Preconditioned Iterative Methods for Nonselfadjoint or Indefinite Elliptic Boundary Value Problems[1]

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PRECONDITIONED ITERATIVE METHODS FOR NONSELFADJOINT OR INDEFINITE ELLIPTIC BOUNDARY VALUE PROBLEMS

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We consider a Galerkin-Finite Element approximation to a general linear elliptic boundary value problem which may be nonselfadjoint or indefinite. We show how to precondition the equations so that the resulting systems of linear algebraic equations lead to iteration procedures whose iterative convergence rates are independent of the number of unknowns in the solution.

1. INTRODUCTION.

In recent years, the application of iterative methods to preconditioned linear systems has been extremely successful in a variety of complex physical applications [3,16]. Many articles are available in the literature which report on the favorable performance of such methods [3,6,10,12].

The two aspects of a resulting algorithm consist of the preconditioner and the underlying iterative method [1,8,12]. Various iterative methods, the most popular being the conjugate gradient (CG) and certain normal forms of the CG method, have been considered extensively both from a theoretical and an experimental viewpoint (see [10] and the references therein). It has been demonstrated that, in general, iterative algorithms with the same theoretical convergence rates

converge, in practice, at about the same rate[1]. The question of choosing an appropriate preconditioner is much more difficult. The preconditioner must in some way be similar to the inverse of the system which is being solved. Consequently, the evaluation of the preconditioner usually requires the solution of a system of equations and so if the method is to result in an improvement of computational efficiency, the preconditioner must have some property which makes it easier to solve than the original system. The iterative convergence rate of the algorithm is extremely sensitive to the choice of preconditioner. Indeed, the choice of a more appropriate preconditioner may reduce the number of iterations by an order of magnitude or more in a given problem.

In this paper we illustrate some techniques for analysing preconditioned iterative methods for nonsymmetric problems. We will discuss the problem of choosing an appropriate preconditioner and study two different iterative algorithms. Typical finite element discretization of an elliptic boundary value problem leads to a matrix problem

$$Mc = d. \tag{1.1}$$

where M is the "stiffness" matrix associated with the discretization and is nonsingular and c and d are vectors. We seek a preconditioner M_1^{-1} such that M_1 is symmetric positive definite, $(M_1)^{-1}$ is easier to compute than $(M)^{-1}$, and $(M_1)^{-1}$ "approximates in some sense" $(M)^{-1}$. System (1.1) can of course be replaced by the equivalent system

$$M^t M_1^{-1} M_1^{-1} Mc = M^t M_1^{-1} M_1^{-1} d . \tag{1.2}$$

The matrix $M' \equiv M^t M_1^{-1} M_1^{-1} M$ is symmetric positive definite and the first algorithm is defined by applying the conjugate gradient method to (1.2). Alternatively, (1.1) is equivalent to the problem

$$M_1^{-1} M^t M_1^{-1} Mc = M_1^{-1} M^t M_1^{-1} d. \tag{1.3}$$

1) The number of iterations to reach a desired accuracy may vary by at most a factor of five [6,10].

The matrix $M'' \equiv M_1^{-1} M^t M_1^{-1} M$ although not usually symmetric, is a symmetric operator with respect to the inner product defined by

$$\langle\langle w,v \rangle\rangle = (M_1 w) \cdot v \ .$$

The CG method can be applied to (1.3) in the $\langle\langle \cdot,\cdot \rangle\rangle$ inner product and leads to Algorithm II of Section 2. Our analysis suggests that the preconditioned iterative method based on (1.3) is more robust than that based on (1.2) since results for (1.2) require additional hypotheses. In fact, we have not been able to obtain results for the scheme based on (1.2) unless the elements used in the methods are of "quasi-uniform" size.

We shall present two general theorems which can be used to derive certain discrete stability estimates. Such estimates lead to bounds on the iterative convergence rates of algorithms for finding the solution of matrix equations resulting from the finite element discretization of elliptic boundary value problems which may be nonsymmetric and/or indefinite. We show how these general results can be applied in a finite element approximation to the Poincaré problem. Both strategies depend upon a priori stability estimates for the continuous problem and use the approximation properties of the discretization to derive the stability estimate for the matrix problems.

The first theorem leads to a strategy which uses a positive definite symmetric problem as a preconditioner for a more complicated nonsymmetric and/or indefinite problem. The problem of the efficient solution of positive definite problems, although not completely solved, has been extensively researched. For example, matrices corresponding to positive definite symmetric problems often have certain diagonal dominance properties which imply that various sparse matrix packages [9,11] can be used for their solution. Also, there are "fast solver" algorithms available for certain elliptic problems on a variety of domains [5,14,15]. Our analytical results guarantee that the iterative convergence rate for our algorithms is independent of the number of unknowns in the system. Thus the cost of convergence to a given accuracy grows linearly with the size of the problem.

The first strategy is applicable to, for example, problems where the differential operator A can be decomposed into a symmetric positive definite operator L and a compact (but not small) perturbation B. The

operators A, L, and B are approximated by discrete operators A_h, L_h, and B_h derived by finite elements. The discrete approximation to the solution u of the original problem is defined as the solution of

$$(L_h + B_h)U = F. \tag{1.4}$$

Problem (1.4) can be replaced by the equivalent problem

$$L_h^{-1}(L_h + B_h)U = L_h^{-1} F . \tag{1.5}$$

We derive the appropriate stability estimates for (1.5) which guarantee that the CG method applied, with respect to $\langle\langle \cdot,\cdot \rangle\rangle$, to (1.3) converges at a rate independent of the number of unknowns in the discretization. In addition, the stability results yield immediately estimates for the discretization error $u-U$.

We give a second theorem which, under additional hypotheses, provides another stability estimate. This estimate, under a further restriction, can be used to show that the CG method applied to (1.2) converges to the solution of (1.2) at a rate which is independent of the number of unknowns in the discretization.

An outline of the remainder of the paper is as follows. In Section 4 we describe two conjugate gradient algorithms for matrix problems. Section 3 gives some preliminaries and notation to be used in the paper. In Section 4 we state the type of estimates needed to guarantee rapid convergence for some iterative methods for solving nonsymmetric and/or indefinite problems. Two theorems used to derive the stability estimates are given in Section 5. In Section 6 we apply the theorems to a finite element approximation of a general elliptic boundary value problem. Finally in Section 7 we apply a stability estimate to bound the discretization error.

2. CONJUGATE GRADIENT ALGORITHMS.

We describe the algorithms which result from applying the conjugate gradient method to the preconditioned systems (1.2) and (1.3). In either case we assume that we are given an initial approximation c_0 to the solution c of (1.1) and the iterative algorithm produces a sequence of

iterates c_i for $i > 0$. We stop the iterative procedure when the residual error $d-Mc$ becomes sufficiently small. We note that applying the conjugate gradient method to preconditioned systems as illustrated in the following algorithms is not-novel however we include the details for completeness.

Applying the conjugate gradient method to (1.2) gives the following algorithm:

ALGORITHM I. $M' = M^t M_1^{-1} M_1^{-1} M$

(1) Define $r_0 = p_0 = M^t M_1^{-1} M_1^{-1} (d-Mc_0)$.

(2) For $i \geq 0$ define

$$\alpha_i = \frac{r_i \circ p_i}{(M' p_i) \circ p_i}$$

$$c_{i+1} = c_i + \alpha_i p_i$$

$$r_{i+1} = r_i - \alpha_i M' p_i$$

$$\beta_i = \frac{(M' r_{i+1}) \circ p_i}{(M' p_i) \circ p_i}$$

$$p_{i+1} = r_{i+1} - \beta_i p_i \quad .$$

Applying the conjugate gradient method in the $\langle\langle \cdot, \cdot \rangle\rangle$ inner product to (1.3) gives the following algorithm:

ALGORITHM II. $M'' = M_1^{-1} M^t M_1^{-1} M$.

(1) Define $r_0 = p_0 = M_1^{-1} M^t M_1^{-1}(d-Mc_0)$.

(2) For $i \geq 0$ define

$$\alpha_i = \frac{(M_1 r_i) \circ p_i}{(M_1 p_i) \circ (M'' p_i)}$$

$$c_{i+1} = c_i + \alpha_i p_i$$

$$r_{i+1} = r_i - \alpha_i M'' p_i$$

$$\beta_i = \frac{(M^{-1} r_{i+1}) \circ (M'' p_i)}{(M^{-1} p_i) \circ (M'' p_i)}$$

$$p_{i+1} = r_{i+1} - \beta_i p_i \ .$$

3. PRELIMINARIES AND NOTATION.

Throughout this paper we shall be concerned with solving boundary value problems on a bounded domain Ω contained in R^2 with boundary Γ. To state our stability estimates, we shall make use of various spaces of functions defined on Ω. The space $L^2(\Omega)$ is the collection of square integrable functions on Ω; that is, a function $f(x)$ defined for (x,y) in Ω is in $L^2(\Omega)$ if

$$\int_\Omega f(x,y)^2 \, dxdy < \infty \ .$$

The $L^2(\Omega)$ inner product is defined by

$$(f,g) \equiv \int_\Omega f(x,y)\, g(x,y) dxdy \quad \text{for} \quad f, g \in L^2(\Omega).$$

We shall also use the Sobolev space $H^1(\Omega)$. Loosely, a function f is in $H^1(\Omega)$ if f, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are all in $L^2(\Omega)$. Thus for functions in $H^1(\Omega)$, we can define the Dirichlet form by

$$D(f,g) \equiv \int_\Omega (\frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y}) dxdy \ .$$

We shall also denote the $L^2(\Gamma)$ inner product by

$$\langle f,g \rangle \equiv \int_\Gamma fg \, ds \ .$$

For any positive integer r, the Sobolev space of $L^2(\Omega)$-functions whose r^{th} order partial derivatives belong to $L^2(\Omega)$ will be denoted by $H^r(\Omega)$.

We also let C and C_i for $i \geq 0$ denote positive constants. The values of C and C_i may be different in different places however C and C_i shall always be independent of the mesh parameter h defining

the approximation method. Thus C and C_i will always be independent of the number of unknowns in the discretization.

To define the approximation of later sections we shall need a collection of finite element approximation subspaces $\{S_h\}$, $0 < h \leq 1$, contained in $H^1(\Omega)$. Typically, finite element approximation subspaces are defined by partitioning the domain Ω into subregions of size h and defining S_h to be the set of functions which are continuous on Ω and piecewise polynomial when restricted to the subregions (see [4,7,17] for details). For example, one could partition Ω into triangles of size h and define S_h to be the functions which are continuous on Ω and linear on each of the triangles. Alternatively, Ω could be partitioned into rectangles and S_h could be defined to be the functions which are continuous on Ω and bilinear on each of the rectangles.

4. ESTIMATES FOR THE CONJUGATE GRADIENT METHOD.

Our analysis of iterative algorithms for preconditioned systems is based on stability estimates for the continuous or nondiscrete problem and the error estimates between the continuous solutions and their discrete approximations. To study the properties of the solutions of boundary value problems in partial differential equations, it is natural to consider operators in their basis free representations since complete sets of basis functions are usually too complex to be of much practical value. Consequently, it is natural to think of the process of solving for the discrete solution of the finite element equations as a basis free operator on the finite element subspace S_h of $H'(\Omega)$. We represent differential and solution operators by the notation A, B, L, or T whereas their discrete counterparts shall be respectively denoted A_h, B_h, L_h and T_h.

The CG method can be applied to find the solution X of the problem

$$L_h X = Y \tag{4.1}$$

where L_h is a symmetric positive definite operator with respect to some inner product (cf. [13]). The CG algorithm requires an initial guess X_0 and produces an approximation X_n to X after n iterative steps. It is

well known that

$$\|X-X_n\|_H \leq 2\left(\frac{\sqrt{\gamma}-1}{\sqrt{\gamma}+1}\right)^n \|X-X_n\|_H \quad (4.2)$$

where γ is the condition number for L_h and is defined to be the ratio of the largest eigenvalue of L_h to the smallest. We note that if L_h satisfies the inequality

$$C_0 \|W\|_H^2 \leq (L_h W, W)_H \leq C_1 \|W\|_H^2 \quad \text{for all } W \in S_h, \quad (4.3)$$

where $(\cdot,\cdot)_H$ denotes the H-inner product, then the condition number γ is bounded by C_1/C_0. Thus estimates of the type (4.3) in conjunction with (4.2) lead to convergence estimates for the CG method applied to (4.1).

The problem of finding the finite element solution in the examples of later sections can be reduced to solving for the solution X of a nonsingular operator equation

$$A_h X = Y \quad (4.4)$$

where A_h is a nonsymmetric and/or nonpositive operator on S_h. We shall first precondition the system, multiply by the adjoint and then apply the CG method in the appropriate inner product.

We assume that we have a symmetric positive definite operator T_h defined on S_h for a preconditioner. The types of preconditioners for which we can get analytic results will be described in later sections.

We note that problem (4.4) can be replaced by the problem of finding X in S satisfying

$$A_h^* T_h T_h A_h X = A_h^* T_h T_h Y \quad (4.5)$$

where A_h^* is the $L^2(\Omega)$ - adjoint of A_h. The CG method with respect to the $L^2(\Omega)$ inner product can be used to solve (4.5). The convergence rate of the resulting algorithm is bounded by (4.2) in the $L^2(\Omega)$ norm where γ is bounded by C_1/C_0 for any C_0 and C_1 satisfying

$$(4.6) \qquad C_0\|W\|^2_{L^2(\Omega)} \le \|T_h A_h W\|^2_{L^2(\Omega)} \le C_1\|W\|^2_{L^2(\Omega)} \quad \text{for all } W \in S_h .$$

In certain applications, estimate (4.6) can be used to derive bounds on the iterative convergence rate of Algorithm I.

Alternatively, problem (4.4) is also equivalent to the problem of finding X in S_h satisfying

$$(4.7) \qquad T_h A_h^* T_h A_h X = T_h A_h^* T_h Y .$$

The operator $B \equiv T_h A_h^* T_h A_h$ is symmetric positive definite in the inner product $(T_h^{-1} W, V)$. Applying the CG method to the solution of (4.7) in this inner product gives an algorithm which converges at a rate described by (4.2) where $\gamma \le C_1/C_0$ for any C_0 and C_1 satisfying

$$(4.8) \quad C_0(T_h^{-1} W,W) \le (T_h A_h W, A_h W) \le C_1(T_h^{-1} W,W) \quad \text{for all } W \in S_h .$$

In applications, estimate (4.8) is used to derive iterative convergence rates for Algorithm II.

5. STABILITY THEOREM.

In this section we give general results which can be used to derive estimates of the form (4.6) and (4.8).

Theorem 1. Let R be a continuous operator and R_h be its discrete approximation. Assume that the following stability and error estimates hold:

$$(5.1) \quad \|\theta\|_{H^1(\Omega)} \le C\{\|(I+R_h)\theta\|_{H^1(\Omega)} + \|\theta\|_{L^2(\Omega)}\} \quad \text{for all } \theta \in S_h.$$

For any $\varepsilon > 0$ there exists C_ε such that

$$(5.2) \quad \|\phi\|_{L^2(\Omega)} \le C_\varepsilon \|(I+R)\phi\|_{L^2(\Omega)} + \varepsilon\|\phi\|_{H^1(\Omega)} \quad \text{for all } \phi \in H^1(\Omega).$$

$$(5.3) \quad \|(R-R_h)\phi\|_{L^2(\Omega)} \le Ch \|\phi\|_{H^1(\Omega)} \quad \text{for all } \phi \in H^1(\Omega).$$

Then there exists $h_0 > 0$ such that for $h < h_0$

$$\|\theta\|_{H^1(\Omega)} \leq C\|(I+R_h)\theta\|_{H^1(\Omega)} \quad \text{for all} \quad \theta \in S_h . \tag{5.4}$$

Remark 1. Estimate (5.4) combined with

$$\|(I+R_h)\theta\|_{H^1(\Omega)} \leq C\|\theta\|_{H^1(\Omega)} \quad \text{for all} \quad \theta \in S_h \tag{5.5}$$

guarantees a uniform (independent of h) iterative convergence rate for the CG iteration for the solution of

$$(I+R_h)^* (I+R_h)U = F$$

where $*$ denotes the adjoint with respect to the $H^1(\Omega)$ inner product. In our finite element applications, $I+R_h = T_h A_h$ and

$$C_0\|\theta\|^2_{H^1(\Omega)} \leq (T_h^{-1}\theta,\theta) \leq C_1\|\theta\|^2_{H^1(\Omega)} \quad \text{for all} \quad \theta \in S_h .$$

Thus (5.4) and (5.5) will imply (4.8) for the particular examples of the next section.

Theorem 2. Let T^1 and T^2 be continuous operators and T_h^1 and T_h^2 be their corresponding discrete approximations. Assume that the following three estimates hold:

$$C_0\|T^1 u\|_{L^2(\Omega)} \leq \|T^2 u\|_{L^2(\Omega)} \leq C_1\|T^1 u\|_{L^2(\Omega)} \quad \text{for all} \quad u \in L^2(\Omega). \tag{5.6}$$

$$\|(T^i - T_h^i) u\|_{L^2(\Omega)} \leq Ch^2 \|u\|_{L^2(\Omega)} \quad \text{for all} \quad u \in L^2(\Omega) . \tag{5.7}$$

$$\|(T_h^i)^{-1} U\|_{L^2(\Omega)} \leq Ch^2\|U\|_{L^2(\Omega)} \quad \text{for all} \quad U \in S_h . \tag{5.8}$$

for $i = 1,2$. Then

$$C_0\|U\|^2_{L^2(\Omega)} \leq \|T_h^2(T_h^1)^{-1} U\|^2_{L^2(\Omega)} \leq C_1\|U\|^2_{L^2(\Omega)} \quad \text{for all} \quad U \in S_h . \tag{5.9}$$

Remark 2. Estimate (5.8) is an inverse property for the operator T_h^1 and in applications is derived from the hypothesis that the mesh elements are of "quasi uniform" size. Estimate (5.9) coincides with (4.6) when $A_h = \left(T_h^1\right)^{-1}$.

Remark 3. The proofs of the above two theorems are simple and consequently will not be included.

6. THE POINCARÉ PROBLEM.

To illustrate our approach we consider a finite element approximation of the Poincaré problem in this section. We consider the following model problem:

$$
\begin{aligned}
&-\Delta u + \frac{\partial u}{\partial x} + Ku = f \quad \text{in} \quad \Omega \\
&\frac{\partial u}{\partial \eta} + \beta \frac{\partial u}{\partial \tau} + \gamma u = 0 \quad \text{on} \quad \Gamma
\end{aligned}
\tag{6.1}
$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, η and τ are respectively the normal and tangential directions along Γ. For simplicity we have considered constant coefficients in defining the differential equation as well as the boundary condition. Our results and iterative algorithms extend to variable coefficient problems without any complications. We also assume that the solution of (6.1) exists and is unique.

The finite element approximation to (6.1) can then be defined by the Galerkin technique. Multiplying (6.1) by an arbitrary function ϕ and integrating by parts shows that the solution u satisfies

$$
D(u,\phi) + \left(\frac{\partial u}{\partial x},\phi\right) + K(u,\phi) + \left\langle \beta \frac{\partial u}{\partial \tau} + \gamma u, \phi \right\rangle = (f,\phi) \,. \tag{6.2}
$$

The finite element approximation U to u is then defined to be the function U in S_h which satisfies

$$
D(U,\theta) + \left(\frac{\partial U}{\partial x},\theta\right) + K(U,\theta) + \left\langle \beta \frac{\partial U}{\partial \tau} + \gamma U, \theta \right\rangle = (f,\theta) \quad \text{for all} \quad \theta \in S_h . \tag{6.3}
$$

Equation (6.3) can be used to derive a system of equations of the form (1.1) defining the discrete solution U, i.e., using a basis for S_h, (6.3) gives N equations for the N unknowns defining U in that basis.

To describe iterative methods for the solution of (6.3) and/or the corresponding matrix system, we shall need to use some operator notation. First, we consider the Neumann problem

$$\begin{aligned} w - \Delta w &= f \quad \text{in} \quad \Omega \\ \frac{\partial w}{\partial \eta} &= 0 \quad \text{on} \quad \Gamma \end{aligned} \tag{6.4}$$

Given a function f in $L^2(\Omega)$, the solution w of (6.4) is in $H^2(\Omega)$ if as we shall assume, Γ is sufficiently smooth. We denote the solution operator T as the map which takes f to $Tf \equiv w$. T is a bounded map of $L^2(\Omega)$ into $H^2(\Omega)$. The finite element approximation to (6.4) is the function W in S_h satisfying

$$D(W,\theta) + (W,\theta) = (f,\theta) \quad \text{for all} \quad \theta \in S_h \,. \tag{6.5}$$

The discrete solution operator T_h can then be defined as the map which takes f to $T_h f \equiv W$. T_h is a map from $L^2(\Omega)$ onto S_h and the following convergence estimate is well known (cf. [2]):

$$\|(T_h - T)f\|_{H^1(\Omega)} \leq Ch\|f\|_{L^2(\Omega)} \,. \tag{6.6}$$

In a similar manner, we can define solution operators for the following variational problems:

$$D(\chi,\phi) + (\chi,\phi) = (\frac{\partial z}{\partial x},\phi) + (k-1)(z,\phi)$$

and

$$D(\psi,\phi) + (\psi,\phi) = \langle \beta \frac{\partial \omega}{\partial \tau} + \gamma\omega, \phi \rangle \,.$$

We define the solution operators $R^1 z \equiv \chi$ and $R^2 \omega \equiv \psi$. The corresponding

finite element approximations are given by the solutions X and Y in S_h satisfying

$$D(X,\theta) + (X,\theta) = (\frac{\partial z}{\partial x},\theta) + (K-1)(z,\theta) \quad \text{for all} \quad \theta \in S_h$$

and

$$D(Y,\theta) + (Y,\theta) = \langle \beta \frac{\partial \omega}{\partial \tau} + \gamma\omega,\theta\rangle \quad \text{for all} \quad \theta \in S_h ,$$

respectively. The discrete solution operators are then defined by $R_h^1 z \equiv X$ and $R_h^2 \omega \equiv Y$ and the following convergence estimates hold:

$$\|(R_h^1 - R^1)z\|_{L^2(\Omega)} \leq Ch\|z\|_{H^1(\Omega)} . \tag{6.7}$$

and

$$\|(R_h^2 - R^2)\omega\|_{L^2(\Omega)} \leq Ch\|\omega\|_{H^1(\Omega)} . \tag{6.8}$$

In terms of operators, problem (6.1) is equivalent to

$$(I + R^1 + R^2)u \equiv TA\, u = Tf .$$

The existence and uniqueness properties of solutions of (6.1) can be used to show that for any $\varepsilon > 0$ there is a constant C_ε such that

$$\|\phi\|_{L^2(\Omega)} \leq C_\varepsilon \|(I+R^1+R^2)\phi\|_{L^2(\phi)} + \varepsilon\|\phi\|_{L^2(\Omega)} . \tag{6.9}$$

The discrete estimate

$$\|\theta\|_{H^1(\Omega)} \leq C\{\|(I+R_h^1+R_h^2)\theta\|_{H^1(\Omega)} + \|\theta\|_{L^2(\Omega)}\} \quad \text{for all} \quad \theta \in S_h \tag{6.10}$$

is immediate from the definition of R_h^i . Problem (6.3) can be stated in terms of operators as

$$(I + R_h^1 + R_h^2)U \equiv T_h A_h\, U = T_h\, f .$$

Applying Theorem 1 we get the following stability estimate:

$$(6.11) \qquad C_0\|W\|^2_{H^1(\Omega)} \le \|T_h A_h W\|^2_{H^1(\Omega)} \le C_1\|W\|^2_{H^1(\Omega)} \quad \text{for all } W \in S_h.$$

The second inequality in (6.11) can be easily derived from the definitions. The constants C_0 and C_1 in (6.10) are independent of the mesh size h. Now it is easy to check that

$$(6.12) \qquad (T_h^{-1} W,V) = D(W,V) + (W,V) \quad \text{for all } W,V \in S_h .$$

Comparing (6.12), (6.11), (4.7) and (4.8) implies that the CG method applied to

$$(6.13) \qquad T_h A_h^* T_h A_h U = T_h A_h^* T_h f$$

converges with a reduction per iteration which can be bounded independently of the number of unknowns.

Let M and M_1 respectively denote the "stiffness" matrices corresponding to (6.3) and (6.5) in a given basis $\mathcal{B} = \{\phi_i\}_{i=1}^N$ for S_h. If the coefficients of a function W in S_h in terms of the basis $\mathcal{B}$ are represented by the vector c then

$$d = M_1^{-1} M^t M_1^{-1} Mc$$

gives the coefficients of $T_h A_h^* T_h A_h W$ in terms of $\mathcal{B}$. Consequently, the sequence of vectors c_i generated by Algorithm II gives the coefficients of the sequence of functions generated by the CG method applied to (6.13). Thus the iterative convergence estimates for the CG method applied to (6.13) imply iterative convergence rates for Algorithm II.

The above procedure is an example of an iterative convergence analysis in $H^1(\Omega)$. We also note that if T_h^1 is another discrete operator on S_h which is spectrally equivalent to T_h in the sense that

$$(6.14) \qquad C_0(T_h W,W) \le (T_h^1 W,W) \le C_1(T_h W,W) \quad \text{for all } W \in S_h$$

then T_h can be replaced by T_h^1 in (6.11).

We next consider an iterative analysis in $L^2(\Omega)$ based on Theorem 2. Let $T^1: L^2(\Omega) \to H^2(\Omega)$ denote the solution operator for problem (6.1) with $\beta = 0$, i.e., $T^1 f \equiv \tilde{u}$. The solution operator T^1 satisfies an estimate of the form

$$C_0 \|T^1 f\|_{L^2(\Omega)} \leq \|Tf\|_{L^2(\Omega)} \leq C_1 \|T^1 f\|_{L^2(\Omega)} . \tag{6.15}$$

We have restricted to the case of $\beta = 0$ since (6.15) is well known in that case. Assume that both T^1 and T can be approximated in the same finite element subspaces and let T_h^1 and T_h denote the corresponding discrete solution operators. The following convergence estimates are well known for a wide class of finite element applications [27]:

$$\|(T_h^i - T^i)f\|_{L^2(\Omega)} \leq Ch^2 \|f\|_{L^2(\Omega)} . \tag{6.16}$$

We finally assume that the inverse properties

$$\|(T_h^i)^{-1}\theta\|_{L^2(\Omega)} \leq Ch^{-2} \|\theta\|_{L^2(\Omega)}, \quad \theta \in S_h, \tag{6.17}$$

are also satisfied. Estimates of the type (6.17) can usually be derived from inverse assumptions for the subspaces. Applying Theorem 2 gives that

$$C_0 \|W\|_{L^2(\Omega)} \leq \|T_h (T_h^1)^{-1} W\|_{L^2(\Omega)} \leq C_1 \|W\|_{L^2(\Omega)} \quad \text{for all } W \in S_h . \tag{6.18}$$

Estimate (6.18) guarantees that the CG method applied in $L^2(\Omega)$ for the solution of

$$A_h^* T_h T_h A_h X = A_h^* T_h T_h f \tag{6.19}$$

where $A_h = (T_h^1)^{-1}$, will converge to the solution X at a rate which is independent of the number of unknowns in S_h. The resulting algorithm does not however correspond to Algorithm I. To guarantee rapid iterative

convergence rates for Algorithm I we must make additional assumptions. Again we use the basis $\mathfrak{B}$ for S_h. If $W \in S_h$ we denote by C_W the coefficients of W in the basis $\mathfrak{B}$. We require that

$$C_0(C_W \cdot C_W) \leq (W,W)_{L^2(\Omega)} \leq C_1(C_W \cdot C_W) \quad \text{for all} \quad W \in S_h . \tag{6.20}$$

Estimate (6.20) states that the Gram or mass matrix is "equivalent" to the coordinate inner product. Combining (6.19) and (6.20) implies

$$C_0 |c|^2 \leq |M_1^{-1} M c|^2 \leq C_1 |c|^2 \tag{6.21}$$

for all N dimensional vectors c. Estimate (6.21) is finally an estimate which can be applied to guarantee uniform iterative convergence rates for Algorithm I.

7. AN ESTIMATE FOR THE DISCRETIZATION ERROR.

In order to estimate the discretization error $u-U$ with u and U defined by (6.2) and (6.3) respectively, we introduce the $H^1(\Omega)$-projection P_h onto S_h. It is defined for $v \in H^1(\Omega)$ by

$$D(P_h v,\theta) + (P_h v,\theta) = D(v,\theta) + (v,\theta), \quad \text{for all} \quad \theta \in S_h . \tag{7.1}$$

It is well known that P_h satisfies

$$\|(I-P_h)v\|_{H^1(\Omega)} \leq Ch^{r-1}\|v\|_{H^r(\Omega)} \tag{7.2}$$

for $v \in H^r(\Omega)$ and some $r > 1$ which depends on the choice of S_h(cf. [2,7]). In view of (7.2), to estimate $u-U$ we need only consider $P_h u-U$. Hence we apply (5.4) to obtain

$$\|P_h u-U\|_{H^1(\Omega)} \leq C\|(I+R_h)(P_h u-U)\|_{H^1(\Omega)},$$

with $R_h = R_h^1 + R_h^2$. From the definitions of R^1, R_h^1, R^2 and R_h^2 we see that

$$(I+R_h)(P_h u-U) = P_h(R^1+R^2)(P_h-I)u .$$

Hence

$$\|P_h u - U\|_{H^1(\Omega)} \leq C\|P_h(R^1+R^2)(P_h-I)u\|_{H^1(\Omega)}$$

from which it follows immediately that

$$\|P_h u - U\|_{H^1(\Omega)} \leq C\|(I-P_h)u\|_{H^1(\Omega)} \quad (7.3)$$

Thus using (7.2) we obtain the estimate for the discretization error,

$$\|u-U\|_{H^1(\Omega)} \leq Ch^{r-1}\|u\|_{H^r(\Omega)}$$

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