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THE MATHEMATICAL FOUNDATIONS OF THE
FINITE ELEMENT METHOD WITH APPLICATIONS
TO PARTIAL DIFFERENTIAL EQUATIONS

#11

COMPUTATIONAL ASPECTS OF THE FINITE ELEMENT METHOD*

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1. Introduction.

In this paper we consider a particular implementation of the finite element procedure for approximating the solution $u(x,y)$ of the boundary value problem

$$(1.1)\qquad -D_x[p(x,y)D_xu] - D_y[q(x,y)D_yu] + c(x,y)u = f(x,y)$$

for all $(x,y) \in S \equiv \{(x,y) \mid 0 < x , y < 1\}$, and

$$(1.2)\qquad u(x,y) = 0$$

for all $(x,y) \in \partial S \equiv$ the boundary of S . We assume that the functions $p(x,y)$, $q(x,y)$, $c(x,y)$, and $f(x,y)$ are smooth and that there exists a positive constant γ such that

$$(1.3)\qquad \gamma \leq p(x,y) , \quad \gamma \leq q(x,y) , \quad 0 \leq c(x,y)$$

for all $(x,y) \in S$.

Our finite element procedure uses basis functions consisting of piecewise bicubic Hermite polynomials defined

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on a mesh which is refined (in a well-defined manner) in a neighborhood of each corner. The coefficients and right-hand side of the resulting linear algebraic system of equations involve integrals (over two-dimensional rectangular elements) which are approximated by the local nine-point product Gaussian quadrature scheme (the tensor product of the one-dimensional three-point Gaussian quadrature schemes). Finally, the approximate linear algebraic system of equations is symmetric and positive definite and is solved by either the band Cholesky or profile Cholesky decomposition procedure.

In sections 2 and 3, we will give theoretical justifications for our procedure. We will show that asymptotically our procedure is far more efficient than the combination of the five-point central difference approximation and SOR (successive overrelaxation) cf. [2]. Moreover, for constant coefficient problems, i.e., $p(x,y)$, $q(x,y)$, and $c(x,y)$ are constants in S, it is asymptotically more efficient than the five-point central difference approximation coupled with the fast direct methods, cf. [2]. In fact, a version of our procedure with a slightly less efficient quadrature scheme has been shown to be asymptotically optimal in a well-defined sense. In section 4, we will present the results of some numerical experiments which demonstrate the effectiveness of our procedure for even modest sized problems. The results of this paper can be directly generalized to other boundary value problems and other piecewise polynomial basis functions. The complete details will be given elsewhere.

We introduce the tensor product mesh $\rho = \Delta_x \times \Delta_y$ where $\Delta_x : 0 = x_0 < x_1 < \dots < x_n = 1$ and $\Delta_y : 0 = y_0 < y_1 < \dots < y_n = 1$. Moreover, for each $1 \leq i \leq n-1$, we define the basis functions

$$(1.4) \qquad \phi_i(x) \equiv \begin{cases} -2(x_i-x_{i-1})^{-3}(x-x_{i-1})^3 \\ \quad + 3(x_i-x_{i-1})^{-2}(x-x_{i-1})^2 , & x \in [x_{i-1},x_i] , \\ 2(x_{i+1}-x_i)^{-3}(x-x_i)^3 \\ \quad - 3(x_{i+1}-x_i)^{-2}(x-x_i)^2 + 1 , & x \in [x_i,x_{i+1}] , \\ 0 , & x \in [0,x_{i-1}] \cup [x_{i+1},1] , \end{cases}$$

$$(1.5) \qquad \xi_i(x) \equiv \begin{cases} (x_i-x_{i-1})^{-2}(x-x_{i-1})^2(x-x_i) , & x \in [x_{i-1},x_i] , \\ (x_{i+1}-x_i)^{-2}(x-x_i)(x_{i+1}-x)^2 , & x \in [x_i,x_{i+1}] , \\ 0 , & x \in [0,x_{i-1}] \cup [x_{i+1},1] , \end{cases}$$

$$(1.6) \qquad \xi_0(x) \equiv \begin{cases} x_1^{-2}x(x_1-x)^2 , & x \in [0,x_1] , \\ 0 , & x \in [x_1,1] , \end{cases}$$

and

$$(1.7)\qquad \xi_n(x) \equiv \begin{cases} (1-x_{n-1})^{-2}(x-x_{n-1})^2(x-1) \ , & \\ & x \in [x_{n-1},1] \ , \\ 0 \ , & x \in [0,x_{n-1}] \ . \end{cases}$$

Similarly, we define $\phi_i(y)$, $1 \leq i \leq n-1$, and $\xi_i(y)$, $0 \leq i \leq n$. Our piecewise bicubic Hermite basis functions are defined to be $\{\phi_i(x)\phi_j(y)$, $\phi_i(x)\xi_k(y)$, $\xi_\ell(x)\phi_j(y)$, $\xi_\ell(x)\xi_k(y) \mid 1 \leq i , j \leq n-1$ and $0 \leq k , \ell \leq n\}$.

There are $4n^2$ such basis functions. Each interior mesh point (x_i,y_j) , $1 \leq i , j \leq n-1$, has four basis functions associated with it; each corner mesh point has one basis function associated with it; and each of the other boundary mesh points has two basis functions associated with it.

If we consecutively order the basis functions associated with each mesh point of ρ and consecutively order the mesh points along rows, we may rename our basis functions as $\{B_i(x,y) \mid 1 \leq i \leq 4n^2 \equiv m\}$ and seek our approximation of the form

$$w(x,y) \equiv \sum_{i=1}^{m} \beta_i B_i(x,y) \quad ,$$

We determine the vector of coefficients $\underline{\beta} \in R^m$ by means of the Rayleigh-Ritz-Galerkin procedure, cf. [12]. This leads to a characterization of $\underline{\beta}$ as the solution of the $m \times m$ linear system

$$(1.8)\qquad A\underline{\beta} = \underline{k} \quad ,$$

where

$$A \equiv [a_{ij}] \equiv \left[\int_o^1 \int_o^1 \{p(x,y) D_x B_i D_x B_j + q(x,y) D_y B_i D_y B_j + c(x,y) B_i B_j\} dxdy\right]$$

and

$$\underline{k} \equiv [k_i] \equiv \left[\int_o^1 \int_o^1 f(x,y) B_i \, dxdy\right] .$$

The matrix A is symmetric, positive definite and hence the linear system has a unique solution.

Because of the local nature of the basis functions, each entry of A and $\underline{k}$ is the sum of integrals over at most four contiguous elements. In place of (1.8), we consider the approximate system

$$\tilde{A}\underline{\tilde{\beta}} = \underline{\tilde{k}} \quad , \tag{1.9}$$

where the entries of $\tilde{A}$ and $\underline{\tilde{k}}$ are obtained from the corresponding entries of A and $\underline{k}$ by using the nine-point product Gaussian quadrature scheme over each element $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $0 \leq i$, $j \leq n - 1$. To be precise, every integral of the form

$$\int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} g(x,y) dxdy$$

is approximated by

$$\frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{4} \sum_{k=1}^{3} \sum_{\ell=1}^{3} w_k w_\ell \times$$

$$\times\ g\left(\frac{1}{2}[x_{i+1}-x_i]\theta_k + \frac{1}{2}[x_{i+1}+x_i], \frac{1}{2}[y_{j+1}-y_j]\theta_\ell + \frac{1}{2}[y_{j+1}+y_j]\right)$$

where

$$(1.10) \qquad w_1 = w_3 = 5/9 \ , \quad w_2 = 8/9 \ ,$$

and

$$(1.11) \qquad \theta_1 = -0.774597 \ , \quad \theta_2 = 0 \ , \quad \theta_3 = 0.774597 \ .$$

We will show in section 2 that the matrix $\tilde{A}$ is symmetric positive definite and hence that (1.9) has a unique solution.

The matrix $\tilde{A}$ is a band matrix with band width $4(n + 2)$. We solve (1.9) by the profile Cholesky decomposition procedure.

2. The asymptotic rate of convergence.

In this section we study the asymptotic rate of convergence of the approximation u_n generated by our finite element procedure to the solution u of the boundary value problem (1.1)-(1.2).

The standard analysis of the finite element method shows that the L^2 error in the finite element approximation is of the same order as the L^2 error in the best approximation to the solution u from the subspace, or equivalently in the interpolant. Thus, for piecewise bicubic Hermite polynomials:

Theorem (cf. [12]): If $u \in W^{k,p}(S)$ $(1 \leq k \leq 4,$ $1 \leq p \leq 2)$, then

$$\|u - u_n\|_{0,2} \leq K\bar{h}^{t+1-(2/p)}$$

where $\bar{h} = \max_{i,j} (x_{i+1} - x_i, y_{j+1} - y_j)$.

In particular, if $u \in W^{4,2}(S)$, as is generally assumed in the literature, then $\|u - u_n\|_{0,2} = O(\bar{h}^4)$. However, even for Poisson's equation

$$\Delta u = f \quad \text{in} \quad S, \quad u = 0 \quad \text{on} \quad \partial S,$$

the solution is likely to have logarithmic singularities in the second derivative at each corner (cf. [9], [13]). In such a case, $u \in W^{3,2-\varepsilon}(S)$ whence $\|u - u_n\|_2 = O(\bar{h}^{3-\varepsilon})$ for any $\varepsilon > 0$, but $u \notin W^{3,2}(S)$ whence $\|u - u_n\|_2 \neq O(\bar{h}^3)$; convergence is no better than third order (see Table I).

Several ideas have been proposed to overcome this loss of accuracy. Babuska [1] has given an overlay technique to locally refine the mesh near each corner, and Fix [4] and Schultz [11] have suggested adding singular corner solutions to the piecewise polynomial basis. But the implementation of these algorithms is rather complicated, the approximate quadrature made more difficult, and the condition number and nonzero structure of the finite element matrix greatly increased. Instead, we use a non-uniform tensor product mesh appropriately graded near the corners which does not create implementation problems versus using a uniform mesh. Under mild assumptions on the rate of growth of high derivatives of the solution u near the corners, fourth order accuracy is restored. We state our result for the case where there is only one singular corner:

Theorem: Let $r = \sqrt{x^2 + y^2}$ and assume that $\|r^{k+\varepsilon-2} D^k u\|_{0,2} \leq K$ for $2 \leq k \leq 4$ $(\varepsilon > 0)$. Then for the mesh $\rho = \Delta_x \times \Delta_y$ with

$$x_i = \left(\frac{i}{n}\right)^{3/2} , \quad y_j = \left(\frac{j}{n}\right)^{3/2} , \quad i,j = 0,\ldots,n$$

we have

$$\|u - u_n\|_{0,2} \leq K'\left(\frac{1}{n}\right)^{4-\varepsilon} .$$

The condition number $K(A)$ of the finite element matrix A is not greatly increased:

Theorem: If the coordinate basis functions $\phi_i(x)$, $\xi_i(x)$, $\phi_i(y)$, $\xi_i(y)$ are locally orthonormalized such that

$$\int_0^1 \phi_i^2 = 1 , \quad \int_0^1 \xi_i^2 = 1 , \quad \int_0^1 \phi_i \xi_i = 0$$

then the condition number of the Gram matrix $G \equiv [g_{ij}]$

$$\equiv \left[\iint_S B_i(x,y) B_j(x,y)\, dxdy\right]$$

is bounded independent of the mesh. Consequently,

$$K(A) \leq K\underline{h}^{-2}$$

where $\underline{h} = \min_{i,j}(x_{i+1}-x_i, y_{j+1}-y_j)$.

Thus for the uniform mesh, $K(A) \sim \mathcal{O}(n^2)$; whereas for the graded mesh, $K(A) \sim \mathcal{O}(n^3)$.

The remaining question to be answered is how to compute the integrals appearing in the coefficients of the finite element matrix A and the right-hand side $\underline{k}$. The integrals must be done numerically but the quadrature rule must be efficient and the approximate finite element matrix $\tilde{A}$ positive definite, yet fourth order convergence must be maintained. Our procedure uses the nine-point product Gaussian quadrature scheme.

<u>Theorem</u>: The matrix $\tilde{A}$ is positive definite.

<u>Proof</u>: Let Q denote the approximate quadrature operator. Given $\underline{\beta} \neq \underline{\theta}$, let

$$w(x,y) = \sum_{i=1}^{4n^2} \beta_i B_i(x,y) \quad .$$

Then

$$\underline{\beta}^T \tilde{A} \underline{\beta} = Q(p(D_x w)^2 + q(D_y w)^2 + cw^2) \geq \gamma Q((D_x w)^2 + (D_y w)^2) \quad .$$

If $\underline{\beta}^T \tilde{A} \underline{\beta} = 0$, then the piecewise bicubic Hermite polynomials $D_x w$, $D_y w$ vanish at the nine quadrature points in each mesh element. From this and the fact that w vanishes on ∂S , we conclude that $w \equiv 0$, a contradiction.

Q.E.D.

A nine point quadrature seems to be minimal; any lower degree scheme experimentally seems not to preserve the high order accuracy. Yet our scheme maintains fourth order convergence: following Fix [5] we introduce the approximate bilinear forms

$$\tilde{a}(u_n,v_n) = Q(pD_x u_n D_x v_n + qD_y u_n D_y v_n + c u_n v_n) \quad ,$$

$$\tilde{b}(f,v_n) = Q(f v_n) \quad ,$$

and note that $\tilde{A}\underline{\tilde{\beta}} = \underline{\tilde{k}}$ if and only if

$$\tilde{a}(\tilde{u}_n,v_n) = \tilde{b}(f,v_n)$$

for all v_n in the subspace, where

$$\tilde{u}_n = \sum_{i=1}^{4n^2} \tilde{\beta}_i B_i \quad .$$

The form $\tilde{a}(u_n,u_n)$ is strongly coercive

$$\tilde{a}(v_n,v_n) \geq c\|v_n\|^2_{1,2}$$

and

$$\begin{aligned}\tilde{a}(u_n-\tilde{u}_n,u_n-\tilde{u}_n) &= \{\tilde{a}(u_n,u_n-\tilde{u}_n) - a(u_n,u_n-\tilde{u}_n)\} \\ &\quad + \{b(f,u_n-\tilde{u}_n) - \tilde{b}(f,u_n-\tilde{u}_n)\} \\ &\leq O(n^{-4})\|u_n - \tilde{u}_n\|_{1,2}\end{aligned}$$

at least for a wide class of nonuniform meshes. Thus

$$\|u_n - \tilde{u}_n\|_{0,2} \leq \|u_n - \tilde{u}_n\|_{1,2} \sim O(n^{-4}) \quad .$$

The result is not true for an arbitrary nonuniform mesh.

3. Operation counts.

In this section, we give operation counts for our finite element procedure. In particular, we analyze the number of arithmetic operations needed (1) to generate the finite element matrix $\tilde{A}$ and the right-hand side $\underline{\tilde{k}}$ by numerical quadrature and (2) to solve the resulting linear system $\tilde{A}\underline{\tilde{\beta}} = \underline{\tilde{k}}$. Throughout this section we should keep in mind the comparable results for the five-point central difference scheme coupled with either SOR (for variable coefficient problems) or the fast direct methods (for constant coefficient problems). By the results of section 2 we know that the number N of mesh points in each coordinate for the five-point difference scheme is comparable to n^2 for comparable accuracies.

We consider the quadratures first. The quadrature question is particularly important since the quality of the implementation of this part of the procedure can easily make a difference of a factor of forty or fifty in the number of arithmetic operations. By symmetry we need compute at most $80n^2$ nonzero entries in $\tilde{A}$ and each such entry is on the average the sum of quadratures over at most two elements. Hence, it suffices to bound the number of arithmetic operations needed for the quadratures over the individual elements. For each fixed element, $[x_i,x_{i+1}] \times [y_j,y_{j+1}]$, let $\alpha_1(x) \equiv \phi_i(x)$, $\alpha_2(x) \equiv \xi_i(x)$, $\alpha_3(x) \equiv \phi_{i+1}(x)$, $\alpha_4(x) \equiv \xi_{i+1}(x)$, $\alpha_5(y) \equiv \phi_j(y)$, $\alpha_6(y) \equiv \xi_j(y)$, $\alpha_7(y) \equiv \phi_{j+1}(y)$, and $\alpha_8(y) \equiv \xi_{j+1}(y)$.

Then, by symmetry, the quantities which we must compute are

$$(3.1)\quad \tilde{a}(\alpha_s\alpha_t, \alpha_r\alpha_z) \ , \quad 1 \leq s \leq r \leq 4 \quad \text{and} \quad 5 \leq t \leq z \leq 8 \ .$$

To compute the quantities (3.1), we first compute for each $1 \leq \ell \leq 3$ and $1 \leq s \leq r \leq 4$

$$(3.2)\quad \psi^p_{sr}(y_{j\ell}) \equiv \left(\frac{x_{i+1} - x_i}{2}\right) \times \sum_{k=1}^{3} w_k p(x_{ik}, y_{j\ell}) D_x\alpha_s(x_{ik}) D_x\alpha_r(x_{ik}) \ ,$$

$$(3.3)\quad \psi^q_{sr}(y_{j\ell}) \equiv \left(\frac{x_{i+1} - x_i}{2}\right) \times \sum_{k=1}^{3} w_k q(x_{ik}, y_{j\ell}) \alpha_s(x_{ik}) \alpha_r(x_{ik}) \ ,$$

and

$$(3.4)\quad \psi^c_{sr}(y_{j\ell}) \equiv \left(\frac{x_{i+1} - x_i}{2}\right) \times \sum_{k=1}^{3} w_k c(x_{ik}, y_{j\ell}) \alpha_s(x_{ik}) \alpha_r(x_{ik}) \ ,$$

where $x_{ik} \equiv \frac{1}{2}[x_{i+1} - x_i]\theta_k + \frac{1}{2}[x_{i+1} + x_i]$, $1 \leq k \leq 3$, and $y_{j\ell} \equiv \frac{1}{2}[y_{j+1} - y_j]\theta_\ell + \frac{1}{2}[y_{j+1} + y_j]$. The computations require approximately a total of $3(10 \cdot 3 \cdot 3 \cdot 2)n^2 = 540n^2$ arithmetic operations for all the elements, where we assume that we have already done the $\mathcal{O}(n)$ operations needed to evaluate the quantities

$$\left(\frac{x_{i+1}-x_i}{2}\right) w_k D_x\alpha_s(x_{ik}) D_x\alpha_r(x_{ik}) \ , \quad \left(\frac{x_{i+1}-x_i}{2}\right) w_k \alpha_s(x_{ik}) \alpha_r(x_{ik})$$

$$1 \leqslant s \leqslant r \leqslant 4 \ , \quad 1 \leqslant k \leqslant 3 \ , \quad 1 \leqslant i \leqslant n-1 \ .$$

Finally we compute

$$\tilde{a}(\alpha_s\alpha_t, \alpha_r\alpha_z) = \frac{(y_{j+1} - y_j)}{2} \times$$

$$\times \sum_{\ell=1}^{3} \{ w_\ell (\psi^p_{sr}(y_{j\ell}) + \psi^c_{sr}(y_{j\ell}))\alpha_t(y_{j\ell})\alpha_z(y_{j\ell})$$

$$+ w_\ell \psi^q_{sr}(y_{j\ell}) D_y\alpha_t(y_{j\ell}) D_y\alpha_z(y_{j\ell}) \}$$

which requires $10 \cdot 10 \cdot 3 \cdot 4 = 1200$ arithmetic operations for each element or $1200n^2$ arithmetic operations for all the elements. Likewise, the number of operations needed to compute $\underline{k}$ is bounded by $180n^2$. Thus, the total of all arithmetic operations to do the quadratures is bounded by $2000n^2$.

Using the results of George, cf. [7], the number of arithmetic operations for the profile Cholesky decomposition procedure is bounded by the number for the band Cholesky decomposition procedure which in turn is bounded by $\sim (4n^2)(4n)^2 = 64n^4$. Thus we have the following table:

Finite Element Procedure

n	operations for quadrature = $2000n^2$	operations for solution = $64n^4$	total
5	50,000	38,000	88,000
10	200,000	640,000	840,000
20	800,000	10,240,000	11,040,000

We see that for coarse meshes most of the operations are used for the quadratures while for moderate or fine meshes most of the operations are used for solving the linear system. To give comparable work estimates for the five-point difference scheme and SOR, we assume perfect knowledge of the optimal relaxation factor and that we iterate until the initial error is reduced by a factor of N^2.

Comparable Five-Point Central Difference Schemes

$N \approx n^2$	SOR† $\left(\frac{2}{3} N^3 \log_2 N\right)$	fast methods $\left(\frac{9}{2} N^2 \log_2 N\right)$
32	100,000+	23,040
128	9,000,000+	504,000
512	1,000,000,000+	16,000,000

cf. [2].

We can see from these figures that our finite element algorithm is much more efficient than the five-point difference scheme and SOR even for coarse meshes and that the factor of improvement becomes larger as the mesh becomes finer.

† Better results can be obtained using SSOR, cf. [14].

Moreover, for constant coefficient problems, we need do only the quadratures to form $\tilde{\underline{k}}$ and we have essentially only the arithmetic operations for solving the system. Hence, our finite element algorithm is more efficient than the fast methods for fine meshes and the factor of improvement becomes larger as the mesh becomes finer.

We should point out that the quadrature work estimates appear to be relatively sharp. However, we can substantially improve the solution work estimates by improving our direct method; e.g., by using George's nested ordering scheme, cf. [8], or by using a fast iterative method, cf. [6].

4. Numerical results.

In this section we present the results of some numerical experiments which illustrate the preceding theoretical results. All computations were performed on a PDP-10 in single precision arithmetic (27 bit mantissa).

Let S denote the unit square $[0,1] \times [0,1]$ in the (x,y)-plane with boundary ∂S. We introduce the tensor product mesh ρ

$$\rho = \Delta \times \Delta \ ; \quad \Delta : 0 = x_0 < x_1 < \dots < x_n = 1$$

where

$$x_k = \frac{1}{2}\left(\frac{2k}{n}\right)^{\beta} , \quad x_{n-k} = 1 - x_k , \quad k = 0,1,\dots, \left[\frac{n+1}{2}\right] .$$

For $\beta = 1$, ρ is the uniform $n \times n$ mesh; for $\beta = \frac{3}{2}$, ρ is the graded mesh referred to in §2. Our basis functions are the piecewise bicubic Hermite polynomials described in §1. The integrals in the finite element matrix $\tilde{A}$ and the right-hand side $\underline{\tilde{k}}$ were computed numerically using the nine-point Gaussian quadrature scheme on each mesh element as described in §1. The system of linear equations $\tilde{A}\underline{\tilde{\beta}} = \underline{\tilde{k}}$ were solved using a band Cholesky routine with the natural ordering of unknowns. The L^2 error estimates were computed numerically using the 25-point product Gaussian quadrature scheme on each mesh element; the L^∞ error estimate is the maximum error at the quadrature points. The estimated rate of convergence is given by

$$\alpha_n = -\frac{\ell n(\varepsilon_n/\varepsilon_{n-1})}{\ell n(n/(n-1))}$$

where ε_n is the estimated error for an $n \times n$ mesh.

Our first example is the Poisson equation:

$$\Delta u = 1 \quad \text{in} \quad S \ ; \quad u = 0 \quad \text{on} \quad \partial S \ .$$

There is no closed form representation for the solution† but the second derivatives are known to have logarithmic singularities at each corner. Thus we would expect $\sim \mathcal{O}(n^{-3})$ convergence using a uniform mesh (see Table I). By grading the mesh near the corners $\left(\beta = \frac{3}{2}\right)$, we improve the convergence to $\sim \mathcal{O}(n^{-4})$ (see Table II).

† A numerical solution was generated by the method of particular solutions (cf. [3], [10]). Using singular particular solutions and taking advantage of the fourfold symmetry, ten terms were sufficient for $\sim 10^{-9}$ accuracy.

By way of comparison, we also present the results of solving the familiar five-point finite difference approximation to the differential equation using the Fast Fourier Transform (FFT) method to solve the difference equations (see Table III). This example is somewhat biased in favor of the finite difference method; the $\mathcal{O}(h^2)$ convergence follows from the fact that the finite difference equations are precisely equivalent to the finite element equations for piecewise linear splines on a triangular mesh and not from the classical analysis. Yet with a graded mesh, the error in the finite element approximation on a 10 × 10 mesh is slightly better than the error in the finite difference approximation on a 128 × 128 mesh.

Our second example is a variable coefficient problem:

$$D_x(e^{xy}D_x u) + D_y(e^{x+y}D_y u) + \frac{1}{1+x+y}\,u = f \quad \text{in} \quad S \quad ,$$

$$u = 0 \quad \text{on} \quad \partial S \quad ,$$

where the right-hand side f is chosen to make the solution

$$u(x,y) = \sin \pi x \cdot \sin \pi y \quad .$$

Both the coefficients and the solution are analytic so that it is not necessary to grade the mesh. The results (see Table IV) indicate fourth order convergence, confirming the fact that the nine-point product Gaussian quadrature scheme is sufficient.

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TABLE I

n	$\|u - u_n\|_2$	α_n	$\|u - u_n\|_\infty$	α_n
1	1.88E-03		4.45E-03	
2	2.81E-04	2.7	6.27E-04	2.7
3	8.03E-05	3.1	2.20E-04	3.1
4	3.24E-05	3.2	1.14E-04	3.2
5	1.61E-05	3.1	7.15E-05	3.1
6	9.18E-06	3.1	4.93E-05	3.1
7	5.72E-06	3.1	3.62E-05	3.1
8	3.81E-06	3.0	2.77E-05	3.0
9	2.66E-06	3.0	2.19E-05	3.0
10	1.94E-06	3.0	1.77E-05	3.0

TABLE II

n	$\|u - u_n\|_2$	α_n	$\|u - u_n\|_\infty$	α_n
1	1.88E-03		4.45E-03	
2	2.81E-04	2.7	6.27E-04	2.7
3	9.73E-05	2.6	2.78E-04	2.6
4	3.86E-05	3.2	9.54E-05	3.2
5	1.83E-05	3.4	5.14E-05	3.4
6	9.51E-06	3.6	2.60E-05	3.6
7	5.43E-06	3.6	1.59E-05	3.6
8	3.30E-06	3.7	9.55E-06	3.7
9	2.12E-06	3.8	6.39E-06	3.8
10	1.42E-06	3.8	4.35E-06	3.8

TABLE III

n	$\|u - u_n\|_2$	$\|u - u_n\|_\infty$
4	1.95E-03	3.36E-03
8	5.27E-04	8.89E-04
16	1.35E-04	2.26E-04
32	3.39E-05	5.67E-05
64	8.87E-06	1.50E-05
128	2.34E-06	4.00E-06

TABLE IV

n	$\|u - u_n\|_2$	α_n	$\|u - u_n\|_\infty$	α_n
1	3.31E-02		6.88E-02	
2	2.41E-03	3.8	8.27E-03	3.8
3	5.98E-04	3.4	2.15E-03	3.4
4	2.10E-04	3.6	8.41E-04	3.6
5	9.05E-05	3.8	3.54E-04	3.8
6	4.50E-05	3.8	1.84E-04	3.8
7	2.47E-05	3.9	9.93E-05	3.9
8	1.47E-05	3.9	6.01E-05	3.9
9	9.24E-06	3.9	3.74E-05	3.9
10	6.10E-06	3.9	2.50E-05	3.9