$ (of nonnegative x and positive ν) lie within the strip

$$0 \le v \le \pi. \tag{28}$$

Graphs of Debye contours can be found, for example, in Ch. 8 of [4].

2.5 The Debye asymptotic expansions

In this subsection we present formulae for the Debye asymptotic expansions that can be found (in a slightly different form) in Ch. 10 of [5].

For $\nu < x$,

$$H_\nu^{(1)}(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{(x^2-\nu^2)^{\frac{1}{4}}} \exp(i\eta_1) \left(\sum_{n=0}^{N}(-1)^n \frac{u_n(ip)}{\nu^n} + \tilde{\theta}_{N+1,2}(\nu,p)\right), \tag{29}$$

where

$$\eta_1 = (x^2-\nu^2)^{\frac{1}{2}} - \nu \ \arccos\left(\frac{\nu}{x}\right) - \frac{\pi}{4}, \tag{30}$$

$\tilde{\theta}_{N+1,2}(\nu,p)$ is the error term, polynomials $u_n(t)$ are defined in (35), (36) below, and

$$p = \frac{\nu}{|x^2-\nu^2|^{\frac{1}{2}}}. \tag{31}$$

For $x < \nu$,

$$J_\nu(x) = \frac{1}{1+\theta_{N+1,1}(\nu,0)} \frac{\exp(-\eta_2)}{(2\pi)^{\frac{1}{2}}\ (\nu^2-x^2)^{\frac{1}{4}}} \left(\sum_{n=0}^{N} \frac{u_n(p)}{\nu^n} + \theta_{N+1,1}(\nu,p)\right), \tag{32}$$

$$Y_\nu(x) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\exp(\eta_2)}{(\nu^2-x^2)^{\frac{1}{4}}} \left(\sum_{n=0}^{N}(-1)^n \frac{u_n(p)}{\nu^n} + \theta_{N+1,2}(\nu,p)\right), \tag{33}$$

where

$$\eta_2 = \nu \ \ln\left(\frac{\nu}{x} + \left(\left(\frac{\nu}{x}\right)^2 - 1\right)^{\frac{1}{2}}\right) - (\nu^2-x^2)^{\frac{1}{2}}, \tag{34}$$

and $\theta_{N+1,1}(\nu,p)$ and $\theta_{N+1,2}(\nu,p)$ are the error terms.

Polynomials $u_n(t)$ are defined by the formulae

$$u_0(t) = 1, \quad u_1(t) = \frac{1}{24}\ (3t - 5t^3), \tag{35}$$

$$u_{n+1}(t) = \frac{1}{2}\, t^2\, (1-t^2)\, \frac{du_n(t)}{dt} + \frac{1}{8} \int_0^t (1-5\tau^2)\, u_n(\tau) d\tau,$$
$$(n = 0, 1, \cdots). \tag{36}$$

The error terms in the Debye expansions satisfy the inequalities

$$|\tilde{\theta}_{N,2}(\nu,p)| \leq 2 \exp\left(\frac{2\tilde{V}_{0,p}(u_1)}{\nu}\right) \frac{\tilde{V}_{0,p}(u_{N+1})}{\nu^{N+1}}, \tag{37}$$

$$|\theta_{N+1,1}(\nu,p)| \leq 2 \exp\left(\frac{2V_{1,p}(u_1)}{\nu}\right) \frac{V_{1,p}(u_{N+1})}{\nu^{N+1}}, \tag{38}$$

and

$$|\theta_{N+1,2}(\nu,p)| \leq 2 \exp\left(\frac{2V_{0,p}(u_1)}{\nu}\right) \frac{V_{0,p}(u_{N+1})}{\nu^{N+1}}. \tag{39}$$

In (37) - (39), symbols $V_{a,b}(f)$ and $\tilde{V}_{a,b}(f)$ denote the so called total variations of functions $f(x)$ and $f(ix)$, respectively (see, for example, Ch. 1 of [5]):

$$V_{a,b}(f) = \int_a^b \left|\frac{df(x)}{dx}\right| dx, \tag{40}$$

$$\tilde{V}_{a,b}(f) = \int_a^b \left|\frac{df(ix)}{dx}\right| dx. \tag{41}$$

3. Error terms of the Debye asymptotic expansions

In this section we obtain estimates of the error terms of the Debye asymptotic expansions (29), (32) and (33). We start with a more detailed analysis of the polynomials $u_n(t)$ defined in (35), (36).

3.1 The polynomials $u_n(t)$

Lemma 3.1

For any $n \geq 1$,

$$u_n(t) = t^n\, \tilde{u}_n(t), \tag{42}$$

where

$$\tilde{u}_n(t) = \sum_{k=0}^{n} a_k^n\, t^{2k}. \tag{43}$$

The coefficients a_k^n are defined by the formulae

$$a_0^0 = 1, \tag{44}$$

$$a_k^n = 0 \quad \text{if } k < 0 \text{ or } k > n, \quad (n = 0, 1, \cdots), \tag{45}$$

$$\begin{aligned} a_k^{n+1} \quad &= \quad a_k^n \left(\frac{n+2k}{2} + \frac{1}{8\,(2k+n+1)} \right) - \\ &\quad a_{k-1}^n \left(\frac{n+2k-2}{2} + \frac{5}{8\,(2k+n+1)} \right), \\ &\qquad (n = 0, 1, \cdots; \quad k = 0, 1, \cdots, n+1). \end{aligned} \tag{46}$$

Proof

We will prove the lemma by induction. For $n = 1$ the formulae (42)- (46) immediately follow from (35). Suppose now that the formulae (42)-(46) are satisfied for certain $n = m > 1$. Then

$$\begin{aligned} &t^2\,(1-t^2)\,\frac{du_m(t)}{dt} = \\ &\qquad t^{m+1} \sum_{k=0}^{m+1} ((m+2k)a_k^m - (m+2k-2)a_{k-1}^m) \cdot t^{2k}, \end{aligned} \tag{47}$$

and

$$\begin{aligned} &\int_0^t (1-5\tau^2)\, u_m(\tau) d\tau = \\ &\qquad t^{m+1} \sum_{k=0}^{m+1} \frac{1}{2k+m+1} (a_k^m - 5a_{k-1}^m) \cdot t^{2k}. \end{aligned} \tag{48}$$

Now substituting (47) and (48) into (36) we observe that (42)-(46) hold for $n = m+1$ which concludes the proof of the lemma. □

The following corollary is an obvious consequence of the lemma and the formulae (12), (13).

Corollary 3.1

For all $n \geq 0$,

$$a_0^n = b_n(0), \tag{49}$$

where the coefficients $b_n(\nu)$ are defined in (12), (13).

Lemma 3.2

For any $t \geq t_1 \geq 0$,

$$\tilde{u}_n(it) \geq \tilde{u}_n(it_1) > 0, \tag{50}$$

$$\frac{\tilde{u}_n(it)}{t^{2n}} \leq \frac{\tilde{u}_n(it_1)}{t_1^{2n}}, \quad \text{if } t \neq 0 \text{ and } t_1 \neq 0, \tag{51}$$

$$\frac{d\tilde{u}_n(it)}{dt} \geq 0, \tag{52}$$

$$\tilde{u}_n(it) \geq \tilde{u}_n(t), \tag{53}$$

$$\frac{d\tilde{u}_n(it)}{dt} \geq \left| \frac{d\tilde{u}_n(t)}{dt} \right|, \tag{54}$$

$$\left| \frac{du_n(it)}{dt} \right| \geq \left| \frac{du_n(t)}{dt} \right|. \tag{55}$$

Proof

It immediately follows from (44)-(46) that for all $n \geq 0$ and $k \leq n$,

$$a_k^n = (-1)^k \tilde{a}_k^n, \tag{56}$$

where

$$\tilde{a}_k^n > 0. \tag{57}$$

Substituting (56) and (57) into (43), we observe that for any real t all the coefficients of the polynomials $\tilde{u}_n(it)$ are positive and therefore the inequalities (50)-(54) are satisfied. The inequality (55) follows from (42), (53) and (54).□

Remark 3.1

While many recurrence relations occurring in mathematical physics are numerically unstable, the recursion (46) is numerically stable since according to (56) and (57), both terms in this relation have the same sign.

The following lemma is an immediate consequence of (42), (52) and (55).

Lemma 3.3

For any $p > 0$ and $n \geq 1$,

$$\tilde{V}_{0,p}(u_n) = \int_0^p \left| \frac{du_n(it)}{dt} \right| dt = p^n \cdot \tilde{u}_n(ip). \tag{58}$$

Furthermore,

$$V_{0,p}(u_n) = \int_0^p \left| \frac{du_n(t)}{dt} \right| dt \leq \tilde{V}_{0,p}(u_n). \tag{59}$$

3.2 Region $x > \nu$

Theorem 3.1

For any

$$x > \nu \geq 0 \tag{60}$$

the error term $\tilde{\theta}_{N+1,2}(\nu, p)$ in the expansion (29) satisfies the inequality

$$|\tilde{\theta}_{N+1,2}(\nu, p)| \leq 2 \exp\left(\frac{2}{3 \cdot g_1^{\frac{3}{2}}} \right) \frac{\tilde{u}_{N+1}(i)}{g_1^{\frac{3}{2}(N+1)}}, \tag{61}$$

where

$$g_1 = \frac{x - \nu}{x^{\frac{1}{3}}}. \tag{62}$$

Proof

The inequality (60) and the definition (62) show that

$$\frac{g_1}{x^{\frac{2}{3}}} \leq 1, \tag{63}$$

and therefore

$$\frac{1}{(x^2 - \nu^2)^{\frac{1}{2}}} \leq \frac{1}{g_1^{\frac{1}{2}} \cdot x^{\frac{2}{3}}}. \tag{64}$$

Now combining (31) with (60) and (64) we observe that

$$p = \frac{\nu}{(x^2 - \nu^2)^{\frac{1}{2}}} \leq \frac{x^{\frac{1}{3}}}{g_1^{\frac{1}{2}}}, \tag{65}$$

and substituting (65) into (50) we obtain

$$\tilde{u}_n(ip) \leq \tilde{u}_n\left(i\frac{x^{\frac{1}{3}}}{g_1^{\frac{1}{2}}}\right). \tag{66}$$

Combining the inequalities (51), (63) and (66) we have

$$\tilde{u}_n(ip) \leq \tilde{u}_n(i)\left(\frac{x^{\frac{2}{3}}}{g_1}\right)^n. \tag{67}$$

Substitution of (31), (64) and (67) into (58) yields

$$\frac{1}{\nu^n}\,\tilde{V}_{0,p}(u_n) = \frac{\tilde{u}_n(ip)}{(x^2-\nu^2)^{\frac{1}{2}n}} \leq \frac{\tilde{u}_n(i)}{g_1^{\frac{3}{2}n}}. \tag{68}$$

Observing that

$$\tilde{u}_1(i) = \frac{1}{3}, \tag{69}$$

and substituting (68) and (69) into (37) we immediately obtain the inequality (61). □

Observation 3.1

Obviously, function $H_\nu^{(1)}(x)$ can be viewed not as a function of x and ν, but as a function of x and the parameter g_1 defined in (62). Then the estimate (61) shows that for $g_1 > 0$ (*i.e.* $x > \nu$) the Debye asymptotic expansion (29) *is not* an asymptotic expansion in inverse powers of (large) parameter ν but it turns out to be *a uniform* (with respect to x) asymptotic expansion in inverse powers of (large) parameter g_1. Moreover, as follows from (61), the error term $\tilde{\theta}_{N+1,2}(\nu,p)$ may be small even if ν is not large. The following theorem describes the behavior of the Debye expansion (29) in the limit $\nu \to 0$.

Theorem 3.2

For any $x > 0$ and $\nu = 0$ the Debye asymptotic expansion (29) *and the Hankel asymptotic expansion* (10) *are identical.*

Proof

From the definitions (11) and (30) we have

$$\lim_{\nu\to 0}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{(x^2-\nu^2)^{\frac{1}{4}}}\exp(i\eta_1) = \lim_{\nu\to 0}\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\exp(i\chi(x,\nu)) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\exp(i(x-\frac{\pi}{4})). \tag{70}$$

Next, combining (31) with (42) and (43) we obtain

$$\lim_{\nu\to 0}\frac{(-1)^n}{\nu^n}u_n(ip)=\lim_{\nu\to 0}\frac{(-i)^n}{(x^2-\nu^2)^{\frac{1}{2}n}}\sum_{k=0}^{n}a_k^n\left(\frac{-\nu^2}{x^2-\nu^2}\right)^k=a_0^n\left(-\frac{i}{x}\right)^n. \tag{71}$$

Now substituting (70) and (71) into (29) and taking into account (49) we see that (for $\nu=0$) the expansions (10) and (29) are identical. □

3.3 Region $x<\nu$

Theorem 3.3

For any

$$\nu>x\geq 0 \tag{72}$$

the error terms $\theta_{N+1,2}(\nu,p)$ of the expansion (32) *satisfies the inequality*

$$|\theta_{N+1,2}(\nu,p)|\leq 2\exp\left(\frac{2}{3\cdot g_2^{\frac{3}{2}}}\right)\frac{\tilde{u}_{N+1}(i)}{g_2^{\frac{3}{2}(N+1)}}, \tag{73}$$

where

$$g_2=\frac{\nu-x}{\nu^{\frac{1}{3}}}. \tag{74}$$

Furthermore, the error term $\theta_{N+1,1}(\nu,p)$ of the expansion (33) *is bounded by*

$$|\theta_{N+1,1}(\nu,p)|\leq 2\exp\left(\frac{2}{3\cdot g_2^{\frac{2}{3}}}\right)\frac{\tilde{u}_{N+1}(i)}{g_2^{\frac{3}{2}(N+1)}}, \tag{75}$$

Proof

Substitution of (59) into (38) yields

$$|\theta_{N,2}(\nu,p)|\leq 2\exp\left(\frac{2\tilde{V}_{0,p}(u_1)}{\nu}\right)\frac{\tilde{V}_{0,p}(u_{N+1})}{\nu^{N+1}}. \tag{76}$$

Now the proof of the inequality (73) becomes almost identical to that of the inequality (61) (see Theorem 3.1) and we omit it.

To prove (75) we observe that for $x<\nu$,

$$p=\frac{\nu}{(\nu^2-x^2)^{\frac{1}{2}}}\geq 1. \tag{77}$$

Therefore

$$V_{1,p}(u_n) = \int_1^p \left| \frac{du_n(t)}{dt} \right| dt \le \int_0^p \left| \frac{du_n(t)}{dt} \right| dt = V_{0,p}(u_n). \tag{78}$$

Substitution of (78) into (38) produces

$$|\theta_{n,1}(\nu, p)| \le 2 \exp\left(\frac{2V_{0,p}(u_1)}{\nu} \right) \frac{V_{0,p}(u_n)}{\nu^n}. \tag{79}$$

Comparing (39) and (79) we see that the proof of (75) is reduced to the proof of (73). □

Observation 3.2

Parallel to Observation 3.1, we can view function $H_\nu^{(1)}(x)$ not as a function of x and ν but as a function of ν and the parameter g_2 defined in (74). Then the estimates (73) and (75) mean that the Debye asymptotic expansions (32) and (33) *are not* expansions in inverse powers of (large) parameter ν but turn out to be *uniform* (with respect to ν) asymptotic expansions in inverse powers of the (large) parameter g_2 (compare with Observation 3.1).

Observation 3.3

In Appendix B it is shown that for $x \gg 1$ and $\nu > x + \text{const} \cdot x^{\frac{1}{3}}$ function $J_\nu(x)$ becomes small whereas function $|Y_\nu(x)|$ becomes large. Therefore, for sufficiently large x the most important part (in terms of applications of Bessel functions) of the region $x < \nu$ can be estimated as

$$x < \nu < x + \text{const} \cdot x^{\frac{1}{3}}, \tag{80}$$

Combining (62), (74) and (80) we have

$$g_2 = |g_1| \left(1 + O\left(x^{-\frac{2}{3}} \right) \right). \tag{81}$$

The estimate (81) and Observation 3.2 show that in the region (80) for $x \gg 1$ the Debye asymptotic expansions (32) and (33) behave almost like uniform expansions in inverse powers of (large) parameter

$$\frac{\nu - x}{x^{\frac{1}{3}}} = -g_1. \tag{82}$$

4. Certain properties of the Sommerfeld integral

In this section we discuss certain analytical properties of the Sommerfeld integral (15) relevant to its numerical evaluation.

4.1 Region $x > \nu$

It immediately follows from Ch. 8 of [4] that the explicit representation of the Sommerfeld integral (15) on the Debye contours defined by (19) and (20) has the form

$$H_\nu^{(1)}(x) = \frac{1}{\pi i} \exp(i(x \sin(\beta) - \nu\beta)) \int_0^\pi \left(\frac{du}{dv} + i\right) \exp(x\phi_1)\, dv, \tag{83}$$

where

$$\phi_1 \equiv \phi_1(v, u(v), \beta) = \cos(v) \cdot \sinh(u) - \cos(\beta) \cdot u, \tag{84}$$

and

$$\frac{du}{dv} = \frac{1}{\sinh(u)\ \sin^2(v)} \cdot (\sin(v - \beta) - (v - \beta) \cdot \cos(\beta) \cdot \cos(v)). \tag{85}$$

In (83)-(85) function $u = u(v)$ is evaluated via (19), (20) and the parameter β is defined in (21).

Theorem 4.1(see §8.31 of [4])

Function $\phi_1(v, u(v), \beta)$, defined in (84), is a nonpositive decreasing function of $|v - \beta|$. It has the only maximum at $v = \beta$ where

$$\phi_1(\beta, u(\beta), \beta) = 0. \tag{86}$$

The following corollary is an immediate consequence the theorem.

Corollary 4.1

The equation

$$x\ \phi_1(\beta, u(\beta), \beta) = \ln(\epsilon). \tag{87}$$

with $\epsilon \in (0, 1)$, $v \in (0, \pi)$ and $x > 0$ has two and only two solutions β_1 and β_2 such that

$$\beta_1 < \beta < \beta_2. \tag{88}$$

Theorem 4.2

For any $\beta \neq 0$,

$$\phi_1 = -\sin(\beta)\ (v-\beta)^2 + O((v-\beta)^3). \tag{89}$$

Proof

From (20) for any $\beta \neq 0$ we have

$$u = (v-\beta) + O((v-\beta)^2). \tag{90}$$

Substituting (90) into (84) we immediately obtain (89). $\Box$

In the rest of the subsection we will estimate the domain on the $x-\nu$ plane where the integral (83) can be evaluated at a constant CPU time. We start with the following remark.

Remark 4.1

For any $0 < v \ll 1$,

$$\exp(x\phi_1) \sim \exp\left(-\frac{f(x,\nu)}{v}\right), \tag{91}$$

where function $f(x,\nu) > 0$. This formula is an immediate consequence of (19) and (20).

As follows from (19), (20), (84), (85) and Theorems 4.1 and 4.2 the integrand of (83) is a nonoscillatory function of v. Moreover, for small $|v-\beta|$,

$$\begin{aligned}&\left(\frac{du}{dv} + i\right)\ \exp(x\phi_1) = \\ &\qquad \left(\left.\frac{du}{dv}\right|_{v=\beta} + i\right) \exp\left(-x\sin(\beta)(v-\beta)^2\right)\ (1+\theta_1(v-\beta)),\end{aligned} \tag{92}$$

where $\theta_1(v-\beta) = O(v-\beta)$ is the error term.

Next, suppose that

$$x \cdot \sin(\beta) \gg 1, \tag{93}$$

and

$$\beta > \frac{C_1}{(x\sin(\beta))^{\frac{1}{2}}}. \tag{94}$$

where

$$C_1 \sim 1 \tag{95}$$

is a (positive) constant.

Observation 4.1

The condition (93) means that the domain where the integrand of (83) is (numerically) not zero is sufficiently small. The condition (94) shows that the distance between the maximum of the integrand of (83) $v = \beta$ and its singularity at $v = 0$ (see Remark 4.1) is larger than several standard deviations of the Gaussian in (92). Therefore, if the inequalities (93) and (94) are satisfied then the error term $\theta_1(v - \beta)$ in (92) can be approximated (with high accuracy) by a low-degree polynomial of $(v - \beta)$ in the domain, where the integrand of (83) is (numerically) not zero. Thus in this case the evaluation of the integral (83) by means of the trapezoidal rule becomes a superalgebraically convergent procedure. Moreover, the number of nodes of this quadrature formula is (asymptotically for large $x \sin(\beta)$ and C_1) independent of ν and x.

Theorem 4.3

For any

$$x \gg 1 \tag{96}$$

and

$$\nu < x - D_1\, x^{\frac{1}{3}}, \tag{97}$$

where $D_1 \sim 1$ is a (positive) constant, the inequalities (93) *and* (94) *are satisfied.*

Proof

We will prove the inequality (93) first. From the definition (21) we have

$$\sin(\beta) = \left(1 - \frac{\nu^2}{x^2}\right)^{\frac{1}{2}}, \tag{98}$$

and thus

$$x \sin(\beta) = \left(x^2 - \nu^2\right)^{\frac{1}{2}}. \tag{99}$$

Combining (96), (97) and (99) we have

$$x\sin(\beta) > (2D_1)^{\frac{1}{2}} \cdot x^{\frac{2}{3}} \left(1 + O\left(x^{-\frac{2}{3}}\right)\right) \gg 1, \tag{100}$$

which completes the proof of (93). Now we turn to the proof of the inequality (94).

Owing to (22) we have $\beta \geq \sin(\beta)$ and therefore we can replace the inequality (94) by a stronger one

$$\sin(\beta) > \frac{C_1}{(x\sin(\beta))^{\frac{1}{2}}}. \tag{101}$$

Substituting (98) into (101) we have

$$x^{\frac{1}{2}} \cdot \left(1 - \frac{\nu^2}{x^2}\right)^{\frac{3}{4}} > C_1. \tag{102}$$

Using (97) we can estimate the left-hand side of (102) as

$$x^{\frac{1}{2}} \cdot \left(1 - \frac{\nu^2}{x^2}\right)^{\frac{3}{4}} > x^{\frac{1}{2}} \cdot \left(2D_1 \cdot x^{-\frac{2}{3}} - D_1^2 \cdot x^{-\frac{4}{3}}\right)^{\frac{3}{4}} \geq (2D_1)^{\frac{3}{4}}. \tag{103}$$

Now it follows from (101)-(103) that the inequality (94) is satisfied if

$$D_1 > \frac{C_1^{\frac{4}{3}}}{2} + \epsilon \sim 1, \tag{104}$$

where $\epsilon > 0$ is any (small) number. □

The inequalities (96), (97) and the estimate (104) define the domain on the $x - \nu$ plain where the conditions (93) and (94) are satisfied and therefore numerical integration of (83) by means of the trapezoidal formula can be done at a CPU time independent of x and ν (see Observation 4.1).

4.2 Region $x < \nu$

It follows from Ch. 8 of [4] that the explicit representation of the Sommerfeld integral (15) on the contours defined by (19), (23) is

$$I_1 = \frac{1}{\pi i} \int_{-\infty}^{\alpha} \exp(x\phi_2)du, \tag{105}$$

whereas on the contours (19), (24) we have

$$I_2 = \frac{1}{\pi i} \int_0^{\pi} \exp(x\phi_3) \left(i + \frac{du}{dv} \right) dv. \tag{106}$$

Obviously,

$$H_\nu^{(1)}(x) = I_1 + I_2. \tag{107}$$

In (105) and (106),

$$\phi_2 \equiv \phi_2(u, \alpha) = \sinh(u) - \cosh(\alpha)\ u, \tag{108}$$

$$\phi_3 \equiv \phi_3(v, u(v), \alpha) = \sinh(u)\ \cos(v) - \cosh(\alpha)\ u, \tag{109}$$

and

$$\frac{du}{dv} = \cosh(\alpha)\ \frac{(\sin(v) - v\ \cos(v))}{\sin^2(v)\ \sinh(u)}. \tag{110}$$

Function $u = u(v)$ in (106), (109) and (110) is evaluated via (19), (24) and the parameter α is defined in (25).

Observation 4.2

Numerical evaluation of the integral (105) can be performed by means of the Gauss-Legendre quadrature formula with the number of nodes independent of ν and x because its integrand is an analytic nonoscillatory function.

In Theorem 4.4 (see below) it is shown that for any $\alpha > 0$ the integrand of (106) has singularities in the complex v-plane. It turns out (see the estimate (111) below) that for $\alpha \ll 1$ (*i.e.* $\nu/x \sim 1$) these singularities lie close to the domain of integration in (106) which impedes numerical evaluation of this integral. Now we will establish a domain on the $x - \nu$ plane where numerical evaluation of (106) (for example, by means of the Gauss-Legendre quadrature formula) can be performed at a CPU time independent of x and ν; the estimate of this domain is obtained in Theorem 4.7 below.

Theorem 4.4

For any $\alpha > 0$ the integrand of (106) *has two imaginary complex conjugate branch points $\alpha_{\pm}$. Moreover, for $\alpha \ll 1$,*

$$\alpha_{\pm} = \pm(3)^{\frac{1}{2}}\alpha i + O(\alpha^3). \tag{111}$$

Proof

For imaginary $v = is$ (s is real) equations (19), (24) become

$$\cosh(u(is)) = \cosh(\alpha)\ \frac{s}{\sinh(s)}. \tag{112}$$

Therefore, for any $\alpha > 0$ there exists a parameter $s = s_0$ such that

$$\cosh(u(is_0)) = 1 \tag{113}$$

and

$$0 < \cosh(u(is)) < 1, \quad |s| > s_0. \tag{114}$$

As follows from (113) and (114) the points $s_{\pm} = \pm s_0$ (or, equivalently, the points $\alpha_{\pm} = \pm i s_0$) are the branch points of the function $\sinh(u) = \Big(\cosh^2(u) - 1\Big)^{\frac{1}{2}}$ and therefore (see (106) and (109)) of the integrand of (106).

To prove (111) we must solve equation (113). Combining (19), (24) and (113) we have

$$\cosh(\alpha) = \frac{\sin(\alpha_{\pm})}{\alpha_{\pm}}. \tag{115}$$

Expanding both parts of (115) in (small) parameters α and $\alpha_{\pm}$ we immediately obtain estimate (111). □

Theorem 4.5 (see §8.31 of [4]).

Function ϕ_3 from (109) *on the contours defined by* (19), (24) *is a monotonically decreasing function of v with the only maximum at $v = 0$.*

Theorem 4.6

For any $\alpha \neq 0$,

$$\phi_2 = \sinh(\alpha) - \alpha\cosh(\alpha) - \frac{1}{2}\sinh(\alpha)\ v^2 + O(v^4). \tag{116}$$

Proof

For small v and $\alpha \neq 0$ we have from (19), (24)

$$u = \frac{1}{6}\coth(\alpha)v^2 + O(v^4). \tag{117}$$

Substituting (117) into (109) we immediately obtain (116). □

As follows from (19), (24), (109), (110) and Theorems 4.5 and 4.6 the integrand of (106) is a nonoscillatory function of v. Moreover, for small v,

$$\left(\frac{du}{dv} + i\right)\exp(x\phi_3) = \exp(-x\cdot(\alpha\cosh(\alpha) - \sinh(\alpha))) \times \\ \left(\left.\frac{du}{dv}\right|_{v=0} + i\right)\exp\left(-\frac{1}{2}x\sinh(\alpha)v^2\right)(1+\theta_2(v)), \tag{118}$$

where $\theta_2(v) = O(v)$ is the error term.

Observation 4.3

The local behavior of the integrands of (106) and (83) is essentially the same (compare (118) and (92)). Therefore the conditions under which the integral (106) can be numerically evaluated at a CPU time, independent of x and ν, are equivalent to the conditions (93) and (94) (see Observation 4.1). These conditions are

$$x\sinh(\alpha) \gg 1, \tag{119}$$

and

$$|\alpha_\pm| > 2^{\frac{1}{2}}\frac{C_2}{(x\sinh(\alpha))^{\frac{1}{2}}}, \tag{120}$$

where the parameters $\alpha_\pm$ are estimated in (111) and

$$C_2 \sim 1 \tag{121}$$

is a (positive) constant. The condition (119) means that the domain where the integrand of (106) is (numerically) not zero is sufficiently small. The condition (120) shows that the distance between the singularities of the integrand of (106) $\alpha_\pm$ and the real axis is larger than several standard deviations of the Gaussian in (118).

Now the proof of the Theorem 4.3 can be repeated almost verbatim yielding the following result:

Theorem 4.7

For any

$$x \gg 1 \tag{122}$$

and

$$\nu > x + D_2\, x^{\frac{1}{3}}, \tag{123}$$

where

$$D_2 \sim 1, \tag{124}$$

is a (positive) constant, the inequalities (119) *and* (121) *are satisfied.*

4.3 Region $x \approx \nu$

In order to construct an algorithm whose complexity does not depend on x and ν in the region

$$|x - \nu| < \text{const} \cdot x^{\frac{1}{3}}, \quad \text{const} \sim 1, \tag{125}$$

(*i.e.* when the conditions (97) and (123) are violated) we will consider numerical integration of (15) along the Debye contours defined in (19), (26) and (27). We note in passing that these contours were extensively used for the derivation of asymptotic expansions of function $H_\nu^{(1)}(x)$ for $x \approx \nu$ (see, for example, Ch.8 of [4], [7]).

Denoting the integral along the contour (19), (26) by J_1 and the integral along the contour (19), (27) by J_2 it can be shown that

$$J_1 = \frac{1}{\pi i} \int_{-\infty}^{0} \exp(x \sinh(u) - \nu u) du, \tag{126}$$

$$J_2 = \frac{1}{\pi i} \int_{0}^{\pi} \exp(x\phi_4)\; (i + \frac{du}{dv}) dv, \tag{127}$$

and

$$H_\nu^{(1)}(x) = J_1 + J_2, \tag{128}$$

where

$$\phi_4 \equiv \phi_4(v, u(v), \tau) = \sinh(u)\ \cos(v) - u + \tau\ (u + i\ v), \tag{129}$$

$$\tau = 1 - \frac{\nu}{x}, \tag{130}$$

$$\frac{du}{dv} = \frac{(\sin(v) - v\ \cos(v))}{\sin^2(v)\ \sinh(u)}. \tag{131}$$

In (127), (129) and (131) function $u = u(v)$ is evaluated via (19), (27).

Observation 4.4

The integral (126) is merely the integral (105) with $\alpha = 0$ and therefore is can be evaluated at a CPU time independent of x and ν (see Observation 4.2).

Observation 4.5

Obviously, the integrand of (127) is an analytic and (for sufficiently small $|\tau|$) nonoscillatory function of v. Therefore the integral (127) can be computed (for example by means of the Gauss-Legendre quadrature formula) at a CPU time independent of ν and x. We will show, however, that for sufficiently large x and $\tau > 0$ the integral (127) can not be evaluated without an unavoidable round-off error (see Observation 4.6 below). The inequalities (141) and (142) below estimate the range of parameters x and ν where this error is small (see also Remark 4.2 below).

Theorem 4.8

For any $0 < \tau \ll 1$ function $Re\phi_4$ has the maximum at

$$v_{max} = \frac{1}{2}(3\tau)^{\frac{1}{2}}(1 + O(\tau)). \tag{132}$$

Moreover,

$$\phi_{max} \equiv Re\ \phi_4(v_{max}, u(v_{max}), \tau) = \frac{1}{3}\ \tau^{\frac{3}{2}}(1 + O(\tau)). \tag{133}$$

Proof

From equations (19), (27) for small v we have

$$u = \frac{1}{3^{\frac{1}{2}}}\ v + \frac{2}{45 \cdot 3^{\frac{1}{2}}}\ v^3 + O(v^5). \tag{134}$$

Therefore

$$\sinh(u)\ \cos(v) = \frac{1}{3^{\frac{1}{2}}}\ v - \frac{2}{5 \cdot 3^{\frac{1}{2}}} v^3 + O(v^5), \tag{135}$$

and

$$\sinh(u)\ \cos(v) - u = -\frac{4}{9 \cdot 3^{\frac{1}{2}}}\ v^3 + O(v^5). \tag{136}$$

Substitution of (134) and (136) into (129) produces

$$Re\ \phi_4 = \frac{1}{3^{\frac{1}{2}}}\ \tau\ v - \frac{4}{9 \cdot 3^{\frac{1}{2}}}\ v^3 + O(\tau\ v^3) + O(v^5). \tag{137}$$

The formulae (132) and (133) are a consequence of (137). □

Theorem 4.9 (see, for example, Ch.8 of [4]).

In the region (125) *for* $x \gg 1$,

$$|H_\nu^{(1)}(x)| = O(x^{-\frac{1}{3}}). \tag{138}$$

Observation 4.6

It follows from the estimate (133) that for sufficiently large x the integrand of (127) may be large whereas, according to Theorem 4.9, the integral itself is (asymptotically) small. Therefore in this case there exist cancellations that account for the round-off errors.

We do not expect significant cancellations if

$$x\phi_{max} < C_3, \tag{139}$$

where

$$C_3 \sim 1 \tag{140}$$

is a (positive) constant and ϕ_{max} is estimated in (133). The condition (139) means that in order to avoid cancellations the maximum of the integrand (127) must be of order $O(1)$.

Using (130) and (133) it is easy to show that for

$$x \gg 1 \tag{141}$$

the condition (139) is equivalent to the condition

$$\nu < x - D_3\, x^{\frac{1}{3}} + O(x^{-\frac{1}{3}}), \tag{142}$$

where

$$D_3 = (3\, C_3)^{\frac{2}{3}} \sim 1, \tag{143}$$

In fact, owing to (138), there must be another condition of the absence of cancellations in addition to (139). Namely, the domain in v where the integrand of (127) is not small must be of order $O(x^{-\frac{1}{3}})$. It can be shown, however, that this condition holds if (142) is satisfied.

We will briefly discuss the case of $\tau < 0$. Now the integrand of (127) does not have a maximum for $v \neq 0$. However, in the vicinity of $v = 0$ this integrand is of order $O(1)$. On the other hand, function $J_\nu(x)$ becomes small if roughly speaking (see Appendix B)

$$\nu > x + \text{const}\; x^{\frac{1}{3}}. \tag{144}$$

Therefore function $J_\nu(x)$ (*i.e.* the real parts of $H_\nu^{(1)}(x)$) can not be evaluated by means of (127) without an unavoidable round-off error if ν satisfies (144) with const $\gg 1$. It can be shown, however, that for

$$\nu < x + D_4 x^{\frac{1}{3}}, \tag{145}$$

where

$$D_4 \sim 1 \tag{146}$$

is a (positive) constant, both real and imaginary parts of function $H_\nu^{(1)}(x)$ can be evaluated without a significant round-off error.

Remark 4.2

The numerical computation of the integrals (126) and (127) provide a method for the evaluation of function $H_\nu^{(1)}(x)$ at a constant CPU time in the region (125) *i.e.* where the algorithms discussed in Subsections 4.1 and 4.2 fail (compare the inequalities (97), (123) and (142), (145), as well as the estimates (104), (124) and (143), (146)). Moreover, for $x \sim 1$ and $x \sim \nu$ the integral (127) can be evaluated without a significant round-off error even if (142) is violated because in this case the maximum of its integrand is of order $O(1)$ (see Theorem 4.8 and inequality (139)).

5. Implementation of the algorithm

In this section we present certain details of the implementation of the algorithm for the evaluation of function $H_\nu^{(1)}(x)$ via the Debye asymptotic expansions (see Section 3) or the contour integration (see Section 4). This scheme was tested to provide double precision accuracy (at least thirteen digits) for

$$2 \leq x \leq 100000 \tag{147}$$

and

$$0 \leq \nu \leq 100000 + 16 \cdot 100000^{\frac{1}{3}}. \tag{148}$$

For

$$\begin{aligned} x &< 2, \\ \nu &\geq 0 \end{aligned} \tag{149}$$

function $H_\nu^{(1)}(x)$ can be evaluated by means of Taylor expansions (see Subsection 2.2). Discussion of the round-off errors for both Taylor expansions and the contour integration is presented in Appendix A.

5.1 Implementation of the Debye asymptotic expansions

Formulae (42)-(46) make numerical evaluation of the Debye asymptotic expansions (29), (32) and (33) fairly straightforward. However, we must estimate the values of parameters x and ν

for which these expansions provide a specified (in our case double precision) accuracy. It is necessary to point out that the computation of a N-term Debye expansion involves evaluation of n-order polynomials $u_n(t)$ for $n = 0, 1, \cdots, N$ (see Lemma 3.1). In other words, the complexity of this procedure is of order $O(N^2)$ operations and therefore for given x and ν it is desirable to choose N as small as possible. The following considerations provide a simple recipe for the optimal (for given x and ν) truncation of the Debye expansions.

The estimates (61), (73), (75) and Observation 3.3 show that for any fixed $x \gg 1$ and given N there exists a constant $g_N \gg 1$ such that the error terms of the expansions (29), (32) and (33) (truncated after N terms) become small if x and ν satisfy the inequality

$$|\nu - x| > g_N \, x^{\frac{1}{3}}. \tag{150}$$

Our numerical experiments showed that for

$$x > 17 \tag{151}$$

and ν satisfying (150) the Debye expansions (29), (32) and (33) provide double precision accuracy. The estimates of constants g_N are presented in Table 1; these values were obtained by means of both analyzing the inequalities (61), (73), (75), and comparing the estimates provided by the Debye expansions with that by contour integration.

The optimal (for given x and ν) truncation of the Debye expansions can be done by the following procedure. First we compute the parameter $g = |x - \nu|/x^{\frac{1}{3}}$, then, using Table 1, find $g_N < g$ closest to g and retain N terms corresponding to this g_N. For example, if $g = 8$ then $g_N = 7$ and thus $N = 13$. As we see from Table 1 the Debye expansions fail to provide double precision accuracy if $g < 6.5$.

Remark 5.1

In addition to the error term $\theta_{N+1,1}(\nu, p)$, estimated in (75), the expansion (32) contains the error term $\theta_{N+1,1}(\nu, 0)$. It can be shown, however, that if the inequalities (150) and (151) are satisfied then (with double precision accuracy) this term can be neglected.

5.2 Numerical integration for $x > \nu$

As follows from (92), for large $x\sin(\beta)$ the integrand of (83) is sharply peaked at $v = \beta$, and thus the interval in v where this integrand is not small may be much narrower than the actual interval of integration $v \in (0, \pi)$. To estimate numerically meaningful domain of integration in (83) we must find (unique) solutions of equation (87) β_1 and β_2 (see Corollary 4.1) with ϵ approximately equal to the absolute value of the (specified) error of the evaluation of (83). After that numerical integration in (83) is restricted to the interval $v \in (\beta_1, \beta_2)$.

In accordance with Observation 4.1 and Theorem 4.3 it was found that for

$$x > 10 \tag{152}$$

and

$$\nu < x - d_1\, x^{\frac{1}{3}} \tag{153}$$

with

$$d_1 \approx 5, \tag{154}$$

the minimal number of nodes of the trapezoidal formula (see Observation 4.1) needed to evaluate (83) with double precision accuracy is

$$n = 25 \tag{155}$$

independent of x and ν. If (152) or (153) are violated then to obtain the same accuracy we have to increase n. Changing the variable of integration (see, for example, [8])

$$v = t^4, \tag{156}$$

it is possible to somewhat improve the estimate (154). It was found that after the change of variable (156) we can evaluate (83) with double precision accuracy in the region (153) where

$$d_1 \approx 1.5, \tag{157}$$

with the number of nodes (of the trapezoidal formula)

$$n = 37 \tag{158}$$

independent of x and ν.

5.3 Numerical integration for $x < \nu$

We will first consider the integral (105) (see Observation 4.2). Function ϕ_2, defined in (108) on the interval $-\infty < u < \alpha$ has the only maximum at $u = -\alpha$. Moreover, for large x this function is not small only for $u \approx -\alpha$. Therefore, we must first estimate the interval $u \in (\alpha_1, \alpha_2)$ ($\alpha_1 < -\alpha < \alpha_2 \leq \alpha$) where the integrand of (105) is (with given accuracy) not zero and restrict the numerical integration to this interval. We evaluated (105) by means of the Gauss-Legendre quadrature formula. It was found that for

$$x > 1 \tag{159}$$

the minimal number of nodes required to evaluate this integral with double precision accuracy is

$$n = 45 \tag{160}$$

independent of x and ν.

Now we will discuss the computation of the integral (106). As follows from (118) for sufficiently large $x\sinh(\alpha)$ the integrand of (106) is sharply peaked at $v = 0$. Therefore we must first estimate the interval $v \in (0, \alpha_3)$ ($\alpha_3 < \pi$) where the integrand of (106) is (with given precision) not zero and restrict the numerical integration to this interval (compare with Subsection 5.1).

The integral (106) was evaluated by means of the Gauss-Legendre quadrature formula. In accordance with Observation 4.3 and Theorem 4.7 it was found that this integral can be evaluated with double precision accuracy with the number of nodes

$$n = 45 \tag{161}$$

independent of x and ν, if x satisfies the inequality (159) and

$$\nu > x + d_2\, x^{\frac{1}{3}}, \tag{162}$$

where

$$d_2 \approx 1.5. \tag{163}$$

5.4 Numerical integration for $x \approx \nu$

The problem of the numerical evaluation of the integrals (126) and (127) is essentially the same as that of the integrals (105) and (106) (see Subsection 5.2) and here we will briefly discuss the results of our numerical experiments.

The integrals (126) and (127) were evaluated by means of the Gauss-Legendre quadrature formula. In accordance with Observation 4.6 and the estimates (142), (144) it was found that both real and imaginary parts of these integrals can be computed with double precision accuracy with the number of nodes

$$n = 33 \tag{164}$$

independent of x and ν, if

$$x > 10, \tag{165}$$

and

$$|x - \nu| < d_3 \cdot x^{\frac{1}{3}}, \tag{166}$$

where

$$d_3 \approx 2. \tag{167}$$

In addition (see Remark 4.2), this scheme provides double precision accuracy with n defined in (164) for

$$1 < x < 10 \tag{168}$$

and

$$x \geq \nu. \tag{169}$$

6. Description of the algorithm

In this section we describe the algorithm for the evaluation of an individual function $H_\nu^{(1)}(x)$ (and therefore, according to (4) - (6), all other Bessel functions) of nonnegative x and ν. The final algorithm consists of two parts: algorithm A that combines contour integration and Taylor expansion, and algorithm B that is based on the Debye asymptotic expansions.

6.1 Algorithm A

if $x \le 2$ **then** compute $H_\nu^{(1)}(x)$ via Taylor expansions (see Subsection 2.2).

end if

if $x > 2$ **and** $\nu > x + 1.5 \cdot x^{\frac{1}{3}}$ **then** compute $H_\nu^{(1)}(x)$ by means of evaluation of the Sommerfeld integral along contours (19), (23), (24) (see Subsections 4.2 and 5.3).

end if

if $x > 9$ **and** $\nu < x - 1.5 \cdot x^{\frac{1}{3}}$ **then** compute $H_\nu^{(1)}(x)$ by means of evaluation of the Sommerfeld integral along contours (19), (20). (see Subsections 4.1 and 5.2).

end if

if $x \le 9$ **or** $|\nu - x| \le 1.5 \cdot x^{\frac{1}{3}}$ **then** compute $H_\nu^{(1)}(x)$ by means of evaluation of the Sommerfeld integral along contours (19), (26), (27) (see Subsections 4.3 and 5.4).

end if

6.2 Algorithm B

if $x \ge 17$ **and** $\nu \le x - 6.5 \cdot x^{\frac{1}{3}}$ **then** compute $H_\nu^{(1)}(x)$ by means of evaluation of the Debye expansion (29) (see Subsections 4.1 and 5.1).

end if

if $x \ge 17$ **and** $\nu \ge x + 6.5 \cdot x^{\frac{1}{3}}$ **then** compute $H_\nu^{(1)}(x)$ by means of evaluation of the Debye expansions (32), (33) (see Subsections 4.1 and 5.1).

end if

6.3 The final algorithm

if $x \ge 17$ **and** $|x - \nu| \ge 6.5 \cdot x^{\frac{1}{3}}$ **then** compute $H_\nu^{(1)}(x)$ by means of the algorithm B

else compute $H_\nu^{(1)}(x)$ by means of the algorithm A

end if

6.4 Timing

A computer (FORTRAN) program using the algorithm described in the preceding sections was implemented and tested on Sun SPARCstation 1. This program consists of approximately 2000 executable lines.

We compared the time for the evaluation of function $H_\nu^{(1)}(x)$ by our algorithm with the time required to compute this function by means of recurrence relation (1) (for increasing orders) in case of integer ν. It was found that in the range of validity of the Debye expansions our algorithm catches up with the recursion for $\nu \approx 10$; in the region where the contour integration is used the same happens for $\nu \approx 800$. For arguments $x < 17$ the algorithm is approximately 20 times slower than the recursion.

8. Conclusions

In the present paper we have shown that the Debye asymptotic expansions, contrary to what appears to be a popular belief, are not expansions in inverse powers of (large) parameter ν but turn out to be uniform expansions in inverse powers of (large) parameter $g_1 = (x-\nu)/x^{\frac{1}{3}}$ for $x > \nu$ and (large) parameter $g_2 = (\nu - x)/\nu^{\frac{1}{3}}$ for $x < \nu$ (see Theorems 3.1, 3.3 and Observations 3.1 and 3.3). For x and ν such that both Taylor and Debye expansions do not provide a specified accuracy we have demonstrated that function $H_\nu^{(1)}(x)$ can be computed at a constant CPU time via (numerical) evaluation of the Sommerfeld integral along contours of steepest descents (the Debye contours). Obviously, numerical integration along contours of steepest descents can be applied for the evaluation of other functions of mathematical physics. In particular, it can be used for the computation of Bessel functions of complex arguments and orders (the classification of the Debye contours in case of complex x and ν can be found, for example, in Ch. 8 of [4]).

In addition, we have obtained new estimates concerning decay of functions $J_\nu(x)$ and $-1/Y_\nu(x)$ of fixed $x > 0$ and large positive ν (see Appendix B). Finally, we have shown that

Bessel functions of the first kind and integer orders provide a solution to a system of differential equations for a chain of coupled harmonic oscillators (see Appendix C).

The author would like to thank Professor Vladimir Rokhlin for useful discussions and for his interest and support.

Appendix A

Round-off errors

In this Appendix we briefly discuss round-off errors that appear (for certain values of x and ν) when function $H_\nu^{(1)}(x)$ is evaluated via either Taylor expansions or contour integration.

A.1 Taylor expansions

It is well known (see, for example, Ch. 3 of [4]) that for

$$\nu = m + \sigma, \tag{170}$$

where m is an integer and $|\sigma| \ll 1$,

$$J_\nu(x)\ \cos(\pi\ \nu) \approx J_{-\nu}(x), \tag{171}$$

and thus in this case formula (8) produces significant round-off error. However, for any fixed $x > 0$ function $Y_\nu(x)$ is an analytic function of ν and therefore it can be evaluated by means of interpolation with respect to the order.

Our experiments showed that that for $|\sigma| > 5 \cdot 10^{-4}$ no significant error occurred. For smaller $|\sigma|$ we used Chebychev interpolation of function $Y_\nu(x)$ on the interval

$$\nu \in [m - 10^{-3}, m + 10^{-3}] \tag{172}$$

with the number of nodes k=6; this number of nodes proves to be sufficient for the evaluation of this function with at least thirteen digits for ν from (172) and $x \leq 2$.

A.2 Numerical integration for $x > \nu$

Our experiments showed that for $x \gg 1$ it is impossible to compute the integral (83) without a round-off error unless function ϕ_1, defined in (84), is carefully evaluated. This error occurs because, as follows from (83), for $x \gg 1$ small errors of the evaluation of function ϕ_1 produce large errors of the integrand of (83).

First of all we observe that for small $|v-\beta|$ there appears a loss of accuracy if we evaluate $\sinh(u)$ by means of the formula

$$\sinh(u) = ((\cosh(u) - 1)(\cosh(u) + 1))^{\frac{1}{2}} \tag{173}$$

with $\cosh(u)$ computed via (19), (20). Writing

$$\cosh(u) - 1 = \frac{f_1(v,\beta) - f_2(v,\beta)}{\sin(v)} \tag{174}$$

where

$$f_1(v,\beta) = (v-\beta)\ \cos(\beta) \tag{175}$$

$$f_2(v,\beta) = \sin(v) - \sin(\beta) \tag{176}$$

we see that for small $|v-\beta|$ each of the functions $f_1(v,\beta)$ and $f_2(v,\beta)$ are of order $O(v-\beta)$ whereas the numerator of (174) is of order $O((v-\beta)^2)$.

To avoid the round-off error, cased by the cancellation of the leading terms of the Taylor expansions of functions (175) and (176), we can first evaluate the numerator of (174) using its Taylor expansion in (small parameter) $(v-\beta)$ and after that compute $\sinh(u)$ by means of (173), and u via

$$u = \ln(\sinh(u) + \cosh(u)). \tag{177}$$

In addition, a round-off error appears if we evaluate function ϕ_1 itself via the formula (84) Rewriting (84) in an equivalent form

$$\phi_1 = (\cos(v) - \cos(\beta))\ \sinh(u) + (\sinh(u) - u)\ \cos(\beta), \tag{178}$$

we see, that in order to avoid cancellations (and therefore the lost of significant digits in (178)) we can evaluate functions $(\cos(v) - \cos(\beta))$ and $(\sinh(u) - u)$ via their Taylor expansions in (small parameters) $(v-\beta)$ and u, consequently.

Our experiments showed that the integral (83) can be computed without significant round-off error if we use Taylor expansions in (174) and (178) for $|v-\beta| \leq 0.1$.

A.3 Numerical integration for $x \approx \nu$

It was found that the integral (126) can be computed without round-off error for any x and ν. However it turns out that we cannot evaluate of the integral (127) without a round-off error which becomes large for $x \gg 1$. Like in case $x > \nu$, the main source of this error is the sensitivity of the computation of the integrand of (127) (for large x) to small errors of the evaluation of function ϕ_4 defined in (129). To analyze this effect we observe that as follows from (134) and (135) for small v functions u and $\sinh(u) \cdot \cos(v)$ are of order $O(v)$ whereas their difference (136) is of order $O(v^3)$. Therefore the round-off error of the evaluation of (136) (and thus of function ϕ_4 from (129)) appears (for sufficiently small v) due to cancellation of the leading terms of the Taylor expansions of (134) and (135).

It follows from our numerical experiments that if the left-hand sides of (134)-(136) are evaluated via their Tailor expansions for $v < 1$ then the integral (127) can be computed without significant round-off error. These expansions are:

$$
\begin{aligned}
u \; = \; & 0.57735026918962576\, v + 0.025660011963983367\, v^3 + \\
& 0.0014662863979419067\, v^5 + 0.000097752426529460445\, v^7 + \\
& 0.74525058224720925 \cdot 10^{-5}\, v^9 + 0.61544207267743328 \cdot 10^{-6}\, v^{11} + \\
& 0.52905118464628039 \cdot 10^{-7}\, v^{13} + \cdots,
\end{aligned} \tag{179}
$$

$$
\begin{aligned}
\sinh(u) \; = \; & 0.57735026918962576\, v + 0.057735026918962576\, v^3 + \\
& 0.0062775386411887880\, v^5 + 0.00065524673408028954\, v^7 + \\
& 0.000066970892258993254\, v^9 + 0.67971226373232793 \cdot 10^{-5}\, v^{11} + \\
& 0.68878126140038453 \cdot 10^{-6}\, v^{13} + \cdots,
\end{aligned} \tag{180}
$$

$$
\begin{aligned}
u - \sinh(u)\, \cos(v) \; = \; & 0.25660011963983367\, v^3 + \\
& 0.00097752426529460445\, v^7 + \\
& 0.000072409204836637368\, v^9 + \\
& 0.74478039260541289 \cdot 10^{-5}\, v^{11} + \\
& 0.74130822294291681 \cdot 10^{-6}\, v^{13} + \cdots.
\end{aligned} \tag{181}
$$

Appendix B

Decay of functions $J_\nu(x)$ and $-1/Y_\nu(x)$ for the large orders and fixed arguments

In this appendix we discuss the behavior of functions $J_\nu(x)$ and $Y_\nu(x)$ for fixed arguments and large orders. Theorems B1 and B2 and formulae (219) and (222) contain the principal results of this appendix.

B1. Statement of the problem

It is well known that for any fixed $x > 0$ and $\nu \to \infty$ function $J_\nu(x)$ decays rapidly (see, for example, Ch. 10 of [5]) and has an asymptotic behavior defined by the formula

$$J_\nu(x) = \frac{1}{\Gamma(\nu+1)} \cdot \left(\frac{x}{2}\right)^\nu (1 + O(\nu^{-1})). \tag{182}$$

However, this approximation is numerically meaningful only for $\nu \gg x^2/4$ (see, for example, Ch. 10 of [5]). On the other hand, often it is necessary to have an accurate estimate of an order $\nu_j > x > 0$ such that

$$J_\nu(x) < \epsilon, \tag{183}$$

for all $\nu > \nu_j$. In (183) x is fixed and $\epsilon > 0$ is supposed to be sufficiently small. This problem arises, for example, when one implements Miller's algorithm (see, for example, [1], [2]), or sums a Neumann series

$$S_\mu(x) = \sum_{n=0}^{\infty} a_n \, J_{\mu+n}(x). \tag{184}$$

It is well known that the behavior of function $J_\nu(x)$ of fixed positive arguments and large orders is close to that of function $-1/Y_\nu(x)$. For example, for any fixed $x > 0$ and $\nu \to \infty$ this function decays rapidly (see, for example, [1]) and has the following asymptotic representation

$$-\frac{1}{Y_\nu(x)} = \pi \cdot \frac{1}{\Gamma(\nu)} \cdot \left(\frac{2}{x}\right)^\nu (1 + O(\nu^{-1})). \tag{185}$$

In this appendix we prove that functions $J_\nu(x)$ and $-1/Y_\nu(x)$ of any fixed $x > 0$ are monotonically decreasing functions of $\nu > x$ (see Theorems B1 and B2 below). In addition, in case of large fixed $x > 0$ and sufficiently small $\epsilon > 0$ we present approximate solutions of equations

$$J_{\nu_j}(x) = \epsilon, \tag{186}$$

and

$$-\frac{1}{Y_{\nu_y}(x)} = \epsilon, \tag{187}$$

with respect to ν_j and ν_y for $\nu_j, \nu_y > x$.

B2. Certain properties of function $J_\nu(x)$ for $\nu > x$.

It was proved in Ch. 10 of [4] that the for any $0 < z \leq 1$,

$$J_\nu(\nu z) > 0, \tag{188}$$

and

$$\frac{\partial J_\nu(\nu z)}{\partial \nu} < 0. \tag{189}$$

Similarly, one can prove the following

Theorem B1

For any $\nu \geq x > 0$,

$$\frac{\partial J_\nu(x)}{\partial \nu} < 0. \tag{190}$$

Proof

We start with Schläfli's representation of function $J_\nu(x)$ (see, for example, Ch. 6 of [4]):

$$J_\nu(x) = \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+\pi i} \exp(x\sinh(w) - \nu w)dw. \tag{191}$$

Differentiating (191) with respect to ν we have

$$\frac{\partial J_\nu(x)}{\partial \nu} = -\frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+\pi i} w\ \exp(x\sinh(w) - \nu w)dw. \tag{192}$$

Deforming the contour of integration in (192) into the contour (19), (24) with $v \in (-\pi, \pi)$ we obtain

$$\frac{\partial J_\nu(x)}{\partial \nu} = -\frac{1}{2\pi i}\int_{-\pi}^{\pi}(u+iv)\left(i+\frac{du}{dv}\right)\exp(x\sinh(u)\ \cos(v)-\nu u)dv, \qquad (193)$$

where the derivative du/dv is defined in (110). On this contour,

$$u > 0, \qquad (194)$$

$$\frac{du(v)}{dv} = -\frac{du(-v)}{dv}. \qquad (195)$$

Combination of (193), (194) and (195) yields

$$\frac{\partial J_\nu(x)}{\partial \nu} = -\frac{1}{\pi}\int_{0}^{\pi}\left(u+v\frac{du}{dv}\right)\exp(x\sinh(u)\ \cos(v)-\nu u)dv. \qquad (196)$$

Finally, for $v \in (0, \pi)$ we have from (110)

$$\frac{du}{dv} \geq 0. \qquad (197)$$

The conclusion of the theorem follows from (194), (196) and (197). □

Observation B1

It follows from formula (188) and Theorem B1 that for any $\nu \geq x > 0$ (x is fixed) equation (186) has at most one real solution. Moreover, if such a solution exists, then the inequality (183) is satisfied for all $\nu > \nu_j$.

B3. Certain properties of function $Y_\nu(x)$ for $\nu > x$.

Turning to the discussion of the behavior of function $Y_\nu(x)$ for $\nu > x > 0$ we will first prove the following

Lemma B1

For any $\nu \geq x > 0$,

$$Y_\nu(x) < 0. \qquad (198)$$

Proof

Combining formulae (5), (105) (106) and (107) we have

$$Y_\nu(x) = -\frac{1}{\pi} \int_{-\infty}^{\alpha} \exp(x\phi_2) du - \frac{1}{\pi} \int_0^{\pi} \exp(x\phi_3) \frac{du}{dv} dv, \tag{199}$$

where the parameter α and functions ϕ_2 and ϕ_3 are defined in (25), (108) and (109), respectively, and function $u = u(v)$ is evaluated via (19), (23). Now the conclusion of the lemma follows from (197) and (199). □

We will now prove the analogue of Theorem B1 for function $Y_\nu(x)$.

Theorem B2

For any $\nu \geq x > 0$,

$$\frac{\partial Y_\nu(x)}{\partial \nu} < 0. \tag{200}$$

Proof

We start with Nicholson's formula (see, for example, Ch. 9 of [5]):

$$N_\nu(x) \equiv \frac{8}{\pi^2} \int_0^\infty K_0(2x \sinh(t)) \cdot \cosh(2\nu t) dt = J_\nu^2(x) + Y_\nu^2(x), \tag{201}$$

where

$$K_0(z) = \int_0^\infty \exp(-z \cosh(t)) dt \tag{202}$$

is Macdonald's function of zero order. It immediately follows from (201) and (202) that for any $x > 0$,

$$\frac{\partial N_\nu(x)}{\partial \nu} > 0. \tag{203}$$

Next, formula (201) yields

$$\frac{\partial Y_\nu(x)}{\partial \nu} = \frac{1}{2Y_\nu(x)} \cdot \left(\frac{\partial N_\nu(x)}{\partial \nu} - 2J_\nu(x) \frac{\partial J_\nu(x)}{\partial \nu} \right). \tag{204}$$

The conclusion of the theorem is a consequence of (188), (190), (198), (203) and (204). □

The following observation is closely related to Observation B1 above.

Observation B2

Lemma B1 and Theorem B2 show that for any fixed positive $x < \nu$ equation (187) has at most one real solution. Moreover, if such a solution exists,

$$-\frac{1}{Y_\nu(x)} < \epsilon \tag{205}$$

for all $\nu > \nu_y$.

In the rest of the appendix we derive approximate (asymptotic) solutions of equations (186) and (187).

B4. Asymptotic solution of equation (186).

We will first prove the following simple

Lemma B2

For any $|z| < 1$,

$$\ln(1+z) = \sum_{n=1}^{\infty} \frac{z^{2n-1}}{2n-1} + \frac{1}{2}\ln(1-z^2). \tag{206}$$

Proof

Expanding left-hand side of (206) into Taylor series we have

$$\ln(1+z) = \sum_{n=1}^{\infty}(-1)^{(n+1)}\frac{z^n}{n} = \sum_{n=1}^{\infty}\frac{z^{2n-1}}{2n-1} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{z^{2n}}{n}, \tag{207}$$

which concludes the proof of the lemma. □

An approximation to ν_j from (186), hereafter denoted by $\tilde{\nu_j}$, will be sought as a solution of the equation

$$\frac{\exp(-\tilde{\eta}_2)}{(2\pi)^{\frac{1}{2}}\,(\tilde{\nu}_j^2 - x^2)^{\frac{1}{4}}} = \epsilon, \tag{208}$$

where the function on the left-hand side of (208) is the leading term of the Debye asymptotic expansion (32) with the change of notation $\nu \to \tilde{\nu_j}$.

Introducing the notation

$$y_j = \frac{\tilde{\nu}_j}{x}, \tag{209}$$

and substituting (209) into (34) we obtain

$$\tilde{\eta}_2 = x \cdot y_j \cdot \left[\ln(y_j) + \ln\left(1 + (1 - y_j^{-2})^{\frac{1}{2}}\right) - (1 - y_j^{-2})^{\frac{1}{2}}\right]. \tag{210}$$

Noticing that for $z = (1 - y_j^{-2})^{\frac{1}{2}}$ equation (206) becomes

$$\ln\left(1 + (1 - y_j^{-2})^{\frac{1}{2}}\right) = (1 - y_j^{-2})^{\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{1}{2n+1}(1 - y_j^{-2})^{\frac{2n+1}{2}} - \ln(y_j), \tag{211}$$

and substituting (211) into (210) we obtain

$$\tilde{\eta}_2 = x\ y_j\ (1 - y_j^{-2})^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\left(1 - y_j^{-2}\right)^n}{2n+1}. \tag{212}$$

Now, substituting (209) and (212) into (208), we have

$$x \cdot y_j \cdot (1 - y_j^{-2})^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{(1 - y_j^{-2})^n}{2n+1} + \frac{1}{4}\ln(y_j^2 - 1) = -\ln\left((2\pi x)^{\frac{1}{2}}\ \epsilon\right). \tag{213}$$

Finally, introducing a new (unknown) function q_j which is a positive solution of the equation

$$y_j = x^{-\frac{2}{3}}\ q_j^2 + 1, \tag{214}$$

and substituting (214) into (213) we have

$$\frac{2^{\frac{3}{2}}}{3}\ q_j^3\ (1 - \frac{1}{20}\ x^{-\frac{2}{3}}\ q_j^2 + O(x^{-\frac{4}{3}}\ q_j^4)) + \frac{1}{2}\ln q_j + \frac{1}{4}\ln(1 + \frac{1}{2}x^{-\frac{2}{3}}\ q_j^2) =$$
$$-\ln(2^{\frac{3}{4}}\ \pi^{\frac{1}{2}}\ x^{\frac{1}{3}}\ \epsilon). \tag{215}$$

We seek the asymptotic solution of (215) under the condition

$$x \gg -\ln(2^{\frac{3}{4}}\ \pi^{\frac{1}{2}}\ x^{\frac{1}{3}}\ \epsilon) \gg 1. \tag{216}$$

Taking into account (216), equation (215) immediately yields the leading term of the asymptotic expansion of q_j:

$$q_j \sim \delta_j, \tag{217}$$

where

$$\delta_j = 3^{\frac{1}{3}}\ 2^{-\frac{1}{2}} \left(-\ln(2^{\frac{3}{4}}\ \pi^{\frac{1}{2}}\ x^{\frac{1}{3}}\ \epsilon)\right)^{\frac{1}{3}}. \tag{218}$$

Corrections to (217) can be found by standard methods (see, for example, Ch. 1 of [5]). After some algebra we obtain

$$\begin{aligned} \frac{\tilde{\nu_j}}{x} - 1 \quad &= \quad \left(\delta_j\ x^{-\frac{1}{3}}\right)^2 \left(1 - \frac{1}{8^{\frac{1}{2}}}\ \ln(\delta_j)\ \delta_j^{-3} + \frac{1}{30}\left(\delta_j\ x^{-\frac{1}{3}}\right)^2 + \right. \\ &\quad \left. O\left(\ln(\delta_j)\ \delta_j^{-6}\right) + O\left(x^{-\frac{2}{3}}\ \delta_j^{-1}\right)\right). \end{aligned} \tag{219}$$

B5. Asymptotic solution of equation (187).

In this subsection we briefly discuss the derivation of the asymptotic solution of equation (187); this approximate solution will be denoted by $\tilde{\nu_y}$. Like in case of equation (186) we seek an approximation to ν_y as a solution (for fixed positive $x < \nu$) of equation

$$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \exp(-\tilde{\eta}_2)\ (\tilde{\nu}_y^2 - x^2)^{\frac{1}{4}} = \epsilon, \tag{220}$$

where the function on the left-hand side of (220) is the modulus of the inverse of the leading term of the Debye asymptotic expansion (33) with the change of notation $\nu \to \tilde{\nu_y}$. The technique employed for solving equation (220) is the same as that for equation (208) and we omit the computational details. It can be shown that the asymptotic solution of (220) derived under the condition

$$x \gg -\ln(2^{\frac{1}{4}}\ \pi^{-\frac{1}{2}}\ x^{-\frac{1}{3}}\ \epsilon) \gg 1, \tag{221}$$

has the form

$$\begin{aligned} \frac{\tilde{\nu_y}}{x} - 1 \quad &= \quad \left(\delta_y\ x^{-\frac{1}{3}}\right)^2 \left(1 + \frac{1}{8^{\frac{1}{2}}}\ \ln(\delta_y)\ \delta_y^{-3} + \frac{1}{30}\left(\delta_y\ x^{-\frac{1}{3}}\right)^2 + \right. \\ &\quad \left. O\left(\ln(\delta_y)\ \delta_y^{-6}\right) + O\left(x^{-\frac{2}{3}}\ \delta_y^{-1}\right)\right), \end{aligned} \tag{222}$$

where

$$\delta_y = 3^{\frac{1}{3}}\ 2^{-\frac{1}{2}} \left(-\ln(2^{\frac{1}{4}}\ \pi^{-\frac{1}{2}}\ x^{-\frac{1}{3}}\ \epsilon)\right)^{\frac{1}{3}}. \tag{223}$$

B6. Comparison of exact and asymptotic solutions of equations (186) *and* (187).

Tables 2 and 3 contain approximations $\tilde{\nu_j}$ and $\tilde{\nu_y}$, obtained via (219) and (222), as well as exact numerical solutions of equations (186) and (187) for several values of x and ϵ (when solving equations (186) and (187) their left-hand sides were computed via contour integration for $x > 2$ and Taylor expansion for $x \leq 2$). It is interesting to note, that the formulae (219) and (222) provide reasonable approximations to ν_j and ν_y even for $x \approx 1$, *i.e.* when the conditions (216), (221) are violated.

Remark B1

It is easy to see from (219) and (222), that up to logarithmic (in x) corrections the parameters $\tilde{\nu_j}$ and $\tilde{\nu_y}$ can be estimated as

$$\tilde{\nu_j} \approx x + c_j(\epsilon)\, x^{\frac{1}{3}}, \tag{224}$$

$$\tilde{\nu_y} \approx x + c_y(\epsilon)\, x^{\frac{1}{3}}, \tag{225}$$

where $c_j(\epsilon) \approx c_y(\epsilon) > 0$ are independent of x. In other words, the approximations (224), (225) provide a (rough) estimate of a domain on the $x - \nu$ plane where functions $J_\nu(x)$ and $-1/Y_\nu(x)$ of any fixed $x \gg 1$ are small for all $\nu > \tilde{\nu_j} \approx \tilde{\nu_y}$.

Appendix C

Bessel functions and a chain of harmonic oscillators

In this appendix we show that functions $J_\nu(x)$ of integer ν describe displacements of coupled harmonic oscillators on a line.

Differentiating the formula (see, for example, [1])

$$\frac{df_\nu(x)}{dx} = \frac{1}{2}(f_{\nu-1}(x) - f_{\nu+1}(x)), \tag{226}$$

where $f_\nu(x)$ has the same meaning as in formula (1) we obtain

$$\frac{d^2 f_\nu(x)}{dx^2} = \frac{1}{4}(f_{\nu-2}(x) - 2\, f_\nu(x) + f_{\nu+2}(x)). \tag{227}$$

Equation (227) can be compared with the differential equations for a system of equal point masses on a line which interact with their nearest neighbors via the elastic force. The displacement from the equilibrium $u_n(t)$ (as a function of time t) of an n-th such oscillator satisfies equation (see, for example, [9])

$$M\ \frac{d^2u_n(t)}{dt^2} = G\ (u_{n-1}(t) - 2\ u_n(t) + u_{n+1}(t)), \tag{228}$$

where M is the mass and G is the elastic constant.

The analogy between (227) and (228) becomes especially transparent if we rewrite (227) for Bessel functions of the first kind of integer order $\nu = n$. It follows from (227) that these functions of even orders satisfy the system of differential equations

$$\frac{d^2J_{2n}(x)}{dx^2} = \frac{1}{4}(J_{2n-2}(x) - 2\ J_{2n}(x) + J_{2n+2}(x)), \quad (n = 0, \pm 1, \pm 2, \cdots); \tag{229}$$

with the initial conditions (see, for example, [1])

$$\begin{aligned} &J_0(0) = 1, \quad J_{2n}(0) = 0, \quad (n = \pm 1, \pm 2, \cdots); \\ &\left.\frac{dJ_{2n}(x)}{dx}\right|_{x=0} = 0, \quad (n = 0, \pm 1, \pm 2, \cdots). \end{aligned} \tag{230}$$

For odd orders we have the same system of equations

$$\frac{d^2J_{2n+1}(x)}{dx^2} = \frac{1}{4}(J_{2n-1}(x) - 2\ J_{2n+1}(x) + J_{2n+3}(x)), \quad (n = 0, \pm 1, \pm 2, \cdots); \tag{231}$$

but the initial conditions in this case are different (see, for example, [1]):

$$\begin{aligned} &J_{2n+1}(0) = 0, \quad (n = 0, \pm 1, \pm 2, \cdots); \\ &\left.\frac{dJ_1(x)}{dx}\right|_{x=0} = \frac{1}{2}, \qquad \left.\frac{dJ_{-1}(x)}{dx}\right|_{x=0} = -\frac{1}{2}, \\ &\left.\frac{dJ_{2n+1}(x)}{dx}\right|_{x=0} = 0, \qquad (n = 1, \pm 2, \cdots). \end{aligned} \tag{232}$$

Comparing (228) with (229) and (231) we see that function $J_{2n}(x)$ (or, equivalently, $J_{2n+1}(x)$) can be viewed as a displacement of an n-th oscillator at a 'time' x in a chain (228) with parameters

$$\begin{aligned} &M = 1, \\ &G = \frac{1}{4}. \end{aligned} \tag{233}$$

It is interesting to observe, that as follows from (229) and (231) the chains of even and odd Bessel function do not interact.

This analogy enables us to give a mechanical interpretation of certain properties of Bessel functions.

1. A zero of a Bessel function can be interpreted as a 'time' at which a corresponding oscillator passes the equilibrium. Therefore, the well known result, that function $J_n(x)$ has infinitely many zeros as $x \to \infty$ reflects the obvious physical property that any oscillator (in a chain with zero friction) passes the equilibrium infinitely many times (as time goes to infinity).

2. The identity (see, for example, [1])

$$\sum_{n=-\infty}^{\infty} J_n^2(x) = J_0^2(x) + 2\sum_{n=1}^{\infty} J_n^2(x) = 1 \tag{234}$$

means that the oscillators in the chains (229) and (231) under the initial conditions (230) and (232) oscillate in such a way that the sum of squares of the displacements in both of them does not depend on 'time' x.

3. In terms of the mechanical model the approximation (219) (or its simplified version (224)) estimates the range of propagation of the initial perturbation (230) (or (232)) at the 'time' x.

4. It is easy to show that the well known relation (see, for example, [1])

$$\sum_{n=-\infty}^{\infty} J_{2n}(x) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x) = 1 \tag{235}$$

is a consequence of the conservation of momentum in the even chain (229). To prove this we observe that according to the initial conditions (230) the total initial momentum of the even chain is zero. Because this system is isolated we can write

$$\sum_{n=-\infty}^{\infty} \frac{dJ_{2n}(x)}{dx} = 0 \tag{236}$$

for any 'time' x. Integrating (236) and observing that due to (230) the constant of integration is equal to unity we immediately obtain (235).

5. An interesting formula can be obtained from the law of conservation of energy in the odd chain (231). Using (233) we find the kinetic K and the potential Π energy of the odd chain (231) (see, for example, [9]):

$$K = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(\frac{dJ_{2n+1}(x)}{dx} \right)^2 , \tag{237}$$

$$\Pi = \frac{1}{8} \sum_{n=-\infty}^{\infty} \left(J_{2n+1}(x) - J_{2n-1}(x) \right)^2 . \tag{238}$$

Combining (226) and (238) we obtain

$$\Pi = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(\frac{dJ_{2n}(x)}{dx} \right)^2 . \tag{239}$$

As follows from (232) and (233) the total initial energy of the odd chain is equal to $1/4$ and, the system being isolated, it remains the same at any 'time' x. Therefore from (237) and (239) we have

$$\sum_{n=-\infty}^{\infty} \left(\frac{dJ_n(x)}{dx} \right)^2 = \frac{1}{2}. \tag{240}$$

The formula (240) means that the sum of kinetic energies of the chains (229) and (231) is independent of 'time' x.

Remark C1

As follows from the preceding analysis Bessel functions are a (rare) example of a discrete dynamic interacting system where the coordinate of any particle can be computed at a CPU time independent of the physical time.

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Table 1: Numerical estimates of the parameter g_N.

N	5	9	13	17
g_N	23	10	7	6.5

Table 2: Comparison of the numerical solution of (186) with the approximation (219).

	ϵ	$x=1$	$x=2$	$x=10$	$x=50$	$x=100$	$x=1000$	$x=10000$	$x=100000$
ν_0	10^{-5}	6.25	8.37	20.10	66.38	120.2	1040.8	10082.1	100163.9
$\tilde{\nu}$	10^{-5}	6.63	8.61	20.22	66.44	120.3	1040.8	10082.1	100164.0
ν_0	10^{-10}	10.28	13.15	27.60	78.79	135.8	1074.5	10156.0	100326.7
$\tilde{\nu}$	10^{-10}	11.20	13.74	27.83	78.88	135.9	1074.6	10156.0	100326.7
ν_0	10^{-20}	17.18	21.20	39.76	98.33	160.1	1126.2	10267.8	100569.4
$\tilde{\nu}$	10^{-20}	19.73	22.85	40.33	98.53	160.2	1126.2	10267.8	100569.4
ν_0	10^{-30}	23.37	28.34	50.25	114.8	180.5	1169.0	10359.7	100768.1
$\tilde{\nu}$	10^{-30}	28.14	31.47	51.32	115.2	180.7	1169.0	10359.7	100768.1

Table 3: Comparison of the numerical solution of (187) with the approximation (222).

	ϵ	$x=1$	$x=2$	$x=10$	$x=50$	$x=100$	$x=1000$	$x=10000$	$x=100000$
ν_y	10^{-5}	7.42	9.89	23.02	72.26	128.2	1062.5	10139.0	100309.4
$\tilde{\nu_y}$	10^{-5}	7.53	9.88	22.96	72.20	128.1	1062.4	10139.0	100309.4
ν_y	10^{-10}	11.43	14.59	30.18	83.76	142.5	1092.2	10202.1	100443.3
$\tilde{\nu_y}$	10^{-10}	12.06	14.91	30.19	83.70	142.4	1092.2	10202.1	100443.3
ν_y	10^{-20}	18.31	22.57	42.07	102.6	165.7	1140.7	10305.2	100663.6
$\tilde{\nu_y}$	10^{-20}	20.59	23.96	42.40	102.6	165.8	1140.7	10305.2	100663.6
ν_y	10^{-30}	24.48	29.68	52.43	118.7	185.6	1181.9	10392.9	100851.4
$\tilde{\nu_y}$	10^{-30}	29.00	32.56	53.26	118.9	185.6	1181.9	10392.9	100851.4