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Finite Element Methods
for Singular Two-Point Boundary Value Problems

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ABSTRACT

FINITE ELEMENT METHODS FOR SINGULAR TWO-POINT BOUNDARY VALUE PROBLEMS

Robert Samuel Schreiber
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Until quite recently, few effective numerical solution techniques were known for solving two-point boundary value problems for the equation

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + qu = f, \quad 0 < x < 1, \quad p(0) = 0.$$

In this dissertation we analyze several new finite element methods for approximating the solution of this problem, and present new analyses for some known methods.

Two classes of singular problem may be distinguished, depending on whether or not $p(x)^{-1}$ $L^1(0,1)$. In the first of these, $p(x)$ behaves like x^σ, $0 \le \sigma < 1$, near 0. The solution $u(x)$ has a singularity (like $x^{1-\sigma}$) at 0, so that its derivatives there are infinite. This frustrates all the usual theory. The finite element techniques proposed so far require that the basis include functions which mimic the behavior of the solution.

We investigate the idea of approximating the solution with piecewise polynomials on a nonuniform mesh adapted to the singularity. Given knowledge of the singularity (i.e., σ), it is possible to

construct a sequence of graded meshes such that the rate of convergence in the L^2-norm is the best possible. We prove upper and lower bounds on the extent to which the mesh must be graded.

The solution can also be approximated by a function of the form $x^{-\sigma}s(x)$, with $s(x)$ a piecewise polynomial. We obtain error bounds and numerical results for these "weighted splines" which indicate that they are the best for practical computation. For a third subspace (due to Crouzeix and Thomas), we improve known error bounds by using a mildly graded mesh.

Problems of the second kind, where $p(x)$ behaves like x^k near 0, with k a positive integer, arise from spherically symmetric elliptic boundary value problems in $n = k+1$ dimensions. The solutions are smooth, indicating that piecewise polynomials on uniform or nearly uniform mesh will be effective approximators.

We improve previous results by removing any restriction on n, by adding the requirement that the approximating spline functions be smooth at the center of the n-dimensional domain, and by obtaining error bounds (of the best possible order) in the natural norms for the problem.

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PART I

CHAPTER 1

INTRODUCTION

1.1 An Example

Consider the two-point boundary value problem

$$(1.1) \qquad -\frac{d}{dx}(\sqrt{x}\,\frac{du}{dx}) = \frac{3}{2}, \qquad 0 < x < 1,$$

with boundary conditions

$$(1.2) \qquad u(0) = u(1) = 0.$$

The unique solution, $u(x) = x^{1/2} - x^{3/2}$, is (uniformly) continuous on $[0,1]$ and continuously differentiable (indeed analytic) in $(0,1)$. However, its derivative grows arbitrarily large as x approaches 0; consequently, while $u \in L^2(0,1)$, none of its derivatives is in $L^2(0,1)$. Therefore, analyses of numerical solution techniques for two-point boundary value problems which require that certain derivatives of u be bounded don't apply to this problem.

Equation (1.1) falls outside the realm of problems treated in most discussions of numerical methods for boundary value problems (both for ordinary and elliptic partial differential equations) because the coefficient $\sqrt{x}$ vanishes at a point of the boundary of the domain. This "singular point" is precisely where the derivatives of the solution blow up, making it impossible to apply well-known numerical techniques to the problem. The goal of this research is to extend numerical methods, specifically the finite element method, to apply to singular two-point boundary value problems. We hope to develop new methods for approximating the solution and to justify those methods with rigorous error bounds.

1.2 Summary of Results Obtained

We shall treat equations of the form

$$(1.3) \qquad -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = f(x), \qquad 0 < x < 1,$$

where $p(x) > 0$ for $x \in (0,1]$, and $p(0) = 0$. (In the nonsingular case, it is assumed that $p(x) > 0$ for all $x \in [0,1]$.) Two classes of singular problem may be distinguished, depending on whether or not the integral $\int_0^1 \frac{dt}{p(t)}$ is convergent, i.e., finite.

Part II of the dissertation is concerned with the first of these two types of singular two-point boundary value problem. Such problems arise in potential theory. A function u which satisfies the equation

$$L_\sigma[u] \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\sigma}{y}\frac{\partial u}{\partial y} = 0$$

is said to be a generalized axially symmetric potential in $n = \sigma+2$ dimensions; σ need not be integral (see [W1]). Parter, in his early treatment of numerical methods for generalized axially symmetric potentials in a rectangle, arrived at the equation

$$\frac{d^2v}{dx^2} + \frac{\sigma}{x}\frac{dv}{dx} - (\pi n)^2 v = 0, \quad 0 < x < 1,$$

$$v(0) = 1, \quad v(1) = 0, \quad |\sigma| < 1,$$

by separation of variables [P1]. Jamet [J1] later considered the more general problem

$$(1.4) \quad -D(x^\sigma \rho(x)Du) + q(x)u = f(x), \quad 0 < x < 1, \quad 0 \leq \sigma < 1,$$

with the boundary conditions (1.2), where ρ is a smooth function strictly positive in the interval [0,1]. For a more complete account of earlier work on numerical methods for this problem, see Section 4.1.

The difficulty with these problems is that their solutions are not smooth in the usual sense of having two more derivatives than f; in fact, their derivatives are unbounded at the origin. Nevertheless we are able to define and analyze two new numerical approximation schemes of high-order accuracy.

As approximators of the solutions of (1.3), we shall use functions which are splines (piecewise polynomials) or are in some way related to such functions. A function is a spline on, say [0,1], if it is a polynomial of some fixed degree on each interval of a given partition of [0,1]. In addition, various continuity requirements may be imposed; thus we may speak of C^1-cubics -- functions which are cubic polynomials in each interval and, together with their first derivatives, are continuous at the "knots" of the partition.

The essential point about the solution u of (1.4) - (1.2) is that $u(x)$ behaves like $x^{1-\sigma}$ near 0. (Thus, for example, $v(x) \equiv x^{\sigma}u(x)$ is smooth and vanishes at 0 (Theorem 5.1)). This leads immediately to the following idea. It has been shown that piecewise polynomials can approximate nonsmooth functions of the form x^{α}, α not an integer, essentially as well as smooth functions, simply by using a nonuniform mesh. Thus, it should be possible to approximate the solution of (1.4) - (1.2) with piecewise polynomials on a nonuniform mesh adapted to the singularity. We show how, given knowledge of the singularity (i.e., σ), to construct a sequence of meshes such that the rate of convergence is the best possible.

As an aside, we give a problem and a subspace (one involving a partition which is not sufficiently skewed towards 0) for which the RRG approximation does not converge at the best possible rate in the L^2-norm; it is worse by an order of magnitude than the L^2-projection of the solution. This appears to be the first such example reported. It has been shown (Eisenstat, Schreiber, and Schultz [E1]) that this cannot happen for a wide class of <u>nonsingular</u> problems.

We next consider subspaces of "weighted splines," functions of the form $x^{-\sigma}s(x)$ with $s(x)$ a spline. Since we know that $u(x) = x^{-\sigma}v(x)$ where v is smooth, it seems likely that weighted splines will be good approximators of u. We obtain error bounds which show that this is indeed the case. These spaces appear to be the best for practical computation.

Work on this problem using the finite element method was begun by Ciarlet, Natterer, and Varga [C1], who used subspaces of functions which are piecewise elements of the null space of the differential operator. These "L-splines" were generalized by Crouzeix and Thomas [C4], who investigated functions which are mapped into polynomials (not just the zero polynomial) by the operator. Applying these spaces to the problem (1.4), Crouzeix and Thomas obtained energy-norm and L^2-norm error bounds which are high-order, but not quite optimal. We present their theory (with some simplifications) in Chapter 8. We then improve their results by showing that if a mildly graded partition is used, then the error in their procedure is of optimal order.

Part III of the dissertation is concerned with singular problems (1.3) in which $p(x) = x^k$ where k is a positive integer. The boundary conditions are not those of (1.2), but rather,

$$(1.5) \qquad u(1) = 0, \quad Du(0) = 0 \quad \text{(alternatively, } u(0) \text{ finite).}$$

These problems arise from multidimensional elliptic boundary value problems (for example, the Dirichlet problem for Poisson's equation $-\Delta U = F$) which possess spherical symmetry. The solution is a function only of distance r from the origin in $\mathbb{R}^n$ and can be obtained by solving a singular two-point boundary value problem of this type. Unlike problems of the first type, the solution is smooth. Nevertheless, the usual theory for two-point boundary value problems breaks down when applied here. A discussion of earlier work on numerical methods for this problem is given in Section 9.1.

We propose to approximate the solution of such an equation by spherically symmetric functions which are splines in the variable r (distance from the center of the domain). This procedure has been analyzed previously ([D6], [J3]) for the special cases $n = 2$ and 3, with error bounds obtained in the usual Sobolev norms on the interval [0,1]. We improve these results in three directions: by removing any restriction on n, by adding the requirement that the approximating spline functions be smooth at the center of the n-dimensional domain, and by obtaining error bounds (of the best possible order) in the "natural" norms for the problem -- the Sobolev norms on the original domain in $\mathbb{R}^n$ instead of the interval to which the problem has been reduced.

1.3 Outline of the Dissertation

There are three main parts to the dissertation, the first of which is devoted to mathematical preliminaries. The numerical methods we employ are all cases of a general class of approximation techniques for solving boundary value problems: the Rayleigh-Ritz-Galerkin (RRG) method. Chapter 2 is a discussion of the Rayleigh-Ritz-Galerkin procedure as applied to singular two-point boundary value problems. Chapter 3 introduces the spaces of spline functions to be used as approximate solutions.

In Part II, we consider the problem (1.4) - (1.2). In Chapter 4, we introduce the variational form of the problem in an appropriate class of function spaces. One of the underlying ideas of Part II is the use of splines on a class of graded nonuniform partitions of [0,1]. These "β-graded" meshes have previously been shown to be useful in approximating functions with a singularity. We define the β-graded meshes and summarize several of their properties in Section 4.3. Chapter 5 is a compendium of results on the properties of the solution which will be used throughout Part II. In Chapter 6, we consider the approximation of singular functions (elements of the spaces introduced in Chapter 4) by piecewise polynomials on a β-graded mesh, and in Chapter 7, we consider approximation by weighted splines. Numerical results are included in both chapters. In Chapter 8, we deal with the generalized L-splines of Crouzeix and Thomas.

Part III of the dissertation is concerned with singular problems (1.3) - (1.5) in which $p(x)$ behaves like x^k near 0, where k is a positive integer. In Chapter 9, we obtain the variational form of the problem, and in Chapter 10, we define and analyze several spline approximation schemes. Numerical results are included.

CHAPTER 2

THE RAYLEIGH-RITZ-GALERKIN PROCEDURE FOR SINGULAR TWO-POINT BOUNDARY VALUE PROBLEMS

2.1 Notation

In this section we introduce notation, definitions, and results which will be used throughout this dissertation.

Let $I \equiv [0,1]$. For each integer $t \geq 0$, let $D^t f(x) \equiv \frac{d^t f}{dx^t}(x)$. For S a finite set, let $|S|$ denote the number of elements of S.

All the functions we deal with will be real-valued. The support of a function f, $\text{supp}(f)$, is the closure of the set of points x such that $f(x) \neq 0$. If f is a bounded function on (a,b), we write

$$\|f\|_{L^\infty(a,b)} \equiv \sup_{a<x<b} |f(x)|.$$

For m a non-negative integer, let $H^m(a,b)$ (respectively, $H_0^m(a,b)$) be the closure of the C^∞ functions (respectively, the C^∞ functions with compact support in (a,b)) with respect to the norm

$$\|f\|_{H^m(a,b)} = \left(\sum_{j=0}^{m} \int_a^b (D^j f(x))^2 \, dx \right)^{1/2}.$$

These are known as Sobolev spaces. Note that $H^m(a,b)$ may be identified with the space of C^{m-1} functions with absolutely continuous $(m-1)^{th}$-derivative and m^{th}-derivative in $L^2(a,b)$, while $H_0^m(a,b)$ may be identified with the subspace of $H^m(a,b)$ consisting of functions which vanish, along with their first m-1 derivatives, at a and b. Thus, $H^0(a,b) = L^2(a,b)$. $H^m(a,b)$ is a Hilbert space with inner product

$$(f,g)_{H^m(a,b)} = \sum_{j=0}^{m} \int_a^b D^j f \, D^j g \, dx.$$

For a complete discussion of Sobolev spaces, see Adams [A1].

We also define $W^{m,\infty}(a,b)$ (respectively, $W_0^{m,\infty}$) as the closure of $C^\infty(a,b)$ (respectively, $\{f \in C^\infty(a,b) \mid f$ has compact support in $(a,b)\}$) with respect to the norm

$$\|f\|_{W^{m,\infty}(a,b)} = \sum_{j=0}^{m} \|D^j f\|_{L^\infty(a,b)}.$$

We denote by H^m (respectively, H_0^m, L^∞, $W^{m,\infty}$) the space $H^m(0,1)$ (respectively, $H_0^m(0,1)$, $L^\infty(0,1)$, $W^{m,\infty}(0,1)$). Moreover, we denote the norm and inner product of H^m by $\|\cdot\|_m$ and $(\cdot,\cdot)_m$, and let $\|f\|_{L^\infty} \equiv \|f\|_{L^\infty(0,1)}$.

<u>Theorem 2.1</u> (Rayleigh-Ritz inequality) [B1]: If $f \in H_0^1(a,b)$, then

$$\pi^2 \int_a^b f^2 \, dx \le (b-a)^2 \int_a^b (Df)^2 \, dx.$$

Theorem 2.2 (Markov's inequality) [M1]: For all polynomials p_n of degree n and all real a, b with a < b,

$$\| Dp_n \|_{L^\infty(a,b)} \leq n^2 (b - a)^{-1} \| p_n \|_{L^\infty(a,b)} .$$

In Part III, we will be concerned with functions defined on a bounded, open subset B of $\mathbb{R}^n$. We define the space $H^m(B)$ by taking the closure of the C^∞-functions with respect to the norm

$$\| f \|_{H^m(B)} = \left(\sum_{|\alpha|<m} \int_B (D^\alpha f)^2 \, d\underline{x} \right)^{1/2},$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \, \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}, \qquad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \qquad |\alpha| = \sum_{i=1}^{n} \alpha_i .$$

Sobolev's lemma [F3, p. 283] is concerned with bounds on the value of a function at an arbitrary point in terms of L^2-norms of its derivatives. While the following theorem can be proved for a more general class of domains, we do not require anything more than a ball in $\mathbb{R}^n$ (an interval in $\mathbb{R}$). Let $\lfloor x \rfloor$ denote the largest integer not exceeding x.

Theorem 2.3: Let B be a ball in $\mathbb{R}^n$. If $u \in H^m(B)$, where $m = \lfloor \frac{n}{2} \rfloor + 1$, then u can be identified with a uniformly continuous function u(x) in B such that

$$\| u \|_{L^\infty(B)} \leq C_n \| u \|_{H^m(B)} ,$$

where C_n is a constant which depends only on n.

In the one-dimensional case, Sobolev's lemma shows that $H^1(a,b) \subset L^\infty(a,b)$ and

$$\| f \|_{L^\infty(a,b)} \leq C_1 \| f \|_{H^1(a,b)} \quad \text{for all } f \in H^1(a,b).$$

If, in addition, f(a) = 0, we have a stronger bound.

Lemma 2.1: If $f \in H^1(a,b)$ and $f(a) = 0$, then

$$\| f \|_{L^\infty(a,b)} \leq (b - a)^{1/2} \| Df \|_{L^2(a,b)}. \tag{2.1}$$

Proof: Since f is absolutely continuous,

$$|f(x)| = \left| \int_a^x Df(t)\, dt \right|.$$

Using the Cauchy-Schwarz inequality,

$$\begin{aligned} |f(x)| &\leq (x - a)^{1/2} \left(\int_a^x (Df(t))^2\, dt \right)^{1/2} \\ &\leq (b - a)^{1/2} \| Df \|_{L^2(a,b)}. \end{aligned}$$

□

We will sometimes use the letter C to denote a generic constant, not the same at each occurrence. Within each chapter we define "local" constants c_0, c_1, ..., to be used and referred to only within that chapter.

2.2 The Rayleigh-Ritz-Galerkin Procedure

We consider the equation

$$(2.2) \qquad Lu = f,$$

where L is a densely defined linear operator on a (real) Hilbert space H and $f \in H$. The development follows Mikhlin [M2], to which the reader is referred for further details and proofs. In the sequel, we shall be concerned with Hilbert spaces of functions defined on a real interval (a,b). In Part II, $L^2(a,b)$ plays the role of H, while in Part III, where we study elliptic problems in a ball in $\mathbb{R}^n$, H is the space of functions square integrable on (a,b) with respect to the weight function x^{n-1}. L is a symmetric differential operator with principal part $-D(p(x)Du)$ and $p(0) = 0$.

We assume that L is defined on a dense subspace M of H and that L is symmetric and positive definite: for every $u,v \in M$,

$$(2.3) \qquad a(u,v) \equiv (Lu,v) = (u,Lv)$$

and

$$(2.4) \qquad \|v\|_E^2 \equiv a(v,v) \geq \frac{1}{\delta^2}\|v\|_H^2,$$

where δ is a constant independent of v. It is well-known that the closure S of M with respect to the norm $\|\cdot\|_E$ is a Hilbert space (and a subspace of H) with inner product $a(u,v)$ and corresponding norm $\|\cdot\|_E$.

Theorem 2.4 [S4]: For every $f \in H$, there exists a unique element $u \in S$, called the generalized solution of (2.2), satisfying

(2.5) $$a(u,v) = (f,v)_H \quad \text{for all } v \in S.$$

Moreover, $u \in S$ satisfies (2.5) if and only if it minimizes the quadratic functional

$$F[u] \equiv a(u,u) - 2(f,u)_H, \quad u \in S.$$

The solution u is bounded in terms of the data f. For, by (2.5) and the Cauchy-Schwarz inequality,

$$\begin{aligned} \|u\|_E^2 = a(u,u) &= (f,u)_H \\ &\leq \|f\|_H \|u\|_H \\ &\leq \|f\|_H \, \delta \, \|u\|_E, \end{aligned}$$

whence

(2.6) $$\|u\|_E \leq \delta \|f\|_H,$$

and by (2.4),

(2.7) $$\|u\|_H \leq \delta^2 \|f\|_H.$$

Let S_n be a finite-dimensional subspace of S. The Galerkin approximation to u in S_n is the unique element $\tilde{u}$ of S_n satisfying

(2.8) $$a(\tilde{u},v_n) = (f,v_n)_H \quad \text{for all } v_n \in S_n.$$

The Rayleigh-Ritz approximation is the unique element of S_n which minimizes the functional $F[u]$ over S_n. Just as in the case of the generalized solution, these definitions are equivalent; we call this element the Rayleigh-Ritz-Galerkin (RRG) approximation to u in S_n.

<u>Theorem 2.5</u>: The Rayleigh-Ritz-Galerkin approximation $\tilde{u}$ is the best possible approximation to u in the subspace S_n with respect to the norm $\|\cdot\|_E$, i.e.,

$$\|u - \tilde{u}\|_E = \inf_{v_n \in S_n} \|u - v_n\|_E. \tag{2.9}$$

<u>Proof</u>: Subtracting (2.8) from (2.5) shows that the error $u - \tilde{u}$ is orthogonal to S_n, i.e.,

$$a(u - \tilde{u}, v_n) = 0 \quad \text{for all } v_n \in S_n.$$

The bound (2.9) then follows by a standard argument (see [S3]).

□

Suppose we want to determine the coefficients of the RRG approximation with respect to a basis $\{B_1, B_2, \ldots, B_n\}$ for S_n. By (2.8),

$$a(\tilde{u}, B_j) = (f, B_j)_H, \quad 1 \leq j \leq n.$$

Thus, if

$$\tilde{u} = \sum_{i=1}^{n} \xi_i B_i,$$

then $\underline{\xi}$ is the solution of the linear system of equations

$$A\underline{\xi} = \underline{f}, \quad A = [a(B_i, B_j)], \quad \underline{f} = [(f, B_i)_H]. \tag{2.10}$$

The equations (2.10) have a unique solution, since the matrix A is symmetric and positive definite. In fact, its symmetry is apparent, and if $\underline{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)^T \in \mathbb{R}^n$, then

$$\underline{\eta}^T A \underline{\eta} = a\left(\sum_{i=1}^{n} \eta_i B_i, \sum_{i=1}^{n} \eta_i B_i\right) = \left\| \sum_{i=1}^{n} \eta_i B_i \right\|_E^2 \geq 0$$

with equality if and only if $\underline{\eta} = 0$. We will later introduce subspaces with computationally attractive local bases (the functions B_i are zero over most of (a,b)). The resulting linear systems will be sparse and, for moderate sized systems, well-conditioned.

When H is a function space on (a,b) and L is a differential operator of the form (1.3), the coefficients p and q of L enter into the energy norm, which makes it cumbersome to use in approximation-theoretic arguments. Therefore, we will work with an equivalent norm $\|\cdot\|_S$; we assume that

$$(2.11) \qquad \lambda \|v\|_S \leq \|v\|_E \leq \Lambda \|v\|_S \quad \text{for all } v \in S,$$

where λ and Λ positive constants independent of v. (In particular,

$$\|v\|_S \equiv \int_a^b w(x)(Dv)^2\, dx,$$

where in Part II, $w(x) = x^\sigma$, and in Part III, $w(x) = x^{n-1}$). The *a priori* bound

$$(2.12) \qquad \|u\|_S \leq \frac{\delta}{\lambda} \|f\|_H$$

then follows from (2.6); by (2.9) we obtain the error bound

$$\|u - \tilde{u}\|_S \leq \frac{1}{\lambda} \|u - \tilde{u}\|_E$$

(2.13)
$$\leq \frac{1}{\lambda} \inf_{v_n \in S_n} \|u - v_n\|_E$$

$$\leq \frac{\Lambda}{\lambda} \inf_{v_n \in S_n} \|u - v_n\|_S.$$

2.3 Error Bounds for the RRG Approximation

In this section, we think of the subspaces S_n as being piecewise polynomial spaces with respect to a mesh of N subintervals. For such spaces, the dimension n is a linear polynomial in N.

The RRG approximation $\tilde{u}$ is the best possible energy-norm approximation to u in the subspace S_n (equation (2.9)). Therefore, the first step in bounding $u - \tilde{u}$ is to find upper bounds for $\inf_{v_n \in S_n} \|u - v_n\|_E$. This is generally done by defining an approximation mapping from S into S_n (interpolation is a typical example) for which error bounds can be obtained _a priori_. Any such error bound implies a corresponding bound for $\|u - \tilde{u}\|_E$.

By (2.4), any bound on $\|u - \tilde{u}\|_E$ automatically induces a bound on $\|u - \tilde{u}\|_H$. However, the resulting bound is generally not sharp, in the sense that the dependence on N (the _order_ of the approximation) is not the best possible. When optimal-order error bounds in the H-norm are desired, more machinery must be developed. The following argument (Nitsche's trick) was first used by Nitsche [N2].

We make the following "approximation hypothesis" concerning the ability of elements of S_n to approximate solutions of the problem (2.2).

(A1) There exists an integer $k \geq 2$, subspaces $X_k \subset X_{k-1} \subset \ldots \subset X_2 \subset S$, and a constant A_1 such that, if $v \in X_\ell$, then

$$\inf_{v_n \in S_n} \| v - v_n \|_S \leq A_1 N^{-(\ell-1)} \| v \|_{X_\ell}.$$

In Part II, we take $\| u \|_{X_\ell} = \| D(x^\sigma Du) \|_{\ell-2}$, while in Part III, $\| u \|_{X_\ell} = \| x^{(n-1)/2} D^\ell u \|_0$, which is just the usual L^2-norm of $D^\ell u$ over the unit ball in $\mathbb{R}^n$.

We also need a "regularity hypothesis", giving bounds on the X_2-norm of a generalized solution of (2.2) in terms of the H-norm of the right-hand side.

(A2) There exists a positive constant Γ such that, if $f \in H$, then the generalized solution $u \in X_2$ and

$$\| u \|_{X_2} \leq \Gamma \| f \|_H.$$

In Part II, regularity is provided by Theorem 5.2; in Part III, by Lemma 9.2.

Theorem 2.6 (Nitsche): If S_n satisfies the approximation hypothesis (A1), the problem satisfies the regularity hypothesis (A2), and $u \in X_\ell$, then

$$(2.14) \qquad \|u - \tilde{u}\|_S \leq \frac{\Lambda}{\lambda} A_1 N^{-(\ell-1)} \|u\|_{X_\ell}$$

and

$$(2.15) \qquad \|u - \tilde{u}\|_H \leq (\Lambda A_1)^2 \Gamma N^{-\ell} \|u\|_{X_\ell}.$$

Proof: (2.14) follows immediately from the approximation hypothesis and the S-norm quasi-optimality (2.13) of the RRG approximation.

Let $\Psi \equiv u - \tilde{u}$ and let Φ be the generalized solution of (2.2) with right-hand side Ψ. By the definition (2.5) of the generalized solution,

$$\|u - \tilde{u}\|_H^2 = (\Psi, u - \tilde{u})_H = a(\Phi, u - \tilde{u}).$$

Let $\Phi_n \in S_n$ be a best E-norm approximation to Φ. Since $u - \tilde{u}$ is orthogonal (in the a(.,.) inner product) to S_n,

$$a(\Phi_n, u - \tilde{u}) = 0.$$

Adding the two previous equations and using the Cauchy-Schwarz inequality,

$$(2.16) \qquad \|u - \tilde{u}\|_H^2 = a(\Phi - \Phi_n, u - \tilde{u}) \leq \|\Phi - \Phi_n\|_E \|u - \tilde{u}\|_E.$$

By the approximation and regularity hypotheses,

$$\begin{aligned} \|\Phi - \Phi_n\|_E &\leq \Lambda \|\Phi - \Phi_n\|_S \\ &\leq \Lambda A_1 N^{-1} \|\Phi\|_{X_2} \\ &\leq \Lambda A_1 \Gamma N^{-1} \|\Psi\|_H. \end{aligned}$$

By the approximation hypothesis and the optimality of the RRG approximation in the E-norm (2.9),

$$\begin{aligned}\|u - \tilde{u}\|_E &= \inf_{v_n \in S_n} \|u - v_n\|_E \\ &\leq \Lambda \inf_{v_n \in S_n} \|u - v_n\|_S \\ &\leq \Lambda A_1 N^{-(\ell-1)} \|u\|_{X_\ell} .\end{aligned}$$

Using these two inequalities to bound the right-hand side of (2.16), we obtain (2.15).

□

CHAPTER 3

ON PIECEWISE POLYNOMIALS

3.1 Introduction

In this chapter, we introduce a broad class of piecewise polynomial (spline) subspaces, including those used in practical computation. After defining the well-known "B-spline" basis functions and stating several of their properties, we discuss approximation by splines, specifically, the quasiinterpolant. The main result is a local error bound for the quasiinterpolant, due to de Boor and Fix [D3]. We also show that the quasiinterpolant can be made to interpolate at the end points, a fact we will later find useful in showing the existence of spline approximations satisfying various boundary conditions.

3.2 Piecewise Polynomial Approximation

Let $\Delta{:}a = x_0 < x_1 < \dots < x_N = b$ be a partition of (a,b), and define

$$I_i \equiv (x_{i-1}, x_i), \quad 1 \leq i \leq N,$$

$$h_i \equiv x_i - x_{i-1}, \quad 1 \leq i \leq N,$$

$$h \equiv \max_{1 \leq i \leq N} h_i,$$

and the local mesh ratio

$$M(\Delta) \equiv \max_{|i-j|=1} \frac{h_i}{h_j} .$$

Let k be a positive integer and let $\underline{z} = (z_1, z_2, \ldots, z_{N-1})$, the incidence vector associated with Δ, have positive integer components, each less than or equal to k-1; i.e., $1 \leq z_i \leq k-1$, $1 \leq i \leq N-1$.

Definition 3.1: The real-valued function s(x) is a spline of order k for Δ and $\underline{z}$ if s(x) coincides with a polynomial of degree < k (i.e., degree < k) on each open subinterval I_i of Δ and has $k-1-z_i$ continuous derivatives at each x_i, i.e.,

$$D^j s(x_i-) = D^j s(x_i+), \quad 0 \leq j \leq k-1-z_i, \quad 1 \leq i \leq N-1.$$

□

The class of all splines of order k for Δ and $\underline{z}$ is denoted by $S^k(\Delta,\underline{z})$. The dimension of $S^k(\Delta,\underline{z})$ is $d = k + \sum_{i=1}^{N-1} z_i$.

Let $\underline{e} = (1,1, \ldots, 1)^T$ be the (N-1)-dimensional "one-vector". We denote by $S^k(\Delta)$ the space $S^k(\Delta,\underline{e})$ and call this the space of smooth splines of order k for Δ. If $\underline{z}$ is any incidence vector, then $S^k(\Delta) \subset S^k(\Delta,\underline{z})$. We denote by $S_0^k(\Delta,\underline{z})$ (respectively, $S_0^k(\Delta)$) the subspace of functions $s \in S^k(\Delta,\underline{z})$ (respectively, $S^k(\Delta)$) such that $s(a) = s(b) = 0$.

The utility of splines for practical computation stems largely from the existence of a well-conditioned local basis, the B-splines. In order to define the B-splines, we need one additional concept. A vector $\underline{t} = \{t_j\}_{j=1}^{d+k}$ is a <u>k-extended partition</u> of (a,b) provided

(3.1)

a) $t_1 = t_2 = \ldots = t_k = a,$

b) $t_{d+1} = t_{d+2} = \ldots = t_{d+k} = b,$

c) $t_j \le t_{j+1}, \quad 1 \le j \le d+k-1,$

d) $t_j < t_{j+k}, \quad 1 \le j \le d.$

Given Δ and $\underline{z}$, let $\underline{t}(\Delta,\underline{z})$ be the k-extended partition such that

i) $t_j \in \{x_i \mid 0 \le i \le N\},$

ii) the multiplicity of x_i in $\{t_j\}$ is z_i.

Let $f(x_0, x_1, \ldots, x_k)$ denote the k^{th} divided difference of the function f on the points $\{x_0, x_1, \ldots, x_k\}$. The set of normalized B-splines on $\underline{t}$ is defined by

$$N_{j,k}(x) = (t_{j+k} - t_j) g_k(t_j, \ldots, t_{j+k}; x), \quad 1 \le j \le d,$$

where

$$g_k(s; x) = (s-x)_+^{k-1} \equiv \begin{cases} (s-x)^{k-1} & s \ge x \\ 0 & s < x \end{cases}.$$

We note the following properties of $N_{j,k}(x)$ [D2]:

$$
(3.2)\quad
\begin{aligned}
&\text{a)}\quad \operatorname{supp}(N_{j,k}) = [t_j,\, t_{j+k}], \qquad 1 \le j \le d;\\
&\text{b)}\quad |V_i| \equiv |\{j \mid \operatorname{supp}(N_{j,k}) \cap I_i \ne \emptyset\}| = k;\\
&\text{c)}\quad 0 \le N_{j,k}(x) \le 1, \qquad a \le x \le b;\\
&\text{d)}\quad \begin{aligned} D^i N_{j,k}(a) &= 0 \quad \text{for all } j \ge 2+i,\\ D^i N_{j,k}(b) &= 0 \quad \text{for all } j \le d-1-i; \end{aligned}\\
&\text{e)}\quad \{N_{j,k} \mid 1 \le j \le d\} \text{ is a basis for } S^k(\Delta,\underline{z}).
\end{aligned}
$$

We shall require a bound on the derivatives of $N_{j,k}$.

__Lemma 3.1__: For all $r \ge 0$ and all $1 \le j \le d$,

$$
(3.3)\qquad |D^r N_{j,k}(x)| \le B_r h_i^{-r}, \qquad \text{all } x \in I_i,
$$

where $B_r = \prod_{\ell=1}^{r} (k-\ell)^2$.

__Proof__: By (3.2)(c), (3.3) holds for $r = 0$. (We adopt the convention that $\prod_{\ell=1}^{0} \ell = 1$.) To prove the general result (by induction on r), we use Markov's inequality (Theorem 2.3). Since $D^r N_{j,k}$ is a polynomial of degree k-r-1 in each of the intervals I_i,

$$
\begin{aligned}
|D^{r+1} N_{j,k}(x)| &\le (k-r-1) h_i^{-1} \| D^r N_{j,k} \|_{L^\infty(I_i)}\\
&\le (k-r-1) B_r h_i^{-(r+1)}\\
&= B_{r+1} h_i^{-(r+1)}, \qquad x \in I_i.
\end{aligned}
$$

□

Birkhoff [B2] devised a projection mapping P from spaces of smooth functions into $S^k(\Delta,\underline{z})$ with the property that, in the interval I_i, Pf depends on f only in some small neighborhood of I_i and the rate of convergence of Pf to f is optimal. His work was generalized to n dimensions (and simplified in one dimension) by de Boor and Fix [D3], who named Pf the quasiinterpolant of f.

Definition 3.2 (de Boor and Fix [D3]): For each integer j, $1 \leq j \leq d$, let τ_j be a point in the support of $N_{j,k}$, i.e., $t_j \leq \tau_j \leq t_{j+k}$. Let f have ℓ-1 continuous derivatives, $1 \leq \ell \leq k$. Then the quasiinterpolant $F_\Delta f = F_{\Delta,\underline{z},\ell}\, f$ is given by

(3.4) $$F_\Delta f \equiv \sum_{j=1}^{d} \lambda_j f\, N_{j,k},$$

where the linear functionals λ_j are given by

(3.5) $$\lambda_j f = \lambda_{j,\ell} f \equiv \sum_{r<\ell} \omega_{j,r} D^r f(\tau_j),$$

(3.6) $$\omega_{j,r} \equiv (-1)^{k-1-r} \frac{D^{k-1-r}\psi_j(\tau_j)}{(k-1)!},$$

and

(3.7) $$\psi_j(x) \equiv (t_{j+1} - x) \ldots (t_{j+k-1} - x)$$

(i.e., ψ_j is the polynomial of degree k-1 vanishing at the knots within the support of $N_{j,k}$).

□

Lemma 3.2 (de Boor and Fix [D3]): If p is a polynomial of degree $< \ell$, then

(3.8) $$F_{\Delta,\underline{z},\ell}\, p \;=\; p.$$

Let θ_i be the smallest interval containing both I_i and $\{\tau_j \mid j \in V_i\}$. Clearly, $\theta_i \subset [x_{i-k}, x_{i+k-1}]$ (see Figure 3.1).

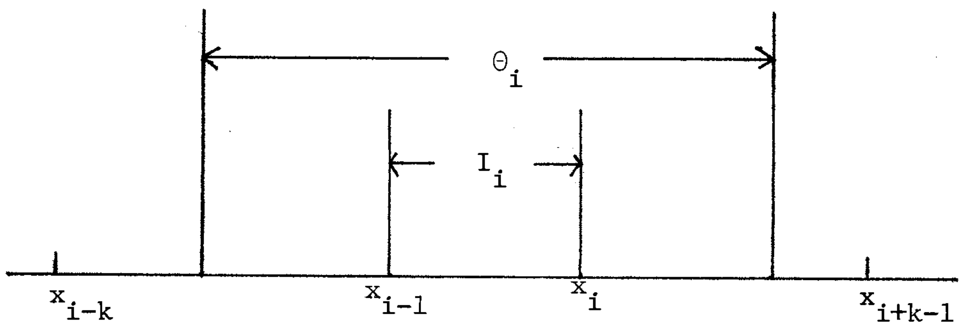


Figure 3.1: θ_i and I_i.

In I_i, $F_\Delta f$ depends only on the values of f in θ_i; thus we have local error bounds.

Theorem 3.1 (de Boor and Fix [D3]): Let $f \in H^\ell(\theta_i)$. There exists a positive constant $K = K(k,\ell,j,M(\Delta))$, such that

(3.9) $$\| D^j(f - F_\Delta f) \|_{L^2(I_i)} \;\le\; K\, |\theta_i|^{\ell-j}\, \| D^\ell f \|_{L^2(\theta_i)},$$

where $|\theta_i|$ = length of θ_i.

An important fact about the quasiinterpolant is that it can be made to interpolate f, and its derivatives of orders 1 through $\ell-1$, at a and b. This will be useful in proving the existence of spline approximations satisfying various boundary conditions.

Lemma 3.3: Let the first (respectively, last) $m \leq \ell$ quasiinterpolation points $\{\tau_j\}$ be placed at a (respectively, b). Then the quasiinterpolant $F_{\Delta,\underline{z},\ell} f$ interpolates f and its first m-1 derivatives at that point.

Proof: We prove the result for interpolation at a; the result for b follows by symmetry. According to (3.2)(d), $D^i N_{j,k}(a) = 0$ for $j > i+1$, so that

$$D^i F_\Delta f(a) = \sum_{j=1}^{i+1} \lambda_j f \, D^i N_{j,k}(a). \tag{3.13}$$

Let $T_a f(x)$ be the first m terms of the Taylor series for f about a, i.e.,

$$T_a f(x) = f(a) + x\, Df(a) + \ldots + x^{m-1} \frac{D^{m-1} f(a)}{m-1!} .$$

We claim that $\lambda_j f = \lambda_j T_a f$ for $1 \leq j \leq m$. If so, then (by (3.13)) $D^i F_\Delta f(a) = D^i F_\Delta T_a f(a)$, $0 \leq i \leq m-1$. Moreover (Lemma 3.2), since $T_a f$ is a polynomial of degree $m-1 < \ell$, $F_\Delta T_a f = T_a f$. Therefore

$$\begin{aligned} D^i F_\Delta f(a) &= D^i F_\Delta T_a f(a) \\ &= D^i T_a f(a) \\ &= D^i f(a), \qquad 0 \leq i \leq m-1. \end{aligned}$$

It remains only to verify that $\lambda_j f = \lambda_j T_a f$ for $1 \le j \le m$. According to (3.7) and (3.1)(a),

$$\psi_j(x) = (a-x)^{k-j} p_{j-1}(x)$$

with p_{j-1} a polynomial of degree j-1. Since we assume that $\tau_1 = \dots = \tau_m = a$,

$$\psi_j^{(k-1-r)}(\tau_j) = \psi_j^{(k-1-r)}(a) = 0 \quad \text{for all } j \le r < k,$$

Therefore, by (3.6),

$$\omega_{j,r} = 0 \quad \text{for all } j \le r < k,$$

and $\lambda_j f$ depends only on the first j derivatives of f at a. But, for $1 \le j \le m$, the value of these derivatives is the same whether $T_a f$ or f is used.

□

Corollary: If $f(a) = f(b) = 0$, $\tau_1 = a$, and $\tau_d = b$, then $F_\Delta f \in S_0^k(\Delta, \underline{z})$.

Later, we will need the bound on the weights $\omega_{j,r}$ of (3.6) given by the following lemma.

Lemma 3.4: For each interval I_i, $1 \le i \le N$, for all $j \in V_i$,

$$|\omega_{j,r}| \le C(k,M(\Delta))\, h_i^r, \quad 0 \le r < k,$$

where $C(k,M(\Delta))$ depends only on k and $M(\Delta)$.

Proof: For $j \in V_i$, the interval (t_j, t_{j+k}) contains I_i. Thus, its length can be bounded in terms of the local mesh ratio and h_i. (see Figure 3.2). In fact,

$$t_{j+k} - t_j \le h_i \, (1 + M(\Delta) + M(\Delta)^2 + \dots + M(\Delta)^{k-1})$$

$$= h_i \frac{M(\Delta)^k - 1}{M(\Delta) - 1}$$

$$\equiv c_1 \, h_i .$$

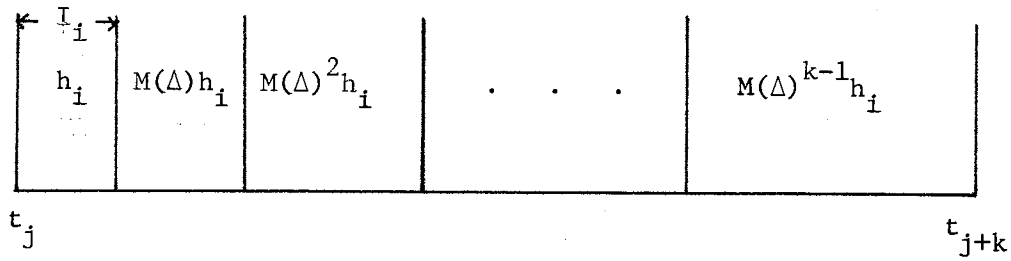


Figure 3.2: I_i and (t_j, t_{j+k}) --- the worst case.

From the definition (3.7) of ψ_j,

$$\| \psi_j \|_{L^\infty(t_j, t_{j+k})} \le (t_{j+k} - t_j)^{k-1} .$$

Thus, by the definition (3.6) of $\omega_{j,r}$, the assumption that $\tau_j \in (t_j, t_{j+k})$, and Markov's inequality (Theorem 2.2),

$$|\omega_{j,r}| \le \frac{1}{(k-1)!} \| D^{(k-1-r)} \psi_j \|_{L^\infty(t_j, t_{j+k})}$$

$$\le \frac{[(k-1)\,(k-2)\,\dots\,(r+1)]^2}{(k-1)!} (t_{j+k} - t_j)^{r+1-k} \, \| \psi_j \|_{L^\infty(t_j, t_{j+k})}$$

$$\le \frac{[(k-1)\,(k-2)\,\dots\,(r+1)]^2}{(k-1)!} (t_{j+k} - t_j)^{r}$$

$$\le \frac{[(k-1)\,(k-2)\,\dots\,(r+1)]^2}{(k-1)!} (c_1 \, h_i)^{r}$$

□

PART II

CHAPTER 4

PROBLEMS WITH $0 \leq \sigma < 1$

4.1 Introduction

In Chapters 4 - 8, we consider the singular two-point boundary value problem

$$(4.1) \qquad - D(p(x)Du) + q(x)u = f(x), \qquad 0 < x < 1,$$

$$(4.2) \qquad u(0) = u(1) = 0,$$

where the coefficients p and q satisfy

(4.3)

a) $p(x) = x^{\sigma}\rho(x)$ and $q(x) = x^{\sigma}\gamma(x)$, where $0 \leq \sigma < 1$,

b) $\rho(x) \geq \rho_{min} > 0$ for all $x \in I$,

c) $\rho \in W^{1,\infty}(I)$,

d) $\gamma \in W^{0,\infty}(I)$.

Parter [P1] and Greenspan [G2] considered finite-difference methods for the problem

$$(4.4)\qquad -D^2u + \frac{\sigma}{x}Du + qu = 0,$$
$$u(0) = 1, \quad u(1) = 0,$$

which arises by separation of variables in the equation for generalized axially symmetric potentials in a rectangle. Problems of this type can be replaced by equivalent problems of the form (4.1) - (4.2). Jamet [J1] also developed finite-difference methods for (4.4) and obtained L^∞-norm error estimates of $O(h^{1-\sigma})$ using a uniform mesh of size h, the exponent $1-\sigma$ being sharp. Gusman and Oganesyan [G3] considered finite-difference methods derived from variational principles for the more general elliptic problem

$$Lu = -\frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(y^\sigma q(x,y)\frac{\partial u}{\partial y}\right) = f(x,y)$$

in a rectangular domain $S = \{(x,y) \mid 0 \leq x \leq a,\ 0 \leq y \leq b\}$ and also obtained low-order error estimates using a uniform mesh. Dershem [D5] devised second-order accurate three-point finite-difference approximations for another singular ordinary differential operator,

$$Lu = D(xDu) - \frac{\alpha^2}{x}u .$$

Using the Rayleigh-Ritz-Galerkin method with a singular (L-spline) subspace on a uniform mesh, Ciarlet, Natterer, and Varga [C1] obtained L^∞-error estimates of order $h^{2-\sigma}$ (see also Chapter 8). Natterer [N1] later developed a very general theory for approximations of this type by converting the problem to an appropriate singular first-order system.

He treated problems with singularities at either endpoint and, using an appropriate analogue of a β-graded mesh (with grading at both ends), obtained improved error bounds of order $N^{-2}\ln(N)^c$ with c a positive constant independent of N. Other subspaces of singular functions have been considered by Crouzeix and Thomas [C4] (see also Chapter 8), Jerome and Pierce [J2], and Dailey and Pierce [D1]. The use of piecewise polynomial spaces on a graded mesh was also suggested by Fried and Yang [F2].

4.2 Variational Form of the Problem

In this section we instantiate the definitions and results of Section 2.2 in the context of the problem (4.1) - (4.2). As the underlying Hilbert space H, we use $H^0 = L^2(I)$. The space S, in which the solution of (4.1) - (4.2) lies, is given in the following definition.

Definition 4.1: Let S be the linear space of absolutely continuous real-valued functions u on I such that u(0) = u(1) = 0 and

$$\sqrt{x^\sigma}\, Du \in L^2(I).$$

□

For u, v ∈ S, we define the inner product

$$a(u,v) \equiv \int_0^1 [p(x)DuDv + q(x)uv]\, dx$$

and the norm

$$\| u \|_E \equiv a(u,u)^{1/2}.$$

The space S is an example of a weighted Sobolev space and may be identified with the closure of the C^∞ functions with compact support in I with respect to the norm $\|\cdot\|_E$. If the problem (4.1) - (4.2) has a classical solution u, then u is in S [C1], but is not necessarily in $H^1(I)$. For example, let $p(x) = x^\sigma$, $q(x) = 0$, and $f(x) = 2-\sigma$. Then $u(x) = x^{1-\sigma} - x^{2-\sigma} \in H^1$ if and only if $\sigma < \frac{1}{2}$.

We will find it convenient to work with a different norm on S,

$$(4.5) \qquad \|u\|_S \equiv \left(\int_0^1 x^\sigma (Du)^2 \, dx \right)^{1/2}, \quad u \in S.$$

We assume that the S-norm and energy norm are equivalent, i.e., that

$$(4.6) \qquad \lambda \|v\|_S \le \|v\|_E \le \Lambda \|v\|_S \quad \text{for all } v \in S,$$

where λ and Λ positive constants independent of v. This assumption will be satisfied provided q(x) is not too small, in particular, if -q(x) is less than the smallest (positive) eigenvalue of -D(pDu). For $(a,b) \subset I$, we define the seminorm

$$\|f\|_{S(a,b)} \equiv \left(\int_a^b x^\sigma (Df)^2 \, dx \right)^{1/2}.$$

The main result of this section is that the form a(.,.) is positive definite.

<u>Theorem 4.1</u>: For every $v \in S$,

$$\begin{aligned} \|v\|_0 &\le \|v\|_{L^\infty} \\ &\le \mu \|v\|_S \\ &\le \delta \|v\|_E, \end{aligned}$$

where $\mu = (1-\sigma)^{-1/2}$ and $\delta = \frac{\mu}{\lambda}$.

Proof: Let $v \in S$. By (4.6),

$$|v(x)| \leq \int_0^x |Dv(t)|\, dt$$

$$= \int_0^x \left| \frac{t^{\sigma/2} Dv}{t^{\sigma/2}} \right| dt$$

$$\leq \left(\int_0^x t^{-\sigma}\, dt\right)^{1/2} \left(\int_0^x t^{\sigma}(Dv)^2\, dt\right)^{1/2}$$

$$= \mu \, \| v \|_S$$

$$\leq \delta \, \| v \|_E.$$

□

In order to obtain higher-order error bounds for the solution of (4.1) - (4.2), we will need to assume that the right-hand side is smooth (e.g., $f \in H^m$, $m \geq 0$). We will later show (Theorem 5.2) that $D(x^{\sigma}Du)$ is as smooth as f and therefore, that u is in one of the following spaces.

Definition 4.2: For $m \geq 0$, let

$$S^m \equiv \{u \in S \mid D(x^{\sigma}Du) \in H^m\},$$

$$S^{-1} \equiv \{u \in S \mid x^{\sigma}Du \in L^{\infty}\},$$

$$\| D(x^{\sigma}Du) \|_{-1} \equiv \| x^{\sigma}Du \|_{L^{\infty}} \quad \text{for } u \in S^{-1}.$$

□

The role of the space X_{ℓ} in Section 2.2 will be played by $S^{\ell-2}$.

Let $u \in S^0$. By Rolle's theorem, there exists $x_0 \in I$ such that $Du(x_0) = 0$. Thus, by the Cauchy-Schwarz inequality,

$$x^\sigma Du(x) = \int_{x_0}^{x} D(t^\sigma Du)(t)\, dt$$

$$\leq \| D(x^\sigma Du) \|_0.$$

Therefore, for $u \in S^m$,

(4.7) $$\| D(x^\sigma Du) \|_{-1} \leq \| D(x^\sigma Du) \|_0 \leq \cdots \leq \| D(x^\sigma Du) \|_m.$$

4.3 On β-graded Meshes

One particular kind of mesh dominates our discussion of singular two-point boundary value problems in Part II. These are the "β-graded" meshes.

Definition 4.3: Let $\beta \geq 1$ and N a positive integer. The β-graded mesh $\Delta_{\beta,N} \equiv \{x_i\}_{i=0}^N$ is the partition of I given by

(4.8) $$x_i = \left(\frac{i}{N}\right)^\beta, \quad 0 \leq i \leq N.$$

□

β-graded meshes were introduced by Rice [R2], who considered spline interpolation of the function x^α, α not an integer. He proved that, for β appropriately chosen as a function of α, the L^∞-error in interpolation on a β-graded mesh was no larger than that obtained when interpolating a smooth function, despite the singularity of x^α at 0. Eisenstat and

Schultz [E2] have applied β-graded meshes to a two-dimensional partial differential equation in which the solution has a singularity due to a corner in the domain. They showed that convergence at the best possible rate in L^2 occurs, despite the singularity, when using a tensor product mesh which is graded in both independent variables. Natterer [N1] employed a β-graded mesh for a singular two-point boundary value problem for a first order system of equations, while Fried and Yang [F2] advocated the use of what amounts to a β-graded mesh for problems like those we consider, but proved no error bounds.

For $\beta = 1$, the mesh $\Delta_{\beta,N}$ is a uniform mesh with $h = N^{-1}$. As β increases, the mesh very rapidly becomes skewed towards 0; to illustrate, we depict $\Delta_{2,6}$ in Figure 4.1.

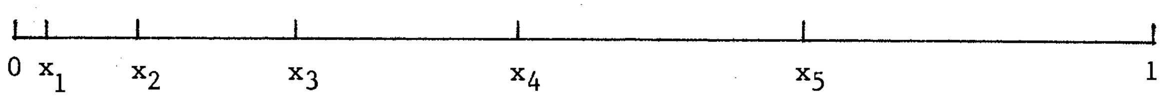


Figure 4.1: $\Delta_{2,6}$

The following lemma summarizes several important properties of β-graded meshes.

Lemma 4.1: There exists a constant $C(\beta) = C(\beta,k)$, independent of N, such that for every $\beta > 1$ and all $N > 0$, $\Delta_{\beta,N}$ satisfies

a) $h_i \le \beta i^{\beta-1} N^{-\beta}$;

b) $h \le \beta N^{-1}$;

(4.9) c) $M(\Delta_{\beta,N}) \le C(\beta)$;

d) $|\theta_i| \le C(\beta) h_i$;

e) the weights $\omega_{j,r}$ of (3.6) satisfy

$$|\omega_{j,r}| \le C(\beta) h_i^r,$$

for all $j \in V_i$.

Proof: By the mean value theorem, there exists $c \in (i-1,i)$ such that

$$i^\beta - (i-1)^\beta = \beta c^{\beta-1}.$$

Thus,

$$h_i = \frac{i^\beta - (i-1)^\beta}{N^\beta} \le \beta c^{\beta-1} N^{-\beta} \le \beta i^{\beta-1} N^{-\beta},$$

which proves (a). By the same argument,

$$h_i \ge \beta (i-1)^{\beta-1} N^{-\beta} \ge h_{i-1}. \tag{4.10}$$

Thus, the sequence $\{h_i\}$ increases monotonically and

$$h = h_N \le \beta N^{\beta-1} N^{-\beta} = \beta N^{-1},$$

which proves (b). Clearly, since $h_1 = x_1 = N^{-\beta}$,

$$\frac{h_2}{h_1} = \frac{2^\beta - 1^\beta}{1^\beta} = 2^\beta - 1,$$

while for $2 < i \leq N$, (4.10) implies that

$$\frac{h_i}{h_{i-1}} \leq \frac{i^{\beta-1}}{(i-2)^{\beta-1}} \leq 3^{\beta-1}.$$

Therefore

$$M(\Delta_{\beta,N}) \equiv \max_{2 \leq i \leq N} \frac{h_i}{h_{i-1}} = \max(2^\beta - 1,\ 3^{\beta-1})$$

(which of these is larger depends on β), which proves (c).

Next, we consider $|\theta_i|$. For all $1 \leq i \leq k$,

$$\theta_i \subset [0, x_{i+k-1}],$$

so that

$$|\theta_i| \leq x_{i+k-1} = \left(\frac{i+k-1}{N}\right)^\beta \leq (i+k-1)^\beta h_i$$

$$\leq (2k-1)^\beta h_i.$$

For $i > k$, θ_i is bounded away from 0, i.e.,

$$\theta_i \subset [x_{i-k}, x_{i+k-1}] \cap [x_1, 1],$$

so by the mean value theorem,

$$|\theta_i| \leq x_{i+k-1} - x_{i-k}$$

$$= \frac{(i+k-1)^\beta - (i-k)^\beta}{N^\beta}$$

$$\leq (2k-1)\beta(i+k-1)^{\beta-1} N^{-\beta}.$$

Using (4.10),

$$|\Theta_i| \leq (2k-1)\beta \left(\frac{i+k-1}{i-1}\right)^{\beta-1} (i-1)^{\beta-1} N^{-\beta}$$

$$\leq (2k-1)\, 2^{\beta-1}\, h_i,$$

since $\frac{i+k-1}{i-1} \leq 2$ for $i > k$. This proves (d). Finally, (e) follows from (d) and Lemma 3.4.

□

CHAPTER 5

PROPERTIES OF THE SOLUTION

5.1 Introduction

In this chapter, we derive several properties of the generalized solution u of (4.1) - (4.2). Our goals are to show that

1) there exists a positive constant Γ_1 such that, if $u \in S^m$, then $x^\sigma u \in H^{m+2}$ and

$$\| D^\ell (x^\sigma u) \|_0 \leq \Gamma_1 \| D^{\ell-1}(x^\sigma Du) \|_0, \quad 1 \leq \ell \leq m+2;$$

2) there exists a positive constant Γ such that, if $f \in H^m$ and the coefficients ρ and γ are sufficiently smooth, then $u \in S^m$ and

$$\| D(x^\sigma Du) \|_m \leq \Gamma \| f \|_m;$$

3) if $u \in S^m$, then there exist positive constants κ_ℓ such that,

$$\| x^{\sigma+\ell} D^{\ell+1} u \|_{L^\infty} \leq \kappa_\ell \| D(x^\sigma Du) \|_\ell, \quad -1 \leq \ell \leq m,$$

and constants $K_\ell = K(\ell,\varepsilon)$ such that, for any $\varepsilon > 0$,

$$\| x^{(2\ell-3+2\sigma+\varepsilon)/2} D^{\ell+2} u \|_0 \leq K_\ell \| D(x^\sigma Du) \|_{\ell-2}, \quad 1 \leq \ell \leq m+2.$$

5.2 A Generalization of Hardy's Inequality

Let

$$C_B^m \equiv \{g \in C^m(I) \mid g(0) = 0\}$$

and

$$H_B^m \equiv \{g \in H^m(I) \mid g(0) = 0\}.$$

We need to establish several properties of the function g/x when $g \in C_B^m$ or $g \in H_B^m$, $m \geq 1$. In Lemmas 5.1 and 5.3 we show that, near the origin, g/x behaves essentially like Dg.

Lemma 5.1: If $g \in C_B^{m+1}$, $m \geq 0$, then $g/x \in C^m$ and

$$\lim_{x \to 0+} D^{\ell}(g/x) = \frac{D^{\ell+1}g(0)}{\ell+1} \tag{5.1}$$

for all $0 \leq \ell \leq m$.

Proof: Clearly $g/x \in C^m(0,1]$; we need only show that, for $0 \leq \ell \leq m$, $\lim_{x \to 0+} D^{\ell}(g/x)$ exists and has the value given by (5.1). For $x > 0$,

$$D^{\ell}(g/x) = \sum_{j=0}^{\ell} \binom{\ell}{j} D^{\ell-j}(x^{-1})\, D^j g$$

$$= \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^{\ell-j}(\ell-j)!\, x^{-(\ell-j)-1}\, D^j g$$

$$= x^{-(\ell+1)} \sum_{j=0}^{\ell} \frac{\ell!}{j!} (-1)^{\ell-j}\, x^j\, D^j g.$$

As $x \to 0$, the last sum approaches 0. Thus, using L'Hôpital's rule,

$$\lim_{x\to 0+} D^{\ell}(g/x)$$

$$= \lim_{x\to 0+} \frac{\sum_{j=0}^{\ell} \frac{\ell!}{j!} (-1)^{\ell-j} x^j D^{j+1} g + \sum_{j=1}^{\ell} \frac{\ell!}{j!} (-1)^{\ell-j} j x^{j-1} D^j g}{(\ell + 1)x^{\ell}}$$

$$= \lim_{x\to 0+} \frac{x^{\ell} D^{\ell+1} g + \sum_{j=1}^{\ell} \frac{\ell!}{(j-1)!} (-1)^{\ell-j+1} x^{j-1} D^j g + \sum_{j=1}^{\ell} \frac{\ell!}{(j-1)!} (-1)^{\ell-j} x^{j-1} D^j g}{(\ell + 1)x^{\ell}}$$

$$= \lim_{x\to 0+} \frac{D^{\ell+1} g}{\ell+1} = \frac{D^{\ell+1} g(0)}{\ell+1} .$$

□

In order to extend Lemma 5.1 to the case $g \in H^m_B$, we need the following result.

<u>Lemma 5.2</u>: For all $m \geq 1$, C^m_B is dense in H^m_B.

<u>Proof</u>: Let $h \in H^m_B \subset H^m$. Given $\varepsilon > 0$, we can (since C^m is dense in H^m) find a $g \in C^m$ such that $\| h - g \|_m < \frac{\varepsilon}{1+C_1}$. Furthermore, by the Sobolev lemma (Theorem 2.3),

$$\| h - g \|_{L^\infty} \leq C_1 \| h - g \|_m < \frac{C_1 \varepsilon}{1+C_1},$$

since $m \geq 1$. Therefore $|g(0)| < \frac{C_1 \varepsilon}{1+C_1}$. Let $\bar{g} = g - g(0)$. Then $\bar{g} \in C^m_B$ and

$$\| \bar{g} - h \|_m \leq \| \bar{g} - g \|_m + \| g - h \|_m < \varepsilon.$$

□

Hardy's inequality [H1] states that, if $g \in H^1_B$, then

$$\| g/x \|_0 \leq 2 \| Dg \|_0.$$

The following lemma generalizes this result to $g \in H^{m+1}_B$, $m \geq 0$.

<u>Lemma 5.3</u>: If $g \in H^{m+1}_B$, $m \geq 0$, then $g/x \in H^m$ and

$$(5.2) \qquad \| D^\ell (g/x) \|_0 \leq \frac{2}{2\ell+1} \| D^{\ell+1} g \|_0$$

for all $0 \leq \ell \leq m$.

<u>Proof</u>: We demonstrate (5.2) for $g \in C^{m+1}_B$. The result then follows from the denseness of C^{m+1}_B in H^{m+1}_B.

We first show, by induction on ℓ, that

$$(5.3) \qquad xD^\ell (g/x) = D^\ell g - \ell D^{\ell-1}(g/x) \quad \text{for } 1 \leq \ell \leq m.$$

The case $\ell = 1$ is trivial. Assuming $\ell > 1$,

$$(5.4) \qquad D(xD^{\ell-1}(g/x)) = xD^\ell (g/x) + D^{\ell-1}(g/x),$$

so that

$$xD^\ell (g/x) = D(xD^{\ell-1}(g/x)) - D^{\ell-1}(g/x).$$

Using the induction hypothesis,

$$\begin{aligned} xD^\ell (g/x) &= D\left(D^{\ell-1} g - (\ell-1) D^{\ell-2}(g/x)\right) - D^{\ell-1}(g/x) \\ &= D^\ell g - \ell D^{\ell-1}(g/x). \end{aligned}$$

This completes the induction.

Next, note that for $\ell = 0$, the result is identical to Hardy's inequality. For $\ell \geq 1$, we prove (5.2) by computing $\|D^\ell(g/x)\|_0$. Integrating by parts,

$$\|D^\ell(g/x)\|_0^2 = \int_0^1 x^{-2}[xD^\ell(g/x)]^2\,dx$$

$$= -x^{-1}[xD^\ell(g/x)]^2 \Big|_0^1$$

$$+ 2\int_0^1 x^{-1}[xD^\ell(g/x)]D[xD^\ell(g/x)]\,dx.$$

The integrated term is non-positive, for

$$-(D^\ell(g/x)|_{x=1})^2 \leq 0,$$

while

$$\lim_{x\to 0} x(D^\ell(g/x))^2 = 0,$$

since, by Lemma 5.1, $D^\ell(g/x)$ is bounded on $[0,1]$. Thus

$$\|D^\ell(g/x)\|_0^2 \leq 2\int_0^1 D^\ell(g/x)\,D(xD^\ell(g/x))\,dx.$$

Using the identity (5.3), we have

$$\|D^\ell(g/x)\|_0^2 \leq 2\int_0^1 D^\ell(g/x)\,(D^{\ell+1}g - \ell D^\ell(g/x))\,dx$$

$$= 2\int_0^1 (D^\ell(g/x)D^{\ell+1}g)\,dx - 2\ell\|D^\ell(g/x)\|_0^2.$$

Hence, by the Cauchy-Schwarz inequality,

$$(2\ell+1)\|D^\ell(g/x)\|_0^2 \leq 2\|D^\ell(g/x)\|_0\,\|D^{\ell+1}g\|_0.$$

$\square$

5.3 Regularity of the Generalized Solution

We turn now to one of the principal results of this chapter, which will later be used to show that solutions of (4.1) - (4.2) are of the form $x^{-\sigma}v(x)$, with v a smooth function. This result is crucial to the theory of Chapter 7, in which we consider approximations to u of the form $x^{-\sigma}\hat{v}$, with $\hat{v}$ an approximation to v. Moreover, it provides the means for proving a regularity result, Theorem 5.2.

Theorem 5.1: If $u \in S^m$, $m \geq 0$, then $x^\sigma u \in H^{m+2}$ and there exists a positive constant Γ_1 independent of u such that

$$(5.5) \qquad \| D^\ell(x^\sigma u) \|_0 \leq \Gamma_1 \| D^{\ell-1}(x^\sigma Du) \|_0, \quad 1 \leq \ell \leq m+2.$$

Proof: Define

$$v_0(t) = t^\sigma Du(t), \quad 0 \leq t \leq 1,$$

$$(5.6) \qquad v_{i+1}(t) = (v_i(t) - v_i(0))/t, \quad 0 < t \leq 1, \quad 0 \leq i \leq m.$$

By hypothesis, $v_0 \in H^{m+1}$; furthermore, it follows from Lemma 5.3 by an obvious inductive argument that $v_i \in H^{m+1-i}$ and

$$(5.7) \qquad \| D^{\ell-i-1} v_i \|_0 \leq \left(\prod_{j=\ell-i-1}^{\ell-2} \left(\frac{2}{2j+1}\right) \right) \| D^{\ell-1} v_0 \|_0$$

provided $i < \ell \leq m + 2$.

We now show by induction on ℓ that

$$(5.8) \qquad D^\ell(x^\sigma u) = D^{\ell-1}v_0 + c_1 D^{\ell-2}v_1 + \ldots + c_{\ell-1}D^0 v_{\ell-1}$$
$$+ c_\ell x^{\sigma-\ell} \int_0^x t^{\ell-1-\sigma} v_{\ell-1}(t)\, dt,$$

where $c_k = c_k(\sigma) = \prod_{i=0}^{k-1} (\sigma - i)$. Since $u \in S$ (and in particular, u is absolutely continuous),

$$x^\sigma u(x) = x^\sigma \int_0^x Du(t)\, dt$$

(5.9)

$$= x^\sigma \int_0^x t^{-\sigma}(t^\sigma Du(t))\, dt.$$

Differentiating (5.9),

$$D(x^\sigma u) = v_0 + \sigma x^{\sigma-1} \int_0^x t^{-\sigma} v_0(t)\, dt,$$

which is (5.8) for $\ell = 1$. For the induction step, first note that the last term on the right-hand side of (5.8) satisfies

$$x^{\sigma-\ell} \int_0^x t^{\ell-1-\sigma}\, v_{\ell-1}(t)\, dt$$

$$= x^{\sigma-\ell} \int_0^x t^{\ell-1-\sigma}\, [v_{\ell-1}(0) + t v_\ell(t)]\, dt$$

$$= \frac{1}{\ell-\sigma} v_{\ell-1}(0) + x^{\sigma-\ell} \int_0^x t^{\ell-\sigma}\, v_\ell(t)\, dt.$$

Now assume (5.8) holds for ℓ. Differentiating, we obtain

$$D^{\ell+1}(x^\sigma u) = D^\ell v_0 + c_1 D^{\ell-1} v_1 + \ldots + c_{\ell-1} D v_{\ell-1}$$

$$+ c_\ell v_\ell + (\sigma-\ell) c_\ell\, x^{\sigma-\ell-1} \int_0^x t^{\ell-\sigma}\, v_\ell(t)\, dt,$$

which is (5.8) for $\ell+1$.

Inequality (5.5) now follows from (5.7) and (5.8) provided we can show that

$$(5.10) \qquad \| x^{\sigma-\ell} \int_0^x t^{\ell-1-\sigma} v_{\ell-1}(t)\, dt \|_0 \leq C \| v_{\ell-1} \|_0$$

for all $1 \leq \ell \leq m+2$. Let

$$I(x) \equiv \int_0^x t^{\ell-1-\sigma} v_{\ell-1}(t)\, dt.$$

Integrating by parts,

$$\| x^{\sigma-\ell} I(x) \|_0^2 = \int_0^1 x^{2\sigma-2\ell} [I(x)]^2\, dx$$

$$= \frac{x^{2\sigma-2\ell+1}}{2\sigma-2\ell+1} [I(x)]^2 \Big|_0^1$$

$$+ \frac{2}{2\ell-2\sigma-1} \int_0^1 x^{2\sigma-2\ell+1}\, I(x)\, x^{\ell-1-\sigma} v_{\ell-1}(x)\, dx.$$

Suppose we could show that the integrated term is not positive. Then, by the Cauchy-Schwarz inequality,

$$\| x^{\sigma-\ell} I(x) \|_0^2 \leq \left| \frac{2}{2\ell-2\sigma-1} \right| \quad \| x^{\sigma-\ell} I(x) \|_0 \| v_{\ell-1} \|_0$$

which is just what we need.

As for the integrated term, at $x = 1$ it is negative, except (possibly) when $\ell = 1$. But when $\ell = 1$,

$$I(x) = \int_0^x t^{-\sigma} v_0(t)\, dt$$

$$= \int_0^x Du(t)\, dt,$$

whence

$$I(1) = \int_0^1 Du(t)\, dt = u(1) - u(0) = 0;$$

so the integrated term vanishes.

At $x = 0$ it vanishes, too. For by the Cauchy-Schwarz inequality,

$$(I(x))^2 = \left(\int_0^x t^{\ell-1-\sigma} v_{\ell-1}(t)\,dt \right)^2$$

$$\leq \left(\int_0^x t^{2(\ell-1-\sigma)}\,dt \right) \left(\int_0^x v_{\ell-1}^2\,dt \right)$$

$$\leq \frac{x^{2\ell-2\sigma-1}}{2\ell-2\sigma-1} \int_0^x v_{\ell-1}^2(t)\,dt,$$

whence, since $v_{\ell-1} \in L^2(I)$,

$$\lim_{x\to 0+} x^{2\sigma-2\ell+1}(I(x))^2 \leq \frac{1}{2\ell-2\sigma-1} \lim_{x\to 0+} \int_0^x v_{\ell-1}^2(t)\,dt = 0.$$

□

We now show that Theorem 5.1 applies to the generalized solution of (4.1) - (4.2) provided the functions f, ρ, and γ are sufficiently smooth.

Theorem 5.2: Let $u \in S$ be the generalized solution of (4.1)-(4.2) with $f \in H^m$, $m \geq 0$. Let the coefficients p and q satisfy (4.3) and, for $m > 0$, the additional hypotheses

(5.11) $$\rho \in W^{m+1,\infty}, \quad \gamma \in W^{m,\infty}.$$

Then $u \in S^m$ and there exists a positive constant Γ, independent of f, such that

(5.12) $$\| D(x^\sigma Du) \|_\ell \leq \Gamma \| f \|_\ell, \quad 0 \leq \ell \leq m.$$

Proof: We start by showing that $u \in S^m$ and

$$\| D(x^{\sigma}Du) \|_{\ell} \leq \bar{C} \; \| f - qu \|_{\ell}, \qquad 0 \leq \ell \leq m,$$

where $\bar{C} > 0$ is independent of f.

Following Reddien [R1], we explicitly construct the generalized solution. First, let

$$(5.13) \qquad g(x) = \int_0^x (qu-f)(t)\, dt.$$

Clearly $g \in L^{\infty}$ and

$$(5.14) \qquad \| g \|_{L^{\infty}} \leq \| f - qu \|_0.$$

Next, let

$$(5.15) \qquad h(x) = \frac{g(x)}{p(x)} = x^{-\sigma} \frac{g(x)}{\rho(x)}$$

and

$$(5.16) \qquad \hat{w}(x) = \int_0^x h(t)\, dt.$$

Finally, let

$$(5.17) \qquad w(x) = \hat{w}(x) - \hat{w}(1) Y(x),$$

where Y is the solution of the problem

$$(5.18) \qquad -D(pDY) = 0$$

$$(5.19) \qquad Y(0) = 0, \quad Y(1) = 1,$$

i.e., $$Y(x) = \frac{\int_0^x \frac{dt}{p(t)}}{\int_0^1 \frac{dt}{p(t)}}.$$

We claim that $w \in S$ and, moreover, $w = u$, the generalized solution. First, $w(0) = w(1) = 0$, as is clear from (5.16), (5.17), and (5.19). Second, by (5.16) and (5.17),

$$Dw = D\hat{w} - \hat{w}(1)DY = \frac{g}{p} - \hat{w}(1)DY,$$

so that

$$x^{\sigma/2}Dw = x^{-(\sigma/2)}\frac{g}{\rho} - \hat{w}(1)x^{\sigma/2}DY.$$

Thus

$$\| x^{\sigma/2}Dw \|_0 \leq \frac{1}{\rho_{min}} \| g \|_{L^\infty} \| x^{-\sigma/2} \|_0 + |\hat{w}(1)| \, \| x^{\sigma/2}DY \|_0,$$

which is finite; hence $w \in S$.

We now show that $\| u - w \|_E = 0$, and hence that $u = w$. Let $v \in S$. Integrating by parts, using (5.13) and (5.15) - (5.18),

$$\int_0^1 pDw(x)Dv(x)\, dx = \int_0^1 -D(pDw)v\, dx$$

$$= \int_0^1 [f-qu](x)v(x)\, dx$$

$$= \int_0^1 pDu(x)Dv(x)\, dx,$$

whence taking $v = u - w$,

$$\| u - w \|_E^2 = \int_0^1 p(D(u - w))^2\, dx = 0.$$

This shows that $u = w$.

By (5.17),

$$\| D(x^\sigma Du) \|_\ell = \| D(x^\sigma Dw) \|_\ell$$

$$\leq \| D(x^\sigma D\hat{w}) \|_\ell + |\hat{w}(1)| \, \| D(x^\sigma DY) \|_\ell .$$

By (5.15) and (5.16),

$$D^{\ell+1}(x^\sigma D\hat{w}) = D^{\ell+1}(x^\sigma h) = D^{\ell+1}(\tfrac{g}{\rho}), \quad 0 \leq \ell \leq m,$$

whence

$$\| D(x^\sigma D\hat{w}) \|_\ell \leq c_0 \| g \|_{\ell+1}$$

$$\leq c_1 \| f - qu \|_\ell, \quad 0 \leq \ell \leq m,$$

where c_1 depends only on ℓ and $\| \rho \|_{W^{\ell,\infty}}$. Also, by (5.14) - (5.16),

$$|\hat{w}(1)| \leq \int_0^1 |h(t)| \, dt$$

$$\leq \frac{1}{\rho_{min}} \| g \|_{L^\infty} \int_0^1 x^{-\sigma} \, dt$$

$$\leq \frac{1}{(1-\sigma)\rho_{min}} \| g \|_{L^\infty}$$

$$\leq \frac{1}{(1-\sigma)\rho_{min}} \| f - qu \|_0 .$$

Thus

$$\| D(x^\sigma Du) \|_\ell \leq \| D(x^\sigma D\hat{w}) \|_\ell + |\hat{w}(1)| \, \| D(x^\sigma DY) \|_\ell$$

$$\leq \bar{C} \| f - qu \|_\ell ,$$

where $\bar{C} = \max(c_1, \frac{\| D(x^\sigma DY) \|_\ell}{(1-\sigma)\rho_{min}})$. (Note that $\| D(x^\sigma DY) \|_\ell$ is a constant, independent of u.)

Now we show that there exist constants M_ℓ such that

$$\| f - qu \|_\ell \leq M_\ell \| f \|_\ell . \tag{5.21}$$

We proceed by induction on ℓ. For $\ell = 0$, the <u>a priori</u> estimate of (2.7) yields

$$\| u \|_0 \leq \delta^2 \| f \|_0,$$

whence

$$\| f - qu \|_0 \leq (1 + \delta^2 \| q \|_{L^\infty}) \| f \|_0$$
$$\equiv M_0 \| f \|_0 .$$

Since $D(x^\sigma Du) \in H^0$, we can apply Theorem 5.1 which yields

$$u = x^{-\sigma} v, \quad v \in H^2$$

and

$$\| v \|_2 \leq \Gamma_1 \| D(x^\sigma Du) \|_0 \leq \Gamma_1 \bar{C} M_0 \| f \|_0 .$$

Thus $qu = (x^\sigma \gamma)(x^{-\sigma} v) = \gamma v$ and

$$\| f - qu \|_2 \leq \| f \|_2 + \| \gamma v \|_2$$
$$\leq \| f \|_2 + c_2 \| v \|_2$$
$$\leq M_2 \| f \|_2 ,$$

where c_2 depends on $\| \gamma \|_{W^{2,\infty}}$ and $M_2 = 1 + c_2 \Gamma_1 \bar{C} M_0$).

We reiterate this argument for $\ell = 2, 4, \ldots, m$ (if m is even) or $\ell = 2, 4, \ldots, m-1$ (if m is odd). To obtain (5.21) for odd values of ℓ, possibly including m, we apply Theorem 5.1. Since $D(x^\sigma Du) \in H^{\ell-1}$, $u = x^{-\sigma} v$, where $v \in H^{\ell+1}$, and

$$\|v\|_{\ell+1} \leq \Gamma_1 \|D(x^\sigma Du)\|_{\ell-1}$$

$$\leq \Gamma_1\bar{C} \|f - qu\|_{\ell-1}$$

$$\leq \Gamma_1\bar{C}M_{\ell-1} \|f\|_{\ell-1}.$$

Thus

$$\|f - qu\|_\ell \leq \|f\|_\ell + \|\gamma v\|_\ell$$

$$\leq \|f\|_\ell + c_\ell \|v\|_\ell$$

$$\leq \|f\|_\ell + c_\ell \|v\|_{\ell+1}$$

$$\leq M_\ell \|f\|_\ell,$$

where c_ℓ depends on $\|\gamma\|_{W^{\ell,\infty}}$ and $M_\ell = 1 + c_\ell\Gamma_1\bar{C}M_{\ell-1})$.

□

5.4 Bounds on Weighted Norms of the Generalized Solution

The last two results of this chapter establish bounds on two "weighted norms" of the derivatives of the solution.

Lemma 5.4: There exist constants κ_ℓ which depend only on ℓ and σ such that, if $u \in S^m$, $m \geq -1$, then $x^{\sigma+\ell}D^{\ell+1}u \in L^\infty(I)$ and

(5.22) $$\|x^{\sigma+\ell}D^{\ell+1}u\|_{L^\infty} \leq \kappa_\ell \|D(x^\sigma Du)\|_\ell$$

for all $-1 \leq \ell \leq m$.

Proof: First take $\ell = -1$. By the definition of S,

$$|u(x)| \leq \int_0^x |Du(t)|\, dt = \int_0^x \frac{t^\sigma |Du(t)|}{t^\sigma}\, dt$$

$$\leq \frac{x^{1-\sigma}}{1-\sigma} \| x^\sigma Du \|_{L^\infty}$$

Therefore

$$|x^{\sigma-1} u(x)| \leq \frac{1}{1-\sigma} \| x^\sigma Du \|_{L^\infty} = \frac{1}{1-\sigma} \| D(x^\sigma Du) \|_{-1}.$$

For $\ell \geq 0$, we use induction on ℓ. Since

$$D^{\ell+1}(x^\sigma u) = x^\sigma D^{\ell+1} u + \sum_{j=1}^{\ell+1} c_j\, x^{\sigma-j} D^{\ell+1-j} u,$$

where $c_j = \binom{\ell+1}{j} \prod_{i=0}^{j-1} (\sigma - i)$, it follows from the induction hypothesis that

$$|x^{\sigma+\ell} D^{\ell+1} u| \leq |x^\ell D^{\ell+1}(x^\sigma u)| + \sum_{j=1}^{\ell+1} |c_j\, x^{\sigma+\ell-j} D^{\ell+1-j} u|$$

(5.23)

$$\leq \| D^{\ell+1}(x^\sigma u) \|_{L^\infty} + \sum_{j=1}^{\ell+1} |c_j|\, \kappa_{\ell-j} \| D(x^\sigma Du) \|_{\ell-j}.$$

By Sobolev's lemma (Theorem 2.3),

$$\| D^{\ell+1}(x^\sigma u) \|_{L^\infty} = \| D^\ell (D(x^\sigma u)) \|_{L^\infty}$$

$$\leq C_1 \| D(x^\sigma u) \|_{\ell+1},$$

and by Theorem 5.1,

$$\| D(x^\sigma u) \|_{\ell+1} \leq \Gamma_1 \| D(x^\sigma Du) \|_\ell.$$

Thus

$$\| D^{\ell+1}(x^\sigma u) \|_{L^\infty} \leq C_1 \Gamma_1 \| D(x^\sigma Du) \|_\ell.$$

Together with (4.7) and (5.23), this yields (5.22) with

$$\kappa_\ell = C_1\Gamma_1 + \sum_{j=1}^{\ell+1} |c_j|\kappa_{\ell-j}.$$

□

Lemma 5.5: For each positive integer ℓ and every $\varepsilon > 0$, there exists a constant $K_\ell = K(\sigma,\varepsilon,\ell)$, independent of u, such that, if $u \in S^m$, $m \geq -1$, then

$$(5.24) \qquad \| x^{(2\ell-3+2\sigma+\varepsilon)/2} D^\ell u \|_0 \leq K_\ell \| D(x^\sigma Du) \|_{\ell-2}$$

for all $1 \leq \ell \leq m+2$.

Proof: Again we use induction on ℓ. For $\ell = 1$,

$$\int_0^1 x^{2\sigma-1+\varepsilon}(Du)^2\, dx \leq \| x^\sigma Du \|^2_{L^\infty} \int_0^1 x^{\varepsilon-1}\, dx$$

$$= \varepsilon^{-1} \| D(x^\sigma Du) \|^2_{-1}.$$

For $\ell > 1$, we differentiate to obtain

$$D^{\ell-1}(x^\sigma Du) = x^\sigma D^\ell u + \sum_{i=1}^{\ell-1} \binom{\ell-1}{i} D^i(x^\sigma) D^{\ell-i}u,$$

whence

$$x^\sigma D^\ell u = D^{\ell-2}[D(x^\sigma Du)] - \sum_{i=1}^{\ell-1} \binom{\ell-1}{i} D^i(x^\sigma) D^{\ell-i}u.$$

Let

$$c_i = \binom{\ell-1}{i} \prod_{j=0}^{i-1} (\sigma-j).$$

By the triangle inequality,

$$\| x^{(2\ell-3+2\sigma+\varepsilon)/2} D^{\ell} u \|_0$$

$$\leq \| x^{(2\ell-3+\varepsilon)/2} D^{\ell-2}(D(x^{\sigma}Du)) \|_0$$

$$+ \sum_{i=1}^{\ell-1} |c_i| \; \| x^{(2(\ell-i)-3+2\sigma+\varepsilon)/2} D^{\ell-i} u \|_0 .$$

Since $\ell \geq 2$, $2\ell - 3 + \varepsilon > 0$, so that

$$\| x^{(2\ell-3+\varepsilon)/2} D^{\ell-2}(D(x^{\sigma}Du)) \|_0 \leq \| D^{\ell-2}(D(x^{\sigma}Du)) \|_0 .$$

Thus, by the inductive hypothesis,

$$\| x^{(2\ell-3+2\sigma+\varepsilon)/2} D^{\ell} u \|_0$$

$$\leq (1 + \sum_{i=1}^{\ell-1} |c_i| K_{\ell-i}) \; \| D(x^{\sigma}Du) \|_{\ell-2} ,$$

whence the result follows with

$$K_{\ell} = 1 + \sum_{i=1}^{\ell-1} |c_i| K_{\ell-i} .$$

□

CHAPTER 6

SPLINES ON A β-GRADED MESH

6.1 Introduction

In this chapter we explore the use of splines on β-graded meshes to approximate the solution of (4.1) - (4.2). Subspaces of piecewise polynomials have been studied extensively, and their properties as approximators of smooth functions are well-known (see, e.g., [S3], [D2]). Our aim here is to find analogues of these results for the case of functions, elements of S^m, with a singularity at 0.

Given $u \in S^m$, we wish to show that with an appropriately chosen mesh the quasiinterpolant is a "good" approximation to u, even though $u \notin H^m$ for any $m \geq 2$. We cope with the singularity in u by grading the mesh so that the intervals near zero are small. We employ a β-graded mesh, the definition and properties of which are given in Section 4.3.

In Section 6.2 we prove error bounds for the quasiinterpolant of functions in S^m. S-norm bounds are obtained in Theorem 6.1 and L^2-norm bounds in Theorem 6.2. In both theorems, we specify a β-grading which yields optimal-order accuracy. Let $k \geq 2$ be the order of the polynomials, $\ell \equiv \min(k, m+2)$, and

$$\beta_1 \equiv \frac{2(\ell-1)}{1-\sigma},$$

$$\beta_2 \equiv \frac{\ell}{1-\sigma},$$

$$\beta_3 \equiv \frac{2\ell}{3-2\sigma} = \frac{\ell}{(3/2)-\sigma}.$$

We obtain the following optimal-order error bounds for the quasiinterpolant $F_\Delta u \in S_0^k(\Delta_{\beta,N},\underline{z})$:

$$\| u - F_\Delta u \|_S \leq C\, N^{-(\ell-1)} \, \| D(x^\sigma Du) \|_{\ell-2} \quad \text{if } \beta > \beta_1,$$

$$\| u - F_\Delta u \|_0 \leq C\, N^{-\ell} \, \| D(x^\sigma Du) \|_{\ell-2} \quad \text{if } \beta > \beta_3.$$

Obviously, since $\ell \geq 2$, $\beta_3 < \beta_2 \leq \beta_1$. Moreover, for fixed ℓ, β_1 and $\beta_2 \to \infty$ as $\sigma \to 1$, while β_3 remains bounded for all $0 \leq \sigma < 1$. Thus, far less β-grading is required to prove the L^2-error bound.

As a corollary to these approximation-theoretic results we obtain (in Theorem 6.3) L^2-norm and S-norm error bounds for the RRG approximation to the generalized solution of (4.1) - (4.2), provided the heavy β-grading of Theorem 6.1 ($\beta > \beta_1$) is used. In Section 6.3 we show that the RRG approximation will _not_ achieve the optimal rate of convergence in the L^2-norm in the spaces $S_0^2(\Delta_{\beta_3,N})$. We conjecture that while $\beta = \beta_3$ is too small, $\beta = \beta_2$ suffices for optimal L^2-norm accuracy of the RRG approximation, while $\beta = \beta_1$ is required for optimal S-norm accuracy. The numerical results in Section 6.4 include an empirical study of the effect of the grading parameter β on the error and rate of convergence of the RRG approximation which tends to substantiate this conjecture. Other experiments confirm the approximation-theoretic results of Section 6.2. In Section 6.5 we consider questions of computational complexity.

6.2 The Quasiinterpolant in S^m

Our proof that splines can approximate a function u(x) which behaves like $x^{1-\sigma}$ depends on the existence of local approximation mappings (for example, the quasiinterpolant). For such an approximation mapping, the error in the interval I_i depends only on the behavior of u in a small number of neighboring intervals. Thus, by using a β-graded mesh, we overcome the growth of the derivatives of u near 0.

We want the quasiinterpolant to satisfy the boundary conditions of (4.2). By the Corollary to Lemma 3.3, if we choose

$$\tau_1 = 0 \quad \text{and} \quad \tau_d = 1, \tag{6.1a}$$

then $F_\Delta u \in S_0^k(\Delta,\underline{z})$ for all $u \in S$. Here, and throughout the rest of the chapter, $\underline{z}$ may be any incidence vector.

We would like to make the quasiinterpolant independent of u near 0 for all but the first subinterval. Therefore, we require that

$$\tau_j \geq x_1, \quad 2 \leq j \leq d. \tag{6.1b}$$

Thus, in the intervals I_i, $2 \leq i \leq N$, $\theta_i \subset [x_1,1]$. The error bounds of Theorem 3.1 then apply, since u is smooth in θ_i.

In the first interval, Theorem 3.1 is of no help, since $D^{\ell}u \notin L^2(0,x_1)$ for any $\ell \geq 1$. But since the mesh is β-graded, so that $x_1 = N^{-\beta}$ is very small, it is possible to bound $\| u - F_{\Delta}u \|_{S(I)}$ by bounding first $\| u \|_{S(I)}$, then $\| F_{\Delta}u \|_{S(I)}$, and using the triangle inequality.

We first obtain bounds on the linear functionals $\lambda_j u$, for $j \in V_1$.

Lemma 6.1: There exists a positive constant c_1 independent of u and N such that, if $u \in S^{\ell-2}$, $\ell \geq 2$, $\Delta = \Delta_{\beta,N}$, and the quasiinterpolation points $\{\tau_j\}$ satisfy (6.1), then

(6.2) $$|\lambda_{j,\ell}u| \leq c_1 x_1^{1-\sigma} \| D(x^{\sigma}Du) \|_{\ell-2}$$

for all $2 \leq j \leq k$.

Proof: Using (6.1b) and the bound of (4.9)(e),

$$\begin{aligned} |\lambda_j u| &= \sum_{r<\ell} \left|\omega_{j,r}\, D^r u(\tau_j)\right| \\ &\leq C(\beta) \sum_{r<\ell} h_1^r \left|D^r u(\tau_j)\right| \\ &\leq C(\beta)\, x_1^{1-\sigma} \sum_{r<\ell} \left|\tau_j^{\sigma+r-1}\, D^r u(\tau_j)\right|, \end{aligned}$$

since $h_1 = x_1$. Now, using Lemma 5.4 and (4.7),

$$\begin{aligned} |\lambda_j u| &\leq C(\beta)\, x_1^{1-\sigma} \sum_{r<\ell} \kappa_{r-1} \| D(x^{\sigma}Du) \|_{r-1} \\ &\leq C(\beta) \left(\sum_{r<\ell} \kappa_{r-1}\right) x_1^{1-\sigma} \| D(x^{\sigma}Du) \|_{\ell-2}. \end{aligned}$$

□

Using this result, we obtain S-norm and L^2-norm bounds on $F_\Delta u$ in the first interval.

Lemma 6.2: There exists a positive constant c_2 independent of u and N such that, if $u \in S^{\ell-2}$, $\ell \geq 2$, $\Delta = \Delta_{\beta,N}$, and the quasiinterpolation points $\{\tau_j\}$ satisfy (6.1), then $F_\Delta u = F_{\Delta,\underline{z},\ell}\, u$ satisfies

$$\|F_\Delta u\|_{S(I_1)} \leq c_2\, x_1^{(1-\sigma)/2}\, \|D(x^\sigma Du)\|_{\ell-2} \tag{6.3}$$

$$\|F_\Delta u\|_{L^2(I_1)} \leq c_2\, x_1^{3/2-\sigma}\, \|D(x^\sigma Du)\|_{\ell-2}. \tag{6.4}$$

Proof: In the proof of Lemma 3.3, it is shown that $\lambda_1 u = u(0) = 0$. Therefore, in I_1, $F_\Delta u$ is given by

$$F_\Delta u = \sum_{j=2}^{k} (\lambda_j u) N_{j,k}(x).$$

Thus,

$$\|F_\Delta u\|_{S(I_1)} \leq \sum_{j=2}^{k} |\lambda_j u|\ \|N_{j,k}\|_{S(I_1)}$$

and

$$\|F_\Delta u\|_{L^2(I_1)} \leq \sum_{j=2}^{k} |\lambda_j u|\ \|N_{j,k}\|_{L^2(I_1)}.$$

Using the result of Lemma 3.1,

$$\begin{aligned}\|N_{j,k}\|_{S(I_1)} &= \left(\int_0^{x_1} x^\sigma (DN_{j,k})^2\, dx\right)^{1/2} \\ &\leq \|DN_{j,k}\|_{L^\infty(I_1)} \left(\frac{x_1^{1+\sigma}}{1+\sigma}\right)^{1/2} \\ &\leq \frac{(k-1)^2}{(1+\sigma)^{1/2}}\, x_1^{(\sigma-1)/2},\end{aligned}$$

while

$$\| N_{j,k} \|_{L^2(I_1)} = \left(\int_0^{x_1} (N_{j,k})^2 \, dx \right)^{1/2}$$
$$\leq \| N_{j,k} \|_{L^\infty(I_1)} x_1^{1/2}$$
$$\leq x_1^{1/2}.$$

Thus, by (6.2),

$$\| F_\Delta u \|_{S(I_1)} \leq \frac{(k-1)^3 c_1}{(1+\sigma)^{1/2}} x_1^{(1-\sigma)/2} \| D(x^\sigma Du) \|_{\ell-2}$$

and

$$\| F_\Delta u \|_{L^2(I_1)} \leq (k-1) c_1 \, x_1^{3/2-\sigma} \| D(x^\sigma Du) \|_{\ell-2}.$$

□

We now obtain S-norm error bounds for the quasiinterpolant.

Theorem 6.1: If $u \in S^m$, $m \geq 0$, $\Delta = \Delta_{\beta,N}$ with $\beta > \beta_1 \equiv \frac{2(\ell-1)}{1-\sigma}$, and the evaluation points $\{\tau_j\}$ satisfy (6.1), then $F_\Delta u = F_{\Delta,\underline{z},\ell}\, u \in S_0^k(\Delta,\underline{z})$ and

(6.5) $$\| u - F_\Delta u \|_S \leq c_3 N^{-(\ell-1)} \| D(x^\sigma Du) \|_{\ell-2},$$

where $\ell \equiv \min(k, m+2)$ and c_3 is independent of u and N.

Proof: We consider I_1 separately from the other intervals. By Lemma 5.4,

$$\| u \|^2_{S(I_1)} = \int_0^{x_1} x^\sigma (Du)^2 dx = \int_0^{x_1} \frac{(x^\sigma Du)^2}{x^\sigma} dx$$
$$\leq \frac{x_1^{1-\sigma}}{1-\sigma} \| x^\sigma Du \|^2_{L^\infty}$$
$$\leq (\mu\kappa_0)^2 x_1^{1-\sigma} \| D(x^\sigma Du) \|^2_0.$$

By (4.8), $x_1^{(1-\sigma)/2} = N^{-\beta(1-\sigma)/2} < N^{-(\ell-1)}$, and by the triangle inequality and (6.3),

$$\begin{aligned} \| u - F_\Delta u \|_{S(I_1)} &\leq \| u \|_{S(I_1)} + \| F_\Delta u \|_{S(I_1)} \\ &\leq (\mu\kappa_0 + c_2)\, x_1^{(1-\sigma)/2} \| D(x^\sigma Du) \|_{\ell-2} \\ &< (\mu\kappa_0 + c_2)\, N^{-(\ell-1)} \| D(x^\sigma Du) \|_{\ell-2} \\ &\equiv c_4\, N^{-(\ell-1)} \| D(x^\sigma Du) \|_{\ell-2}. \end{aligned} \tag{6.6}$$

Now we consider the remaining intervals I_i, $2 \leq i \leq N$. The error bounds of Theorem 3.1 refer to the neighborhood Θ_i of I_i. Let j_i be the largest integer such that $\Theta_i \subset [x_{j_i}, 1]$. By the definition of Θ_i and the conditions (6.1) on the points $\{\tau_j\}$,

$$j_i \geq \max(1, i-k)$$

(see Figure 6.1).

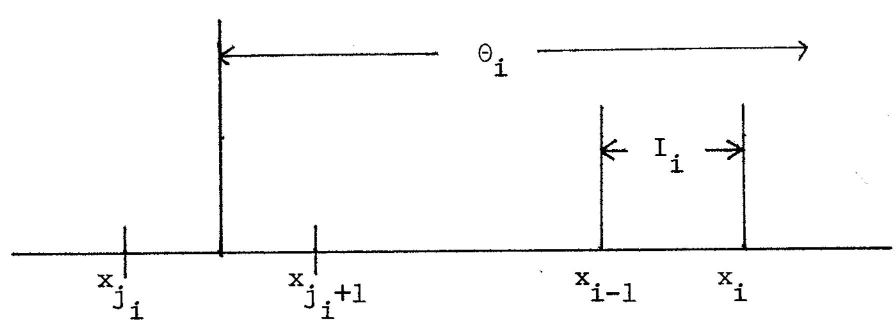


Figure 6.1: Definition of j_i.

Using the error bounds of Theorem 3.1,

$$\|u - F_\Delta u\|^2_{S(I_i)} = \int_{x_{i-1}}^{x_i} x^\sigma (D(u - F_\Delta u))^2\,dx$$

$$\leq x_i^\sigma \int_{x_{i-1}}^{x_i} (D(u - F_\Delta u))^2\,dx$$

$$\leq K\, x_i^\sigma\, |\Theta_i|^{2(\ell-1)} \int_{\Theta_i} (D^\ell u)^2\,dx.$$

Let $\varepsilon \equiv (1-\sigma) - \frac{2}{\beta}(\ell-1)$. Since $\beta(1-\sigma) > 2(\ell-1)$, $\varepsilon > 0$. By inequality (4.9)(d),

$$\|u - F_\Delta u\|^2_{S(I_i)}$$

$$\leq K\, C(\beta)^{2(\ell-1)} \frac{x_i^\sigma\, h_i^{2(\ell-1)}}{x_{j_i}^\zeta} \int_{\Theta_i} x^\zeta\, (D^\ell u)^2\,dx,$$

where $\zeta \equiv 2\ell - 3 + 2\sigma + \varepsilon$. (We introduced the term x^ζ so that Lemma 5.5 can be used later to bound the integral.) Using the definition (4.8) of x_i and the bound (4.9)(a) on h_i,

$$\frac{x_i^\sigma\, h_i^{2(\ell-1)}}{x_{j_i}^\zeta} \leq \beta^{2(\ell-1)} \frac{i^{\beta\sigma + 2(\ell-1)(\beta-1)}}{j_i^{\beta\zeta}} N^{-\beta(\sigma+2(\ell-1)-\zeta)}.$$

Moreover, $\beta(\sigma + 2(\ell-1) - \zeta) = \beta(1-\sigma-\varepsilon) = 2(\ell-1)$,

$$\begin{aligned}
\beta\zeta &= \beta(2\ell - 3 + 2\sigma + \varepsilon) \\
&= 2\ell\beta - \beta(1-\sigma-\varepsilon) - 2\beta + \beta\sigma \\
&= 2\ell\beta - 2(\ell-1) - 2\beta + \beta\sigma \\
&= \beta\sigma + 2(\ell-1)(\beta-1),
\end{aligned}$$

and $\frac{i}{j_i} \leq k$, since $j_i \geq \max(1, i-k)$. Thus

$$\| u - F_\Delta u \|^2_{S(I_i)}$$

$$(6.7) \qquad \leq K \, (\beta C(\beta))^{2(\ell-1)} k^{\beta\zeta} \, N^{-2(\ell-1)} \int_{\theta_i} x^\zeta (D^\ell u)^2 \, dx.$$

$$\equiv c_5^2 \, N^{-2(\ell-1)} \int_{\theta_i} x^\zeta (D^\ell u)^2 \, dx.$$

The intervals θ_i overlap, but no more than 2k of them at any point. Thus, by Lemma 5.5,

$$\sum_{i=2}^{N} \| x^{\zeta/2} D^\ell u \|^2_{L^2(\theta_i)} \leq 2k \, \| x^{\zeta/2} D^\ell u \|^2_0$$

$$\leq 2k \, K_\ell^2 \, \| D(x^\sigma Du) \|^2_{\ell-2}.$$

(By the assumptions on β, $\varepsilon > 0$ as required by Lemma 5.5.) Together with (6.6) and (6.7), this yields

$$\| u - F_\Delta u \|^2_S = \sum_{i=1}^{N} \| u - F_\Delta u \|^2_{S(I_i)}$$

$$\leq (c_4^2 + 2k(K_\ell c_5)^2) \, N^{-2(\ell-1)} \, \| D(x^\sigma Du) \|^2_{\ell-2}.$$

□

We now consider L^2-norm error bounds.

<u>Theorem 6.2</u>: If $u \in S^m$, $m \geq 0$, $\Delta = \Delta_{\beta,N}$ with $\beta > \beta_3 \equiv \frac{2\ell}{3 - 2\sigma}$, and the evaluation points $\{\tau_j\}$ satisfy (6.1), then $F_\Delta u = F_{\Delta,\underline{z},\ell} \, u \in S_0^k(\Delta,\underline{z})$ and

$$(6.8) \qquad \| u - F_\Delta u \|_0 \le c_6 N^{-\ell} \| D(x^\sigma Du) \|_{\ell-2},$$

where $\ell \equiv \min(k, m+2)$ and c_6 is independent of u and N.

Proof: Again we consider I_1 separately from the other intervals. By (4.8), $x_1^{3-2\sigma} = N^{-\beta(3-2\sigma)} < N^{-2\ell}$, and by Lemma 5.4 and (4.7),

$$\| u \|^2_{L^2(I_1)} = \int_0^{x_1} u^2 \, dx$$

$$= \int_0^{x_1} (x^{\sigma-1} u)^2 \, x^{2-2\sigma} \, dx$$

$$(6.9) \qquad \le \frac{1}{3-2\sigma} \, x_1^{3-2\sigma} \, \| x^{\sigma-1} u \|^2_{L^\infty}$$

$$\le \frac{1}{3-2\sigma} \, x_1^{3-2\sigma} \, \kappa_{-1}^2 \, \| D(x^\sigma Du) \|^2_{-1}$$

$$\le \frac{\kappa_{-1}^2}{3-2\sigma} \, N^{-2\ell} \, \| D(x^\sigma Du) \|^2_0 .$$

By the triangle inequality, (6.4), and (6.9),

$$(6.10) \qquad \| u - F_\Delta u \|_{L^2(I_1)} \le \left(c_2 + \frac{\kappa_{-1}}{\sqrt{3-2\sigma}}\right) N^{-\ell} \, \| D(x^\sigma Du) \|_{\ell-2}$$

$$\equiv c_7 \, N^{-\ell} \, \| D(x^\sigma Du) \|_{\ell-2} .$$

We now consider I_i, $2 \le i \le N$. According to Theorem 3.1 and (4.9)(d),

$$\int_{x_{i-1}}^{x_i} (u - F_\Delta u)^2 \, dx \le K^2 \, |\theta_i|^{2\ell} \int_{\theta_i} (D^\ell u)^2 \, dx$$

$$\le K^2 \, C(\beta)^{2\ell} \, h_i^{2\ell} \int_{\theta_i} (D^\ell u)^2 \, dx .$$

Because $\beta(3 - 2\sigma) > 2\ell$,

$$\varepsilon \equiv (3 - 2\sigma) - 2\ell/\beta > 0$$

and

$$\beta(3 - 2\sigma - \varepsilon) = 2\ell.$$

Let x_{j_i} and ζ be defined as in the proof of Theorem 6.1. Then

$$\int_{x_{i-1}}^{x_i} (u - F_\Delta u)^2\, dx \le K^2\, C(\beta)^{2\ell} \frac{h_i^{2\ell}}{x_{j_i}^{\zeta}} \int_{\Theta_i} x^\zeta (D^\ell u)^2\, dx$$

$$\le K^2\, (\beta C(\beta))^{2\ell} \frac{i^{2\ell(\beta-1)}}{j_i^{\beta\zeta}} N^{-\beta(2\ell-\zeta)} \int_{\Theta_i} x^\zeta (D^\ell u)^2 dx$$

(6.11)

$$\le K^2\, (\beta C(\beta) k^{\beta-1})^{2\ell}\, N^{-2\ell} \int_{\Theta_i} x^\zeta (D^\ell u)^2\, dx$$

$$\equiv c_8\, N^{-2\ell} \int_{\Theta_i} x^\zeta (D^\ell u)^2\, dx,$$

since $i/j_i \le k$, $\beta(2\ell-\zeta) = \beta(3 - 2\sigma - \varepsilon) = 2\ell$, and

$$\beta\zeta = \beta(2\ell - 3 + 2\sigma + \varepsilon) = 2\ell\beta - \beta(3 - 2\sigma - \varepsilon) = 2\ell(\beta - 1).$$

From (6.10) and (6.11),

$$\int_0^1 (u - F_\Delta u)^2\, dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (u - F_\Delta u)^2\, dx$$

$$\le N^{-2\ell} \left(c_7^2\, \| D(x^\sigma Du) \|_{\ell-2}^2 + c_8 \sum_{i=2}^{N} \int_{\Theta_i} x^\zeta (D^\ell u)^2\, dx \right).$$

No more than 2k of the Θ_i overlap at any point, so by Lemma 5.5,

$$\sum_{i=2}^{N} \int_{\Theta_i} x^{\zeta}(D^{\ell}u)^2\,dx \le 2k \int_0^1 x^{\zeta}(D^{\ell}u)^2\,dx$$

$$\le 2kK_{\ell-2} \ \| D(x^{\sigma}Du) \|_{\ell-2}^2 .$$

This yields (6.8), with $c_6 = (c_7^2 + 2kK_{\ell-2}c_8)^{1/2}$.

□

We now obtain error bounds for the Rayleigh-Ritz-Galerkin procedure for approximating the solution of the singular problem (4.1) - (4.2).

Theorem 6.3: Let $\tilde{u} \in S_0^k(\Delta_{\beta,N},\underline{z})$ be the RRG approximation to the generalized solution u of (4.1) - (4.2), where $\beta > \beta_1 \equiv \frac{2(\ell-1)}{1-\sigma}$. If p, q, and f satisfy the hypotheses of Theorem 5.2, then

(6.12) $$\| u - \tilde{u} \|_S \le \frac{\Lambda}{\lambda} c_3 \Gamma N^{-(\ell-1)} \| f \|_{\ell-2}$$

and

(6.13) $$\| u - \tilde{u} \|_0 \le (\Lambda c_3 \Gamma)^2 N^{-\ell} \| f \|_{\ell-2},$$

where $\ell \equiv \min(k, m+2)$.

Proof: By Theorem 6.1 (the "approximation hypothesis"), Theorem 5.2 (the "regularity hypothesis"), and Nitsche's trick (Theorem 2.6),

$$\| u - \tilde{u} \|_S \le \frac{\Lambda}{\lambda} c_3 N^{-(\ell-1)} \| D(x^{\sigma}Du) \|_{\ell-2}$$

and

$$\| u - \tilde{u} \|_0 \le (\Lambda c_3)^2 \Gamma N^{-\ell} \| D(x^{\sigma}Du) \|_{\ell-2}.$$

We now use Theorem 5.2 to bound the right-hand side of these inequalities.

□

6.2 A Counterexample

To the best of our knowledge, no example has yet been reported in which the Rayleigh-Ritz-Galerkin method failed to produce an optimal-order L^2-norm approximation. Eisenstat, Schultz, and the author [E1] have shown that no such example exists for nonsingular problems. For the differential equation (4.1) - (4.2) in which $p \geq p_{min} > 0$ and a family $\{S_h\}_{h>0}$ of C^1-piecewise polynomial subspaces with respect to a quasiuniform mesh,

$$\|u - \tilde{u}\|_0 \leq C \inf_{v_h \in S_h} \|u - v_h\|_0,$$

where $\tilde{u}$ is the RRG approximation to u in S_h and the constant C is independent of u and h. However, for the differential equation which concerns us here, there do exist subspaces for which the L^2-norm of the error of the RRG approximations do not decrease as fast, as a function of h, as the error in the best L^2-approximations. We shall now give such an example.

We restrict our attention to the simple case of continuous, piecewise linear splines (the subspace $S^2(\Delta)$, which we hereafter refer to as $L(\Delta)$). Let u be the generalized solution of (4.1) - (4.2). According to Theorem 6.3, if $D(x^\sigma Du) \in H^0$ and $\tilde{u} \in L(\Delta_{\beta,N})$ is the RRG approximation to u, where $\beta > \beta_1 = \frac{2}{1-\sigma}$, then

$$\|u - \tilde{u}\|_0 \leq CN^{-2} \|D(x^\sigma Du)\|_0.$$

This much β-grading is not required for a 2^{nd}-order L^2-norm approximation to exist. The result of Theorem 6.2 is the error bound

$$\| u - u_I \|_0 \le CN^{-2} \| D(x^{\sigma} Du) \|_0,$$

where $u_I \in L(\Delta_{\beta,N})$ is the quasiinterpolant of u and $\beta > \beta_3 = \frac{4}{3-2\sigma}$.

If we choose as the quasiinterpolation points $\{\tau_j\}$ of (3.5) - (3.6)

(6.14) $$\tau_j = x_{j-1}, \quad 1 \le j \le d = N+1,$$

then the quasiinterpolant interpolates:

$$u_I(x_i) = u(x_i), \quad 0 \le i \le N.$$

In other words, the piecewise linear quasiinterpolant with the "obvious" choice (6.14) for τ_j is the familiar piecewise linear interpolate. To show this, we note that for k = 2, the quasiinterpolation formulae involve only $u(\tau_j)$ and $Du(\tau_j)$. By choosing $\tau_j = x_{j-1}$, we make the coefficients $\omega_{j,1} = 0$ and $\omega_{j,0} = 1$; thus $\lambda_j u = u(x_{j-1})$.

Let us define the "hat functions" $L_i \in L(\Delta)$ by

$$L_i(x_j) = \delta_{ij}, \quad 0 \le i,j \le N$$

(see Figure 6.2).

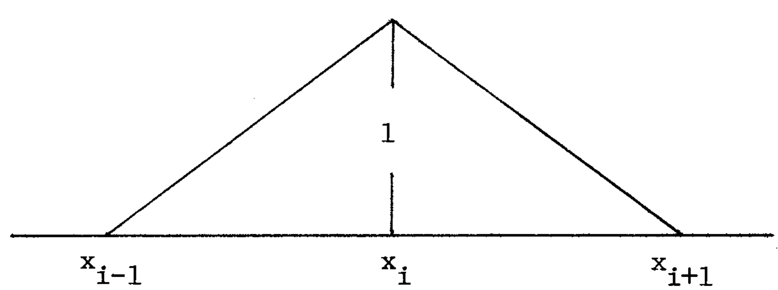


Figure 6.2: The "hat function" $L_i(x)$.

It is easily verified that $L_i = N_{i+1,2}$; i.e., the "hat functions" are the piecewise linear B-splines. We have shown, therefore, that the quasiinterpolant u_I is given by

$$u_I = \sum_{i=1}^{N+1} (\lambda_i u) N_{i,2}(x)$$

$$= \sum_{i=0}^{N} u(x_i) L_i = \sum_{i=1}^{N-1} u(x_i) L_i ,$$

and that

$$u_I(x_i) = u(x_i) L_i(x_i) = u(x_i).$$

In order to show that the RRG approximation is not optimal in this case, we need a lower bound on the error $\| u - \tilde{u} \|_0$. But since

$$\| u - \tilde{u} \|_0 \geq \| \tilde{u} - u_I \|_0 - \| u - u_I \|_0$$

and $\| u - u_I \|_0 = O(N^{-2})$, it suffices to prove that $\| \tilde{u} - u_I \|_0 \geq CN^{-2+\varepsilon}$ for some C, $\varepsilon > 0$. We will in fact show that $\| \tilde{u} - u_I \|_0 \geq CN^{-1}$.

Our plan of attack is as follows. We know that the coefficient vector of $\tilde{u}$ with respect to the basis $\{L_i\}$ is the solution of the linear system $A\underline{\xi} = \underline{f}$ of (2.10). For a specific problem (that is, a specific function f and corresponding solution u), we will compute the elements of A and $\underline{f}$. (Fortunately, A is tridiagonal.) Since we know u, we also can compute the coefficient vector $\underline{\eta}$ of u_I and the vector $\underline{g} = A\underline{\eta}$. It happens that the elements of $\underline{f} - \underline{g}$ are positive and that A^{-1} has positive elements. Thus the elements of $\underline{w} = \underline{\xi} - \underline{\eta} = A^{-1}(\underline{f} - \underline{g})$ are also positive. We then get a lower bound on the norm of $u - \tilde{u}$ by exhibiting a vector $\underline{\zeta}$ with the property that $A\underline{\zeta} < A\underline{w}$ and showing that the L^2-norm of the element of $L(\Delta)$ with coefficient vector $\underline{\zeta}$ decreases at the rate $O(N^{-1})$.

We now consider the specific example

$$- D(x^{1/2}Du) = \frac{3}{2}, \qquad 0 < x < 1,$$

$$u(0) = u(1) = 0,$$

which has the solution $u(x) = x^{1/2} - x^{3/2}$. Let $\Delta = \Delta_{\beta_3,N}$, with $\beta_3 = \frac{2k}{3-2\sigma} = 2$. The $L(\Delta)$-interpolate u_I of u can be written

$$u_I = \sum_{i=1}^{N-1} \eta_i L_i(x)$$

with

(6.15) $$\eta_i = x_i^{1/2} - x_i^{3/2} = \frac{i}{N} - \left(\frac{i}{N}\right)^3.$$

Note that $u_I \in L_0(\Delta) \equiv \{v \in L(\Delta) \mid v(0) = v(1) = 0\}$.

Let $\tilde{u} = \sum_{i=1}^{N-1} \xi_i L_i$. The vector $\underline{\xi} = [\xi_i] = A^{-1}\underline{f}$, where

$$A = \begin{bmatrix} b_1+b_2 & -b_2 & & & & \\ -b_2 & \cdot & \cdot & & & \\ & \cdot & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & & \\ & -b_i & b_i+b_{i+1} & -b_{i+1} & & \\ & & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & -b_{N-1} \\ & & & & -b_{N-1} & b_{N-1}+b_N \end{bmatrix},$$

$$b_i = \int_{x_{i-1}}^{x_i} x^{1/2}(DL_i)^2\,dx = \int_{x_{i-1}}^{x_i} x^{1/2}(h_i)^{-2}\,dx$$

$$= \frac{2}{3} h_i^{-2} (x_i^{3/2} - x_{i-1}^{3/2}) \qquad (6.16)$$

$$= \frac{2}{3} N \frac{i^3 - (i-1)^3}{(2i-1)^2},$$

and

$$f_i = \frac{3}{2}\int_0^1 L_i(x)\,dx = \frac{3}{4}(h_i + h_{i+1}) = 3iN^{-2}.$$

From (6.15) and (6.16),

$$A\underline{\eta} = \underline{g} = [g_i],$$

where

$$g_i = -b_i\eta_{i-1} + (b_i + b_{i+1})\eta_i - b_{i+1}\eta_{i+1}$$

$$= 3iN^{-2}\left(1 - \frac{1/3}{3(16i^4 - 8i^2 + 1)}\right) + \frac{4i}{3(16i^4 - 8i^2 + 1)}.$$

Let $e = u_I - \tilde{u} = \sum_{i=1}^{N-1} (\eta_i - \xi_i)L_i = \sum_{i=1}^{N-1} w_i L_i$. Then

$$A\underline{w} = \underline{g} - \underline{f} = \frac{4i - iN^{-2}}{3(16i^4 - 8i^2 + 1)} > 0.$$

In order to establish a $C(N^{-1}) + o(N^{-1})$ lower bound on $\| e \|_0$, we employ the piecewise linear function $\hat{e} \equiv \sum_{i=1}^{N-1} \zeta_i L_i \in L_0(\Delta)$ given by

$$\zeta_i \equiv \frac{N-i}{6N^2}, \qquad 1 \leq i \leq N-1.$$

We claim that $\|\hat{e}\|_0 = CN^{-1} + O(N^{-3/2})$ and

(6.17) $$0 < \hat{e}(x_i) < e(x_i), \quad 1 \leq i \leq N-1$$

(i.e., $0 < \zeta_i < w_i$, $1 \leq i \leq N-1$).

In order to show (6.17), we make use of the fact that A is an irreducible Stieltjes matrix †. Thus [V1, Corollary 3 of Theorem 3.11], $A^{-1} > 0$. This implies that, for any n-vectors $\underline{\xi}$ and $\underline{\eta}$, if $A\underline{\xi} > A\underline{\eta}$, then $\underline{\xi} > \underline{\eta}$.

First we compute $\|\hat{e}\|_0$. Note that, in the interval $[x_1, 1]$, $\hat{e}$ is monotonically decreasing. Thus

$$\|\hat{e}\|_0^2 \geq \sum_{i=2}^{N} h_i \, \hat{e}(x_i)^2$$

$$= \sum_{i=2}^{N} \frac{2i-1}{N^2} \left(\frac{N-i}{6N^2} \right)^2$$

$$= \frac{1}{36N^6} \sum_{i=2}^{N} (2i-1)(N-i)^2$$

---

† A Stieltjes matrix is a symmetric positive definite matrix with non-positive off-diagonal elements. As in [V1], we say for n x n matrices $A = [a_{ij}]$, $B = [b_{ij}]$ that $A > B$ if $a_{ij} > b_{ij}$ for all $1 \leq i,j \leq n$. We say $A > 0$ if $a_{ij} > 0$ for all $1 \leq i,j \leq n$. These notational definitions also apply to n-vectors, considered as n x 1 matrices.

As we noted above, in order to prove (6.17) we need to compute $A\underline{\zeta}$ and check that $A\underline{\zeta} < A\underline{w} = \underline{g} - \underline{f}$. But

$$\begin{aligned}(A\underline{\zeta})_1 &= (b_1 + b_2)\zeta_1 - b_2\zeta_2 \\ &= \frac{2}{3}N\left[\frac{16}{9}\left(\frac{N-1}{6N^2}\right) - \frac{7}{9}\left(\frac{N-2}{6N^2}\right)\right] \\ &= \frac{1}{9N}\left(N - \frac{2}{9}\right) < \frac{1}{9} < (A\underline{w})_1\end{aligned}$$

since $(A\underline{w})_1 = \frac{4-N^{-2}}{27} \geq \frac{1}{9}$.

For $2 \leq i \leq N-1$, $b_{i+1} < b_i$, whence

$$\begin{aligned}(A\underline{\zeta})_i &= \frac{1}{6N^2}\,[-b_i(N-i+1) + (b_i + b_{i+1})(N-i) - b_{i+1}(N-i-1)] \\ &= \frac{1}{6N^2}\,(b_{i+1} - b_i) < 0 < (A\underline{w})_i.\end{aligned}$$

We have shown that $\|e\|_0 > \|\hat{e}\|_0 = CN^{-1} + 0(N^{-3/2})$. Therefore,

$$\|u - \tilde{u}\|_0 \geq \|\tilde{u} - u_I\|_0 - \|u - u_I\|_0 > CN^{-1} + 0(N^{-3/2}).$$

Thus, in this situation, the RRG procedure does not produce an optimal L^2-approximation.

6.4 Numerical Results

In this section we present the results of some numerical experiments which illustrate the computational procedure analyzed in the previous sections. We consider the problem

$$- D(\sqrt{x}\,Du) = \pi^2 \sin(\pi x) + \frac{1}{2x}\left(\pi \cos(\pi x) - \frac{\sin(\pi x)}{x} \right),$$

$$u(0) = u(1) = 0,$$

which has the solution $u(x) = \dfrac{\sin(\pi x)}{\sqrt{x}}$. The RRG approximations to u from various spline subspaces were computed and the error tabulated below.

All computations were performed in double precision on a PDP-10 (with 54 binary digits). The integrals required were computed using Gaussian quadrature, with k-1 nodes in each interval of the mesh. In the tables below, e_N denotes the error $u - \tilde{u}$, where $\tilde{u}$ is the RRG approximation in $S_0^k(\Delta_{\beta,N})$. We measure the error in the L^2-norm ($\| e_N \|_0$) and the S-norm ($\| e_N \|_S$), which were computed using Gaussian quadrature with k+1 nodes in each interval. We also computed the quantity

$$\| e_N \|_{\infty,\Delta} \equiv \max_{1 \le i \le N} | e_N(x_i) |.$$

In the following tables, we use the shorthand notation

$$5.2\ (-4) \quad \text{for} \quad 5.2 \times 10^{-4}.$$

Next to the error e_N for a given value of N, we list the observed rate of convergence of the error, computed from the norms of e_N and the previous error $e_{N'}$, by the formula

$$r_N = \frac{\log(\| e_N \| / \| e_{N'} \|)}{\log(N' / N)} .$$

For $\sigma = .5$, the values of β_1, β_2, and β_3 are given (for various k) in Table 6.1.

k	β_1	β_2	β_3
2	4	4	2
3	8	6	3
4	12	8	4
6	20	12	6

Table 6.1: Values of β_1, β_2, and β_3 at $\sigma = .5$

First we illustrate the result of using heavily β-graded meshes $\Delta_{\beta_3,N}$, which were analyzed in Theorems 6.1 and 6.3.

N	$\| e_N \|_0$	RATE	$\| e_N \|_{\infty,\Delta}$	RATE	$\| e_N \|_S$	RATE
16	.79 (-02)		.11 (-01)		.12 (+00)	
32	.89 (-03)	3.15	.11 (-02)	3.32	.30 (-01)	1.98
48	.25 (-03)	3.16	.22 (-03)	4.02	.13 (-01)	2.00
64	.10 (-03)	3.09	.64 (-04)	4.22	.75 (-02)	1.98
80	.52 (-04)	3.06	.25 (-04)	4.29	.48 (-02)	1.98
96	.30 (-04)	3.04	.11 (-04)	4.31	.34 (-02)	1.97

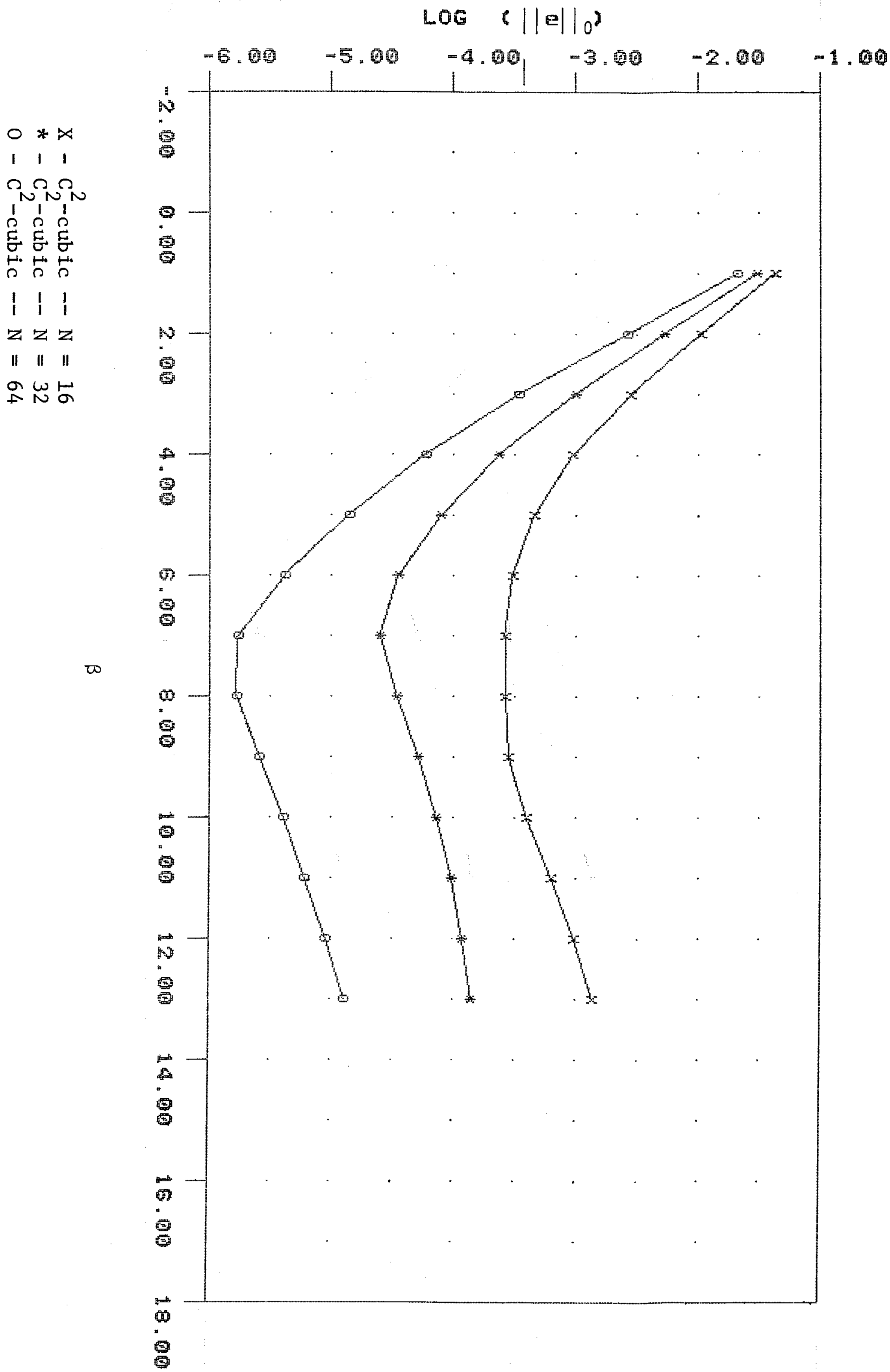
Table 6.2: Error in $S_0^3(\Delta_{8,N})$ -- 8 = β_1

N	$\| e_N \|_0$	RATE	$\| e_N \|_{\infty,\Delta}$	RATE	$\| e_N \|_S$	RATE
16	.97 (-03)		.11 (-02)		.23 (-01)	
32	.12 (-03)	3.01	.20 (-03)	2.36	.40 (-02)	2.50
48	.31 (-04)	3.35	.55 (-04)	3.22	.14 (-02)	2.59
64	.91 (-05)	4.27	.15 (-04)	4.44	.63 (-03)	2.81
80	.34 (-05)	4.45	.54 (-05)	4.75	.33 (-03)	2.85
96	.15 (-05)	4.37	.34 (-05)	2.42	.20 (-03)	2.85

Table 6.3: Error in $S_0^4(\Delta_{12,N})$ -- 12 = β_1

In order to get some idea of the effect of the parameter β on the error $\| e_N \|_0$, we computed the RRG approximations in several cubic spline subspaces (the spaces $S_0^4(\Delta)$). We used meshes with N = 16, 32, and 64 and $\beta = 1, 2, \ldots, 13$. These results are presented in Figure 6.3.

The results of this experiment lead us to believe that the RRG approximation in $S_0^k(\Delta_{\beta_2,N})$ converges in the L^2-norm at the rate N^{-k} and that the error will generally be smaller than for the more heavily graded meshes with $\beta = \beta_1$. The results of an experiment to test this hypothesis are presented in Tables 6.4 - 6.6.



EFFECT OF GRADING ON L2-NORM OF ERROR

N	$\Vert e_N \Vert_0$	RATE	$\Vert e_N \Vert_{\infty,\Delta}$	RATE	$\Vert e_N \Vert_S$	RATE
32	.38 (-03)	3.18	.36 (-03)	3.79	.21 (-01)	1.85
48	.11 (-03)	3.08	.73 (-04)	3.95	.10 (-01)	1.76
64	.46 (-04)	3.04	.32 (-04)	2.87	.61 (-02)	1.71
80	.23 (-04)	3.02	.17 (-04)	2.90	.42 (-02)	1.68
96	.13 (-04)	3.01	.99 (-05)	2.91	.31 (-02)	1.65

Table 6.4: Error in $S_0^3(\Delta_{6,N})$ -- $6 = \beta_2$

N	$\Vert e_N \Vert_0$	RATE	$\Vert e_N \Vert_{\infty,\Delta}$	RATE	$\Vert e_N \Vert_S$	RATE
32	.34 (-04)	2.95	.53 (-04)	3.38	.46 (-02)	1.95
48	.59 (-05)	4.32	.79 (-05)	4.70	.20 (-02)	2.01
64	.17 (-05)	4.36	.36 (-05)	2.78	.11 (-02)	2.00
80	.67 (-06)	4.15	.19 (-05)	2.81	.72 (-03)	2.00
96	.32 (-06)	4.00	.11 (-05)	3.07	.50 (-03)	2.00

Table 6.5: Error in $S_0^4(\Delta_{8,N})$ -- $8 = \beta_2$

N	$\Vert e_N \Vert_0$	RATE	$\Vert e_N \Vert_{\infty,\Delta}$	RATE	$\Vert e_N \Vert_S$	RATE
32	.34 (-05)	4.76	.59 (-05)	4.60	.12 (-02)	2.96
48	.36 (-06)	5.55	.71 (-06)	5.22	.37 (-03)	3.00
64	.59 (-07)	6.26	.11 (-06)	6.35	.16 (-03)	3.00
80	.15 (-07)	6.12	.31 (-07)	5.90	.80 (-04)	3.00
96	.52 (-08)	5.81	.12 (-07)	5.14	.46 (-04)	3.00

Table 6.6: Error in $S_0^6(\Delta_{12,N})$ -- $12 = \beta_2$

Note that the L^2-norm of the error for the spaces $S_0^3(\Delta_{\beta_2,N})$ and $S_0^4(\Delta_{\beta_2,N})$ appears to be smaller than that for the more heavily graded spaces $S_0^3(\Delta_{\beta_3,N})$ and $S_0^4(\Delta_{\beta_3,N})$, while the S-norm of the error is larger. Moreover, it appears that the rate of convergence in the S-norm has been reduced from $O(N^{-(k-1)})$ to $O(N^{-k/2})$. A slight modification of the proof of Theorem 6.1 shows that, in general, using $\beta > \beta_2$ yields this rate of convergence in the S-norm.

In order to illustrate the result of Theorem 6.2, we computed the L^2-projections of u on several spline subspaces using only the grading required in that theorem, $\beta = \beta_3$. The results are given in Tables 6.7 - 6.9. The data confirm that, with $\beta = \beta_3$, the L^2-projection is k^{th}-order accurate (in the L^2-norm); it also seems that the rate of convergence in the S-norm is only 1/4 that of the L^2-rate. Moreover, the convergence at the knots is only at 1/2 the L^2-rate -- the opposite of superconvergence!

N	$\|e_N\|_0$	RATE	$\|e_N\|_{\infty,\Delta}$	RATE	$\|e_N\|_S$	RATE
8	.40 (-02)		.82 (-02)		.19 (+00)	
16	.47 (-03)	3.09	.29 (-02)	1.51	.10 (+00)	0.84
24	.14 (-03)	3.02	.16 (-02)	1.50	.76 (-01)	0.77
32	.58 (-04)	3.00	.10 (-02)	1.50	.61 (-01)	0.76
48	.17 (-04)	2.99	.55 (-03)	1.50	.45 (-01)	0.75

Table 6.7: Error in L^2-projection on $S_0^3(\Delta_{3,N})$ -- $3 = \beta_3$

N	$\|e_N\|_0$	RATE	$\|e_N\|_{\infty,\Delta}$	RATE	$\|e_N\|_S$	RATE
8	.35 (-03)		.27 (-02)		.99 (-01)	
16	.41 (-04)	3.10	.67 (-03)	2.04	.49 (-01)	1.00
24	.76 (-05)	4.13	.29 (-03)	2.00	.33 (-01)	1.00
32	.24 (-05)	4.01	.16 (-03)	2.00	.24 (-01)	1.00
48	.50 (-06)	3.84	.74 (-04)	2.00	.17 (-01)	1.00

Table 6.8: Error in L^2-projection on $S_0^4(\Delta_{4,N})$ -- $4 = \beta_3$

N	$\|e_N\|_0$	RATE	$\|e_N\|_{\infty,\Delta}$	RATE	$\|e_N\|_S$	RATE
8	.17 (-03)		.85 (-03)		.58 (-01)	
16	.60 (-05)	4.81	.10 (-03)	2.98	.20 (-01)	1.49
24	.67 (-06)	5.42	.32 (-04)	2.99	.11 (-01)	1.49
32	.13 (-06)	5.67	.13 (-04)	3.00	.73 (-02)	1.50
48	.13 (-07)	5.67	.40 (-05)	3.00	.40 (-02)	1.50

Table 6.9: Error in L^2-projection on $S_0^6(\Delta_{6,N})$ -- $6 = \beta_3$

6.5 A Comment on the Computational Complexity of the Algorithm

Mesh grading, in effect, puts more of the computational effort into the region where u is badly behaved. The price one pays is the corresponding decrease in effort (and accuracy) elsewhere. As shown in Section 4.3, the largest interval in a β-graded mesh is approximately β times the size of the corresponding interval in a uniform mesh. Thus, to obtain the same accuracy as we obtain in a nonsingular problem using a uniform mesh, we might have to take a mesh with β times as many intervals. The computational complexity of the numerical procedures used in the RRG method is $O(N)$ for a mesh with N intervals. (To set up the linear system of (2.10) takes $O(k^3)$ work in each subinterval. The order of the system, which is equal to the dimension of the spline subspace, grows linearly with N; since it is banded with bandwidth independent of N, its solution requires $O(N)$ work and storage.) Therefore, this method requires on the order of β times as much work for the same accuracy as in the nonsingular case.

CHAPTER 7

WEIGHTED SPLINES

7.1 Introduction

In this chapter we consider approximations of the form $x^{-\sigma}s(x)$, where $s(x)$ is a smooth function vanishing at 0 and 1. When $s(x)$ is a piecewise polynomial, we call these functions "weighted splines". We know, from Theorem 5.1, that $u = x^{-\sigma}v$, where v is smooth. Therefore, we expect that $x^{-\sigma}\hat{v}$ will be a good approximation to u when $\hat{v}$ is a good spline approximation to v.

In Section 7.2, we define weighted splines and obtain a useful bound on the S-norm of a function w, vanishing at 0, in terms of the L^{∞}-norm of $D(x^{\sigma}w)$. In Section 7.3, we prove error bounds for a weighted spline approximation scheme on a weakly β-graded mesh $(1 \leq \beta < 2)$ and use this result to obtain optimal-order error bounds for the RRG approximation. As the approximation $\hat{v}$ we use the quasiinterpolant of v. In contrast to the β-grading used in Chapter 6, here β remains bounded above for fixed k and all $0 \leq \sigma < 1$. Moreover, in Chapter 6, for fixed

σ, $\beta \to \infty$ as $k \to \infty$, but in this chapter, for fixed σ, $\beta \to 1$ as $k \to \infty$. Numerical results are included in Section 7.4.

7.2 Weighted Spline Subspaces

Definition 7.1. If $0 \leq \sigma < 1$ and S_n is a space of functions defined on I, let

$$S_{n,\sigma} \equiv \{x^{-\sigma}s \mid s \in S_n\}.$$

□

If S_n is any of the spaces $S^k(\Delta,\underline{z})$ (respectively, $S_0^k(\Delta,\underline{z})$, $S_0^k(\Delta)$), then we call the elements of $S_{n,\sigma}$ "weighted splines" and denote the space $S_{n,\sigma}$ by $WS^k(\sigma,\Delta,\underline{z})$ (respectively, $WS_0^k(\sigma,\Delta,\underline{z})$, $WS_0^k(\sigma,\Delta)$).

The following result will be used both to show that $WS_0^k(\sigma,\Delta,\underline{z}) \subset S$ and to prove error bounds in Section 7.3.

Lemma 7.1. If $w(x) = x^{-\sigma}v(x)$, $0 < x \leq 1$, where $v \in W^{1,\infty}$ and $v(0) = 0$, then $w \in S$ and

$$\|w\|_{S(0,b)} \leq c_0 b^{(1-\sigma)/2} \|Dv\|_{L^\infty(0,b)} \quad \text{for all } 0 \leq b \leq 1,$$

with $c_0 \equiv \dfrac{1+\sigma}{\sqrt{1-\sigma}}$.

Proof: By L'Hôpital's rule,

$$\lim_{x\to 0} w(x) = \lim_{x\to 0} \frac{v(x)}{x^\sigma}$$

$$= \lim_{x\to 0} \frac{Dv(x)}{\sigma\, x^{\sigma-1}}$$

$$= 0$$

since $x^{\sigma-1}\to\infty$ while $Dv(x)$ is bounded. Thus, $w(0)$ may be taken to be 0. Furthermore,

$$\|w\|^2_{S(0,b)} = \int_0^b x^\sigma (Dw(x))^2\, dx$$

$$= \int_0^b x^\sigma [x^{-\sigma}Dv - \sigma x^{-\sigma} \frac{v}{x}]^2\, dx$$

$$= \int_0^b x^{-\sigma} [(Dv)^2 - 2\sigma(\frac{v}{x})Dv + \sigma^2(\frac{v}{x})^2]\, dx.$$

Since $v(0) = 0$ and $v \in W^{1,\infty}$,

$$\left|\frac{v}{x}\right| = \frac{1}{x}\left|\int_0^x Dv(t)\, dt\right| \leq \|Dv\|_{L^\infty(0,b)}$$

for all $0 < b \leq 1$, so that

$$\left\|\frac{v}{x}\right\|_{L^\infty(0,b)} \leq \|Dv\|_{L^\infty(0,b)}.$$

Therefore

$$\|w\|^2_{S(0,b)} \leq (1 + 2\sigma + \sigma^2)\ \|Dv\|^2_{L^\infty(0,b)} \int_0^b x^{-\sigma}\, dx$$

$$= c_0^2\, b^{1-\sigma}\ \|Dv\|^2_{L^\infty(0,b)} < \infty .$$

$\square$

Corollary: If $S_n \subset W_0^{1,\infty}(I)$, then $S_{n,\sigma} \subset S$.

Proof: If $v \in W_0^{1,\infty}$ and $w = x^{-\sigma}v$, then $w(1) = v(1) = 0$. The preceeding lemma shows that $w(0) = 0$ and that $x^{\sigma/2}Dw \in L^2(I)$.

□

In particular, $WS_0^k(\sigma,\Delta,\underline{z}) \subset S$.

7.3 Weighted Spline Approximation

According to Theorem 5.1, if $u \in S^m$, then $v \equiv x^{\sigma}u \in H^{m+2}$. We now consider using $\hat{u} \equiv x^{-\sigma}\hat{v}$ to approximate u, where $\hat{v} \equiv F_{\Delta}v$ is the quasiinterpolant of v.

Theorem 7.1: If $u \in S^m$, $m \geq 0$, $\Delta = \Delta_{\beta,N}$ with $\beta = \beta_4 \equiv \frac{2(\ell-1)}{2(\ell-1) - \sigma}$, and $\underline{z}$ is any incidence vector, then there exists an element $\hat{u} \in WS_0^k(\sigma,\Delta,\underline{z})$ satisfying

$$\| u - \hat{u} \|_S \leq c_1 N^{-(\ell-1)} \| D^{\ell-1}(x^{\sigma}Du) \|_0,$$

where $\ell \equiv \min(k, m+2)$ and c_1 is independent of u and N.

Proof: Let $\hat{v} = F_{\Delta}v$. We have already shown that if $\tau_1 = 0$ and $\tau_d = 1$, then $\hat{v} \in S_0^k(\Delta,\underline{z})$, i.e., it satisfies the boundary conditions. Moreover, by Lemma 3.3, if we also take $\tau_2 = 0$, then

$$D(v - \hat{v})|_{x=0} = 0.$$

This result enables us to use Lemma 2.1 to bound $\| D(v - \hat{v}) \|_{L^{\infty}(I_1)}$ which, in turn, bounds $\| u - \hat{u} \|_{S(I_1)}$.

We first consider I_1. By Theorem 5.1, $v \in H^{m+2}$. Thus, $v - \hat{v} \in W^{1,\infty}(I_1)$ and $v(0) = \hat{v}(0) = 0$. By Lemma 7.1,

$$\| u - \hat{u} \|^2_{S(I_1)} \leq c_0^2 \, h_1^{1-\sigma} \, \| D(v - \hat{v}) \|^2_{L^\infty(I_1)}.$$

Since the quasiinterpolant interpolates Dv at 0, Lemma 2.1 applies, and

$$\| D(v - \hat{v}) \|_{L^\infty(I_1)} \leq h_1^{1/2} \, \| D^2(v - \hat{v}) \|_{L^2(I_1)}.$$

Now using the error bound (3.9) for the quasiinterpolant and the bound (4.9)(d),

$$\| D(v - \hat{v}) \|_{L^\infty(I_1)} \leq K \, h_1^{1/2} \, |\theta_1|^{\ell-2} \, \| D^\ell v \|_{L^2(\theta_1)}$$

$$\leq K \, C(\beta)^{\ell-2} \, h_1^{(\ell-3/2)} \, \| D^\ell v \|_{L^2(\theta_1)}.$$

(By assumption $\ell \equiv \min(k, m+2) \geq 2$.) Thus,

$$\| u - \hat{u} \|^2_{S(I_1)} \leq (c_0 K C(\beta)^{\ell-2})^2 \, h_1^{2(\ell-1)-\sigma} \, \| D^\ell v \|^2_{L^2(\theta_1)}$$

(7.1)

$$\equiv c_2 \, N^{-2(\ell-1)} \, \| D^\ell v \|^2_{L^2(\theta_1)},$$

since $h_1^{2(\ell-1)-\sigma} = N^{-\beta[2(\ell-1)-\sigma]} = N^{-2(\ell-1)}$.

In the other intervals, we have, by (3.9) and the bound (4.9)(d),

$$\| u - \hat{u} \|^2_{S(I_i)} = \int_{I_i} x^{-\sigma} [(D(v-\hat{v}))^2 - 2\sigma (\frac{v-\hat{v}}{x}) D(v-\hat{v}) + \sigma^2 (\frac{v-\hat{v}}{x})^2] dx$$

(7.2)

$$\leq x_{i-1}^{-\sigma} \Big(\| D(v-\hat{v}) \|^2_{L^2(I_i)} + 2\sigma \| D(v-\hat{v}) \|_{L^2(I_i)} \, x_{i-1}^{-1} \, \| v-\hat{v} \|_{L^2(I_i)} + \sigma^2 \, x_{i-1}^{-2} \, \| v-\hat{v} \|^2_{L^2(I_i)} \Big)$$

$$\leq K^2 x_{i-1}^{-\sigma} h_i^{2(\ell-1)} [1 + 2\sigma \frac{h_i}{x_{i-1}} + \sigma^2 (\frac{h_i}{x_{i-1}})^2] \quad \| D^\ell v \|^2_{L^2(\theta_i)} .$$

But, by (4.9)(c),

$$\frac{h_i}{x_{i-1}} \leq \frac{h_i}{h_{i-1}} \leq M(\Delta_{\beta,N}) \leq C(\beta),$$

and by (4.9)(a),

$$x_{i-1}^{-\sigma} h_i^{2(\ell-1)} \leq \beta^{2(\ell-1)} \frac{i^{2(\ell-1)(\beta-1)}}{(i-1)^{\beta\sigma}} N^{-\beta[2(\ell-1)-\sigma]}$$

$$\leq 2^{2(\ell-1)} 2^{\beta\sigma} N^{-\beta[2(\ell-1)-\sigma]}$$

$$\leq 4^\ell N^{-2(\ell-1)}.$$

(Here we have used the identity

$$2(\beta-1)(\ell-1) = \beta(2(\ell-1)-\sigma) - 2(\ell-1) + \beta\sigma = \beta\sigma$$

and the inequalities $\beta\sigma \leq \beta \leq 2$ and $\frac{i}{i-1} \leq 2$.) Thus, by (7.2),

$$\| u - \hat{u} \|^2_{S(I_i)} \leq 4^\ell (K[1 + \sigma C(\beta)])^2 N^{-2(\ell-1)} \| D^\ell v \|^2_{L^2(\theta_i)}$$

$$\equiv c_3 N^{-2(\ell-1)} \| D^\ell v \|^2_{L^2(\theta_i)} .$$

Together with (7.1) and Theorem 5.1 (again noting that at most 2k of the θ_i overlap), this yields

$$\| u - \hat{u} \|^2_S \leq \sum_{i=1}^{N} \| u - \hat{u} \|^2_{S(I_i)}$$

$$\leq \max(c_2, c_3) N^{-2(\ell-1)} \sum_{i=1}^{N} \| D^\ell v \|^2_{L^2(\theta_i)}$$

$$\leq 2k \max(c_2, c_3) N^{-2(\ell-1)} \| D^\ell v \|^2_0$$

$$\leq 2k\Gamma_1^2 \max(c_2, c_3) N^{-2(\ell-1)} \| D^{\ell-1}(x^\sigma Du) \|^2_0 .$$

□

We now obtain S-norm and L^2-norm error bounds for the RRG approximation to u in weighted spline subspaces. The mild β-grading required in Theorem 7.1 suffices for L^2-norm error bounds of optimal order.

Theorem 7.2: Let $\tilde{u} \in WS_0^k(\sigma, \Delta_{\beta,N}, \underline{z})$ be the RRG approximation to the generalized solution u of (4.1) - (4.2), where $\beta = \beta_4 \equiv \frac{2(\ell-1)}{2(\ell-1) - \sigma}$ and $\underline{z}$ is any incidence vector. If p, q, and f satisfy the hypotheses of Theorem 5.2, then

$$\| u - \tilde{u} \|_S \leq \frac{\Lambda}{\lambda} c_1 \Gamma N^{-(\ell-1)} \| f \|_{\ell-2}$$

and

$$\| u - \tilde{u} \|_0 \leq (\Lambda c_1 \Gamma)^2 N^{-\ell} \| f \|_{\ell-2},$$

where $\ell \equiv \min(k, m+2)$.

Proof: The result follows from Theorem 7.1, the bounds on the solution in terms of the data (Theorem 5.2), and Nitsche's trick (Theorem 2.6).

□

7.4 Computational Considerations and Numerical Results

As a practical matter, the method of this chapter seems to be superior to any other so far presented for solving (4.1) - (4.2), including the techniques of Chapter 6. The rate of convergence in the theory is optimal. No restriction need be made on the smoothness of the piecewise polynomials, as is the case in Chapter 8. Furthermore, the computationally convenient basis of B-splines, used for computing with the spline spaces $S^k(\Delta,\underline{z})$, may be modified in an obvious manner to provide an equally attractive basis for $WS^k(\sigma,\Delta,\underline{z})$. Only the slightest β-grading is required, and our numerical results show that the error actually obtained is markedly less for the weighted spline spaces than for those of Chapter 6. The reason for this was given in the note on computational complexity in Section 6.5.

Using the functions $B_i \equiv x^{-\sigma}N_{i,k}(x)$ as a basis, we are faced with a serious problem when forming the matrix A by numerical quadrature. The integrands in the inner products of (2.10) have a singularity at the origin of the form $x^{-\sigma}$. Using, say, Gaussian quadrature, the error due to this singularity is so great (especially in the first interval) that the rate of convergence of the RRG approximation is seriously affected. Increasing the number of quadrature nodes in the interval does little good (see Table 7.1 for an example using weighted cubic splines on the problem of Section 6.4; the integrals were computed using a 5-point Gaussian quadrature in each interval).

N	$\| e_N \|_0$	RATE	$\| e_N \|_{\infty,\Delta}$	RATE	$\| e_N \|_S$	RATE
8	.56 (-01)		.13 (+00)		.25 (+00)	
16	.24 (-01)	1.24	.58 (-01)	1.16	.13 (+00)	0.96
24	.19 (-01)	0.54	.48 (-01)	0.46	.12 (+00)	0.25
32	.16 (-01)	0.54	.42 (-01)	0.48	.11 (+00)	0.25
40	.14 (-01)	0.54	.38 (-01)	0.49	.10 (+00)	0.26

Table 7.1: Effect of Quadrature Error

Subspace: $WS_0^4(.5,\Delta_{1.09,N})$ -- $1.09 = \beta_4$

The problem is solved by using quadrature formulae which are adapted to the type of integrand encountered in forming A. Thus, we use a formula which is exact for integrands of the form

$$x^{-s}\ p(x)$$

for p a polynomial of maximum degree. These "weighted Gaussian quadrature formulae" may be computed using an algorithm of Golub and Welsch [G1]. Finding a k-point formula for the weight function $\omega(x)$ on the interval (a,b) requires that the first 2k+1 moments of ω,

$$\int_a^b x^j\ \omega(x)\ dx, \qquad 0 \leq j \leq 2k$$

be computed. The procedure then takes the Cholesky decomposition of a k+1 x k+1 matrix and solves a k x k symmetric tridiagonal eigensystem. The eigenvalues and the first components of the eigenvectors are the nodes and weights of the quadrature rule.

If used in every interval of the mesh, this procedure can increase the cost of computing A. However, it suffices to chose a point $\bar{x}$ in (0,1) (say $\bar{x}$ = .05) and use the special quadrature rules only in intervals which intersect $(0,\bar{x})$. Tables 7.2 - 7.4 illustrate the result of this procedure, applied to the problem considered in Section 6.4.

Computations were again performed in double precision (54 binary digits) on a PDP-10. The matrix A and vector $\underline{f}$ were computed using k-1 - point Gaussian quadratures in each interval (weighted or unweighted depending on whether or not the interval intersects (0,.05)). The errors were computed using k+1 - point quadratures, weighted in the same intervals. Further explanation of notational details is given in Section 6.4.

N	$\| e_N \|_0$	RATE	$\| e_N \|_{\infty,\Delta}$	RATE	$\| e_N \|_S$	RATE
8	.45 (-03)		.91 (-03)		.17 (-01)	
16	.55 (-04)	3.03	.13 (-03)	2.84	.42 (-02)	2.01
24	.16 (-04)	3.01	.40 (-04)	2.86	.18 (-02)	2.00
32	.69 (-05)	3.00	.17 (-04)	2.87	.10 (-02)	2.00
40	.35 (-05)	3.00	.92 (-05)	2.87	.67 (-03)	2.00

Table 7.2: Error in $WS_0^3(.5, \Delta_{1.14,N})$ -- $1.14 = \beta_4$

N	$\| e_N \|_0$	RATE	$\| e_N \|_{\infty,\Delta}$	RATE	$\| e_N \|_S$	RATE
8	.27 (-04)		.53 (-04)		.10 (-02)	
16	.16 (-05)	4.08	.32 (-05)	4.07	.13 (-03)	3.04
24	.31 (-06)	4.03	.62 (-06)	4.02	.37 (-04)	3.01
32	.97 (-07)	4.01	.20 (-06)	4.01	.16 (-04)	3.01
40	.40 (-07)	4.01	.80 (-07)	4.01	.80 (-05)	3.00

Table 7.3: Error in $WS_0^4(.5, \Delta_{1.09,N})$ -- $1.09 = \beta_4$

N	$\| e_N \|_0$	RATE	$\| e_N \|_{\infty,\Delta}$	RATE	$\| e_N \|_S$	RATE
8	.11 (-06)		.21 (-06)		.42 (-05)	
16	.15 (-08)	6.12	.30 (-08)	6.09	.12 (-06)	5.09
24	.13 (-09)	6.05	.26 (-09)	6.03	.16 (-07)	5.03
32	.24 (-10)	6.02	.47 (-10)	6.01	.38 (-08)	5.02
40	.62 (-11)	6.00	.12 (-10)	5.95	.12 (-08)	5.01

Table 7.4: Error in $WS_0^6(.5, \Delta_{1.05,N})$ -- $1.05 = \beta_4$

CHAPTER 8

GENERALIZED L-SPLINES

8.1 Introduction

L-splines, functions which are elements of the null space of a differential operator L^*L in each interval of a partition Δ, are a natural generalization of polynomial splines (see [S5]). L-spline subspaces, with $L = \sqrt{p(x)}Du$, were used in the RRG method for the singular problem (4.1) - (4.2) by Ciarlet, Natterer, and Varga [C1], who obtained L^∞-norm error bounds which, for a uniform mesh, are of order $h^{2-\sigma}$. They noted that a suitably chosen non-uniform mesh might improve the rate of convergence; in fact, their approximation scheme is second-order accurate if a β-graded mesh is used with $\beta = \frac{2}{2-\sigma}$. (This is a special case of the main result of this chapter.)

L-splines were subsequently generalized by Crouzeix and Thomas [C4]. Given a partition Δ, they considered functions which, in the interval I_i, are mapped by the operator L^*L into a predetermined finite-dimensional space of functions P_i. In general, this space can differ from one interval to another; however, for the purpose of obtaining approximate solutions of a differential equation in which the right-hand side is smooth, it is no handicap to assume that P_i is the space of polynomials of some given degree. Taking $L = \sqrt{p(x)}Du$ and P_i the space of polynomials of degree < k-2, Crouzeix and Thomas showed that the L^2-norm of the error in the RRG approximation to the solution of (4.1) - (4.2) was $O(h^{k-\sigma})$. They considered only C^0 subspaces, however, and their results do not apply to smoother generalized L-splines.

In this chapter, we redevelop Crouzeix and Thomas's theory. Because we consider only problems of the form (4.1) - (4.2), our presentation is more elementary; moreover, we have simplified the proofs of Lemmas 8.1 and 8.2. We then show that, using a β-graded mesh with $\beta = \frac{2}{2-\sigma}$, the error is of order N^{-k}.

8.2 Generalized L-spline Approximation in S^m

Let Δ be a given partition of I, and let

$$S_0(I_i) \equiv \{s \in S \mid s(x_{i-1}) = s(x_i) = 0\}, \qquad 1 \leq i \leq N.$$

Note that $S_0(I_i)$ is a closed subspace of S. The following result is an analogue of the Rayleigh-Ritz inequality (cf. Theorem 2.1) and is similar to Lemma 5 of [C4].

Lemma 8.1: For all $v \in S_0(I_i)$,

$$(8.1) \qquad \int_{x_{i-1}}^{x_i} v^2\,dx \leq C(I_i)^2 \int_{x_{i-1}}^{x_i} x^{\sigma}(Dv)^2\,dx,$$

where

$$(8.2)\qquad C(I_i)^2 = \begin{cases} \dfrac{h_i^2}{\pi^2\, x_{i-1}^{\sigma}} & i > 1, \\ 4x_1^{2-\sigma} & i = 1. \end{cases}$$

Proof: For $i > 1$, by the Rayleigh-Ritz inequality,

$$\int_{x_{i-1}}^{x_i} v^2\, dx \le \pi^{-2} h_i^2 \int_{x_{i-1}}^{x_i} (Dv)^2\, dx$$

$$\le \pi^{-2} h_i^2\, x_{i-1}^{-\sigma} \int_{x_{i-1}}^{x_i} x^{\sigma}(Dv)^2\, dx.$$

For $i = 1$, an integration by parts shows that

$$\int_0^{x_1} v^2\, dx = xv^2\Big|_0^{x_1} - 2\int_0^{x_1} xv(x)Dv(x)\, dx.$$

Since $v(0) = v(x_1) = 0$, the integrated term vanishes. Therefore, by the Cauchy-Schwarz inequality,

$$\int_0^{x_1} v^2\, dx \le 2\left(\int_0^{1} v^2\, dx\right)^{1/2}\left(\int_0^{x_1} (xDv)^2\, dx\right)^{1/2}.$$

Cancelling the common factor $\|v\|_{L^2(I_1)}$ and squaring the resulting inequality yields

$$\int_0^{x_1} v^2\, dx \le 4\int_0^{x_1} x^2(Dv)^2\, dx$$

$$\le 4x_1^{2-\sigma}\int_0^{x_1} x^{\sigma}(Dv)^2\, dx.$$

□

Let k be an integer, $k \geq 2$. A <u>σ-polynomial</u> (of degree k-1) is a function of the form

$$c_0 + c_1 x^{1-\sigma} + \dots + c_{k-1} x^{k-1-\sigma}.$$

Note that s is a σ-polynomial of degree k-1 if and only if $D(x^\sigma Ds)$ is a polynomial of degree k-3 (the 0 polynomial when k = 2). We denote by $P(\sigma,k)$ the space of σ-polynomials of degree k-1.

<u>Definition 8.1</u>: A function $s(x) \in S$ is said to be a (σ,k)-spline with respect to Δ if it coincides with a σ-polynomial of degree k-1 in each interval of Δ. We denote the set of all (σ,k)-splines with respect to Δ by $S_\sigma^k(\Delta)$.

□

Clearly, if $s \in S_\sigma^k(\Delta)$, then there exist polynomials $p_1, \dots, p_N$ of degree k-3 such that $D(x^\sigma Ds) = p_i$ in I_i, $1 \leq i \leq N$. Integrating by parts, we have that

$$\int_{x_{i-1}}^{x_i} x^\sigma Ds Dv \, dx = \int_{x_{i-1}}^{x_i} -D(x^\sigma Ds) v \, dx$$

(8.3)

$$= \int_{x_{i-1}}^{x_i} -p_i v \, dx, \quad \text{for all } v \in S_0(I_i), \quad 1 \leq i \leq N.$$

The converse also holds. Let $s_{(i)}$ denote the restriction of s to I_i. If $s \in S^0$ satisfies (8.3), then $D(x^\sigma Ds_{(i)}) = p_i$ (i.e., $s_{(i)}$ is a σ-polynomial) in each interval I_i, since $S_0(I_i)$ is dense in $L^2(I_i)$.

It happens that the orthogonal projection of a function u on $S^k_\sigma(\Delta)$ with respect to $(.,.)_{S(I_i)}$ interpolates u at the knots. This result is due to Crouzeix and Thomas [C4].

Lemma 8.2: Given $u \in S$, there exists $\hat{u} \in S^k_\sigma(\Delta)$ satisfying

$$\hat{u}(x_i) = u(x_i), \quad 0 \leq i \leq N \tag{8.4}$$

and

$$\| u - \hat{u} \|_{S(I_i)} = \inf_{s \in P(\sigma,k)} \| u - s \|_{S(I_i)}, \quad 1 \leq i \leq N. \tag{8.5}$$

Proof: Suppose $s \in S^0(I_i) \equiv \{s \in S \mid D(x^\sigma Ds_{(i)}) \in L^2(I_i)\}$ is orthogonal to the subspace $S_0(I_i)$ with respect to the inner product $(.,.)_{S(I_i)}$, i.e.,

$$\int_{x_{i-1}}^{x_i} x^\sigma DvDs \, dx = 0 \quad \text{for all } v \in S_0(I_i).$$

Clearly, s satisfies (8.3) with $p_i = 0$. Therefore, $s_{(i)}$ is a σ-polynomial: $s_{(i)} \in P(\sigma,2) \subset P(\sigma,k)$.

We denote by $X^\perp$ the space of functions orthogonal (with respect to $(.,.)_{S(I_i)}$) to the set X. We have just shown that

$$s \in (S^0(I_i) \cap S_0(I_i)^\perp) \Rightarrow s_{(i)} \in P(\sigma,2) \subset P(\sigma,k),$$

or

$$(S^0(I_i) \cap S_0(I_i)^\perp) \subset P(\sigma,k).$$

Thus $P(\sigma,k)^\perp \subset (S^0(I_i) \cap S_0(I_i)^\perp)^\perp$. Moreover, since $S^0(I_i)$ is dense in $S(I_i)$, $(S^0(I_i) \cap S_0(I_i)^\perp)$ is dense in $S_0(I_i)^\perp$; thus, anything orthogonal to $(S^0(I_i) \cap S_0(I_i)^\perp)$ is orthogonal to $S_0(I_i)^\perp$. Hence,

$$P(\sigma,k)^{\perp} \subset S_0(I_i)^{\perp\perp} = S_0(I_i),$$

since $S_0(I_i)$ is a closed subpsace (see Schecter [S1]).

Now let t be the orthogonal projection of u on $P(\sigma,k)$. Since $u - t \in P(\sigma,k)^{\perp}$, it follows that $u - t \in S_0(I_i)$; i.e., t interpolates u. It is now clear how to construct $\hat{u}$. We simply piece together the best σ-polynomial approximations in each interval; the resulting function clearly satisfies (8.4) and (8.5), is continuous and satisfies the boundary conditions.

□

We now obtain error bounds for $u - \hat{u}$. This result and its corollary are due to Crouzeix and Thomas [C4].

<u>Lemma 8.3</u>: Let $u \in S^0$ and $\hat{u} \in S^k_\sigma(\Delta)$ be the approximation defined in Lemma 8.2. Then for all $1 \leq i \leq N$,

$$\text{(8.6)} \qquad \|u - \hat{u}\|_{S(I_i)} \leq C(I_i) \inf_{p \in P_{k-2}} \|D(x^\sigma Du) - p\|_{L^2(I_i)}$$

$$\text{(8.7)} \qquad \|u - \hat{u}\|_{L^2(I_i)} \leq (C(I_i))^2 \inf_{p \in P_{k-2}} \|D(x^\sigma Du) - p\|_{L^2(I_i)}.$$

<u>Proof</u>: Let $p \in P_{k-2}$ be arbitrary. Let $v_p \in P(\sigma,k)$ be the unique solution to the boundary-value problem

$$D(x^\sigma Dv) = p, \quad x_{i-1} < x < x_i,$$

$$v(x_{i-1}) = u(x_{i-1}), \qquad v(x_i) = u(x_i).$$

Obviously $u - v_p \in S_0(I_i)$. Integrating by parts, using the boundary conditions on v_p, the Cauchy-Schwarz inequality, and (8.1),

$$\|u - v_p\|^2_{S(I_i)} = \int_{x_{i-1}}^{x_i} -(u - v_p)[D(x^\sigma Du) - p]\, dx$$

$$\leq \|u - v_p\|_{L^2(I_i)} \|D(x^\sigma Du) - p\|_{L^2(I_i)}$$

$$\leq C(I_i) \|u - v_p\|_{S(I_i)} \|D(x^\sigma Du) - p\|_{L^2(I_i)}.$$

Thus

$$\|u - v_p\|_{S(I_i)} \leq C(I_i) \|D(x^\sigma Du) - p\|_{L^2(I_i)}.$$

But $\hat{u}$ is the best approximation to u in the norm $\|\cdot\|_{S(I_i)}$, so

$$\|u - \hat{u}\|_{S(I_i)} = \inf_{p \in P_{k-2}} \|u - v_p\|_{S(I_i)}$$

$$\leq C(I_i) \inf_{p \in P_{k-2}} \|D(x^\sigma Du) - p\|_{L^2(I_i)},$$

which proves inequality (8.6). Since $u - \hat{u} \in S_0(I_i)$, inequality (8.7) follows from (8.1) and (8.6).

□

<u>Corollary</u>: Let $u \in S^0$ and $\hat{u} \in S^k_\sigma(\Delta)$ be the approximation defined in Lemma 8.2. Then

(8.8) $$\|u - \hat{u}\|_S \leq \max_{1 \leq i \leq N} \{C(I_i)\} R(u)$$

(8.9) $$\|u - \hat{u}\|_0 \leq \max_{1 \leq i \leq N} \{(C(I_i))^2\} R(u),$$

where

$$(8.10) \qquad R(u) \equiv \left(\sum_{i=1}^{N} \inf_{p \in P_{k-2}} \| D(x^{\sigma} Du) - p \|_{L^2(I_i)}^2 \right)^{1/2}.$$

We now bound the quantities $C(I_i)$ and $R(u)$ in terms of N and the Sobolev norms of $D(x^{\sigma}Du)$.

Lemma 8.4: If $\Delta = \Delta_{\beta,N}$ with $\beta = \frac{2}{2-\sigma}$, then the constants $C(I_i)$ defined in Lemma 8.1 satisfy

$$(8.11) \qquad C(I_i) \le 2N^{-1}, \qquad 1 \le i \le N.$$

Proof: By (8.2) (for i = 1),

$$C(I_1)^2 = 4x_1^{2-\sigma} = 4N^{-\beta(2-\sigma)} = 4N^{-2}.$$

By (8.2) (for i > 1) and (4.9)(a),

$$C(I_i)^2 = \frac{h_i^2}{\pi^2 x_{i-1}^{\sigma}}$$

$$\le \frac{\beta^2 i^{2(\beta-1)}}{\pi^2 (i-1)^{\beta\sigma}} N^{-\beta(2-\sigma)}.$$

Since $\beta(2-\sigma) = 2$, $2(\beta-1) = \beta\sigma$, and $\beta\sigma \le \beta \le 2$,

$$\frac{\beta^2 i^{2(\beta-1)}}{(i-1)^{\beta\sigma}} \le 4\left(\frac{i}{i-1}\right)^{\beta\sigma} \le 16,$$

and

$$C(I_i)^2 \le \frac{16}{\pi^2} N^{-2} < 4N^{-2}, \qquad i \ge 2.$$

□

<u>Lemma 8.5</u> (Ciarlet and Raviart [C2]): If $u \in S^m$, $m \geq 0$, then

(8.12) $$R(u) \leq h^{\ell-2} \| D^{\ell-1}(x^\sigma Du) \|_0,$$

where R(u) is given by (8.10) and $\ell \equiv \min(m+2, k)$.

<u>Corollary</u>: Let $u \in S^m$ and $\hat{u} \in S_\sigma^k(\Delta_{\beta,N})$ be the approximation defined in Lemma 8.2 with $\beta = \frac{2}{2-\sigma}$. Then

(8.13) $$\| u - \hat{u} \|_S \leq 2\beta^{\ell-2} N^{-(\ell-1)} \| D^{\ell-1}(x^\sigma Du) \|_0$$

(8.14) $$\| u - \hat{u} \|_0 \leq 4\beta^{\ell-2} N^{-\ell} \| D^{\ell-1}(x^\sigma Du) \|_0.$$

<u>Proof</u>: Use (8.11) and (8.12) to bound the right-hand sides of (8.8) and (8.9).

□

Crouzeix and Thomas obtain error bounds for the RRG approximation $\tilde{u}$ to u in $S_\sigma^k(\Delta)$ by showing that $\hat{u} - \tilde{u}$ is small. We prefer, however, to use Nitsche's trick (Theorem 2.6).

<u>Theorem 8.1</u>: Let $\tilde{u} \in S_\sigma^k(\Delta_{\beta,N})$ be the RRG approximation to the generalized solution u of (4.1) - (4.2), where $\beta = \frac{2}{2-\sigma}$. If p, q, and f satisfy the hypotheses of Theorem 5.2, then

$$\| u - \tilde{u} \|_S \leq (2 \frac{\Lambda}{\lambda} \beta^{\ell-2}\Gamma) N^{-(\ell-1)} \| f \|_{\ell-2}$$

$$\| u - \tilde{u} \|_0 \leq (2\Lambda \beta^{\ell-2}\Gamma)^2 N^{-\ell} \| f \|_{\ell-2},$$

where $\ell \equiv \min(m+2, k)$.

PART III

CHAPTER 9

SPHERICALLY SYMMETRIC PROBLEMS

9.1 Introduction

In Part III, we consider the numerical solution of spherically symmetric, elliptic partial differential equations. There are numerous applications in engineering and the sciences in which the solution of a spherically symmetric, elliptic equation is desired (see the references in [R3]). Since all the functions involved are spherically symmetric (that is, they depend only on distance from the center of the domain), the problem can be replaced by an equivalent two-point boundary value problem.

When an n-dimensional problem is so simplified, the resulting problem has the form

$$- D(r^{n-1}p(r)Du) + r^{n-1}q(r)u \;=\; r^{n-1}f(r), \quad 0 < r < 1,$$

$$u(1) = 0, \quad Du(0) = 0 \quad \text{(alternatively, } u(0) \text{ finite).}$$

In marked contrast to the singular problems of Part II, the solution is smooth despite the singularity in the equation. It should therefore be possible to approximate the solution accurately using the Rayleigh-Ritz-Galerkin method with a piecewise polynomial subspace, without any sort of mesh grading. In Chapter 10, we obtain optimal-order error bounds, showing that this procedure is theoretically well-founded. Instead of the usual one-dimensional Sobolev norms, we use norms which are appropriate to the original n-dimensional setting of the problem.

In 2 and 3 dimensions, Russell and Shampine [R3] proved error bounds for approximation procedures specially designed to deal with the apparent singularity at the origin. In particular, they treated collocation in which the basis is augmented by singular basis functions, RRG using singular patch bases (L-splines), and a finite-difference scheme of Jamet [J1] designed to handle the singularity. Crouzeix and Thomas [C4] and Reddien [R1] considered this problem as part of a wider class and obtained similar results for subspaces which include singular basis functions.

Dupont and Wahlbin [D6] and Jesperson [J3] analyzed the RRG procedure with piecewise polynomials, and obtained error bounds of optimal order in the one-dimensional Sobolev norms. The import of their results, together with those of these two chapters, is that no special measures are required for this problem: the Rayleigh-Ritz-Galerkin method using high-order piecewise polynomial spaces on a uniform mesh is a highly effective numerical method.

Section 9.2 is a summary of the variational form of the problem and the properties of its solutions. Bounds on solutions of Hermite interpolation problems, required for the error bounds of Chapter 10, are proved in Section 9.3.

9.2 Spherically Symmetric Elliptic Problems

Let $B(b)$ be the open ball of radius b in R^n, i.e.,

$$B(b) \equiv \{\underline{x} \in R^n \mid r(\underline{x}) < b\},$$

where $r(\underline{x}) \equiv \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$, and let $B \equiv B(1)$ be the open unit ball. We say that a function $V(\underline{x})$ defined on B is spherically symmetric if it depends only on distance $r(\underline{x})$ from the origin. If V is spherically symmetric, we call the function $v(r)$ such that $v(r(\underline{x})) = V(\underline{x})$ the radial part of V.

Consider the spherically symmetric, elliptic partial differential equation

$$\text{(9.1)} \qquad -\nabla(p(r(\underline{x}))\nabla U) + q(r(\underline{x}))U = f(r(\underline{x})), \quad \underline{x} \in B,$$

$$\text{(9.2)} \qquad U(\underline{x}) = 0, \quad \underline{x} \in \partial B,$$

where p, q, and f are given functions defined on $[0,1]$. We assume that

(9.3)

a) $p(r(\underline{x})) \in C^1(B)$,

b) $p(r(\underline{x})) \geq p_{min} > 0$ for all $\underline{x} \in B$,

c) $q(r(\underline{x})) \in C(B)$,

d) $q(r(\underline{x})) \geq 0$ for all $\underline{x} \in B$.

Since the data of the problem are spherically symmetric, an obvious symmetry argument shows that U is too. (A change in coordinate systems by rotation around any axis passing through the origin leaves the problem, and hence its solution, unchanged.)

By transforming equation (9.1) to spherical coordinates and setting to 0 all partial derivatives with respect to angle, one obtains a singular two-point boundary value problem for the radial part u of U:

$$(9.4) \qquad - D(r^{n-1}p(r)Du) + r^{n-1}q(r)u = r^{n-1}f(r), \quad 0 < r < 1,$$

$$(9.5) \qquad u(1) = 0, \quad Du(0) = 0 \quad \text{(alternatively, } u(0) \text{ finite).}$$

When n = 3, the change of variables v = xu results in the nonsingular problem

$$- D^2v(r) + q(r)v(r) = rf(r), \quad 0 < r < 1,$$

$$v(0) = v(1) = 0.$$

Unfortunately, we know of no such trick when $n \neq 3$!

Another one-dimensional analogue of (9.1) - (9.2) is

$$(9.4') \quad - D(|x|^{n-1}p(|x|)Du) + |x|^{n-1}q(|x|)u = |x|^{n-1}f(|x|), \quad -1 < x < 1,$$

$$(9.5') \qquad u(-1) = u(1) = 0.$$

The relation to (9.4) - (9.5) is that the solution u(r) of the former problem is the restriction to [0,1] of the solution of the latter.

Suppose that G is spherically symmetric and that g is the radial part of G. Using an $(r,\underline{\theta})$ coordinate system in which $r = r(\underline{x})$ is given by (9.1) and $\underline{\theta}$ represents an n-1 dimensional vector of angles, we have

$$\int_{B(b)} G\, d\underline{x} = \int_0^b \int_{\partial B} r^{n-1} G(r,\underline{\theta})\, d\underline{\theta} dr$$

$$= \Pi_n \int_0^b r^{n-1} g(r)\, dr,$$

where Π_n is the area of the unit hypersphere in $\mathbb{R}^n$ [M3]. This observation motivates the following definition.

<u>Definition</u> <u>9.1</u>: For real-valued functions f,g defined on (0,b), let

$$(f,g)_{B(b)} \equiv \int_0^b r^{n-1} f(r) g(r)\, dr$$

and

$$\| f \|_{B(b)} \equiv (f,f)_{B(b)}^{1/2}.$$

□

<u>Definition</u> <u>9.2</u>: Let $J^m(b)$ (respectively, $J_0^m(b)$) denote the closure of the C^∞ functions with all odd derivatives vanishing at 0 (respectively, the C^∞ functions which vanish in a neighborhood of b with all odd derivatives vanishing at 0) with respect to the norm

$$\| f \|_{m,B(b)} \equiv \left(\sum_{j=0}^{m} \| D^j f \|_{B(b)}^2 \right)^{1/2}.$$

□

Note that $J^m(b)$ (respectively, $J_0^m(b)$) may be identified with the restriction of the spherically symmetric functions in $H^m(B(b))$ (respectively, $H_0^m(B(b))$ to any line segment from the origin to the boundary of $B(b)$.

Our assumptions on p and q imply that there exist positive constants λ and Λ such that

$$(9.6) \qquad \lambda^2 \| Du \|^2_{B(b)} \leq a(u,u)_{B(b)} \leq \Lambda^2 \| Du \|^2_{B(b)}$$

for all $u \in J^1(b)$, where

$$a(u,v)_{B(b)} \equiv \int_0^b r^{n-1} [pDuDv + quv]\, dr.$$

The results of Sections 2.2 and 2.3 apply to (9.4) - (9.5). J^0 is the underlying Hilbert space, and J_0^1 the space S of admissible functions, with norm $\| v \|_S = \| Dv \|_B$.

Our approximation-theoretic results in the spaces J^m will rely on the following basic result.

Lemma 9.1 (Friedrich's inequality [M3]): If $v \in J_0^1(b)$, then

$$(9.7) \qquad \| v \|_{B(b)} \leq b \, \| Dv \|_{B(b)}.$$

Inequalities (9.6) and (9.7) show that the bilinear form $a(\ ,\ )_B$ is positive definite over the space J_0^1. Thus, for each $f \in J^0$, there exists a unique $u \in J_0^1$, the generalized solution of (9.4) - (9.5), such that

$$a(u,v)_B = (f,v)_B \quad \text{for all } v \in J_0^1.$$

Regularity results for the problem (9.1) - (9.2) are well-known, and corresponding results for (9.4) - (9.5) follow immediately.

Lemma 9.2 [F3]: There exists a constant Γ such that for all $m \geq 2$, if $f \in J^{m-2}$, then the generalized solution $u \in J^m$ and

(9.8) $$\| u \|_{m,B} \leq \Gamma \| f \|_{m-2,B} .$$

We now state a variant of the Sobolev lemma. Let $m_0 = m_0(n) \equiv \lfloor \frac{n}{2} \rfloor + 1.$

Lemma 9.3 [F3]: There exists a positive constant C_n such that, if $u \in J^m(b)$, $m \geq m_0$, then

(9.9) $$(u(0))^2 \leq C_n^2 \sum_{j=0}^{m_0} b^{2j-n} \| D^j u \|_{B(b)}^2 .$$

When bounds on the value at a point (other than 0) of a function in $J^m(b)$ are desired, the Sobolev lemma is not the best result. The following stronger result shows that $J^m(b)$ behaves more like a one-dimensional Sobolev space. Let $\bar{m}$ be the largest odd integer less than or equal to $m - m_0$, i.e.,

$$\bar{m} \equiv \begin{cases} m - m_0 - 1 & (m - m_0 \text{ even}) \\ m - m_0 & (m - m_0 \text{ odd}) \end{cases} .$$

Lemma 9.4: If $v \in J^m(b)$ then, for all $0 < a < b$,

a) $v \in H^m(a,b)$;

b) $v \in C^{m-1}[a,b]$;

c) $$\| D^j v \|^2_{L^\infty(a,b)} \leq C_1^2\, a^{1-n}\, [(b-a)^{-1} \| D^j v \|^2_{B(b)} + (b-a) \| D^{j+1} v \|^2_{B(b)}]$$

$0 \leq j \leq m-1$;

d) v has $m - m_0$ continuous derivatives at 0, and its odd derivatives vanish there, i.e.,

$$D^i v(0) = 0, \quad i = 1, 3, \ldots, \bar{m}; \tag{9.10}$$

e) if in addition, $v \in J^m_0$, then

$$D^i v(b) = 0, \quad 0 \leq i \leq m-1.$$

Proof: First, note that

$$\| g \|^2_{m,B(b)} \geq \sum_{i=0}^{m} \int_a^b x^{n-1} (D^i g)^2\, dx$$

$$\geq a^{n-1} \sum_{i=0}^{m} \int_a^b (D^i g)^2\, dx$$

$$= a^{n-1} \| g \|^2_{H^m(a,b)} .$$

Thus, if a sequence of functions $\{v_n\}$ converges to v with respect to the $J^m(b)$-norm, it also converges in the $H^m(a,b)$-norm. This immediately implies (a). Next, (b) follows from (a) by the one-dimensional Sobolev inequality (Theorem 2.3). Moreover,

$$\| D^j v \|^2_{L^\infty(a,b)}$$

$$\leq C_1^2 \, [(b-a)^{-1} \| D^j v \|^2_{L^2(a,b)} + (b-a) \| D^{j+1} v \|^2_{L^2(a,b)}]$$

$$\leq C_1^2 \, a^{1-n} \, [(b-a)^{-1} \| D^j v \|^2_{B(b)} + (b-a) \| D^{j+1} v \|^2_{B(b)}] ,$$

which proves (c).

By the inequality (9.9), convergence in the $J^m(b)$-norm implies convergence at zero of derivatives up to order $m - m_0$. Since v is the limit in this norm of a sequence of smooth function satisfying (9.10), v does too, proving (d). Finally, (e) follows by the same argument, using part (c) rather than Lemma 9.3.

□

9.3 On Hermite Interpolation

In Chapter 10, we will obtain an error bound involving the (spherical) norms of a polynomial characterized by interpolation conditions on the polynomial and its derivatives at two points. In this section we consider bounds on the solutions of such problems.

Let k be a positive integer, e_0, e_1 be integers such that $e_0 + e_1 = k$, and $b > 0$ be given. The two-point Hermite interpolation problem is to determine a polynomial $p_k(x)$ of degree k-1 such that

$$D^j p_k(0) = y_0^{(j)}, \quad 0 \le j < e_0,$$
(9.11)
$$D^j p_k(b) = y_1^{(j)}, \quad 0 \le j < e_1.$$

It is known that (9.11) has a unique solution for arbitrary data $y_i^{(j)}$ [S2].

We shall need bounds on the norm of the solution of an Hermite interpolation problem in terms of the size of the data.

Lemma 9.5: There exists a constant $C_e = C(e_0, e_1)$ independent of b and $y_i^{(j)}$ such that

(9.12) $$\| D^\ell p_k \|_{B(b)} \le C_e \, b^{(\frac{n}{2} - \ell)} \sum_{i=0}^{1} \sum_{j=0}^{e_i - 1} b^j \, |y_i^{(j)}|$$

for all $0 \le \ell \le k$, where p_k is the polynomial which satisfies (9.11) with data $y_i^{(j)}$.

Proof: Let $z_i^{(j)} = b^j \, y_i^{(j)}$ and q_k be the polynomial of degree k-1 such that

$$D^j q_k(0) = z_0^{(j)}, \quad 0 \le j < e_0,$$
(9.13)
$$D^j q_k(1) = z_1^{(j)}, \quad 0 \le j < e_1.$$

Clearly, $p_k(x) = q_k(\frac{x}{b})$. The coefficients $\underline{\gamma} = (\gamma_i)$ of q_k satisfy

$$M\underline{\gamma} = \underline{z},$$

where $M = M(e_1, e_2)$ is a nonsingular k x k matrix and $\underline{z}$ is the data of (9.13), in some given order [S2]. Hence,

$$\|\underline{\gamma}\|_1 \leq \|M^{-1}\|_1 \; \|\underline{z}\|_1,$$

where $\|\underline{z}\|_1 \equiv \sum_i |z_i|$ is the usual vector 1-norm and $\|M^{-1}\|_1$ the corresponding matrix norm. Therefore

$$\|D^\ell p_k\|_{B(b)} = \|D^\ell q_k(\tfrac{x}{b})\|_{B(b)}$$

$$= \|\sum_{i=0}^{k-1} \gamma_i \, D^\ell((\tfrac{x}{b})^i)\|_{B(b)}$$

$$\leq \sum_{i=0}^{k-1} |\gamma_i| \; \|D^\ell((\tfrac{x}{b})^i)\|_{B(b)}$$

$$\leq \|\underline{\gamma}\|_1 \max_{\ell \leq i < k} \left(\int_0^b x^{n-1} \left[\prod_{j=0}^{\ell-1} (i-j) \; b^{-i} \, x^{i-\ell}\right]^2 dx \right)^{1/2}$$

$$= C \, b^{(\frac{n}{2} - \ell)} \; \|\underline{\gamma}\|_1$$

$$\leq C \; \|M^{-1}\|_1 \; b^{(\frac{n}{2} - \ell)} \; \|\underline{z}\|_1,$$

where C depends only on k, n, and ℓ.

□

CHAPTER 10

SPHERICAL SPLINE APPROXIMATION THEORY

10.1 Introduction

If $V \in H^m(B)$ is a spherically symmetric function, we propose to approximate it with smooth spherically symmetric functions. Let $v \in J^m$ be the radial part of V and $\hat{v} \in J^m$ be a spline approximation to v; we take as our approximation to V the function $\hat{V}$ whose radial part is $\hat{v}$. As a measure of the error $D^j(V - \hat{V})$, we use the norm $\| D^j(v - \hat{v}) \|_{B(b)}$.

Spline subpaces $SS^k(\Delta,\underline{z})$ of J^m and $SS_0^k(\Delta,\underline{z})$ of $J^m \cap J_0^1$ are defined in Section 10.2. First, we construct spline spaces $ES^k(\Delta',\underline{z}')$ and $ES_0^k(\Delta',\underline{z}')$ whose elements are even functions on (-1,1), requiring sufficient smoothness at 0 for odd derivatives of the appropriate orders to vanish. The restriction to [0,1] of $s \in ES^k(\Delta',\underline{z}')$ is an element of J^m. we then define the space of spherically symmetric splines, $SS^k(\Delta,\underline{z})$, to be the space of all such functions.

Section 10.3 deals with approximation by spherical splines. As the approximation $\hat{v}$, we use a quasiinterpolant $F_\Delta v$. In Theorem 10.1, we show that for $v \in J^m$,

$$\| D^j(v - F_\Delta v) \|_B \le C h^{m-j} \| D^m v \|_B$$

for a partition with mesh-length h, where the constant C is independent of v and h. We obtain similar error bounds for the Rayleigh-Ritz-Galerkin approximation in Theorem 10.2. Numerical results are included in Section 10.4.

10.2 Spline Subspaces of J^m

We shall construct a space of splines $SS^k(\Delta,\underline{z}) \subset J^m$ (respectively, $SS^k_0(\Delta,\underline{z}) \subset J^m \cap J^1_0$). In view of Lemma 9.4, we must force the derivatives of orders 1, 3, ..., $\bar{m}$ of such splines to vanish at 0. Let k be a positive integer, Δ a partition of [0,1], and $\underline{z}$ an incidence vector. Define the symmetric partition Δ' by

$$\Delta' : -1 = x_{-N} < x_{-N+1} < \dots < x_0 = 0 < x_1 < \dots < x_N = 1,$$

where $x_{-i} \equiv -x_i$, $1 \le i \le N$; and the symmetric incidence vector $\underline{z}'$ by

$$\underline{z}' = (z_{-N+1}, \dots, z_0, \dots, z_{N-1}),$$

where $z_{-i} \equiv z_i$, $1 \le i \le N-1$ (z_0 is as yet arbitrary). The resulting knot vector $\underline{t}(\Delta',\underline{z}')$ is also symmetric.

Definition 10.1: Let $ES^k(\Delta',\underline{z}')$ (respectively, $ES^k_0(\Delta',\underline{z}')$) be the space of even spline functions of order k with respect to Δ' and $\underline{z}'$ (respectively, the even spline functions of order k with respect to Δ' and $\underline{z}'$ which vanish at -1 and 1). Let $SS^k(\Delta,\underline{z})$ (respectively, $SS^k_0(\Delta,\underline{z})$)

be the space of functions on [0,1] which coincide with the restriction to [0,1] of an element of $ES^k(\Delta',\underline{z}')$ (respectively, $ES_0^k(\Delta',\underline{z}')$).

□

To motivate what follows, we shall outline the proof of error bounds given in Section 10.3. Let $\{\tau_j\}$ be quasiinterpolation points for $\underline{t}(\Delta',\underline{z}')$ which, like Δ' and $\underline{z}'$, are symmetric about 0; by the symmetry of $\underline{t}(\Delta',\underline{z}')$ and $\{\tau_j\}$, the quasiinterpolant maps even functions into even functions, i.e., into $ES^k(\Delta',\underline{z}')$. We shall also require that the τ_j be bounded away from 0, because we want to use Lemma 9.4 (part (c)) to bound the value of a function at τ_j. To enforce both this requirement and symmetry, z_0 must be chosen so that the dimension of $S^k(\Delta',\underline{z}')$ (which is the number of quasiinterpolation points) is even.

Another constraint on z_0 is that we shall not be able to prove error bounds for even splines which are forced to be too smooth at 0. In particular, we want z_0 to be sufficiently large so that elements of $ES^k(\Delta',\underline{z}')$ aren't required to have $\bar{m}+1$ continuous derivatives at 0.

Let

$$z_0 = \begin{cases} \max(k-1-\bar{m},\ 1) & (k \text{ odd}) \\ \max(k-1-\bar{m},\ 2) & (k \text{ even}) \end{cases} .$$

Then

i) $1 \le z_0 \le k-1$,

ii) the dimension of $S^k(\Delta',\underline{z}')$ is an even number,

iii) elements of $S^k(\Delta',\underline{z}')$ and $ES^k(\Delta',\underline{z}')$ are not required to have a continuous derivative of order $\bar{m}+1$.

Indeed, (i) is obvious. For (ii), note that

$$d = \dim(S^k(\Delta',\underline{z}')) = k + \sum_{i=1-N}^{N-1} z_i$$

$$= k + z_0 + 2\sum_{i=1}^{N-1} z_i;$$

d is even since $k + z_0$ is even. Finally, the elements of $S^k(\Delta',\underline{z}')$ and $ES^k(\Delta',\underline{z}')$ have

$$k - 1 - z_0 = \begin{cases} \min(\bar{m},\ k-3) & (k \text{ even}) \\ \min(\bar{m},\ k-2) & (k \text{ odd}) \end{cases}$$

$$\le \bar{m}$$

continuous derivatives at 0, which proves (iii).

Because the partition $\underline{t}(\Delta',\underline{z}')$ is symmetric, the B-spline basis functions have the symmetry property

(10.1) $$N_{i,k}(-x) = N_{d+1-i,k}(x), \quad -1 < x < 1, \quad 1 \le i \le d.$$

The functions

$$B_j \equiv N_{j,k} + N_{d+1-j,k}, \quad 1 \le j \le \frac{d}{2},$$

are a basis for $ES^k(\Delta',\underline{z}')$, and their restrictions to [0,1] are a basis for $SS^k(\Delta,\underline{z})$. ($B_1$ must be deleted from bases for $ES_0^k(\Delta',\underline{z}')$ and $SS_0^k(\Delta,\underline{z})$.) Clearly, $SS_0^k(\Delta,\underline{z})$ is a subspace of $J^m \cap J_0^1$.

10.3 Error Bounds for the Quasiinterpolant

We first consider approximation by polynomials in a neighborhood of the origin.

Lemma 10.1: If $v \in J^m(b)$, $m \geq 1$, then there exists a polynomial $T_b v$ of degree m-1 satisfying

$$(10.2) \qquad \| D^j (v - T_b v) \|_{B(b)} \leq b^{m-j} \| D^m v \|_{B(b)}, \qquad 0 \leq j \leq m.$$

Proof: According to Lemma 9.4, v has m-1 continuous derivatives at b. Let $T_b v$ be the first m terms of the Taylor series for v at b, i.e.,

$$T_b v(x) = v(b) + Dv(b)(x-b) + \ldots + \frac{D^{m-1} v(b)}{m-1!}(x - b)^{m-1}.$$

We prove (10.2) inductively, starting with the case j = m and proceeding downwards. For j = m, (10.2) holds trivially. By the interpolation conditions which define $T_b v$, $D^j (v - T_b v) \in J_0^1(b)$ for $0 \leq j \leq m$. Therefore, by (9.7) and the inductive hypothesis,

$$\| D^j (v - T_b v) \|_{B(b)} \leq b \; \| D^{j+1} (v - T_b v) \|_{B(b)}$$

$$\leq b \; [b^{m-(j+1)} \; \| D^m v \|_{B(b)}].$$

□

Lemma 10.2: If $v \in J^m(b)$, $m \geq 0$, then there exists a polynomial $S_b v$ of degree m-1 such that

(10.3) $$\| D^j(v - S_b v) \|_{B(b)} \leq c_1 \, b^{m-j} \, \| D^m v \|_{B(b)}$$

(where c_1 is independent of v and b), and the odd derivatives up to order $\bar{m}$ of $S_b v$ vanish at zero, i.e.,

(10.4) $$D^i S_b v(0) = 0, \quad i = 1, 3, \ldots, \bar{m}.$$

Proof: If $m < m_0$, then (10.4) is vacuous; choosing $S_b v = T_b v$ gives the desired result. Now assume that $m \geq m_0$. According to Lemma 9.4, v has m-1 continuous derivatives at b and $m-m_0$ continuous derivatives at 0. Let $S_b v$ be defined by the $m_0 - 1$ conditions

$$D^i S_b v(b) = D^i v(b), \quad i = 0, 1, \ldots, m_0 - 2,$$

and the $m - m_0 + 1$ conditions

(10.5) $$D^i S_b v(0) = D^i v(0), \quad i = 0, 1, 2, \ldots, m - m_0.$$

$S_b v$ is well-defined since it is the solution of an Hermite interpolation problem. Moreover, (10.4) is satisfied, because by Lemma 9.4 part (d), the odd derivatives up to order $\bar{m}$ of v vanish at 0.

We shall bound $\| D^j(v - S_b v) \|_{B(b)}$ by showing that $\| D^j(T_b v - S_b v) \|_{B(b)}$ is of the same size as $\| D^j(v - T_b v) \|_{B(b)}$. But if $E_b v \equiv T_b v - S_b v$, then

(10.6) $$D^i E_b v(b) = 0, \quad i = 0, 1, \ldots, m_0 - 2,$$

and by the conditions (10.5),

(10.7) $\quad D^i E_b v(0) = D^i(T_b v - v)(0), \quad i = 0, 1, 2, \ldots, m - m_0.$

Since $E_b v$ solves the Hermite interpolation problem (10.6) - (10.7), bounds on $E_b v$ follow from Lemma 9.5. We have

$$\| D^j E_b v \|_{B(b)} \leq C_e \sum_{i=0}^{m-m_0} b^{(n/2)-j+i} |D^i(v - T_b v)(0)|,$$

and by Sobolev's lemma and (10.2),

$$\begin{aligned} \| D^j E_b v \|_{B(b)} &\leq C_e \sum_{i=0}^{m-m_0} b^{(n/2)-j+i} \left(C_n^2 \sum_{\ell=0}^{m_0} b^{2\ell-n} \| D^{\ell+i}(v-T_b v) \|^2_{B(b)} \right)^{1/2} \\ &\leq C_e \sum_{i=0}^{m-m_0} b^{(n/2)-j+i} \left(C_n^2 (m_0+1)\, b^{2(m-i)-n} \| D^m v \|^2_{B(b)} \right)^{1/2} \\ &= C_e C_n (m-m_0+1)\sqrt{(m_0+1)}\, b^{m-j} \| D^m v \|_{B(b)}. \end{aligned}$$

□

We require the following bounds on weighted norms of derivatives of the B-splines.

<u>Lemma 10.3</u>: For $i > 0$ and each integer $\ell \geq 0$,

$$\| x^{(n-1)/2} D^\ell N_{j,k} \|_{L^2(I_i)} \leq c_2 \frac{x_i^{n/2}}{h_i^\ell},$$

where c_2 depends only on n and ℓ.

<u>Proof</u>: By Lemma 3.1,

$$\| x^{(n-1)/2} D^\ell N_{j,k} \|^2_{L^2(I_i)} = \int_{x_{i-1}}^{x_i} x^{n-1} (D^\ell N_{j,k}(x))^2 \, dx$$

$$\leq \quad \| D^{\ell} N_{j,k} \|^2_{L^\infty(I_i)} \frac{x_i^n}{n}$$

$$\leq \quad (B_\ell h_i^{-\ell})^2 \frac{x_i^n}{n} .$$

□

We now show that to every $v \in J^m \cap J_0^1$ there corresponds an m^{th}-order accurate approximation $\hat{v}$ in $SS_0^k(\Delta,\underline{z})$. We first extend v to $x \in (-1,1)$ by setting

$$v(-x) = v(x), \quad 0 < x < 1.$$

Next, we show that the quasiinterpolant $F_{\Delta'} v \in ES_0^k(\Delta',\underline{z}')$ is m^{th}-order accurate; its restriction to [0,1] will be the desired approximation in $SS_0^k(\Delta,\underline{z})$.

Theorem 10.1: There exists a positive constant c_3 which depends on the local mesh ratio $M(\Delta)$ such that, if $v \in J^m \cap J_0^1$, $m \leq k$, then there exists $\hat{v} \in SS_0^k(\Delta,\underline{z})$ such that

$$\| D^{\ell}(v - \hat{v}) \|_B \leq c_3 h^{m-\ell} \| D^m v \|_B$$

for all $0 \leq \ell \leq m$.

Proof: Let $\{\tau_j\}$ be quasiinterpolation points for $\underline{t}(\Delta',\underline{z}')$ satisfying

$$\text{a)} \quad \tau_j = -\tau_{d-j+1}, \quad 1 \leq j \leq d;$$

$$(10.8) \qquad \text{b)} \quad \tau_1 = -1 \text{ and } \tau_d = 1;$$

$$\text{c)} \quad \tau_i \notin (x_{-1}, x_1).$$

The requirements (10.8)(a) and (10.8)(c) are compatible since the number d of basis functions is even. Let $\hat{v} = F_{\Delta'} v = F_{\Delta',\underline{z}',m} v$ be the quasiinterpolant of v with points $\{\tau_j\}$. At this point, all we know is that $F_{\Delta'} v \in S^k(\Delta',\underline{z}')$.

By the symmetry of the function v, the partition $\underline{t}$, and the quasiinterpolation points $\{\tau_j\}$,

$$\lambda_j(v) = \lambda_{d-j+1}(v), \quad 1 \leq j \leq d.$$

Thus,

$$\begin{aligned} F_{\Delta'} v &= \sum_{j=1}^{d} \lambda_j(v)\, N_{j,k} \\ &= \sum_{j=1}^{d/2} \lambda_j\, (N_{j,k} + N_{d-j+1,k}) \\ &= \sum_{j=1}^{d/2} \lambda_j\, B_j, \end{aligned}$$

which shows that $F_{\Delta'} v \in ES^k(\Delta',\underline{z}')$. By (10.8)(b) and Lemma 3.3, $F_{\Delta'} v$ interpolates v at -1 and 1. Thus, $F_{\Delta'} v \in ES_0^k(\Delta',\underline{z}')$ and $\hat{v} \equiv F_{\Delta'} v\big|_{[0,1]} \in SS_0^k(\Delta,\underline{z})$.

We now consider error bounds. Since we are concerned with restrictions to [0,1], we need to bound

$$E_{i,\ell}^2 \equiv \int_{x_{i-1}}^{x_i} x^{n-1} (D^\ell (v - F_\Delta v))^2\, dx, \quad 1 \leq i \leq N.$$

We consider two separate cases, depending on whether or not $\Theta_i \cap (-x_1,x_1) = \emptyset$.

Let i be such that $\Theta_i \cap (-x_1,x_1) \neq \emptyset$. Then $0 < i \leq k$. Let b be the smallest number such that $\Theta_i \subset [-b,b]$ and $b \geq x_2$. Then

$$(10.9) \qquad h_i < b \leq x_{i+k-1} \leq (2k-1)h.$$

Moreover,

(10.10) For all $1 \leq j \leq i$,

$$\frac{b}{h_j} \leq \frac{\sum_{\ell=1}^{2k-1} h_\ell}{\min_{1 \leq \ell \leq i} h_\ell}$$

$$\leq \sum_{\ell=1}^{2k-1} (M(\Delta))^\ell$$

$$\leq \frac{(M(\Delta))^{2k}-1}{M(\Delta)-1} \equiv c_4.$$

(see Figures 10.1 and 3.2).

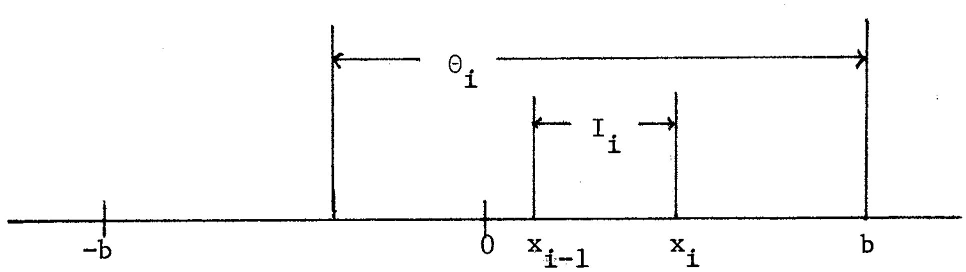


Figure 10.1: The neighborhood (-b,b) of I_i

Let $S_b v$ be the local polynomial approximation to v given by Lemma 10.2, and extend $S_b v$ to (-b,b) by reflection about the y-axis:

$$S_b(-x) = S_b(x), \qquad 0 \leq x \leq b.$$

Then

i) $S_b v$ is a polynomial of degree m-1 in each of the intervals (-b,0) and (0,b);

ii) $S_b v$ is even;

iii) $S_b v$ has $\bar{m} + 1$ continuous derivatives at 0.

For (iii), note that the (one-sided) even derivatives agree at 0 since $S_b v$ is symmetric about the y-axis, while the odd derivatives of appropriate orders all vanish at 0 by (10.4).

Let $S_b v$ be defined for $x \notin (-b,b)$ by extending each of its two polynomial pieces. Clearly $S_b v \in ES^k(\Delta', \underline{z}')$. Let $E_b v \equiv v - S_b v$. Since the quasiinterpolant reproduces elements of $ES^k(\Delta', \underline{z}')$,

$$
\begin{aligned}
E_{i,\ell} &= \| x^{(n-1)/2} D^{\ell}(v - F_{\Delta'} v) \|_{L^2(I_i)} \\
&\leq \| x^{(n-1)/2} D^{\ell} E_b v \|_{L^2(I_i)} + \| x^{(n-1)/2} D^{\ell}(S_b v - F_{\Delta'} v) \|_{L^2(I_i)} \\
&= \| x^{(n-1)/2} D^{\ell} E_b v \|_{L^2(I_i)} + \| x^{(n-1)/2} D^{\ell} F_{\Delta'} E_b v \|_{L^2(I_i)} \qquad (10.11) \\
&\leq \| D^{\ell} E_b v \|_{B(x_i)} + \| x^{(n-1)/2} D^{\ell} F_{\Delta'} E_b v \|_{L^2(I_i)} .
\end{aligned}
$$

We want to get bounds like those of (10.3) for $E_{i,\ell}$. In view of (10.11), we only need to bound $\| x^{(n-1)/2} D^{\ell} F_{\Delta'} E_b v \|_{L^2(I_i)}$. By (3.4) and the triangle inequality,

$$\|x^{(n-1)/2} D^{\ell} F_{\Delta,E_b v}\|_{L^2(I_i)} \le \sum_{j\in V_i} |\lambda_j(E_b v)| \; \|x^{(n-1)/2} D^{\ell} N_{j,k}\|_{L^2(I_i)}.$$

By Lemma 3.4, (10.8)(c), and (10.9),

$$\begin{aligned} |\lambda_j(E_b v)| &\le \sum_{r<m} |\omega_{j,r}| \; |D^r E_b v(\tau_j)| \\ &\le C(k,M(\Delta)) \; h_i^r \; \|D^r E_b v\|_{L^\infty(x_1,b)} \\ &\le C(k,M(\Delta)) \; b^r \; \|D^r E_b v\|_{L^\infty(x_1,b)}. \end{aligned}$$

Since $b \ge x_2$, $\frac{1}{(b - x_1)} \le \frac{1}{h_2} \le \frac{c_4}{b}$. Thus, by Lemma 9.4, Lemma 10.2, and (10.10),

$$\begin{aligned} |\lambda_j(E_b v)| &\le C(k,M(\Delta)) \; b^r \left(C_1^2 \; x_1^{1-n} \; [(b-x_1)^{-1} \|D^r E_b v\|^2_{B(b)} + (b-x_1) \|D^{r+1} E_b v\|^2_{B(b)}] \right)^{1/2} \\ &\le C(k,M(\Delta)) \; b^r \; C_1 c_1 c_4 \; x_1^{(1-n)/2} \; b^{m-r-(1/2)} \; \|D^m v\|_{B(b)}. \end{aligned}$$

(When $\tau_j < 0$, we use the fact that $E_b v$ is even, whence $|D^r E_b v(\tau_j)| = |D^r E_b v(-\tau_j)|$.) Thus, by (10.10),

$$\begin{aligned} |\lambda_j(E_b v)| &\le C(k,M(\Delta)) C_1 c_1 \; c_4^{(n+1)/2} \; b^{m-(n/2)} \; \|D^m v\|_{B(b)} \\ &\equiv c_5 \; b^{m-(n/2)} \; \|D^m v\|_{B(b)}. \end{aligned}$$

Together with (3.2)(b), Lemma 10.3, and (10.10),

$$\| x^{(n-1)/2} D^{\ell} F_{\Delta, E_b} v \|_{L^2(I_i)}$$

$$\leq \sum_{j \in V_i} |\lambda_j(E_b v)| \; \| x^{(n-1)/2} D^{\ell} N_{j,k} \|_{L^2(I_i)}$$

(10.12)

$$\leq kc_2 c_5 \frac{b^m}{h_i^{\ell}} \| D^m v \|_{B(b)}$$

$$\leq kc_2 c_5 c_4^{\ell} \, b^{m-\ell} \; \| D^m v \|_{B(b)} .$$

It follows from (10.11), (10.3), (10.12), and (10.9) that

$$E_{i,\ell} \leq [c_1 + kc_2 c_5 c_4^{\ell}] \; b^{m-\ell} \; \| D^m v \|_{B(b)}$$

(10.13)

$$\leq [c_1 + kc_2 c_5 c_4^{\ell}] \; (2(k-1))^{m-\ell} \; h^{m-\ell} \; \| D^m v \|_{B(b)}$$

$$\equiv c_6 \, h^{m-\ell} \; \| D^m v \|_{B(b)} .$$

We now turn to the case that $\Theta_i \cap (-x_1, x_1) = \emptyset$. Let x_{j_i} be the first knot to the left of Θ_i (see Figure 6.1). Note that

$$|\Theta_i| = \text{length of } \Theta_i \leq 2kh$$

and

$$\frac{x_i}{x_{j_i}} \leq k .$$

We have, using the error bounds of Theorem 3.1,

$$E_{i,\ell}^2 \leq x_i^{n-1} \int_{x_{i-1}}^{x_i} (D^{\ell}(v - F_{\Delta}v))^2 \, dx$$

$$\leq (K(2k)^{m-\ell})^2 \, x_i^{n-1} \, h^{2(m-\ell)} \int_{\theta_i} (D^m v)^2 \, dx$$

$$\leq (K(2k)^{m-\ell})^2 \, (x_i/x_{j_i})^{n-1} \, h^{2(m-\ell)} \int_{\theta_i} x^{n-1} (D^m v)^2 \, dx$$

$$\leq (K(2k)^{m-\ell})^2 \, k^{n-1} \, h^{2(m-\ell)} \int_{\theta_i} x^{n-1} (D^m v)^2 \, dx$$

$$\equiv c_7^2 \, h^{2(m-\ell)} \int_{\theta_i} x^{n-1} (D^m v)^2 \, dx.$$

Together with (10.13) and the usual observation that not more that 2k of the θ_i intersect at any point, we have

$$\| D^{\ell}(v - F_{\Delta}v) \|_B^2 = \sum_{i=1}^{N} E_{i,\ell}^2 \leq c_3^2 \, h^{2(m-\ell)} \, \| D^m v \|_B^2,$$

where $c_3^2 = 2k \max(c_6^2, c_7^2)$.

□

We now consider the error in the RRG approximation to the generalized solution u of (9.4) - (9.5) and show that the RRG approximation $\tilde{u}$ in $SS_0^k(\Delta, \underline{z})$ is an optimal-order approximation to u.

Theorem 10.2: Let u be the generalized solution of (9.4) - (9.5). let $\tilde{u} \in SS_0^k(\Delta,\underline{z})$ be the RRG approximation to u. If $f \in J^{m-2}$, then

$$\| D(u - \tilde{u}) \|_B \le \frac{\Lambda}{\lambda} c_3 \Gamma\, h^{\ell-1} \; \| f \|_{\ell-2,B}$$

$$\| u - \tilde{u} \|_B \le (\Lambda c_3 \Gamma)^2\, h^{\ell} \; \| f \|_{\ell-2,B},$$

where $\ell \equiv \min(m, k)$.

Proof: By Lemma 9.2, the generalized solution $u \in J^m \cap J^1_0$. The error bounds then follow from the approximation results of Theorem 10.1, the regularity theorem (Lemma 9.2), and Nitsche's trick (Theorem 2.6).

□

10.4 Computational Aspects of the Method

Other authors who have treated the problem (9.4) - (9.5) have imposed either no constraint, or, at most, the natural boundary condition $Du(0) = 0$ on the space of approximate solutions ([R3], [J3], [D6]). The resulting approximation to u, while accurate, will not give rise to a smooth function in the original domain $B \subset \mathbb{R}^n$. By using the space $SS_0^k(\Delta,\underline{z})$, this smoothness is obtained.

An alternative is to deal with the problem (9.4′) - (9.5′). In that case, rather than enforcing the requirement of evenness on the subspace, as we did with $ES_0^k(\Delta',\underline{z}')$ (and $SS_0^k(\Delta,\underline{z})$), we take the RRG approximation to u from the space $S_0^k(\Delta',\underline{z}')$. Since the basis functions are no longer even, we would have to take the inner products over (-1,1) instead of (0,1). Surprisingly, the restriction of this spline to [0,1] is the RRG approximation to u from $SS_0^k(\Delta,\underline{z})$!

It suffices to show that the RRG approximation $\tilde{u} \in S_0^k(\Delta',\underline{z}')$ is even. Clearly, by the symmetry property (10.1) of the B-splines, $\tilde{u} = \sum_{j=1}^{d} \xi_j N_{j,k}$ is even if the vector $\underline{\xi}$ is symmetric about its middle, i.e.,

$$\xi_j = \xi_{d+1-j}, \quad 1 \leq j \leq d. \tag{10.14}$$

These coefficients are obtained as the solution of the linear system $A\underline{x} = \underline{f}$ of (2.10). Because of the evenness of the data (functions p, q, and f) and the basis functions, the matrix A will be symmetric about the alternate diagonal and the vector $\underline{f}$ will be symmetric about its middle. Thus, the coefficients of $\tilde{u}$ will satisfy (10.14).

It might appear that by dealing with a two-point boundary value problem on (-1,1) in which all the functions involved are even, we would do twice as much work as is necessary. This is incorrect. It does not cost any more, in work and storage, to use the (-1,1) problem than the (0,1) problem. Suppose, for example, that we use the space $S_0^k(\Delta',\underline{z}')$. Because of the symmetries of the matrix A, only 1/4 of its elements need to be computed. Moreover, using an algorithm of Evans and Hatzopoulos [E3] which takes advantage of symmetry about the alternate diagonal, the equations (2.10) can be solved in half the time required by the usual band Cholesky algorithm.

The effect of numerical quadrature (used to compute the matrix A and the right-hand side vector $\underline{f}$) on the accuracy of the RRG approximation has been analyzed by Fix for nonsingular problems [F1]. He showed that if the integrals are computed using composite Gaussian quadrature with k-1 points in each interval, then the error due to the quadrature is asymptotically as small as the discretization error. We conjecture that this result applies to the singular problem (9.4) - (9.5) and that k-1 points suffice. The numerical results of the next section strongly support this viewpoint.

10.5 Numerical Results

In this section we present the results of a numerical experiment, which illustrates the utility of the computational procedure analyzed in the previous sections. Following Russell and Shampine [R3], we consider the problem

$$- D(x^2 Du) + 4x^2 u = - 20x^2$$

$$u(-1) = u(1) = 0$$

which has the solution $u(x) = \dfrac{5 \sinh 2x}{x \sinh 2} - 5$.

The RRG approximations to u from several of the $ES_0^k(\Delta', \underline{z}')$ spaces were computed, and the error tabulated below. The partition Δ' of

(-1,1) was uniform, with 2N subintervals and mesh-width $h = \frac{1}{N}$. All computations were performed in double precision on a PDP-10 (with 54 binary digits). The integrals required were computed using Gaussian quadrature, with k-1 nodes in each interval of the mesh. We give the norms $\|e\|_B$ and $\|De\|_B$, computed using k+1 - point composite Gaussian quadrature rules, and also the quantity $\|e_N\|_{\infty,\Delta}$. Further details concerning notation and the observed rate of convergence (RATE, below) are given in Section 6.4.

As predicted by the theory, the rate of convergence appears to be h^k for the error and h^{k-1} for the derivative. Moreover, k-1 quadrature nodes per interval are sufficient to maintain the predicted rate of convergence.

N	$\|e_N\|_B$	RATE	$\|e_N\|_{\infty,\Delta}$	RATE	$\|De_N\|_B$	RATE
4	.48 (-01)		.38 (+00)		.42 (+00)	
8	.12 (-01)	1.98	.12 (+00)	1.68	.21 (+00)	1.02
12	.54 (-02)	2.00	.58 (-01)	1.75	.14 (+00)	1.01
16	.31 (-02)	2.00	.35 (-01)	1.78	.10 (+00)	1.00
20	.20 (-02)	2.00	.23 (-01)	1.79	.82 (-01)	1.00
24	.14 (-02)	2.00	.17 (-01)	1.80	.69 (-01)	1.00
28	.10 (-02)	2.00	.13 (-01)	1.81	.59 (-01)	1.00
32	.76 (-03)	2.00	.10 (-01)	1.82	.51 (-01)	1.00

Table 10.1: Error in $ES_0^2(\Delta')$

N	$\|e_N\|_B$	RATE	$\|e_N\|_{\infty,\Delta}$	RATE	$\|De_N\|_B$	RATE
4	.11 (-02)		.15 (-02)		.19 (-01)	
8	.10 (-03)	3.40	.12 (-03)	3.63	.46 (-02)	2.06
12	.28 (-04)	3.20	.26 (-04)	3.82	.20 (-02)	2.02
16	.11 (-04)	3.11	.90 (-05)	3.74	.11 (-02)	2.01
20	.57 (-05)	3.07	.40 (-05)	3.60	.72 (-03)	2.00
24	.33 (-05)	3.05	.21 (-05)	3.63	.50 (-03)	2.00
28	.21 (-05)	3.03	.12 (-05)	3.65	.37 (-03)	2.00
32	.14 (-05)	3.03	.73 (-06)	3.67	.28 (-03)	2.00

Table 10.2: Error in $ES_0^3(\Delta')$

N	$\| e_N \|_B$	RATE	$\| e_N \|_{\infty,\Delta}$	RATE	$\| De_N \|_B$	RATE
4	.41 (-04)		.96 (-04)		.11 (-02)	
8	.28 (-05)	3.83	.64 (-05)	3.91	.15 (-03)	2.86
12	.59 (-06)	3.86	.14 (-05)	3.72	.46 (-04)	2.90
16	.19 (-06)	3.90	.47 (-06)	3.80	.20 (-04)	2.93
20	.81 (-07)	3.93	.20 (-06)	3.84	.10 (-04)	2.94
24	.39 (-07)	3.94	.99 (-07)	3.87	.60 (-05)	2.95
28	.21 (-07)	3.95	.54 (-07)	3.89	.38 (-05)	2.96
32	.13 (-07)	3.96	.32 (-07)	3.90	.26 (-05)	2.96

Table 10.3: Error in $ES_0^4(\Delta')$

N	$\| e_N \|_B$	RATE	$\| e_N \|_{\infty,\Delta}$	RATE	$\| De_N \|_B$	RATE
4	.18 (-06)		.12 (-05)		.42 (-05)	
8	.36 (-08)	5.65	.82 (-08)	7.21	.17 (-06)	4.60
12	.33 (-09)	5.91	.76 (-09)	5.85	.24 (-07)	4.85
16	.60 (-10)	5.95	.14 (-09)	5.86	.59 (-08)	4.91
20	.16 (-10)	5.97	.38 (-10)	5.88	.19 (-08)	4.94
24	.53 (-11)	5.97	.13 (-10)	5.89	.79 (-09)	4.95
28	.21 (-11)	5.98	.52 (-11)	5.90	.37 (-09)	4.96
32	.96 (-12)	5.97	.24 (-11)	5.89	.19 (-09)	4.97

Table 10.4: Error in $ES_0^6(\Delta')$

CONCLUSIONS

For the two important classes of singular two-point boundary value problem, we have shown how finite element methods can accurately approximate the solution. We feel that for linear, one-dimensional problems with a singularity at one of the endpoints, many of the important problems have been solved. Also, our results probably can be extended without great difficulty to time-dependent and mildly nonlinear problems.

In a few areas, we have no satisfactory theoretical results. How much β-grading is required for the subspaces of Chapter 6? It seems that a β_2-graded mesh is the thing to use (if L^2-norm error is important), but we have not been able to prove this. The lack of any local error bound for the RRG approximation has defeated our attempts.

We have said nothing about the collocation method. Nevertheless, theoretical results for nonsingular problems and numerical experiments for singular problems indicate that it is an attractive alternative to RRG for singular two-point boundary value problems. Using splines on a β-graded mesh (as in Chapter 6), collocation produces an optimally accurate L^2-norm approximation, provided we collocate at the Gaussian points. Unfortunately, these methods apply only to C^1 spline spaces.

The results of a numerical experiment on the problem of Section 6.4 are given in Tables C.1 - C.3. It appears that, unlike RRG, collocation succeeds in producing an optimal-order L^2-norm approximation in the spaces $S_0^k(\Delta_{\beta_3,N})$! Other experiments show that collocation at the weighted Gaussian points is effective in the weighted spline spaces of Chapter 7.

We have not given any theoretical justification for the use of numerical quadrature in the RRG method. It appears that, just as in the nonsingular case, k-1 quadrature nodes in each interval are sufficient to preserve the rate of convergence, but we have no proof of this.

The generalized L-spline spaces of Chapter 8 suffer from a severe restriction: beyond continuity, no smoothness my be imposed. For the same partition, a smooth space would have smaller dimension, making the RRG approximation less costly to compute.

For the spherically symmetric problem, we feel that the theory is more solid. Jesperson has already succeeded in obtaining L^∞-error bounds [J3], while de Hoog and Weiss have some results on collocation [D4]. The effect of numerical quadrature, however, still remains to be analyzed.

N	$\|e_N\|_0$	RATE	$\|e_N\|_{\infty,\Delta}$	RATE	$\|e_N\|_S$	RATE
8	.37 (-01)		.46 (-01)		.17 (+00)	
16	.94 (-02)	1.99	.13 (-01)	1.79	.89 (-01)	0.95
24	.42 (-02)	2.00	.72 (-02)	1.51	.64 (-01)	0.80
32	.24 (-02)	2.00	.47 (-02)	1.50	.52 (-01)	0.77
48	.11 (-02)	2.00	.26 (-02)	1.50	.38 (-01)	0.76

Table C.1: Collocation: Error in $S_0^3(\Delta_{3,N})$ -- 3 = β_3

N	$\|e_N\|_0$	RATE	$\|e_N\|_{\infty,\Delta}$	RATE	$\|e_N\|_S$	RATE
8	.70 (-03)		.80 (-02)		.85 (-01)	
16	.76 (-04)	3.22	.20 (-02)	2.00	.43 (-01)	1.00
24	.17 (-04)	3.72	.89 (-03)	2.00	.28 (-01)	1.00
32	.55 (-05)	3.85	.50 (-03)	2.00	.21 (-01)	1.00
40	.23 (-05)	3.89	.32 (-03)	2.00	.17 (-01)	1.00

Table C.2: Collocation: Error in $S_0^4(\Delta_{4,N})$ -- 4 = β_3

N	$\|e_N\|_0$	RATE	$\|e_N\|_{\infty,\Delta}$	RATE	$\|e_N\|_S$	RATE
8	.36 (-04)		.12 (-02)		.40 (-01)	
16	.69 (-06)	5.69	.15 (-03)	3.00	.14 (-01)	1.50
24	.69 (-07)	5.69	.45 (-04)	3.00	.77 (-02)	1.50
32	.13 (-07)	5.72	.19 (-04)	3.00	.50 (-02)	1.50
48	.13 (-08)	5.79	.56 (-05)	3.00	.27 (-02)	1.50

Table C.3: Collocation: Error in $S_0^6(\Delta_{6,N})$ -- 6 = β_3

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