Multi-Grid Algorithms with Applications to Elliptic Boundary-Value Problems

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ABSTRACT

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This work is primarily concerned with solving the large sparse linear systems which arise in connection with finite-element or finite-difference procedures for solving self-adjoint elliptic boundary-value problems. These problems can be expressed in terms of abstract variational problems on Hilbert spaces. Our (multi-grid) schemes involve a sequence of auxiliary finite-dimensional spaces which do not have to be nested. We approximate the solution using the largest (finite-dimensional) space. These schemes are recursive in nature: they combine smoothing iterations in a space with solving one or more correction problems using smaller spaces. Under certain circumstances, the solution to a problem can be approximated well using smaller spaces. Since the smaller spaces are required to have geometrically fewer unknowns than the largest space, the savings in computation can be substantial. In fact, we prove that these procedures are optimal order under appropriate conditions. Our general theory is discretization independent and can be applied to problems which do not arise from partial differential equations.

As examples, we consider three particular discretizations of variable coefficient self-adjoint second order elliptic boundary-value problems. The first is a finite-element discretization on a convex domain in two dimensions. The second is a finite-difference discretization in one dimension. The last is a finite-difference discretization on the unit square.

1: General Theory

In this section, we discuss the approximate solution of an abstract elliptic variational problem. Our scheme involves a sequence of finite-dimensional spaces M_j, $j = 1, 2, \ldots, k$. We approximate the solution using the largest space. Under certain circumstances, the solution to a problem can be approximated well using smaller spaces. Since we require the smaller spaces to have geometrically fewer unknowns than the largest one, the savings in computation can be substantial. In fact, we prove that these procedures are optimal order under appropriate conditions. While this theory is applied to solving the large sparse linear systems which arise in connection with finite-element or finite-difference procedures for solving self-adjoint elliptic boundary-value problems in the next section, it can also be applied to problems which do not arise from partial differential equations.

Our k-level scheme is related to the multi-grid techniques used by Bank and Dupont [6], which is related to the techniques of Brandt [10], Bakhvalov [5], Federenko [14, 15], Nicolaides [25, 26], and Hackbusch [18, 19, 20]. The earlier proofs are for particular discretizations of model elliptic boundary-value problems. Their domains are covered by meshes or triangulations which are refined uniformly. Only Van Rosendale's proof [29] allows nonuniformly refined domains. The proofs here use abstract function space arguments which make no reference to the particular discretization, domain, or method of refinement. Further, we do not require the solution spaces to be nested as in the proofs of the cited references.

Assume we are given a triple,

$$\{H, a(u,v), f(v)\}, \tag{1.1}$$

where H is a Hilbert space with norm $\|\cdot\|$, $a(u,v)$ is a continuous symmetric real valued bilinear form on $H \otimes H$, and $f(v):H \to R$ is a continuous real valued linear functional. Furthermore, we assume that there exists a constant $a_0 > 0$ such that

$$a(v,v) \geq a_0\|v\|^2 \text{ for all } v \in H.$$

The bilinear form $a(\cdot,\cdot)$ induces the *energy norm*

$$|||u|||^2 \equiv a(u,u).$$

We seek an approximation to the solution of

Problem 1.1: Given $\{H, a(u,v), f(v)\}$, find $u \in H$ such that

$$a(u,v) = f(v) \text{ for all } v \in H.$$

Problem 1.1 has a unique solution (see Ciarlet [11]).

We now consider the finite-dimensional approximation of Problem 1.1. Let M_j, $j \geq 1$, be a sequence of N_j-dimensional spaces. Associated with each space M_j is a continuous, symmetric, positive-definite bilinear form $a_j(u,v)$ and a continuous, bounded linear form $f_j(v)$. We require that

$$N_j \sim \sigma N_{j-1}, \text{ for some } \sigma \geq 2. \tag{1.2}$$

We will see that σ is important: when $\sigma > 2$, we can always construct optimal order algorithms to approximate the solution to Problem 1.1.

We assume that linear operators exist which project H onto M_j and inject M_j into H for any $j \geq 1$:

$$\begin{aligned} &p_j\colon\ H \xrightarrow[\text{onto}]{} M_j \qquad \text{and} \\ &i_j\colon\ M_j \xrightarrow[1\text{-}1]{} H. \end{aligned} \tag{1.3}$$

For $j \geq 1$, the linear operators defined by

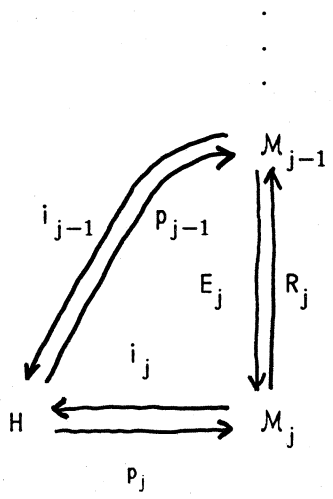
$$\begin{aligned} &E_j\colon\ M_{j-1} \to M_j \quad \text{and} \\ &R_j\colon\ M_j \xrightarrow[\text{onto}]{} M_{j-1} \end{aligned} \tag{1.4}$$

interpolate between adjacent solution spaces. One natural definition of E_j and R_j is

$$E_j = p_j i_{j-1} \quad \text{and} \quad R_j = p_{j-1} i_j. \tag{1.5}$$

It is natural to define R_j as the adjoint of E_j:

Figure 1-1: Space Operators



$$R_j = E_j^*.$$

In this case,

$$E_j^* = i_{j-1}^* p_j^* = p_{j-1} i_j = R_j.$$

If we assume that $p_j = i_j^*$, then all of the operators can be defined in terms of the injection operators i_j. However, we will have occasion to use more general operators than those in (1.5). Figure 1-1 shows the relationship between these operators and the various spaces.

For $j \geq 1$, the finite-dimensional approximations of Problem 1.1 are

Problem 1.2: Given $\{M_j, a_j(u,v), f_j(v)\}$, find $u_j \in M_j$ such that

$$a_j(u_j,v) = f_j(v) \text{ for all } v \in M_j.$$

Associated with each space M_j are eigenvalues $\lambda_i^{(j)}$ and eigenfunctions (eigenvectors) $\psi_i^{(j)}$, $1 \leq i \leq N_j$, satisfying

$$a_j(v,\psi_i^{(j)}) = \lambda_i^{(j)} (v,\psi_i^{(j)})_j \text{ for all } v \in M_j,$$

where $(\cdot,\cdot)_j$ denotes the inner product in M_j. Let δ_{ik} be the Kronecker delta. Without loss of generality,

$$0 < \lambda_1^{(j)} \leq \lambda_2^{(j)} \leq \ldots \leq \lambda_{N_j}^{(j)} \equiv \Lambda_j,$$

$$(\psi_i^{(j)},\psi_k^{(j)})_j = \delta_{ik}, \quad 1 \leq i,k \leq N_j, \text{ and} \tag{1.6}$$

$$a_j(\psi_i^{(j)},\psi_k^{(j)}) = \lambda_i^{(j)}\delta_{ik}.$$

With each space M_j, we define discrete norms

$$|||v|||_s^2 \equiv \sum_{i=1}^{N} c_i^2(\lambda_i^{(j)})^s, \quad \text{for } v \equiv \sum_{i=1}^{N} c_i\,\psi_i^{(j)},$$

where we have suppressed the j subscript on the norm and $-2 \leq s \leq 2$. Note that $|||v|||_1 = |||v|||$ is the usual energy norm on level j. Hereafter, we drop both the superscripts from the eigenvalues and eigenfunctions (eigenvectors) and the subscript from the dimension of the spaces.

We require the bilinear forms on adjacent spaces to have a consistency relationship. As a consequence, the energy norms on two adjacent spaces are uniformly consistent.

Hypothesis 1.3 (Energy Norm Consistency): For all $j > 1$ there exists a positive constant C_1, independent of j, such that

$$a_j(E_jv,E_jw) = C_1\, a_{j-1}(v,w), \text{ for all } v,w \in M_{j-1}.$$

Since $|||v|||^2 = a_j(v,v)$. This hypothesis implies that for any $v \in M_{j-1}$, the energy norm of v on level $j-1$ is equal to a constant times the energy norm of E_jv on level j.

We now define and analyze a k-level iterative procedure for solving Problem 1.2. The process involves solving problems like Problem 1.2 sequentially for j=1, 2,... , k. The *k-level scheme* has three parameters: m and n, which determine the number of smoothing iterations used; and p, which is used in a recursion iteration.

Algorithm 1.4: **MG(k, m, n, p)**

Given an integer $k > 0$ and $\{M_j, a_j(\cdot,\cdot), f_j(\cdot)\}_{j=1}^{k}$, we want to approximate $u_k \in M_k$, where $a_k(u_k,v) = f_k(v)$ for all $v \in M_k$.

(a) If $k = 1$, then solve directly.

(b) If $k > 1$, then one iteration of the k-level scheme takes an initial guess $z_0 \in M_k$ to a final approximation $z_{m+n+1} \in M_k$ in three steps:

(i) if $n > 0$, define z_i, $1 \leq i \leq n$, by

$$(z_i - z_{i-1}, v)_k = \Lambda_k^{-1}[f_k(v) - a_k(z_{i-1},v)], \text{ for all } v \in M_k. \tag{1.7}$$

(ii) Let $q \in M_{k-1}$ be the approximation of $\bar{q} \in M_{k-1}$ obtained by applying p iterations of the (k−1)-level scheme to the residual equation

$$\begin{aligned} a_{k-1}(\bar{q},v) &= C_1^{-1}\{f_k(E_k v) - a_k(z_n, E_k v)\} \\ &\equiv \bar{f}_{k-1}(v), \text{ for all } v \in M_{k-1}, \end{aligned} \tag{1.8}$$

starting from an initial guess zero. Then set

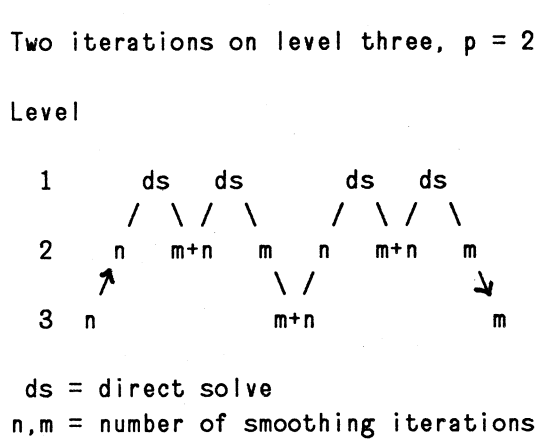
$$z_{n+1} = z_n + E_k q. \tag{1.9}$$

(iii) If $m > 0$, then define z_i, $n+2 \leq i \leq m+n+1$, by (1.7).

In the *correction recursion iteration* (step (ii)), we approximately compute the elliptic projection of the error in M_{k-1} using p iterations of the (k−1)-level scheme applied to a problem of Problem 1.2's form with $j = k - 1$. In the *smoothing iterations* (steps (i) and (iii)), error components whose oscillation are "large" are damped. A simultaneous displacement procedure is used in this step. Later in this section we will see that Λ_k^{-1} can be replaced by a particular type of bound (see Hypothesis 1.6). We will also see that (1.7) can be replaced by other iterations which are computationally more attractive, but do not affect the character of our convergence results. The use of iteration (1.7) simplifies the initial analysis of the convergence. Figure 1-2 contains a three-level, two-iteration example of Algorithm 1.4 with $p = 2$. Note that computation begins with the largest space and uses the smaller ones only to solve correction recursion problems.

There are three cases of note in Algorithm 1.4: (a) when $n > 0$ and $m = 0$, (b) when

Figure 1-2: Three-Level Example of Algorithm 1.4



$n = 0$ and $m > 0$, and (c) when $n > 0$ and $m > 0$. Case (a) is the scheme analyzed by Astrakhantsev [3], Bank and Dupont [6], Hackbusch [18, 19, 20], Nicolaides [24, 25], and Van Rosendale [29] for finite-element discretizations of various elliptic boundary-value problems. Federenko [14, 15] and Bakhvalov [5] analyzed this case for finite-difference discretizations. Each of these authors assumed in their convergence proofs that $p > 1$. Our convergence result is true even when $p = 1$. Brandt [10] gave a heuristic analysis of (b) and (c) for finite-difference discretizations using local mode analysis. The motivation for studying cases (b) and (c) comes from trying to understand the behavior of a large finite-difference program which is described in Chapter 6 of Douglas [12, 13]. It was observed empirically that case (b) sometimes required fewer correction recursions to achieve the same accuracy as case (a).

Before proving a convergence theorem for Algorithm 1.4, we need to state one definition and two more hypotheses. The first hypothesis is a bound for the largest eigenvalues and the other is an error estimate for the correction produced by the (k−1)-level iteration. Finally, we prove a lemma describing the smoothing iteration's effect on the error.

Definition 1.5: The error on level k at the i^{th} stage of Algorithm 1.4 will be denoted by

$$e_i \equiv u_k - z_i.$$

The first hypothesis states what form the bound for the maximum eigenvalue is

assumed to have.

Hypothesis 1.6 (Maximum Eigenvalue): There exist positive constants δ and C_2, each independent of j, such that

$$\Lambda_j \leq C_2 N_j^{2\delta}, \; 1 \leq j \leq k.$$

The use of Λ_k in the smoothing iteration (1.7) may be replaced by any upper bound satisfying Hypothesis 1.6. The last hypothesis is a norm estimate:

Hypothesis 1.7 (Approximating Error Estimate): For some α with $0 < \alpha \leq 1$, there exists a positive constant C_3 such that

$$|||E_k \bar{q} - e_n|||_{1-\alpha} \leq C_3^{\alpha} N_k^{-2\alpha\delta} |||e_n|||_{1+\alpha}.$$

For problems derived from elliptic boundary-value problems, the value of α depends on the spatial domain.

The following lemma is used in the convergence proof to analyze the effect of the smoothing on elements in the solution spaces.

Lemma 1.8: Let $n > 0$ be any integer and $z_0 \in M_k$. Then the smoothing iteration (1.7) is a contraction operator:

$$|||e_n||| \leq |||e_0|||. \tag{1.10}$$

Further, for every fixed $0 \leq \omega \leq 2$ and $0 < \alpha \leq 1$,

$$|||e_n|||_{\omega} \leq C_2^{\alpha/2} N_k^{\alpha\delta} (2n + \alpha)^{-\alpha/2} \, |||e_0|||_{\omega-\alpha}. \tag{1.11}$$

Proof: Let $N \equiv N_k$ and $\Lambda \equiv \Lambda_k$. From (1.7) we can deduce that

$$(e_i - e_{i-1}, v)_k = -\Lambda^{-1} a_k(e_{i-1}, v), \text{ for all } v \in M_k, \; i \geq 1. \tag{1.12}$$

We can expand e_0 in terms of the eigenfunctions:

$$e_0 \equiv \sum_{i=1}^{N} \beta_i \psi_i.$$

Using (1.12) we can show that

$$e_n = \sum_{i=1}^{N} \beta_i (1 - \lambda_i/\Lambda)^n \psi_i. \tag{1.13}$$

Since $(1 - \lambda_i/\Lambda)^n \leq 1$, we have that

$$|||e_n||| \leq |||e_0|||.$$

The proof of (1.11) uses (1.10) and the Maximum Eigenvalue Hypothesis:

$$\begin{aligned}
|||e_n|||_\omega^2 &= \sum_{i=1}^{N} \beta_i^2 \lambda_i^\omega (1 - \lambda_i/\Lambda)^{2n} \\
&= \Lambda^\alpha \sum_{i=1}^{N} \beta_i^2 \lambda_i^{\omega-\alpha} (\lambda_i/\Lambda)^\alpha (1 - \lambda_i/\Lambda)^{2n} \\
&\leq \Lambda^\alpha \max_{x\in[0,1]} |x^\alpha (1-x)^{2n}| \sum_{i=1}^{N} \beta_i^2 \lambda_i^{\omega-\alpha} \\
&\leq \Lambda^\alpha (2n+\alpha)^{-\alpha} |||e_0|||_{\omega-\alpha}^2 \\
&\leq C_2^\alpha N^{2\alpha\delta} (2n+\alpha)^{-\alpha} |||e_0|||_{\omega-\alpha}^2.
\end{aligned}$$

Taking the square root of both sides of this inequality completes the proof.

QED

The convergence of Algorithm 1.4 is established in the Theorems 1.9 and 1.10. In essence, these results say that the error on level k can be reduced by any positive constant less than one provided the correction recursion problem on level k−1 can be solved sufficiently accurately. Theorem 1.9 requires that $p > 1$ and is true for any k. Theorem 1.10 requires that k be finite and is true for any p.

Theorem 1.9 (Convergence of Algorithm 1.4): Assume that Hypotheses 1.3, 1.6, and 1.7 hold. Let $p > 1$ be any fixed integer. For any constant $0 < \gamma < 1$ there exist a nonnegative integer I which depends only on p and γ, such that

$$|||e_{m+n+1}||| \leq \gamma |||e_0|||, \text{ for all } m+n \geq I. \tag{1.14}$$

Proof: This proof is motivated by the work of Bank and Dupont [6]. The basic idea of this proof is to show that the smoothing iteration (1.7) reduces the oscillatory components of the error (corresponding to the larger eigenvalues) while the correction recursion iteration (1.8) reduces the smoother components of the error. The proof is by induction on the index k of the space. Assume the result is true for 1, 2, ... , k−1. We now prove the result for M_k.

By Lemma 1.8 (with $\omega = 1 + \alpha$),

$$|||e_n||| \leq |||e_0|||$$

and

$$|||e_n|||_{1+\alpha} \leq C_2^{\alpha/2} N_k^{\alpha\delta} (2n + \alpha)^{-\alpha/2} |||e_0|||. \tag{1.15}$$

We can estimate the effect of the correction recursion iteration (1.8). Using (1.8) and the Energy Norm Consistency Hypothesis, we have for all $v \in M_{k-1}$,

$$a_{k-1}(\bar{q},v) = C_1^{-1} a_k(e_n, E_k v)$$

or

$$a_k(E_k\bar{q} - e_n, E_k v) = 0. \tag{1.16}$$

This shows that the exact solution $\bar{q}$ of the (k−1)-level correction recursion problem corrects exactly all components in z_n which belong to M_{k-1}. Take $v = \bar{q}$ in (1.16). Then

$$a_k(E_k\bar{q}-e_n,E_k\bar{q}) = 0 \Rightarrow a_k(E_k\bar{q},E_k\bar{q}) = a_k(e_n,E_k\bar{q})$$

$$\Rightarrow |||E_k\bar{q}||| \leq |||e_n||| \leq |||e_0|||.$$

Using this, the Energy Norm Consistency Hypothesis, and the induction hypothesis gives us

$$|||E_k(q-\bar{q})||| = C_1^{-1/2}|||q-\bar{q}||| \leq C_1^{-1/2}\gamma^p|||\bar{q}||| = \gamma^p|||E_k\bar{q}|||$$

$$\leq \gamma^p|||e_0|||. \tag{1.17}$$

We are now ready to estimate $|||e_{m+n+1}|||$. We can define

$$e_{n+1} = E_kq-e_n$$

$$= (E_k\bar{q}-e_n) + E_k(q-\bar{q}).$$

If S^m reflects the effect of m smoothing iterations on any $v \in M_k$, then

$$e_{m+n+1} = S^m e_{n+1}.$$

Define $C_4 \equiv C_2^\alpha C_3^\alpha$. Using Lemma 1.8 (with $\omega = 1$), (1.15), (1.17), and the Approximating Error Estimate Hypothesis yields

$$|||e_{m+n+1}||| \leq |||S^m(E_k\bar{q}-e_n)||| + |||S^mE_k(q-\bar{q})|||$$

$$\leq C_2^{\alpha/2}(2m+\alpha)^{-\alpha/2}\, N_k^{\alpha\delta}\, |||E_k\bar{q}-e_n|||_{1-\alpha} + |||E_k(q-\bar{q})|||$$

$$\leq C_2^{\alpha/2}\, C_3^\alpha (2m+\alpha)^{-\alpha/2}\, N_k^{-\alpha\delta}\, |||e_n|||_{1+\alpha} + \gamma^p|||e_0|||$$

$$\leq [C_4(2m+\alpha)^{-\alpha/2}(2n+\alpha)^{-\alpha/2} + \gamma^p]\, |||e_0|||. \tag{1.18}$$

Choose I such that

$$C_4(2m+\alpha)^{-\alpha/2}(2n+\alpha)^{-\alpha/2} \leq \gamma - \gamma^p, \text{ for all } m+n \geq I. \tag{1.19}$$

Then

$$|||e_{m+n+1}||| \leq \gamma |||e_0|||. \tag{1.20}$$

Clearly, γ and I can be chosen independent of the N_k.

QED

When $p = 1$, the proof of Theorem 1.9 breaks down because the right hand side of (1.19) is zero. If we assume that k is a small natural number, then we can define a sequence $\{\gamma_j\}_{j=1}^{k}$ such that

$\gamma_j - \gamma_{j-1}^p = C_5 > 0, \quad 2 \leq j \leq k$ and $p > 0$,

$\gamma_1 > 0$, and

$\gamma_k = \gamma < 1$.

Corresponding to Theorem 1.9 is

Theorem 1.10: Assume that Hypotheses 1.3, 1.6, and 1.7. Let $p > 0$ be any fixed integer. For any constant $0 < \gamma < 1$ there exist a nonnegative integer I which depends only on p, γ, and C_5, such that

$|||e_{m+n+1}||| \leq \gamma|||e_0|||$, for all $m+n \geq I$.

Proof: The proof follows the one for Theorem 1.9. Equation (1.17) becomes

$$|||E_k(q-\bar{q})||| \leq \gamma_{k-1}^p |||e_0|||.$$

and (1.19) becomes

$$C_4(2m + \alpha)^{-\alpha/2}(2n + \alpha)^{-\alpha/2} \leq \gamma_k - \gamma_{k-1}^p, \text{ for all } m+n \geq I.$$

QED

At first glance, it appears that choosing $m = n$ in (1.18) would be better than choosing either $n = 0$, $m > 0$ or $n > 0$, $m = 0$. However, special case proofs show that this is not necessarily true. Assume that $n > 0$, $m = 0$. By modifying the proof of

Theorem 1.9, we can show that

$$|||e_{n+1}||| \leq [C_4^{1/2}(2n + \alpha)^{-\alpha/2} + \gamma^p] \, |||e_0|||.$$

For n large, this bound for the error reduction looks like

$$C_4^{1/2} (2n)^{-\alpha/2} \leq \epsilon \equiv \gamma - \gamma^p.$$

So

$$n \geq \tfrac{1}{2} C_4^{1/\alpha} \epsilon^{-2/\alpha}$$

is required to reduce the error by a factor of ϵ. If $\tilde{n} = \tilde{m}$, the error reduction (see (1.18)) looks like

$$C_4 (2\tilde{n})^{-\alpha} \leq \epsilon.$$

Once again,

$$\tilde{n} \geq \tfrac{1}{2} C_4^{1/\alpha} \epsilon^{-1/\alpha} = \epsilon^{1/\alpha} \cdot \tfrac{1}{2} C_4^{1/\alpha} \epsilon^{-2/\alpha} = \epsilon^{1/\alpha} n$$

is required to reduce the error by a factor of ϵ. The amount of work, which depends on n and $2\tilde{n}$, depends on the relative sizes of one and $2^\alpha \epsilon$. Thus, we have shown

Theorem 1.11: Assume Theorem 1.9 holds. Define $\epsilon \equiv \gamma - \gamma^p$. Then choosing $n > 0$, $m = 0$ in (1.18) is better than choosing $\tilde{n} = \tilde{m} > 0$ only when $2^\alpha \epsilon > 1$. Alternately, choosing $\tilde{n} = \tilde{m} > 0$ in (1.18) is better only when $2^\alpha \epsilon < 1$.

The practical significance of this is that if $\gamma - \gamma^p$ (or $\gamma_k - \gamma_{k-1}^p$) is large, it is more efficient to do the smoothing at once, rather than splitting it around the correction recursion. Similar analysis holds for the case of $n = 0$, $m > 0$.

We now analyze the cost of one iteration of Algorithm 1.4. Let F(N) be the cost of reducing the error by a factor of γ for a problem with N unknowns. We assume that the cost of the smoothing iterations (1.7) (or an iteration with similar properties) on the finest level, level k, can be bounded by $C_6(m+n)N_k \equiv C_7 N_k$, where C_6 is independent of k. The cost of the correction recursion (1.8) is $pF(N_{k-1})$. Thus,

$$F(N_k) \sim pF(N_{k-1}) + C_7 N_k. \tag{1.21}$$

Since $N_k \sim \sigma N_{k-1}$ (see (1.2)), the solution of (1.21) is

$$F(N_k) \sim C_7 N_k \sum_{i=0}^{k} (p/\sigma)^i.$$

Asymptotically in k,

$$F(N) \leq \begin{cases} C_8 N, & 1 \leq p < \sigma \\ C_8 N \log N, & p = \sigma \\ C_8 N^{\log p}, & p > \sigma, \end{cases} \tag{1.22}$$

where the logarithms are base σ [2]. One of the surprises of multi-grid is that implementations (e.g., Bank and Sherman [7] or Douglas [12, 13]) exhibit the asymptotic convergence rates for very small k's.

Choosing $1 \leq p < \sigma$ leads to an optimal order algorithm, in the sense that the error can be reduce by a fixed factor of γ each iteration with work proportional to the number of unknowns. When k is finite, the cost for the direct solution of the one level problem may not be majorized by $C_7 N_1$}. In this case, the algorithm is not optimal order.

We may want to reduce the initial error by a factor of $N^{-q\delta}$ for some fixed q. The obvious implementation would then require F(N)logN operations. We assume the solutions u_j of Problem 1.2 satisfy

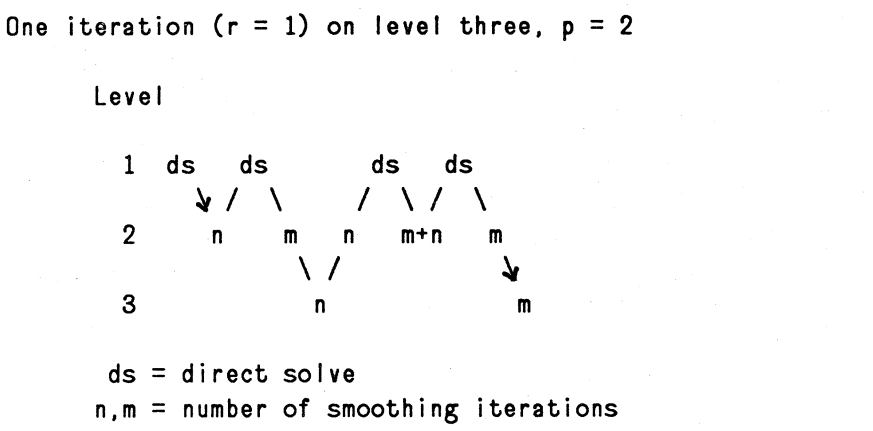
$$|||p_j u - u_j||| \leq K N_j^{-q\delta}, \; j \geq 1,$$

where K is a constant independent of N_j. Denote by $\tilde{u}_j$ the computed solution of the j-level scheme (Algorithm 1.4). To avoid the extra logN factor, we use $E_j \tilde{u}_{j-1}$ as the initial guess to $\tilde{u}_j$, $j > 1$, and prove that the initial error is small. Approximate solutions $\tilde{u}_j$ of finite-dimensional Problem 1.2 are generated using the following:

Algorithm 1.12: **MGN(j, r, m, n, p)**

Given an integer $j > 0$ and $\{M_i, a_i(\cdot,\cdot), f_i(\cdot)\}_{i=1}^{j}$, we want to approximate $u_j \in M_j$, where $a_j(u_j, v) = f_j(v)$ for all $v \in M_j$.

Figure 1-3: Three-Level Example of Algorithm 1.12



(a) If $j = 1$, then solve Problem 1.2 directly.

(b) If $j > 1$, then starting from an initial guess $z_0 \equiv E_j \tilde{u}_{j-1}$, apply r iterations of MG(j, m, n, p) (see Algorithm 1.4) to Problem 1.2 to obtain $\tilde{u}_j$.

This algorithm actually has four parameters. The parameter r determines the number of iterations of MG(j,m,n,p) to use. Recall that m and n are the number of smoothing iterations and that p is the number of correction recursion iterations in Algorithm 1.4.

Figure 1-3 contains a three-level example of one iteration ($r = 1$) of Algorithm 1.12. For the correction recursion problems, $p = 2$. Unlike Algorithm 1.4, computation begins with the smallest space and winds its way "down" to the largest space. It is worth noting that when $n = 0$ and $z_0 = 0$ in Figure 1-2, the two algorithms become much more similar. In this case, the first recursion correction problem (see (1.8)) in Figure 1-2 has $\bar{f}_k(v) \equiv f_k(v)$.

The convergence properties of Algorithm 1.12 are stated in the Theorem 1.13.

Theorem 1.13: Assume that Hypotheses 1.3, 1.6, and 1.7 hold. Let $r \geq 1$ be any fixed integer. Suppose

(i) $|||p_j u - u_j||| \leq KN_j^{-q\delta}, j \geq 1,$

(ii) $|||p_j u - E_j p_{j-1} u||| \leq KN_j^{-q\delta}, j \geq 2,$ and

(iii) $|||u_1 - \tilde{u}_1||| \leq KN_1^{-q\delta}$

Then

(a) For any integer $p \geq 1$, there exists a nonnegative integer I such that for $j \geq 1$,

$|||u_j - \tilde{u}_j||| \leq KN_j^{-q\delta}$ for all $m+n \geq I$.

(b) $|||p_j u - \tilde{u}_j||| \leq 2KN_j^{-q\delta}$

(c) the cost of computing $\tilde{u}_j$ is bounded by $C_g F(N_j)$, where $C_g \equiv \sigma r/(\sigma-1)$ is independent of j.

Proof: (a) The proof is by induction on the index j of the space. Assume the result is true for 1, 2, ... , j−1. We now prove the result for M_j. Define $\tilde{e}_j \equiv |||u_j - \tilde{u}_j|||$ for $j \geq 1$. After r iterations of the j-level scheme,

$$\tilde{e}_j \leq \gamma^r |||u_j - E_j \tilde{u}_{j-1}|||$$

$$\leq \gamma^r \{ |||u_j - p_j u||| + |||p_j u - E_j p_{j-1} u||| +$$

$$|||E_j(p_{j-1} u - u_{j-1})||| + |||E_j(u_{j-1} - \tilde{u}_{j-1})||| \}$$

$$\leq \gamma^r \{ K(2 + C_1^{1/2} + C_1^{-1/2}\sigma^{q\delta}) N_j^{-q\delta} \}.$$

Choose $\gamma \leq (2 + C_1^{1/2} + C_1^{-1/2}\sigma^{q\delta})^{-1/r}$ and I according to Theorem 1.9 or Theorem 1.10. Hence,

$$\tilde{e}_j \leq KN_j^{-q\delta}$$

(b) This is a simple consequence of the triangle inequality:

$$|||p_j u - \tilde{u}_j||| \leq |||p_j u - u_j||| + |||u_j - \tilde{u}_j|||$$

$$\leq 2KN_j^{-q\delta}$$

(c) The cost of computing $\tilde{u}_j$ is bounded by

$$F(N_1) + r \sum_{i=2}^{j} F(N_i) \leq C_8 F(N_j)\, \sigma r/(\sigma - 1) = C_9 F(N_j).$$

QED

It is worth pointing out that r is independent of j. This tells us that Algorithm 1.12 is optimal order whenever $1 \leq p < \sigma$.

In conclusion, we have shown that solutions to problems like Problem 1.1 can be approximated in finite-dimensional spaces using an optimal order procedure. In the next section we verify that Hypotheses 1.3, 1.6, and 1.7 hold for particular discretizations of several elliptic boundary-value problems.

2: Applications

2.1 Model Problems

In this section, we discuss the application of the algorithms of Section 1 to the solution of large sparse linear systems which arise in connection with finite-element and finite-difference procedures for solving self-adjoint elliptic boundary-value problems. We show a few examples in which the algorithms and theorems of Section 1 apply. We will see that these can be optimal order results.

Section 2.2 is concerned with a finite-element procedure. Our model problem is the Neumann problem

$$\begin{cases} -\nabla\cdot(P\nabla u) + Su = f \text{ in } \Omega \\ \\ u_n = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a polygonal domain in R^2. We assume that $P \in C^1(\bar{\Omega})$, $S \in C(\bar{\Omega})$, and that there exist positive constants $\underline{p}$, $\bar{p}$, $\underline{s}$, and $\bar{s}$ such that

$$0 < \underline{p} \leq P(x) \leq \bar{p} \text{ and } 0 < \underline{s} \leq S(x) \leq \bar{s}, \text{ for all } x \in \bar{\Omega}. \tag{2.2}$$

Most of our arguments apply to the Dirichlet problem

$$\begin{cases} -\nabla\cdot(P\nabla u) + Su = f \text{ in } \Omega \\ \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{2.3}$$

with only minor modifications. We comment on the extensions as they arise.

In Section 2.3, we discuss finite-difference approximations to the model problems

$$\begin{cases} -(Pu_x)_x + Su = f \text{ in } \Omega \equiv (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{2.4}$$

and

$$\begin{cases} -(Pu_x)_x - (Pu_y)_y + Su = f \text{ in } \Omega \equiv (0,1)x(0,1) \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.5}$$

We assume that $f \in L^2(\Omega)$ and that P and S are constrained as before.

2.2 Finite-Elements

We seek a weak form solution of (2.1): find $u \in \mathcal{H}^1(\Omega)$ such that

$$a(u,v) = (f,v) \text{ for all } v \in \mathcal{H}^1(\Omega), \tag{2.6}$$

where

$$a(u,v) \equiv \int_\Omega P\nabla u\cdot\nabla v + Suv\, dx \quad \text{and}$$

$$(f,v) \equiv \int_\Omega fv\, dx. \tag{2.7}$$

Then there exists a unique weak solution $u \in \mathcal{H}^1(\Omega)$ for all $f \in L^2(\Omega)$ (see Ciarlet [11]). The spaces $\mathcal{H}^s$, for s a positive integer, will be the usual Sobolev spaces equipped with norms

$$\|u\|_s^2 \equiv \sum_{|\beta|\le s} \|D^\beta u\|_s^2 = \sum_{|\beta|\le s} (D^\beta u, D^\beta u).$$

The spaces $\mathcal{H}^s$ for s positive and non-integral will be defined by interpolation (see Agmon [1] or Lions and Magenes [23]). For s negative, $\mathcal{H}^s$ will be defined as the dual of $\mathcal{H}^{-s}$. The bilinear form $a(\cdot,\cdot)$ induces the energy norm $|||u|||^2 = a(u,u)$.

A modest amount of elliptic regularity for the solution of (2.6) is required.

Hypothesis 2.1 (Regularity): We assume there exists a constant $0 < \alpha \le 1$, such that for all $f \in \mathcal{H}^{\alpha-1}$ there exists a unique solution $u \in \mathcal{H}^{1+\alpha}$ of (2.6) and

$$\|u\|_{1+\alpha} \le C(P,S,\Omega)\, \|f\|_{\alpha-1}.$$

For a complete discussion of what values of α correspond to specific domains Ω, see Kellog [22] and Babuska and Aziz [4].

We now consider a finite-element approximation of (2.6). Let T_j, $j \geq 1$, be a nested sequence of triangulations of Ω. Take T_1 to be a fixed triangulation. For $T \in T_1$, denote the diameter of T by h_T, and let $h_T \cdot d_T$ denote the diameter of the inscribing circle for T. Define

$$h_1 \equiv \max_{T \in T_1} h_T, \quad \delta_0 \equiv \min_{T \in T_1} d_T, \quad \text{and } \delta_1 \equiv \min_{T \in T_1} h_T/h_1.$$

The constant δ_0 is a measure of the shape regularity of the triangles in T_1 and δ_1 is a measure of their uniformity. We construct T_j, $j > 1$, inductively: divide every $T \in T_{j-1}$ into μ^2 congruent triangles, where μ is independent of j. When $\mu = 2$, this means we construct four triangles in T_j by pairwise connecting the midpoints of the edges. Each triangulation T_j will have shape regularity and uniformity constants δ_0 and δ_1 and will have $h_j \equiv \max_{T \in T_j} h_T = \mu^{1-j} h_1$.

With each triangulation T_j, we associate the N_j-dimensional space M_j of C^0-piecewise linear polynomials. Following (1.2), we know that

$$N_{j+1} \sim \sigma N_j, \tag{2.8}$$

where $\sigma = \mu^2$ asymptotically. Since the triangulations are nested, we have that M_j is a subset of M_{j+1}, $j \geq 1$. The spaces M_j satisfy the following standard approximation property [8, 9, 21, 27]: if $u \in \mathcal{H}^s$, $1 \leq s \leq 1+\alpha$, then there exists a $u_j \in M_j$ such that

$$\|u-u_j\|_0 + h_j \|u-u_j\|_1 \leq C(\delta_0, \delta_1, \Omega)\, h_j^s \, \|u\|_s. \tag{2.9}$$

We briefly remark on the Dirichlet model problem (2.3). The definition of T_j, $j \geq 1$, remains the same. Let M_j be a subset of $\mathcal{H}_0^1$ and let it be the space of C^0-piecewise linear polynomial associated with T_j satisfying the Dirichlet boundary conditions. Then M_j is a subset of M_{j+1}, $j \geq 1$, as before. With this modification, the results of this section will remain valid.

Using the notation of Section 1, we define for each space M_j, $j \geq 1$,

$$a_j(u,v) \equiv a(u,v), \quad f_j(v) \equiv (f,v), \quad \text{and} \quad (u,v)_j \equiv (u,v) \text{ for all } u,v \in M_j.$$

The interpolation and projection operators, E_j, p_j, and i_j (see (1.3) and (1.4)), are the natural projections and injections.

Associated with M_j are eigenvalues $\lambda_i^{(j)}$, eigenfunctions $\psi_i^{(j)}$, and a maximum eigenvalue Λ_j of $a(\cdot,\cdot)$ fulfilling the requirements of (1.6), where $1 \leq i \leq N_j$. A simple homogeneity argument shows that

$$\Lambda_j \leq C(P,S,\delta_0,\delta_1,\Omega)h_j^{-2} \tag{2.10}$$

(see Strang and Fix [28]). For $-2 \leq s \leq 2$, we define discrete norms

$$|||v|||_s^2 \equiv \sum_{i=1}^{N} c_i^2(\lambda_i^{(j)})^s, \text{ for } v \equiv \sum_{i=1}^{N} c_i \psi_i^{(j)}, \tag{2.11}$$

where we have suppressed the j subscript on the norm. Note that $|||v|||_1 = |||v|||$ and $|||v|||_0$ is comparable to $\|v\|_0$. In fact, the proof of the following norm equivalence is almost identical to Lemma 1 in Bank and Dupont [6]:

Lemma 2.2: There exists a constant $C = C(P,S,\Omega,\delta_0,\delta_1,\beta)$ such that for $0 \leq s \leq 1$,

$$C^{-1}\|v\|_s \leq |||v|||_s \leq C\|v\|_s.$$

In order to establish the convergence of Algorithms 1.4 and 1.12 for the finite-element case, we must verify Hypotheses 1.3, 1.6, and 1.7. It is immediate that (the Energy Norm Consistency) Hypothesis 1.3 holds. Using (2.10) and the fact that $N_j \sim Ch_j^{-2}$ we can verify that (the Maximum Eigenvalue) Hypothesis 1.6 holds.

A duality argument is used to verify (the Approximating Error Estimate) Hypothesis 1.7. When α is an integer, this is a standard result [11, 28].

Lemma 2.3: Let α be defined by the Regularity Hypothesis. Then

$$|||\bar{q}-e_n|||_{1-\alpha} \leq C\mu^{2\alpha} h_k^{2\alpha} |||e_n|||_{1+\alpha}.$$

Proof: By Lemma 2.2 and duality,

$$|||\bar{q}-e_n|||_{1-\alpha} \leq C\|\bar{q}-e_n\|_{1-\alpha}$$

$$= C \sup_{\rho\in\mathcal{H}^{\alpha-1}} \frac{(\rho,\bar{q}-e_n)}{\|\rho\|_{\alpha-1}} \tag{2.12}$$

For $\rho \in \mathcal{H}^{\alpha-1}$, let $\eta \in \mathcal{H}^{\alpha+1}$ be defined by

$$a(\eta,v) = (\rho,v) \text{ for all } v \in \mathcal{H}^1.$$

Taking $v = \bar{q}-e_n$ gives us

$$(\rho,\bar{q}-e_n) = a(\eta,\bar{q}-e_n)$$

$$= a(\eta-\omega,\bar{q}-e_n), \text{ for any } \omega \in M_{k-1}.$$

By the Regularity Hypothesis and (2.9),

$$(\rho,\bar{q}-e_n) \leq Ch_{k-1}^{\alpha} \|\eta\|_{\alpha+1} |||\bar{q}-e_n|||$$

$$\leq C\mu^{\alpha} h_k^{\alpha} \|\rho\|_{\alpha-1} |||\bar{q}-e_n|||.$$

Combining this with (2.12) yields

$$|||\bar{q}-e_n|||_{1-\alpha} \leq C\mu^{\alpha} h_k^{\alpha} |||\bar{q}-e_n|||. \tag{2.13}$$

However,

$$|||\bar{q}-e_n|||^2 = a(\bar{q}-e_n,\bar{q}-e_n)$$

$$= -a(\bar{q}-e_n,e_n) \tag{2.14}$$

$$\leq |||\bar{q}-e_n|||_{1-\alpha}|||e_n|||_{1+\alpha}.$$

Substituting (2.13) into the right-hand side of (2.14) gives us

$$|||\bar{q}-e_n||| \leq C\mu^{\alpha} h_k^{\alpha} |||e_n|||_{1+\alpha},$$

which we substitute back into (2.13) to complete the proof.

QED

This proves that Algorithm 1.4 converges at the rates specified by Theorems 1.9 and 1.10 for the finite-element case of this section.

One of the advantages of finite-element methods is that the theory of Section 1 can be applied using a variety of norms. As an example, we prove a special case of Theorem 1.9 for the L^2 norm. It is similar to the results of Nicolaides [25] for the l^2 norm and is the analogue of Corollary 1 of Bank and Dupont [6]. To get L^2 results, we assume that the solution u has $\mathcal{H}^2$ regularity, i.e., $\alpha = 1$. This assumption requires that Ω be convex [17, 28].

Theorem 2.4: Assume the Regularity Hypothesis holds for $\alpha = 1$. Let $p > 1$ be any fixed integer. For any constant $0 < \gamma < 1$ there exists a nonnegative integer I, which depends only on p and γ, such that

$$\|e_{m+n+1}\|_0 \leq \gamma \|e_0\|_0 \text{ for all } m+n \geq I.$$

Proof: The proof is by induction on the index k of the space. Assume the result is true for 1, 2, ... , k−1. We now prove the result for M_k. From (1.13) it is immediate that

$$\|e_n\|_0 \leq \|e_0\|_0. \tag{2.15}$$

Using an argument similar to the proof of Lemma 1.8 shows that

$$\|e_n\|_2 \leq C\mu^{-2} h_k^{-2} (2n+1)^{-1} \|e_0\|_0. \tag{2.16}$$

Lemmas 2.2 and 2.3 with $\alpha = 1$ yield

$$\|\bar{q} - e_n\|_0 \leq C\mu^2 h_k^2 \|e_n\|_2. \tag{2.17}$$

The analogue of (1.17) is derived using the induction hypothesis and (2.15) – (2.17):

$$\|q-\bar{q}\|_0 \leq \gamma^p \|\bar{q}\|_0$$

$$\leq \gamma^p \{\|\bar{q}-e_n\|_0 + \|e_n\|_0\}$$

$$\leq \gamma^p \{C\mu^2 h_k^2 \|e_n\|_2 + \|e_0\|_0\}$$

$$\leq \gamma^p \{C(2n+1)^{-1} + 1\}\|e_0\|_0$$

$$\leq C\gamma^p \|e_0\|_0.$$

Using an argument similar to (1.18) gives us

$$\|e_{m+n+1}\|_0 \leq C\{(2m+1)^{-1}(2n+1)^{-1} + \gamma^p\}\|e_0\|_0$$

and Theorem 2.4 follows.

QED

When $p = 1$ we can prove the analogue to Theorem 1.10.

It is useful to associate with M_j a symmetric, positive definite bilinear form $b_j(\cdot,\cdot)$, which is comparable to the L^2 inner product. We assume there exists a constant β, independent of h_j, such that

$$0 < \beta^{-1}(v,v) \leq b_j(v,v) \leq \beta(v,v) \text{ for all } v \in M_j,\ v \neq 0. \tag{2.18}$$

For $b_j(\cdot,\cdot)$, we define generalized eigenvalues $\tilde{\lambda}_i$ and eigenfunctions $\tilde{\psi}_i$, $1 \leq i \leq N_j$, by

$$a(v,\tilde{\psi}_i) = \tilde{\lambda}_i b_j(v,\tilde{\psi}_i) \text{ for all } v \in M_j, \quad 0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \ldots \leq \tilde{\lambda}_{N_j} \equiv \tilde{\lambda}_j,$$

$$b_j(\tilde{\psi}_i,\tilde{\psi}_k) = \delta_{ik}, \quad \text{and} \quad a(\tilde{\psi}_i,\tilde{\psi}_k) = \tilde{\lambda}_i\delta_{ik},\ 1 \leq k \leq N_j.$$

It then follows that

$$\tilde{\lambda}_j = \max_{v\in M_j,\, v\neq 0} \frac{a(v,v)}{b_j(v,v)} \leq \beta \max_{v\in M_j,\, v\neq 0} \frac{a(v,v)}{(v,v)}$$

$$\leq C(P,S,\Omega,\delta_0,\delta_1,\beta)h_j^{-2}.$$

For $-2 \leq s \leq 2$, define $|||v|||_{s,b}^2$ similarly to $|||v|||_s^2$ (see (2.11)). Note that $|||v|||_{1,b} = |||v|||$ and $|||v|||_{0,b}$ is comparable to $\|v\|_0$. In fact, a norm equivalence similar to that in Lemma 2.2 can be proved with $|||v|||_{s,b}$ substituted for $|||v|||_s$.

The smoothing iteration (1.7),

$$(z_i - z_{i-1}, v)_k = \Lambda_k^{-1} [f_k(v) - a_k(z_{i-1}, v)], \text{ for all } v \in M_k,$$

requires the solution of a linear system involving the mass matrix at every step. In practice this is too expensive. We can replace (1.7) with the smoothing iteration

$$b_k(z_i - z_{i-1}, v) = \Lambda_k^{-1} [(f,v) - a(z_{i-1}, v)], \text{ for all } v \in M_k. \tag{2.19}$$

There are numerous choices for $b_j(\cdot,\cdot)$, $j \geq 1$. When the standard nodal basis is used, (2.18) is satisfied by $b_j(\cdot,\cdot)$ corresponding to the diagonal of the mass matrix. This allows smoothing by an under-relaxed Jacobi scheme.

The convergence of the k-level scheme (Algorithm 1.4) is summarized by the following result. Its proof is almost identical to the ones for Theorems 1.9 and 2.4.

Theorem 2.5: Assume the Regularity Hypothesis holds. Define the k-level scheme (Algorithm 1.4) using (2.19) instead of (1.7). Let $\|\cdot\|$ denote either the energy or L^2 norm (and be fixed). Let $p > 1$ be any fixed integer. For any constant $0 < \gamma < 1$ there exists a nonnegative integer I, which depends only on p and γ, such that

$$\|e_{m+n+1}\| \leq \gamma \|e_0\| \text{ for all } m+n \geq I.$$

We can extend this theorem to cover the case of $p = 1$ analagously to Theorem 1.10.

We conclude this section by noting that for the finite-element method of this section, Algorithm 1.12 converges at the rate specified by Theorem 1.13 for either the energy or L^2 norm. Let $\|\cdot\|$ denote either of these norms. Then Theorem 1.13 can be rewritten as follows:

Theorem 2.6: Assume the Regularity Hypothesis holds. Let $r \geq 1$ be any fixed integer. Suppose

(i) $\|u-u_j\| \leq Kh_j^q$, $j \geq 1$, and

(ii) $\|u_1-\tilde{u}_1\| \leq Kh_1^q$.

Then for any integer $p \geq 1$, there exists a nonnegative integer I such that for $j \geq 1$,

$\|u_j-\tilde{u}_j\| \leq Kh_j^q$ for all $m+n \geq I$.

Moreover,

$\|u-\tilde{u}_j\| \leq 2Kh_j^q$,

and the cost of computing $\tilde{u}_j$ is bounded by $C_9F(N_j)$, where
$C_9 \equiv \sigma r/(\sigma-1)$ is independent of j and σ is defined by (2.8).

It is worth pointing out that r is independent of j. This implies that Algorithm 1.12 is optimal order when $1 \leq p < \sigma$.

2.3 Finite-Differences

As in Section 2.2, we seek weak form solutions to both (2.4) and (2.5). We prove that the theorems and analysis of Section 1 apply to a particular discretization of (2.4) and (2.5). Most of this section consists of an analysis of a constant coefficient problem. In Section 2.3.3, we explain how to extend the analysis to the variable coefficient self-adjoint problems (2.4) and (2.5).

2.3.1 One-Dimensional Definitions

Let $k > 1$ and $N_0 > 0$ be fixed integers. We define $N_j \equiv 2N_{j-1} + 1$, $j \geq 1$. Following (1.2), we know that $\sigma = 2$. We define k uniform grids Ω^j, $1 \leq j \leq k$, by

$$h_k \equiv (N_k + 1)^{-1}, \quad h_j \equiv 2^{k-j}h_k, \quad \text{and} \quad \Omega^j \equiv \{ ih_j \mid 1 \leq i \leq N_j \}.$$

For the moment, we assume that $P \equiv 1$ and $S \equiv 0$ in (2.4). The general case will be

analyzed at the end of the section. We discretize (2.4) by approximating the second derivatives by central differences to obtain a system of linear equations

$$A_k u_k = h_k f_k, \tag{2.20}$$

where u_k and f_k approximate the solution u and the right-hand side f on Ω^k. The N_k x N_k matrix A_k is given by

$$A_k \equiv h_k^{-1} \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix} \tag{2.21}$$

(see Varga [30]). Factoring and solving this algebraic system directly requires only $5N_k$ multiplications and $3N_k$ additions [16]. We will show that the multi-grid algorithms of Section 1 require $O(N_k \log N_k)$ operations to solve this problem to accuracy comparable to the discretization error, i.e., $O(N_k^{-2})$.

We associate a *solution space* M_j, $1 \leq j \leq k$, with each grid. Let $N \equiv N_k$. Then

$$M_k \equiv \{ v \mid v \in R^N \} \quad \text{and}$$

$$M_{k-1} \equiv \{ v \mid v \in M_k \text{ and } v = < 0, v_1, 0, v_2, 0, \ldots, 0, v_{N_{k-1}}, 0 > \}. \tag{2.22}$$

The spaces $M_{k-2}, \ldots, M_1$ are defined recursively.

We define *piecewise-linear interpolation* operators between adjacent solution spaces

$$E_2: M_{j-1} \rightarrow M_j \quad \text{and} \quad R: M_j \rightarrow M_{j-1}$$

by means of the matrices

$$E_2 \equiv \begin{bmatrix} 0 & .5 & 0 & 0 & & & & & & & & \\ 0 & 1 & & & & & & & & & & \\ 0 & .5 & 0 & .5 & 0 & 0 & & & & & & \\ . & & & 1 & & & & & & & & \\ . & 0 & 0 & .5 & 0 & .5 & 0 & 0 & & & & \\ . & & & & & 1 & & & & & & \\ & & & 0 & 0 & .5 & 0 & .5 & 0 & 0 & & \\ & & & & & & . & & & & & \\ & & & & & & & . & & & . & \\ & & & & & & & & . & & . & \\ & & & & & & & & 1 & & . & \\ & & & & & & 0 & 0 & .5 & 0 & .5 & 0 \\ & & & & & & & & & & 1 & 0 \\ & & & & & & & & 0 & 0 & .5 & 0 \end{bmatrix} \tag{2.23}$$

and $R \equiv E^T$.

Problems on coarser grids are defined using the interpolation matrices. Linear systems on coarser grids, A_j, $1 \leq j < k$, are defined recursively:

$$A_{k-j} \equiv E_2^T A_{k-j+1} E_2.$$

Let $r \in R^N$ be a right-hand side for level k. Then the *(k−1)-level problem* is defined by

$$A_{k-1} \bar{q} = E_2^T r \equiv f_{k-1}, \ \bar{q} \in M_{k-1}. \tag{2.24}$$

Half the rows and columns of A_{k-1} are zero. We can re-order the matrix so that the nonzero rows and columns are the first N_{k-1} rows and columns. Then A_{k-1} has a submatrix whose form is identical to A_k. The re-ordered solution space M_{k-1} has the form

$$\{ v \in M_k \mid v = < v_1, v_2, \ldots, v_{N_{k-1}}, 0, \ldots, 0 > \}.$$

The (k−i)-level problems are defined similarly.

Before the theory of Section 1 can be applied, we must complete the definitions required for the triples used by Problems 1.1 and 1.2. For $j \geq 1$, a bilinear form $a_j(\cdot,\cdot)$, a linear functional $f_j(v)$, and an inner product $(\cdot,\cdot)_j$ are defined by

$$\begin{aligned} &a_j(u,v) \equiv u^T A_j v, \quad f_j(v) \equiv h_j f_j^T v, \text{ for all } u,v \in M_j, \text{ and} \\ &(u,v)_j \equiv h_j u^T v. \end{aligned} \tag{2.25}$$

The bilinear form $a_j(\cdot,\cdot)$ induces the energy norm $|||u|||^2 \equiv a_j(u,u)$ for all $u \in M_j$. Associated with each $a_j(\cdot,\cdot)$ are $N \equiv N_j$ nonzero eigenvalues $\lambda_i^{(j)}$ and eigenvectors $\psi_i^{(j)}$ satisfying (1.6). Let Λ_j be the largest eigenvalue. For $-2 \leq s \leq 2$, discrete norms are defined by

$$|||v|||_s^2 \equiv \sum_{i=1}^{N} \beta_i^2 (\lambda_i^{(j)})^s \text{ for } v \equiv \sum_{i=1}^{N} \beta_i \psi_i^{(j)}, \tag{2.26}$$

where we have suppressed the j subscript on the norm. Note that $|||v|||_1 = |||v|||$ is the usual energy norm on level j. Hereafter, we drop the superscripts from the eigenvalues and eigenvectors.

2.3.2 Two-Dimensional Definitions

Let $k > 1$ and $\bar{N}_0 > 0$ be fixed integers. We define $\bar{N}_j \equiv 2\bar{N}_{j-1} + 1$, $j \geq 1$. We define k uniform grids Ω_2^j, $1 \leq j \leq k$, as products of the one-dimensional domains Ω^j:

$$h_k \equiv (\bar{N}_k + 1)^{-1}, \quad h_j \equiv 2^{k-j} h_k, \quad \text{and} \quad \Omega_2^j \equiv \Omega^j \otimes \Omega^j.$$

Each grid Ω_2^j covers the interior of Ω with $N_j \equiv \bar{N}_j^2$ points. Following (1.2), we know that $\sigma \equiv 4$.

For the moment, we assume that $P \equiv 1$ and $S \equiv 0$ in (2.5). The general case will be analyzed in Section 2.3.3. We discretize (2.5) by central differences to obtain a system of linear equations

$$B_k u_k = h_k^2 f_k, \tag{2.27}$$

where u_k and f_k approximate the solution u and the right-hand side f on Ω_2^k. The N_j x N_j matrix B_k is given by

$$B_k \equiv h_k \{A_k \otimes I_N + I_N \otimes A_k\},$$

where I_N is the $N_k x N_k$ identity matrix and A_k is defined in (2.21).

We define solution spaces M_j, $1 \leq j \leq k$, as the tensor-product of the one-dimensional solution spaces defined in (2.22). Following the techniques of Section 2.3.1, we define piecewise-bilinear interpolation operators between adjacent solution spaces

$$E_b \equiv E_2 \otimes E_2 \colon M_{j-1} \rightarrow M_j.$$

Problems on coarser grids are defined using the interpolation matrices. Linear systems on coarser grids, B_j, $1 \leq j < k$, are defined recursively:

$$B_{k-j} \equiv E_b^T B_{k-j+1} E_b.$$

Let $N \equiv N_k$ and $r \in R^N \otimes R^N$ be a right-hand side for level k. Then the *(k−1)-level problem* is defined by

$$B_{k-1} \bar{q} = E_b^T r \equiv f_{k-1}, \quad \bar{q} \in M_{k-1}. \tag{2.28}$$

Similar to the one-dimensional case, we define

$$a_j(u,v) \equiv u^T B_j v, \quad f_j(v) \equiv h_j^2 f_j^T v, \text{ for all } u,v \in M_j, \quad \text{and}$$

$$(u,v)_j \equiv h_j^2 u^T v.$$

The bilinear form $a_j(\cdot,\cdot)$ induces the energy norm $|||u|||^2 \equiv a_j(u,u)$ for all $u \in M_j$. Associated with each $a_j(\cdot,\cdot)$ are N_j nonzero eigenvalues $\lambda_{\rho\omega}$ and eigenvectors $\psi_{\rho\omega}$ satisfying (1.6). Let Λ_j be the largest eigenvalue. For $-2 \leq s \leq s$, discrete norms $|||v|||_s$ are defined similarly to (2.26).

2.3.3 Convergence

In order to establish the convergence of Algorithms 1.4 and 1.12 for the finite-difference cases of this chapter, we must verify Hypotheses 1.3, 1.6, and 1.7. In this section, E will refer to either E_2 or E_b. We begin by showing that (the Energy Norm Consistency) Hypothesis 1.3 holds. It is derived using the definition of the linear systems A_{j-1} and B_{j-1}, $j > 1$.

Lemma 2.7: Let $j > 1$ be an integer. Then

$$a_j(Ev,Ew) = a_{j-1}(v,w), \text{ for all } v,w \in M_{j-1}.$$

That (the Maximum Eigenvalue) Hypothesis 1.6 holds is a simple consequence of the

explicit formula for the eigenvalues of A_j and B_{j-1}.

Lemma 2.8: $\Lambda_j \leq 4dh_j^{-2}$ where d is the dimension of the problem.

The proof that Hypothesis 1.7 holds is a simplified version of the proof of Lemma 2.3 with $\alpha = 1$. It requires Lemma 2.9.

Lemma 2.9: Let $k > 1$ be an integer and d the dimension of the probelm. Let $C(s,1) = 4^{(s-1)/2}$ and $C(s,2) = 8^{s/2}$. For $s = 0$, 1, or 2 and $u \in M_k$,

$$\min_{v \in M_{k-1}} |||u-Ev|||_s \leq C(s,d)\, h_k^{2-s}\, |||u|||_2. \tag{2.29}$$

The proof is contained in Douglas [13]. This proves that Algorithm 1.4 converges at the rates specified by Theorems 1.9 and 1.10 for the finite-difference cases of this section. As a consequence of Lemma 2.9 we can estimate the constant C_4 in (1.18), which governs the number of smoothing steps required in Algorithm 1.4. For the one-dimensional problem (2.4), $C_4 = 4$ and for the two-dimensional problem (2.4), $C_4 = 16 \cdot 2^{1/2}$.

Before we can show that the j-level scheme (Algorithm 1.12) holds, we need to define projection and injection operators between the spaces H and M_j, $1 \leq j \leq k$ (see (1.3)):

p_j: $H \xrightarrow[\text{onto}]{} M_j$ as evaluation of $u \in H$ at the grid points of Ω^j and

i_j: $M_j \xrightarrow{1-1} H$ as piecewise-linear interpolation.

These operators have properties which are worth pointing out, namely,

$p_j i_j$ = Identity on Ω^j and $i_j E p_{j-1} = i_{j-1} p_{j-1}$ on Ω^{j-1}.

We can verify that the assumptions of Theorem 1.13 hold using these properties and simple known facts about the model problems. Details are contained in Douglas [13]. This shows that Algorithm 1.12 converges at the rate specified in Theorem 1.13.

We conclude this section by considering (2.4) and (2.5) when they are no longer restricted to $P \equiv 1$ and $S \equiv 0$. We discretize (2.4) and (2.5) by central differences to get $N_k \times N_k$ linear system of equations. As before, we define coarse grid approximations of

the problems using the interpolation matrices. We define inner products $\tilde{a}_j(u,v)$ using the linear systems of equations. Energy norms are defined by $||||u||||^2 \equiv \tilde{a}_j(u,v)$. The discrete norms $||||u||||_s$, $0 \leq s \leq 2$ and $u \in M_j$, are defined as usual. We prove the following theorem in Douglas [13]:

Theorem 2.10: Let $j \geq 1$ be a fixed integer and d the dimension of the problem. Then the following holds:

(a) the linear systems are symmetric and have N_j positive eigenvalues which are bounded by Ch_j^{-2} where C is independent of j.

(b) For $\beta = 0$, 1, or 2, there exist positive constants $C_{1,\beta}$ and $C_{2,\beta}$ such that

$$C_{1,\beta}|||u|||_\beta \leq ||||u||||_\beta \leq C_{2,\beta}|||u|||_\beta \text{ for all } u \in M_j.$$

(c) $||||E_b\bar{q}-e_n||||_0 \leq K(d)\, C_{2,1}\, h_k^2\, ||||e_n||||_2$, where $K(1) = 1$ and $K(2) = 8^{1/2}$.

Theorems 1.9, 1.10, and 1.13 can be rewritten using the $||||\cdot||||$ norm instead of the $|||\cdot|||$ norm. This proves that the variable coefficient Dirichlet problems (2.4) and (2.5) can be solved using the theory of Chapter 1 in $O(N_k)$ operations.

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