Yale Sparse Matrix Package

II. The Nonsymmetric Codes[1]

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1. Introduction

Consider the NxN system of linear equations

(1) $M x = b,$

where the coefficient matrix M is large, sparse, and nonsymmetric. Assume that M can be factored in the form

$$M = L D U,$$

where L is a lower triangular matrix, D is a diagonal matrix, and U is a unit upper triangular matrix. Such systems arise frequently in scientific computation, e.g., in finite difference and finite element approximations to non-self-adjoint elliptic boundary value problems. In this report, we present a package of efficient, reliable, well-documented, and portable FORTRAN subroutines for solving these systems. See [3] for a corresponding package for symmetric problems.

Direct methods for solving (1) are generally variations of Gaussian elimination. We form the LDU decomposition of A, and successively solve the triangular systems

(2) $L y = b, \quad D z = y, \quad U x = z.$

When M is large ($N >> 1$), (dense) Gaussian elimination is prohibitively expensive in terms of both the work ($\sim 2/3\ N^3$ multiplies) and storage (N^2 words) required. But, since M is sparse, most entries of M, L, and U are zero and there are significant advantages to factoring M without storing or operating on these zeroes. Recently, a number of implementations of sparse Gaussian elimination have appeared based on this idea, cf., [2, 6, 7, 8].

In section 2, we describe the scheme used for storing sparse matrices, while, in section 3, we give an overview of the package from the point of view of the user; for further details of the algorithms employed, see [4, 5]. In section 4, we illustrate the performance of the package on a typical model problem. Listings of the three sets of subroutines for factoring and solving the class of sparse nonsymmetric systems under consideration appear in Appendices 1, 2, and 3. These three sets of subroutines have different storage schemes and basically trade-off run-time efficiency for storage. Appendix 4 contains a test driver which sets up a problem and calls all three sets of subroutines for solution. A sample output appears as Appendix 5.

2. Sparse Matrix Storage Schemes

Since the coefficient matrix M and the triangular factors L and U are large and sparse, it is inefficient to store them as dense matrices. The package has two schemes for storing sparse matrices, called the "uncompressed storage scheme" and the "compressed storage scheme." The input matrix M is always stored using the first of these, while the triangular factors L and U may be stored using either one, depending on which subroutines are used. The subroutine NDRV uses the "uncompressed storage scheme" for L and U while the subroutines TDRV and CDRV use the "compressed storage scheme."

The uncompressed storage scheme has been used previously in various forms, cf. [1, 6]. To use it to store the input matrix M requires three one-dimensional arrays: IA, JA, and A. The nonzero entries of M are stored row-by-row in the REAL array A. To identify the individual nonzero entries in a row, we need to know in which column each entry lies. The INTEGER array JA contains the column indices which correspond to the nonzero entries of M, i.e., if A(K) = M(I,J), then JA(K) = J. In addition, we need to know where each row starts and how long it is. The INTEGER array IA contains the index positions in JA and A where the rows of M begin, i.e., if M(I,J) is the first (leftmost) entry of the I-th row and A(K) = M(I,J), then IA(I) = K. Moreover, IA(N+1) is defined as the index in JA and A of the first location following the last element in the last row. Thus, the number of entries in the I-th row is given by IA(I+1) - IA(I), the nonzero entries of the I-th row are stored consecutively in

A(IA(I)), A(IA(I)+1), ..., A(IA(I+1)-1),

and the corresponding column indices are stored consecutively in

JA(IA(I)), JA(IA(I)+1), ..., JA(IA(I+1)-1).

For example, the 5x5 matrix

```
          1.  0.  2.  0.  0.
          0.  3.  0.  0.  0.
M  =      0.  4.  5.  6.  0.
          0.  0.  0.  7.  0.
          0.  0.  0.  8.  9.
```

is stored as

	1	2	3	4	5	6	7	8	9
IA	1	3	4	7	8	10			
JA	1	3	2	2	3	4	4	4	5
A	1.	2.	3.	4.	5.	6.	7.	8.	9. .

The overhead in this storage scheme is the storage required for the INTEGER arrays IA and JA. But since IA has N+1 entries and JA has one entry for each element of A, the total overhead is approximately equal to the number of nonzero entries in M.

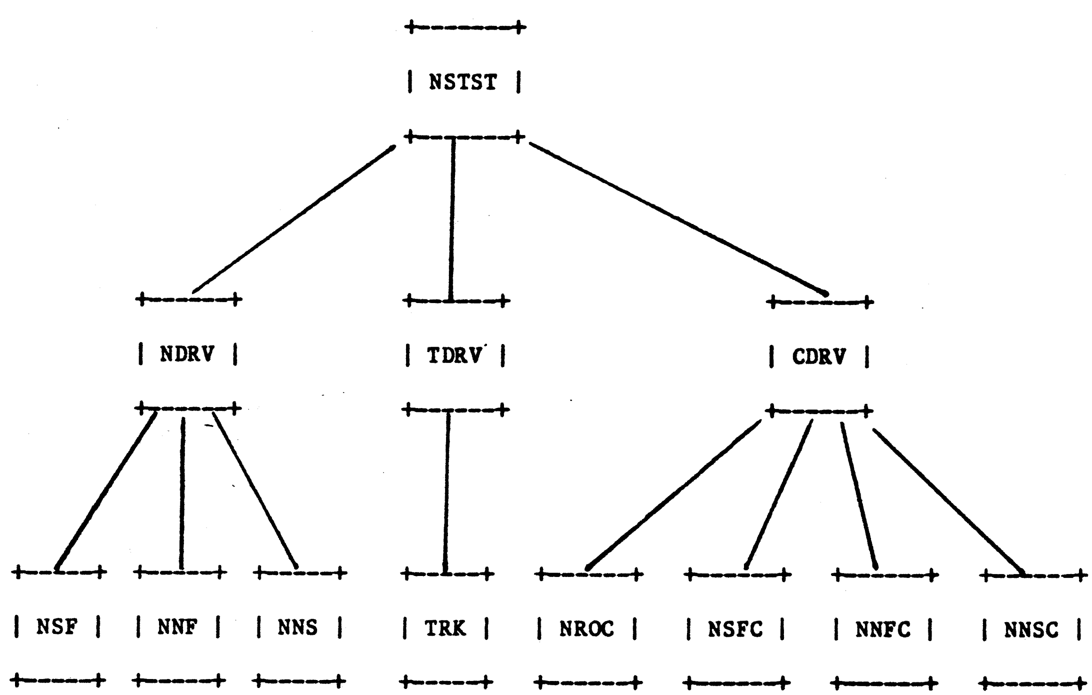
The triangular matrices L and U are stored in basically the same fashion using the arrays IL, JL, L and IU, JU, U respectively, except that the diagonal entries are not stored in these arrays. The diagonal entries of L and U are known to be ones and are not stored and the array D is used to store the reciprocals of the diagonal entries of the diagonal matrix D.

In certain situations, where storage is at a premium, it is essential to reduce storage overhead, even at the cost of decreased runtime efficiency. This can be done by storing L and U with the more complex compressed storage scheme. This scheme incurs more operational overhead than the uncompressed storage scheme, but in many important cases the storage requirement can be substantially reduced. For a detailed description, see [4, 5, 9].

3. A Sparse Nonsymmetric Matrix Package

The package consists of a test driver, three driver subroutines, and eight subroutines (see Figure 1). The three drivers (subroutines NDRV, TDRV, and CDRV) are specific implementation designs which illustrate the space-time tradeoff mentioned in section 2. The test driver (subroutine NSTST) sets up a model sparse nonsymmetric system of linear equations and calls each of the three driver subroutines to solve the linear system. In the remainder of this section, we describe each of these subroutines in somewhat greater detail. The codes themselves are extensively documented; for further details about the algorithms employed see [4, 5].

Figure 1: A schematic overview of the sparse symmetric matrix package



Our basic design for the implementation of sparse elimination follows that of Chang [1], which has proved to be especially robust. The first implementation, NDRV, is designed for speed. It uses the uncompressed storage scheme for M, L, and U because of the smaller operational overhead associated with it. We break the computation up into three distinct phases: symbolic factorization (subroutine NSF), numeric factorization and the solution for one right-hand side (subroutine NNF), and forward and back solution for additional right-hand sides (subroutine NNS). The subroutine NSF computes the zero structures of L and U from that of M (disregarding the numerical entries in M). The subroutine NNF then uses the structural information generated by NSF to compute the numerical entries of L and U and to solve for one right-hand side.

The main advantage of splitting up the computation in this way is flexibility. To solve a single system of equations, it suffices to use NSF and NNF (PATH=1 in NDRV). It should be pointed out here that a one line modification of NNF can be made to allow the solution of a single system without storing L: simply comment out the line

```
L(I) = - LI,
```

as indicated in the code. This change will yield substantial storage savings without the loss in efficiency incurred by TDRV. To solve several systems in which the coefficient matrices have the same zero structure, it suffices to use NSF and NNF only once each for the first system and then to use NNF once for each subsequent system (PATH=2 in

NDRV). Finally, to solve several systems with the same coefficient matrix but different right hand sides, it suffices to use NSF and NNF only once each for the first system, and then to use NNS once for each subsequent system (PATH=3 in NDRV).

A drawback to the multi-phase design of NDRV is that it is necessary to store the description of the zero structures of both L and U. By giving up some flexibility, the second implementation, TDRV, greatly reduces the storage requirements. The entire computation is performed in a single phase (subroutine TRK) to avoid storing either the description or the numerical entries of L. Moreover, U is stored with the compressed storage scheme to reduce the storage overhead. This subroutine incurs more operational overhead than NDRV, and we lose the ability to efficiently solve a sequence of related systems. However the total storage requirements are usually significantly smaller (see Tables 4-6).

Finally, the third implementation, CDRV, attempts to balance the design goals of speed, flexibility, and storage economy. It splits the computation as in NDRV to allow flexibility and efficiency, but it uses the compressed storage scheme as in TDRV to reduce storage overhead. The rows and columns of the original matrix M can be reordered (e.g., to reduce fillin or ensure numerical stability) before calling CDRV. If no reordering is done, then set R(I)=C(I)=IC(I) = I for I=1,...,N. The solution Z is returned in the original order. If the columns have been reordered (i.e., C(I).NE.I for some I), then CDRV will call a subroutine

NROC which rearranges each row of JA and A, leaving the rows in the orginal order, but placing the elements of each row in increasing order with respect to the new ordering. If PATH.NE.1, then NROC is assumed to have been called already.

To solve a single system of equations, it suffices to use NROC (if the columns of M have been reordered), NSFC, and NNFC (PATH=1 in CDRV). It should be pointed out here that a one line modification of NNFC can be made to allow the solution of a single system without storing L: simply comment out the line

```
L(IRL(I)) = - LKI,
```

as indicated in the code. This change will yield substantial storage savings without the loss in efficiency incurred in TDRV. To solve several systems in which the coefficient matrices have the same zero structure, it suffices to use NROC, NSFC, and NNFC only once each for the first system and then to use NNFC once for each subsequent system (PATH=2 in CDRV). Finally, to solve several systems with the same coefficient matrix but different right hand sides, it suffices to use NROC, NSFC, and NNFC only once each for the first system, and then to use NNSC once for each subsequent system (PATH=3 in CDRV).

The test driver (program NSTST) is used to verify the performance of the package on a particular computer system, and may be used as a guide to understanding how to use the package. It generates the coefficient matrix for a nonsymmetric five-point difference equation on a 3x3 grid and chooses the right-hand side so that the solution vector x is (1,2,3,4,5,6,7,8,9) (see Appendix 4). The grid points are given in the natural row-by-row ordering. At each stage the values of all relevant variables are printed out, and a sample output appears as Appendix 5.

4. Performance

One of the most important aspects of any package is its performance in terms of both the time and storage required to solve a typical problem. In Tables 1-6, we present the time and storage required to solve a nonsymmetric five-point difference equation on an nxn grid for several values of n. These computations were performed in single precision on an IBM 370/158 using the FORTRAN IV Level H Extended compiler.

Table 1: Times for 5-point operator on a 20x20 mesh

Code	NSF(C)	NNF(C)	sec/*	NNS(C)	Total
NDRV	0.213	0.560	9.978	0.063	0.773
TDRV			15.561		0.873
CDRV	0.267	0.790	14.077	0.087	1.057

Table 2: Times for 5-point operator on a 30x30 mesh

Code	NSF(C)	NNF(C)	sec/*	NNS(C)	Total
NDRV	0.583	2.050	9.503	0.170	2.633
TDRV			14.046		3.030
CDRV	0.650	2.810	13.026	0.233	3.460

Table 3: Times for 5-point operator on a 40x40 mesh

Code	NSF(C)	NNF(C)	sec/*	NNS(C)	Total
NDRV	1.197	5.430	9.250	0.347	6.626
TDRV			13.134		7.710
CDRV	1.243	7.313	12.459	0.480	8.556

Table 4: Storage for 5-point operator on a 20x20 mesh

Code	A/JA	L	JL	U	JU	Total+	Mults.
NDRV	1,920	3,368	3,368	3,368	3,368	15,47	56,118
TDRV	1,920			3,368	1,889	7,660	56,118
CDRV	1,920	3,368	1,889	3,368	1,889	13,716	56,118

Table 5: Storage for 5-point operator on a 30x30 mesh

Code	A/JA	L	JL	U	JU	Total+	Mults.
NDRV	4,380	9,456	9,456	9,456	9,456	42,327	215,708
TDRV	4,380			9,456	4,538	19,397	215,708
CDRV	4,380	9,456	4,538	9,456	4,538	35,190	215,708

+Total storage required by driver

Table 6: Storage for 5-point operator on a 40x40 mesh

Code	A/JA	L	JL	U	JU	Total[+]	Mults.
NDRV	7,840	19,926	19,926	19,926	19,926	87,707	586,970
TDRV	7,840			19,926	8,423	37,952	586,970
CDRV	7,840	19,926	8,423	19,926	8,423	69,500	586,970

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C                           Appendix 1                       7/31/77
C
C       Subroutines for Solving Sparse Nonsymmetric Systems
C       of Linear Equations  (Uncompressed Pointer Storage)
C
C
C*** Subroutine NDRV
C*** Driver for subroutines for solving sparse nonsymmetric systems of
C       linear equations (uncompressed pointer storage)
C
        SUBROUTINE  NDRV
     *     (N, R,C,IC, IA,JA,A, B, Z, NSP,ISP,RSP,ESP, PATH, FLAG)
C
C    PARAMETERS
C    Class abbreviations are --
C       n - INTEGER variable
C       f - REAL variable
C       v - supplies a VALUE to the driver
C       r - returns a RESULT from the driver
C       i - used INTERNALly by the driver
C       a - ARRAY
C
C Class | Parameter
C ------+----------
C       |
C         The nonzero entries of the coefficient matrix M are stored
C    row-by-row in the array A.  To identify the individual nonzero
C    entries in each row, we need to know in which column each entry
C    lies.  The column indices which correspond to the nonzero entries
C    of M are stored in the array JA;  i.e., if  A(K) = M(I,J),  then
C    JA(K) = J.  In addition, we need to know where each row starts and
C    how long it is.  The index positions in JA and A where the rows of
C    M begin are stored in the array IA;  i.e., if M(I,J) is the first
C    nonzero entry (stored) in the I-th row and A(K) = M(I,J),  then
C    IA(I) = K.  Moreover, the index in JA and A of the first location
C    following the last element in the last row is stored in IA(N+1).
C    Thus, the number of entries in the I-th row is given by
C    IA(I+1) - IA(I),  the nonzero entries of the I-th row are stored
C    consecutively in
C            A(IA(I)),  A(IA(I)+1),  ..., A(IA(I+1)-1),
C    and the corresponding column indices are stored consecutively in
C            JA(IA(I)), JA(IA(I)+1), ..., JA(IA(I+1)-1).
C    For example, the 5 by 5 matrix
C                ( 1. 0. 2. 0. 0.)
C                ( 0. 3. 0. 0. 0.)
C            M = ( 0. 4. 5. 6. 0.)
C                ( 0. 0. 0. 7. 0.)
C                ( 0. 0. 0. 8. 9.)
C    would be stored as
C               | 1  2  3  4  5  6  7  8  9
C            ---+--------------------------
C            IA | 1  3  4  7  8 10
C            JA | 1  3  2  2  3  4  4  4  5
C             A | 1. 2. 3. 4. 5. 6. 7. 8. 9.         .
C
C nv    | N     - number of variables/equations.
C fva   | A     - nonzero entries of the coefficient matrix M, stored
C       |           by rows.
C       |           Size = number of nonzero entries in M.
C nva   | IA    - pointers to delimit the rows in A.
```

```
C       |               Size = N+1.
C nva   | JA     - column numbers corresponding to the elements of A.
C       |               Size = size of A.
C fva   | B      - right-hand side b;  B and Z can the same array.
C       |               Size = N.
C fra   | Z      - solution x;  B and Z can be the same array.
C       |               Size = N.
C
C          The rows and columns of the original matrix M can be
C     reordered (e.g., to reduce fillin or ensure numerical stability)
C     before calling the driver.  If no reordering is done, then set
C     R(I) = C(I) = IC(I) = I  for I=1,...,N.  The solution Z is returned
C     in the original order.
C
C nva   | R      - ordering of the rows of M.
C       |               Size = N.
C nva   | C      - ordering of the columns of M.
C       |               Size = N.
C nva   | IC     - inverse of the ordering of the columns of M;  i.e.,
C       |              IC(C(I)) = I  for I=1,...,N.
C       |               Size = N.
C
C          The solution of the system of linear equations is divided into
C     three stages --
C       NSF -- The matrix M is processed symbolically to determine where
C                fillin will occur during the numeric factorization.
C       NNF -- The matrix M is factored numerically into the product LDU
C                of a unit lower triangular matrix L, a diagonal matrix D,
C                and a unit upper triangular matrix U, and the system
C                Mx = b  is solved.
C       NNS -- The linear system  Mx = b  is solved using the LDU
C                factorization from NNF.
C     For several systems whose coefficient matrices have the same
C     nonzero structure, NSF need be done only once (for the first
C     system);  then NNF is done once for each additional system.  For
C     several systems with the same coefficient matrix, NSF and NNF need
C     be done only once (for the first system);  then NNS is done once
C     for each additional right-hand side.
C
C nv    | PATH   - path specification;  values and their meanings are --
C       |              1  perform NSF and NNF.
C       |              2  perform NNF only  (NSF is assumed to have been
C       |                   done in a manner compatible with the storage
C       |                   allocation used in the driver).
C       |              3  perform NNS only  (NSF and NNF are assumed to
C       |                   have been done in a manner compatible with the
C       |                   storage allocation used in the driver).
C
C          Various errors are detected by the driver and the individual
C     subroutines.
C
C nr    | FLAG   - error flag;  values and their meanings are --
C       |                0     No Errors Detected
C       |                N+K   Null Row in A  --  Row = K
C       |               2N+K   Duplicate Entry in A  --  Row = K
C       |               3N+K   Insufficient Storage in NSF  --  Row = K
C       |               4N+1   Insufficient Storage in NNF
C       |               5N+K   Null Pivot  --  Row = K
C       |               6N+K   Insufficient Storage in NSF  --  Row = K
```

```
C       |          7N+1   Insufficient Storage in NNF
C       |          8N+K   Zero Pivot  --  Row = K
C       |         10N+1   Insufficient Storage in NDRV
C       |         11N+1   Illegal PATH Specification
C
C        Working storage is needed for the factored form of the matrix
C     M plus various temporary vectors.  The arrays ISP and RSP should be
C     the same;  integer storage is allocated from the beginning of ISP
C     and real storage from the end of RSP.
C
C nv    | NSP   - declared dimension of ISP and RSP;  NSP generally must
C       |           be larger than  5N+3 + 2K  (where  K = (number of
C       |           nonzero entries in M)).
C nvira | ISP   - integer working storage divided up into various arrays
C       |           needed by the subroutines;  ISP and RSP should be
C       |           the same array.
C       |           Size = NSP.
C fvira | RSP   - real working storage divided up into various arrays
C       |           needed by the subroutines;  ISP and RSP should be
C       |           the same array.
C       |           Size = NSP.
C nr    | ESP   - if sufficient storage was available to perform the
C       |           symbolic factorization (NSF), then ESP is set to the
C       |           amount of excess storage provided (negative if
C       |           insufficient storage was available to perform the
C       |           numeric factorization (NNF)).
C
      INTEGER  R(1), C(1), IC(1),  IA(1), JA(1),  ISP(1), ESP,
     *   PATH, FLAG,  Q, IM, D, U, ROW, TMP,  UMAX
      REAL  A(1),  B(1),  Z(1),  RSP(1)
C
      IF (PATH.LT.1 .OR. PATH.GT.3)  GO TO 111
C ******  Initialize and divide up temporary storage  *****************
      IL = 1
      IU = IL + N+1
      JL = IU + N+1
      FLAG = 0
C
C ******  Call NSF if flag is set  ************************************
      IF (PATH.GT.1)  GO TO 2
        IM = NSP -   N
        Q  = IM  - (N+1)
        MAX = Q - JL
        IF (MAX.LT.0)  GO TO 110
        JLMAX = MAX/2
        JUTMP = JL + JLMAX
        JUMAX = Q - JUTMP
        CALL  NSF
     *      (N,  R, IC,  IA, JA,
     *       ISP(IL), ISP(JL), JLMAX,  ISP(IU), ISP(JUTMP), JUMAX,
     *       RSP(Q),  RSP(IM),  FLAG)
        IF (FLAG.NE.0)  GO TO 100
C ******  Move JU next to JL  *****************************************
        JLMAX = ISP(IL+N)-1
        JU = JL + JLMAX
        JUMAX = ISP(IU+N)-1
        IF (JUMAX.LE.0)  GO TO 2
        DO 1 J=1,JUMAX
    1     ISP(JU+J-1) = ISP(JUTMP+J-1)
```

```
C
C  ******  Call remaining subroutines  *********************************
   2     JLMAX = ISP(IL+N)-1
         JU   = JL   + JLMAX
         JUMAX = ISP(IU+N)-1
         L    = JU   + JUMAX
         LMAX  = JLMAX
         D    = L    + LMAX
         U    = D    + N
         ROW = NSP - N
         TMP = ROW - N
         UMAX  = TMP - U
         ESP = UMAX - JUMAX
C
         IF (PATH.GT.2)  GO TO 3
           CALL  NNF
     *        (N,  R, C, IC,  IA, JA, A,  Z,  B,
     *         ISP(IL), ISP(JL), RSP(L), LMAX,   RSP(D),
     *            ISP(IU), ISP(JU), RSP(U), UMAX,
     *         RSP(ROW),  RSP(TMP),  FLAG)
           IF (FLAG.NE.0)  GO TO 100
           RETURN
C
   3     CALL  NNS
     *      (N,  R, C,
     *       ISP(IL), ISP(JL), RSP(L),  RSP(D),
     *         ISP(IU), ISP(JU), RSP(U),
     *      Z,  B,  RSP(TMP))
         RETURN
C
C ** ERROR:  Error Detected in NSF, NNF, or NNS
 100     RETURN
C ** ERROR:  Insufficient Storage
 110     FLAG = 10*N + 1
         RETURN
C ** ERROR:  Illegal PATH Specification
 111     FLAG = 11*N + 1
         RETURN
         END
```

```
C
C       ----------------------------------------------------------------
C
C               YALE SPARSE MATRIX PACKAGE - NONSYMMETRIC CODES
C                    SOLVING THE SYSTEM OF EQUATIONS Mx = b
C                        (UNCOMPRESSED POINTER STORAGE)
C
C    I.   SUBROUTINE NAMES
C         Subroutine names are of the form Nxx where --
C    (1) the first letter is N for nonsymmetric matrices;
C    (2) the second letter is either S for symbolic processing or N
C         for numeric processing;
C    (3) the third letter is either F for factorization or S for
C         solution.
C
C    II.  CALLING SEQUENCES
C         The coefficient matrix can be processed by an ordering routine
C    (e.g., to reduce fillin or ensure numerical stability) before using
C    the remaining subroutines.  If no reordering is done, then set
C    R(I) = C(I) = IC(I) = I  for I=1,...,N.  The calling sequence is --
C        (      (matrix ordering))
C         NSF   (symbolic factorization to determine where fillin will
C                 occur during numeric factorization)
C         NNF   (numeric factorization into product LDU of unit lower
C                 triangular matrix L, diagonal matrix D, and unit upper
C                 triangular matrix U, and solution of linear system)
C         NNS   (solution of linear system for additional right-hand
C                 side using LDU factorization from NNF)
C
C    III. STORAGE OF SPARSE MATRICES
C         The nonzero entries of the coefficient matrix M are stored
C    row-by-row in the array A.  To identify the individual nonzero
C    entries in each row, we need to know in which column each entry
C    lies.  The column indices which correspond to the nonzero entries
C    of M are stored in the array JA;  i.e., if  A(K) = M(I,J),  then
C    JA(K) = J.  In addition, we need to know where each row starts and
C    how long it is.  The index positions in JA and A where the rows of
C    M begin are stored in the array IA;  i.e., if M(I,J) is the first
C    nonzero entry (stored) in the I-th row and A(K) = M(I,J),  then
C    IA(I) = K.  Moreover, the index in JA and A of the first location
C    following the last element in the last row is stored in IA(N+1).
C    Thus, the number of entries in the I-th row is given by
C    IA(I+1) - IA(I),  the nonzero entries of the I-th row are stored
C    consecutively in
C            A(IA(I)),  A(IA(I)+1),  ..., A(IA(I+1)-1),
C    and the corresponding column indices are stored consecutively in
C            JA(IA(I)), JA(IA(I)+1), ..., JA(IA(I+1)-1).
C    For example, the 5 by 5 matrix
C                ( 1. 0. 2. 0. 0.)
C                ( 0. 3. 0. 0. 0.)
C            M = ( 0. 4. 5. 6. 0.)
C                ( 0. 0. 0. 7. 0.)
C                ( 0. 0. 0. 8. 9.)
```

```
C    would be stored as
C              | 1  2  3  4  5  6  7  8  9
C           ---+--------------------------
C           IA | 1  3  4  7  8 10
C           JA | 1  3  2  2  3  4  4  4  5
C            A | 1. 2. 3. 4. 5. 6. 7. 8. 9.
C         The strict triangular portions of the matrices L and U are
C    stored in the same fashion using the arrays  IL, JL, L  and
C    IU, JU, U  respectively.  The diagonal entries of L and U are
C    assumed to be equal to one and are not stored.  The array D
C    contains the reciprocals of the diagonal entries of the matrix D.
C
C    IV.  ADDITIONAL STORAGE SAVINGS
C         In NSF, R and IC can be the same array in the calling
C    sequence if no reordering of the coefficient matrix has been done.
C         In NNF, R, C and IC can all be the same array if no reordering
C    has been done.  If only the rows have been reordered, then C and IC
C    can be the same array.  If the row and column orderings are the
C    same, then R and C can be the same array.  Z and ROW can be the
C    same array.
C         In NNS, R and C can be the same array if no reordering has
C    been done or if the row and column orderings are the same.  Z and B
C    can be the same array;  however, then B will be destroyed.
C
C    V.   PARAMETERS
C         Following is a list of parameters to the programs.  Names are
C    uniform among the various subroutines.  Class abbreviations are --
C       n - INTEGER variable
C       f - REAL variable
C       v - supplies a VALUE to a subroutine
C       r - returns a RESULT from a subroutine
C       i - used INTERNALly by a subroutine
C       a - ARRAY
C
C Class | Parameter
C ------+----------
C fva   | A     - nonzero entries of the coefficient matrix M, stored
C       |           by rows.
C       |           Size = number of nonzero entries in M.
C fva   | B     - right-hand side b.
C       |           Size = N.
C nva   | C     - ordering of the columns of M.
C       |           Size = N.
C fvra  | D     - reciprocals of the diagonal entries of the matrix D.
C       |           Size = N.
C nr    | FLAG  - error flag;  values and their meanings are --
C       |            0     No Errors Detected
C       |            N+K   Null Row in A  --  Row = K
C       |           2N+K   Duplicate Entry in A  --  Row = K
C       |           3N+K   Insufficient Storage for JL  --  Row = K
C       |           4N+1   Insufficient Storage for L
C       |           5N+K   Null Pivot  --  Row = K
C       |           6N+K   Insufficient Storage for JU  --  Row = K
C       |           7N+1   Insufficient Storage for U
C       |           8N+K   Zero Pivot  --  Row = K
C nva   | IA    - pointers to delimit the rows in A.
C       |           Size = N+1.
C nva   | IC    - inverse of the ordering of the columns of M;  i.e.,
C       |           IC(C(I) = I  for I=1,...N.
C       |           Size = N.
C nvra  | IL    - pointers to delimit the rows in L.
C       |           Size = N+1.
```

```
C nvra  | IU    - pointers to delimit the rows in U.
C       |           Size = N+1.
C nva   | JA    - column numbers corresponding to the elements of A.
C       |           Size = size of A.
C nvra  | JL    - column numbers corresponding to the elements of L.
C       |           Size = JLMAX.
C nv    | JLMAX - declared dimension of JL;  JLMAX must be larger than
C       |           the number of nonzero entries in the strict lower
C       |           triangle of M plus fillin  (IL(N+1)-1 after NSF).
C nvra  | JU    - column numbers corresponding to the elements of U.
C       |           Size = JUMAX.
C nv    | JUMAX - declared dimension of JU;  JUMAX must be larger than
C       |           the number of nonzero entries in the strict upper
C       |           triangle of M plus fillin  (IU(N+1)-1 after NSF).
C fvra  | L     - nonzero entries in the strict lower triangular portion
C       |           of the matrix L, stored by rows.
C       |           Size = LMAX
C nv    | LMAX  - declared dimension of L;  LMAX must be larger than
C       |           the number of nonzero entries in the strict lower
C       |           triangle of M plus fillin  (IL(N+1)-1 after NSF).
C nv    | N     - number of variables/equations.
C nva   | R     - ordering of the rows of M.
C       |           Size = N.
C fvra  | U     - nonzero entries in the strict upper triangular portion
C       |           of the matrix U, stored by rows.
C       |           Size = UMAX.
C nv    | UMAX  - declared dimension of U;  UMAX must be larger than
C       |           the number of nonzero entries in the strict upper
C       |           triangle of M plus fillin  (IU(N+1)-1 after NSF).
C fra   | Z     - solution x.
C       |           Size = N.
C
C
C       ------------------------------------------------------------
C
C*** Subroutine NSF
C*** Symbolic LDU-factorization of a nonsymmetric sparse matrix
C      (uncompressed pointer storage)
C
        SUBROUTINE  NSF
     *     (N, R,IC, IA,JA, IL,JL,JLMAX, IU,JU,JUMAX, Q, IM, FLAG)
C
C       Input variables:   N, R,IC, IA,JA, JLMAX, JUMAX.
C       Output variables:  IL,JL, IU,JU, FLAG.
C
C       Parameters used internally:
C nia   | Q     - suppose M' is the result of reordering M;  if
C       |           processing of the Kth row of M' (hence the Kth rows
C       |           of L and U) is being done, then Q(J) is initially
C       |           nonzero if M'(K,J) is nonzero;  since values need
C       |           not be stored, each entry points to the next
C       |           nonzero;  for example, if  N=9  and the 5th row of
C       |           M' is
C       |                   0 x x 0 x 0 0 x 0,
```

```
C       |             then Q will initially be
C       |                     a 3 5 a 8 a a 10 a 2        (a - arbitrary);
C       |             Q(N+1) points to the first nonzero in the row and
C       |             the last nonzero points to  N+1;  as the algorithm
C       |             proceeds, other elements of Q are inserted in the
C       |             list because of fillin.
C       |             Size = N+1.
C nia   | IM    - at each step in the factorization, IM(I) is the last
C       |             element in the Ith row of U which needs to be
C       |             considered in computing fillin.
C       |             Size = N.
C
C  Internal variables--
C    JLPTR - points to the last position used in  JL.
C    JUPTR - points to the last position used in  JU.
C
      INTEGER  R(1), IC(1),  IA(1), JA(1),  IL(1), JL(1),
     *   IU(1), JU(1),  Q(1),  IM(1),  FLAG,  QM, VJ
C
C  ******  Initialize pointers  ****************************************
      JLPTR = 0
      IL(1) = 1
      JUPTR = 0
      IU(1) = 1
C
C  ******  For each row of L and U  ************************************
      DO 10 K=1,N
C  ******  Set Q to the reordered row of A  ****************************
        Q(N+1) = N+1
        JMIN = IA(R(K))
        JMAX = IA(R(K)+1) - 1
        IF (JMIN.GT.JMAX)  GO TO 101
        DO 2 J=JMIN,JMAX
          VJ = IC(JA(J))
          QM = N+1
   1      M = QM
          QM = Q(M)
          IF (QM.LT.VJ)  GO TO 1
          IF (QM.EQ.VJ)  GO TO 102
            Q(M) = VJ
            Q(VJ) = QM
   2      CONTINUE
C
C  ******  For each entry in the lower triangle  ***********************
        I = N+1
   3    I = Q(I)
        IF (I.GE.K)  GO TO 7
C  ******  L(K,I) will be nonzero, so add it to JL  ********************
          JLPTR = JLPTR+1
          IF (JLPTR.GT.JLMAX)  GO TO 103
          JL(JLPTR) = I
          QM = I
```

```
C  ******  Inspect Ith row for fillin, adjust IM if possible  **********
            JMIN = IU(I)
            JMAX = IM(I)
            IF (JMIN.GT.JMAX)  GO TO 6
            DO 5 J=JMIN,JMAX
              VJ = JU(J)
              IF (VJ.EQ.K)  IM(I) = J
   4          M = QM
              QM = Q(M)
              IF (QM.LT.VJ)  GO TO 4
              IF (QM.EQ.VJ)  GO TO 5
                Q(M) = VJ
                Q(VJ) = QM
                QM = VJ
   5          CONTINUE
   6        GO TO 3
C
C  ******  Check for null pivot  ****************************************
   7      IF (I.NE.K)  GO TO 105
C  ******  Remaining elements of Q define structure of U(K, )  *********
   8      I = Q(I)
          IF (I.GT.N)  GO TO 9
            JUPTR = JUPTR+1
            IF (JUPTR.GT.JUMAX)  GO TO 106
            JU(JUPTR) = I
            GO TO 8
C  ******  Get ready for next row  **************************************
   9      IM(K) = JUPTR
          IL(K+1) = JLPTR+1
  10      IU(K+1) = JUPTR+1
C
        FLAG = 0
        RETURN
C
C ** ERROR:  Null Row in A
 101    FLAG = N + R(K)
        RETURN
C ** ERROR:  Duplicate Entry in A
 102    FLAG = 2*N + R(K)
        RETURN
C ** ERROR:  Insufficient Storage for JL
 103    FLAG = 3*N + K
        RETURN
C ** ERROR:  Null Pivot
 105    FLAG = 5*N + K
        RETURN
C ** ERROR:  Insufficient Storage for JU
 106    FLAG = 6*N + K
        RETURN
        END
```

```
C
C       -------------------------------------------------------------
C
C*** Subroutine NNF
C*** Numeric LDU-factorization of sparse nonsymmetric matrix and
C      solution of system of linear equations (uncompressed pointer
C      storage)
C
        SUBROUTINE  NNF
     *     (N, R,C,IC, IA,JA,A, Z, B, IL,JL,L,LMAX, D, IU,JU,U,UMAX,
     *      ROW, TMP, FLAG)
C
C       Input variables:   N, R,C,IC, IA,JA,A, B, IL,JL,LMAX, IU,JU,UMAX
C       Output variables:  Z, L,D,U, FLAG
C
C       Parameters used internally:
C fia   | ROW   - holds intermediate values in calculation of L, D, U.
C       |            Size = N.
C fia   | TMP   - holds new right-hand side b' for solution of the
C       |            equation  Ux = b'.
C       |            Size = N.
C
        INTEGER  R(1), C(1), IC(1),  IA(1), JA(1),
     *     IL(1), JL(1), LMAX,  IU(1), JU(1), UMAX,  FLAG
        REAL  A(1), Z(1), B(1),  L(1), D(1), U(1),  ROW(1), TMP(1),  LI
C
C  ******  Check storage  ***********************************************
        IF (IL(N+1)-1 .GT. LMAX)  GO TO 104
        IF (IU(N+1)-1 .GT. UMAX)  GO TO 107
C
C  ******  For each row  ************************************************
        DO 10 K=1,N
C  ******  Set the initial structure of ROW  ****************************
          JMIN = IL(K)
          JMAX = IL(K+1) - 1
          IF (JMIN.GT.JMAX)  GO TO 2
C  ******  If L(K,M) .NE. 0, ROW(M)=0  **********************************
          DO 1 J=JMIN,JMAX
   1        ROW(JL(J)) = 0
   2      ROW(K) = 0
          JMIN = IU(K)
          JMAX = IU(K+1) - 1
          IF (JMIN.GT.JMAX)  GO TO 4
C  ******  If U(K,M) .NE. 0, ROW(M)=0  **********************************
          DO 3 J=JMIN,JMAX
   3        ROW(JU(J)) = 0
   4      JMIN = IA(R(K))
          JMAX = IA(R(K)+1) - 1
C  ******  Set ROW to Kth row of reordered A  ***************************
          DO 5 J=JMIN,JMAX
   5        ROW(IC(JA(J))) = A(J)
C  ******  Initialize SUM  **********************************************
          SUM = B(R(K))
```

```
C
C  ******  Assign the Kth row of L and adjust ROW, SUM  ****************
        IMIN = IL(K)
        IMAX = IL(K+1) - 1
        IF (IMIN.GT.IMAX)  GO TO 8
        DO 7 I=IMIN,IMAX
          LI = - ROW(JL(I))
C  ******  If L is not required, then comment out the following line  **
          L(I) = - LI
          SUM = SUM + LI * TMP(JL(I))
          JMIN = IU(JL(I))
          JMAX = IU(JL(I)+1) - 1
          IF (JMIN.GT.JMAX)  GO TO 7
          DO 6 J=JMIN,JMAX
    6       ROW(JU(J)) = ROW(JU(J)) + LI * U(J)
    7     CONTINUE
C
C  ******  Assign diagonal D and Kth row of U, set TMP(K)  *************
    8   IF (ROW(K).EQ.0)  GO TO 108
        DK = 1 / ROW(K)
        D(K) = DK
        TMP(K) = SUM * DK
        JMIN = IU(K)
        JMAX = IU(K+1) - 1
        IF (JMIN.GT.JMAX)  GO TO 10
        DO 9 J=JMIN,JMAX
    9     U(J) = ROW(JU(J)) * DK
   10   CONTINUE
C
C  ******  Solve  Ux = TMP  by back substitution  **********************
        K = N
        DO 13 I=1,N
          SUM = TMP(K)
          JMIN = IU(K)
          JMAX = IU(K+1) - 1
          IF (JMIN.GT.JMAX)  GO TO 12
          DO 11 J=JMIN,JMAX
   11       SUM = SUM - U(J) * TMP(JU(J))
   12     TMP(K) = SUM
          Z(C(K)) = SUM
   13     K = K-1
C
        FLAG = 0
        RETURN
C
C ** ERROR:  Insufficient Storage for L
  104   FLAG = 4*N + 1
        RETURN
C ** ERROR:  Insufficient Storage for U
  107   FLAG = 7*N + 1
        RETURN
C ** ERROR:  Zero Pivot
  108   FLAG = 8*N + K
        RETURN
        END
```

```
C
C       -----------------------------------------------------------  --  ------
C
C*** Subroutine NNS
C*** Numeric solution of a sparse nonsymmetric system of linear
C      equations given LDU-factorization (uncompressed pointer storage)
C
        SUBROUTINE  NNS
     *     (N, R,C, IL,JL,L, D, IU,JU,U, Z, B, TMP)
C
C       Input variables:   N, R,C, IL,JL,L, D, IU,JU,U, B
C       Output variables:  Z
C
C       Parameters used internally:
C fia   | TMP   - holds new right-hand side b' for solution of the
C       |             equation Ux = b'.
C       |             Size = N.
C
        INTEGER  R(1), C(1),  IL(1), JL(1),  IU(1), JU(1)
        REAL  L(1), D(1), U(1),  Z(1), B(1),  TMP(1)
C
C  ******  Solve LDy = b  by forward substitution  *********************
        DO 2 K=1,N
          SUM = B(R(K))
          JMIN = IL(K)
          JMAX = IL(K+1) - 1
          IF (JMIN.GT.JMAX)  GO TO 2
          DO 1 J=JMIN,JMAX
   1        SUM = SUM - L(J) * TMP(JL(J))
   2      TMP(K) = SUM * D(K)
C
C  ******  Solve  Ux = y  by back substitution  ************************
        K = N
        DO 5 I=1,N
          SUM = TMP(K)
          JMIN = IU(K)
          JMAX = IU(K+1) - 1
          IF (JMIN.GT.JMAX)  GO TO 4
          DO 3 J=JMIN,JMAX
   3        SUM = SUM - U(J) * TMP(JU(J))
   4      TMP(K) = SUM
          Z(C(K)) = SUM
   5      K = K-1
        RETURN
        END
```

```
C                           Appendix 2                           7/31/77
C
C           Subroutines for Solving Sparse Nonsymmetric Systems
C            of Linear Equations (Track Nonzeroes Dynamically)
C
C
C*** Subroutine TDRV
C*** Driver for subroutine for solving sparse nonsymmetric systems of
C           linear equations (track nonzeroes dynamically)
C
            SUBROUTINE  TDRV
       *        (N, R,IC, IA,JA,A, B, Z, NSP,ISP,RSP,ESP, FLAG)
C
C     PARAMETERS
C     Class abbreviations are --
C        n - INTEGER variable
C        f - REAL variable
C        v - supplies a VALUE to the driver
C        r - returns a RESULT from the driver
C        i - used INTERNALly by the driver
C        a - ARRAY
C
C Class | Parameter
C ------+----------
C       |
C          The nonzero entries of the coefficient matrix M are stored
C     row-by-row in the array A.  To identify the individual nonzero
C     entries in each row, we need to know in which column each entry
C     lies.  The column indices which correspond to the nonzero entries
C     of M are stored in the array JA;  i.e., if  A(K) = M(I,J),  then
C     JA(K) = J.  In addition, we need to know where each row starts and
C     how long it is.  The index positions in JA and A where the rows of
C     M begin are stored in the array IA;  i.e., if M(I,J) is the first
C     nonzero entry (stored) in the I-th row and A(K) = M(I,J),  then
C     IA(I) = K.  Moreover, the index in JA and A of the first location
C     following the last element in the last row is stored in IA(N+1).
C     Thus, the number of entries in the I-th row is given by
C     IA(I+1) - IA(I),  the nonzero entries of the I-th row are stored
C     consecutively in
C             A(IA(I)),  A(IA(I)+1),  ..., A(IA(I+1)-1),
C     and the corresponding column indices are stored consecutively in
C             JA(IA(I)), JA(IA(I)+1), ..., JA(IA(I+1)-1).
C     For example, the 5 by 5 matrix
C                 ( 1. 0. 2. 0. 0.)
C                 ( 0. 3. 0. 0. 0.)
C             M = ( 0. 4. 5. 6. 0.)
C                 ( 0. 0. 0. 7. 0.)
C                 ( 0. 0. 0. 8. 9.)
C     would be stored as
C                | 1  2  3  4  5  6  7  8  9
C             ---+--------------------------
C             IA | 1  3  4  7  8 10
C             JA | 1  3  2  2  3  4  4  4  5
C              A | 1. 2. 3. 4. 5. 6. 7. 8. 9.
C
C nv    | N     - number of variables/equations.
C fva   | A     - nonzero entries of the coefficient matrix M, stored
C       |           by rows.
C       |           Size = number of nonzero entries in M.
C nva   | IA    - pointers to delimit the rows in A.
```

```
C          |             Size = N+1.
C nva      | JA     - column numbers corresponding to the elements of A.
C          |             Size = size of A.
C fva      | B      - right-hand side b;  B and Z can the same array.
C          |             Size = N.
C fra      | Z      - solution x;  B and Z can be the same array.
C          |             Size = N.
C
C          The rows and columns of the original matrix M can be
C     reordered (e.g., to reduce fillin or ensure numerical stability)
C     before calling the driver.  If no reordering is done, then set
C     R(I) = C(I) = IC(I) = I  for I=1,...,N.  The solution Z is returned
C     in the original order.
C
C nva      | R      - ordering of the rows of M.
C          |             Size = N.
C nva      | IC     - inverse of the ordering of the columns of M;  i.e.,
C          |             IC(C(I)) = I  for I=1,...,N,  where C is the
C          |             ordering of the columns of M.
C          |             Size = N.
C
C          Various errors are detected by the driver and the individual
C     subroutines.
C
C nr       | FLAG   - error flag;  values and their meanings are --
C          |               0     No Errors Detected
C          |              N+K    Null Row in  A  --  Row = K
C          |             2N+K    Duplicate Entry in A  --  Row = K
C          |             5N+K    Null Pivot  --  Row = K
C          |             8N+K    Zero Pivot  --  Row = K
C          |             10N+1   Insufficient Storage in TDRV
C          |             12N+K   Insufficient Storage in TRK
C
C          Working storage is needed for the factored form of the matrix
C     M plus various temporary vectors.  The arrays ISP and RSP should be
C     the same;  integer storage is allocated from the beginning of ISP
C     and real storage from the end of RSP.
C
C nv       | NSP    - declared dimension of ISP and RSP;  NSP generally must
C          |             be larger than  6N+2 + 2*K  (where  K = (number of
C          |             nonzero entries in the upper triangle of M)).
C nvira    | ISP    - integer working storage divided up into various arrays
C          |             needed by the subroutines;  ISP and RSP should be
C          |             the same array.
C          |             Size = NSP.
C fvira    | RSP    - real working storage divided up into various arrays
C          |             needed by the subroutines;  ISP and RSP should be
C          |             the same array.
C          |             Size = NSP.
C nr       | ESP    - if NSP is sufficiently large to allocate space, then
C          |             ESP is set to the amount of excess storage provided.
C
      INTEGER  R(1), IC(1),  IA(1), JA(1),  ISP(1), ESP,  FLAG,
     *   U, ROW, TMP, Q
      REAL  A(1),  B(1),  Z(1),  RSP(1)
```

```
C
C  ******  Initialize and divide up temporary storage  *****************
        IJU = 1
        IU  = IJU +  N
        Q   = IU  + N+1
        IM  = Q   + N+1
        JU  = IM  +  N
        U   = JU
        ROW = NSP -  N
        TMP = ROW -  N
        MAX = TMP - JU
        IF (MAX.LT.0)  GO TO 110
C
C  ******  Call zero-tracking subroutine  ******************************
        FLAG = 0
        CALL  TRK
     *     (N,  R, IC,  IA, JA, A,  Z,  B,
     *      ISP(IJU), ISP(JU), ISP(IU), RSP(U), MAX,
     *      ISP(Q),  ISP(IM),  RSP(ROW),  RSP(TMP),  FLAG,  ESP)
        IF (FLAG.NE.0)  GO TO 100
        RETURN
C
C ** ERROR:  Error Detected in TRK
 100    RETURN
C ** ERROR:  Insufficient Storage
 110    FLAG = 10*N + 1
        RETURN
        END
```

```
D
C     ----------------------------------------------------------------
C
C                YALE SPARSE MATRIX PACKAGE - ZERO-TRACKING CODE
C                     SOLVING THE SYSTEM OF EQUATIONS Mx = b
C
C   I.    SUBROUTINE NAMES
C         TRK performs an LDU-decomposition of the matrix M, without
C   storing L or D, and solves the linear system of equations.
C
C   II.   CALLING SEQUENCES
C         The coefficient matrix can be processed by an ordering routine
C   (e.g., to reduce fillin or ensure numerical stability) before using
C   the remaining subroutines.  If no reordering is done, then set
C   R(I) = C(I) = IC(I) = I  for I=1,...,N.  The calling sequence is --
C       (       (matrix ordering))
C         TRK   (solution of linear system of equations)
C   (If several systems with the same coefficient matrix but different
C   right-hand sides or several systems whose coefficient matrices have
C   the same nonzero structure are to be solved, and sufficient space
C   is available, other subroutines should be used.)
C
C   III. STORAGE OF SPARSE MATRICES
C         The nonzero entries of the coefficient matrix M are stored
C   row-by-row in the array A.  To identify the individual nonzero
C   entries in each row, we need to know in which column each entry
C   lies.  The column indices which correspond to the nonzero entries
C   of M are stored in the array JA;  i.e., if  A(K) = M(I,J),  then
C   JA(K) = J.  In addition, we need to know where each row starts and
C   how long it is.  The index positions in JA and A where the rows of
C   M begin are stored in the array IA;  i.e., if M(I,J) is the first
C   nonzero entry (stored) in the I-th row and A(K) = M(I,J),  then
C   IA(I) = K.  Moreover, the index in JA and A of the first location
C   following the last element in the last row is stored in IA(N+1).
C   Thus, the number of entries in the I-th row is given by
C   IA(I+1) - IA(I),  the nonzero entries of the I-th row are stored
C   consecutively in
C           A(IA(I)),  A(IA(I)+1),  ...,  A(IA(I+1)-1),
C   and the corresponding column indices are stored consecutively in
C           JA(IA(I)), JA(IA(I)+1), ..., JA(IA(I+1)-1).
C   For example, the 5 by 5 matrix
C               ( 1. 0. 2. 0. 0.)
C               ( 0. 3. 0. 0. 0.)
C           M = ( 0. 4. 5. 6. 0.)
C               ( 0. 0. 0. 7. 0.)
C               ( 0. 0. 0. 8. 9.)
C   would be stored as
C              | 1  2  3  4  5  6  7  8  9
C           ---+--------------------------
C           IA | 1  3  4  7  8 10
C           JA | 1  3  2  2  3  4  4  4  5
C            A | 1. 2. 3. 4. 5. 6. 7. 8. 9.
C         The strict upper triangular portion of the matrix U is stored
C   in a similar fashion using the arrays  IU, JU, U,  except that an
C   additional array IJU is used to compress storage of JU by allowing
C   some of the column indices to be used for more than one row.
C   IJU(K) points to the starting location in JU of entries for the Kth
C   row.  Compression in JU occurs in two ways.  First, if a row I was
C   merged into the current row K, and the number of elements merged in
C   from (the tail portion of) row I is the same as the final length of
```

```
C    row K, then the Kth row and the tail of row I are identical and
C    IJU(K) may point to the start of the tail.  Second, if some tail
C    portion of the (K-1)st row is identical to the head of the Kth row,
C    then IJU(K) may point to the start of that tail portion.  For
C    example, the nonzero structure of the strict upper triangular part
C    of the matrix
C            d 0 x x x
C            0 d 0 x x
C            0 0 d x 0
C            0 0 0 d x
C            0 0 0 0 d
C    would be represented as
C                | 1 2 3 4 5 6
C            ----+------------
C             IU | 1 4 6 7 8 8
C             JU | 3 4 5 4
C            IJU | 1 2 4 3
C
C    IV.  ADDITIONAL STORAGE SAVINGS
C         JU and U should be the same array.  TRK fills JU from the
C    beginning of the array and U from the end of the array.
C         R and IC can be the same array in the calling sequence if no
C    reordering of the coefficient matrix has been done.  Z and ROW can
C    be the same array.
C
C    V.   PARAMETERS
C         Following is a list of parameters to TRK.  Class abbreviations
C    are --
C       n - INTEGER variable
C       f - REAL variable
C       v - supplies a VALUE to a subroutine
C       r - returns a RESULT from a subroutine
C       i - used INTERNALly by a subroutine
C       a - ARRAY
C
C Class | Parameter
C ------+---------
C fva   | A     - nonzero entries of the coefficient matrix M, stored
C       |           by rows.
C       |           Size = number of nonzero entries in M.
C fva   | B     - right-hand side b.
C       |           Size = N.
C nr    | ESP   - if enough storage was provided for JU and U, then ESP
C       |           is set to amount of excess storage provided.
C nr    | FLAG  - error flag;  values and their meanings are --
C       |             0     No Errors Detected
C       |             N+K   Null Row in  A  --  Row = K
C       |            2N+K   Duplicate Entry in A  --  Row = K
C       |            5N+K   Null Pivot  --  Row = K
C       |            8N+K   Zero Pivot  --  Row = K
C       |           12N+K   Insufficient Storage for JU/U  --  Row = K
C nva   | IA    - pointers to delimit the rows in A.
C       |           Size = N+1.
C nva   | IC    - inverse of the ordering of the columns of M;  i.e.,
C       |           IC(C(I)) = I  for I=1,...N,  where C is the
```

```
C        |            ordering of the columns of M.
C        |            Size = N.
C nia    | IJU    - pointers to the first element in each row in JU, used
C        |            to compress storage in JU.
C        |            Size = N.
C nia    | IU     - pointers to delimit the rows in U.
C        |            Size = N+1.
C nva    | JA     - column numbers corresponding to the elements of A.
C        |            Size = size of  A.
C nia    | JU     - column numbers corresponding to the elements of U;
C        |            JU and U should be the same array.
C        |            Size = MAX.
C nv     | MAX    - declared dimension of JU and U;  MAX must be larger
C        |            than the size of U  (the number of nonzero entries
C        |            in the strict upper triangle of M plus fillin)  plus
C        |            the size of JU  (the size of U minus compression).
C nv     | N      - number of variables/equations.
C nva    | R      - ordering of the rows of M.
C        |            Size = N.
C fia    | U      - nonzero entries in the strict upper triangular portion
C        |            of U, stored by rows;  JU and U should be the same
C        |            array.
C        |            Size = MAX.
C fra    | Z      - solution x.
C        |            Size = N.
C
C        ------------------------------------------------------------------
C
C*** Subroutine TRK
C*** Numerical solution of sparse nonsymmetric system of linear
C      equations (track zeroes dynamically)
C
        SUBROUTINE  TRK
     *    (N, R,IC, IA,JA,A, Z, B, IJU,JU,IU,U,MAX,
     *     Q, IM, ROW, TMP, FLAG, ESP)
C
C       Input variables:   N, R,IC, IA,JA,A, B, MAX
C       Output variables:  Z, FLAG
C
C       Parameters used internally:
C nia    | Q      - suppose M' is the result of reordering M;  if
C        |            processing of the Kth row of M' (hence the Kth rows
C        |            of L and U) is being done, then Q(J) is initially
C        |            nonzero if M'(K,J) is nonzero;  since values need
C        |            not be stored, each entry points to the next
C        |            nonzero;  for example, if  N=9  and the 5th row of
C        |            M' is
C        |                    0 x x 0 x 0 0 x 0,
C        |            then Q will initially be
C        |                    a 3 5 a 8 a a 10 a 2       (a - arbitrary);
C        |            Q(N+1) points to the first nonzero in the row and
C        |            the last nonzero points to  N+1;  as the algorithm
C        |            proceeds, other elements of Q are inserted in the
C        |            list because of fillin.
C        |            Size = N+1.
C nia    | IM   - at each step in the factorization, IM(I) is the last
C        |          element of the Ith row of U which needs to be
C        |          considered in computing fillin.
C        |          Size = N.
```

```
C fia   | ROW - holds intermediate values in calculation of U.
C       |         Size = N.
C fia   | TMP - holds new right-hand side b' for solution of the
C       |         equation  Ux = b'.
C       |         Size = N.
C
      INTEGER  R(1), IC(1),  IA(1), JA(1),
     *   IJU(1), JU(1), IU(1),  Q(1),  IM(1),  FLAG,  ESP,  VJ, QM
      REAL  A(1), Z(1), B(1),  U(1),  ROW(1), TMP(1)
C
C  ******  Initialize  ****************************************************
      JUMIN = 1
      JUMAX = 0
      IU(1) = MAX
C
C  ******  For each row  **************************************************
      DO 20 K=1,N
C  ******  Initialize Q and ROW to the Kth row of reordered A  *********
        LUK = 0
        Q(N+1) = N+1
        JMIN = IA(R(K))
        JMAX = IA(R(K)+1) - 1
        IF (JMIN.GT.JMAX)  GO TO 101
        DO 2 J=JMIN,JMAX
          VJ = IC(JA(J))
          QM = N+1
    1     M = QM
          QM = Q(M)
          IF (QM.LT.VJ)  GO TO 1
          IF (QM.EQ.VJ)  GO TO 102
            LUK = LUK+1
            Q(M) = VJ
            Q(VJ) = QM
            ROW(VJ) = A(J)
    2     CONTINUE
C
C  ******  Link through Q  ************************************************
        LMAX = 0
        IJU(K) = JUMAX
        I = N+1
    3   I = Q(I)
        LUK = LUK-1
        IF (I.GE.K)  GO TO 8
          QM = I
          JMIN = IJU(I)
          JMAX = IM(I)
          LUI = 0
          IF (JMIN.GT.JMAX)  GO TO 7
C  ******  and find nonzero structure of Kth row of L and U  ***********
            DO 5 J=JMIN,JMAX
              VJ = JU(J)
              IF (VJ.GT.K)  LUI = LUI+1
    4         M = QM
              QM = Q(M)
              IF (QM.LT.VJ)  GO TO 4
              IF (QM.EQ.VJ)  GO TO 5
                LUK = LUK+1
                Q(M) = VJ
                Q(VJ) = QM
                ROW(VJ) = 0
                QM = VJ
    5         CONTINUE
```

```
C  ******  Adjust IJU and IM  *******************************************
              JTMP = JMAX - LUI
              IF (LUI.LE.LMAX)  GO TO 6
                LMAX = LUI
                IJU(K) = JTMP+1
   6          IF (JTMP.LT.JMIN)  GO TO 7
                IF (JU(JTMP).EQ.K)  IM(I) = JTMP
   7        GO TO 3
C
C  ******  See if JU storage can be compressed  ************************
   8      IF (I.NE.K)  GO TO 105
          IF (LUK.EQ.LMAX)  GO TO 14
            IF (JUMIN.GT.JUMAX)  GO TO 12
              I = Q(K)
              DO 9 JMIN=JUMIN,JUMAX
                IF (JU(JMIN)-I)  9, 10, 12
   9            CONTINUE
              GO TO 12
  10          IJU(K) = JMIN
              DO 11 J=JMIN,JUMAX
                IF (JU(J).NE.I)  GO TO 12
                I = Q(I)
                IF (I.GT.N)  GO TO 14
  11            CONTINUE
              JUMAX = JMIN - 1
C  ******  Store pointers in JU  ****************************************
  12        JUMIN = JUMAX + 1
            JUMAX = JUMAX + LUK
            IF (JUMAX.GT.IU(K))  GO TO 112
            I = K
            DO 13 J=JUMIN,JUMAX
              I = Q(I)
  13          JU(J) = I
            IJU(K) = JUMIN
  14      IU(K+1) = IU(K) - LUK
          IF (JUMAX.GT.IU(K+1))  GO TO 112
          IM(K) = IJU(K) + LUK - 1
C
C  ******  Calculate numerical values for Kth row  *********************
          SUM = B(R(K))
          I = N+1
  15      I = Q(I)
          IF (I.GE.K)  GO TO 18
            AKI = - ROW(I)
            SUM = SUM + AKI * TMP(I)
            JMIN = IU(I+1) + 1
            JMAX = IU(I)
            IF (JMIN.GT.JMAX)  GO TO 17
            MU = IJU(I) - JMIN
            DO 16 J=JMIN,JMAX
  16          ROW(JU(MU+J)) = ROW(JU(MU+J)) + AKI * U(J)
  17        GO TO 15
```

```
C  ******  Store values in TMP and U  ************************************
  18       IF (ROW(K).EQ.0)  GO TO 108
           DK = 1 / ROW(K)
           TMP(K) = SUM * DK
           JMIN = IU(K+1) + 1
           JMAX = IU(K)
           IF (JMIN.GT.JMAX)  GO TO 20
           MU = IJU(K) - JMIN
           DO 19 J=JMIN,JMAX
  19         U(J) = ROW(JU(MU+J)) * DK
  20       CONTINUE
         ESP = IU(N+1) - JUMAX
C
C  ******  Solve  Ux = TMP  by back substitution  ***********************
         K = N
         DO 23 I=1,N
           SUM = TMP(K)
           JMIN = IU(K+1) + 1
           JMAX = IU(K)
           IF (JMIN.GT.JMAX)  GO TO 22
           MU = IJU(K) - JMIN
           DO 21 J=JMIN,JMAX
  21         SUM = SUM - U(J) * TMP(JU(MU+J))
  22       TMP(K) = SUM
  23       K = K-1
         DO 24 K=1,N
  24       Z(K) = TMP(IC(K))
C
         FLAG = 0
         RETURN
C
C ** ERROR:  Null Row in A
 101     FLAG = N + R(K)
         RETURN
C ** ERROR:  Duplicate Entry in A
 102     FLAG = 2*N + R(K)
         RETURN
C ** ERROR:  Null Pivot
 105     FLAG = 5*N + K
         RETURN
C ** ERROR:  Zero Pivot
 108     FLAG = 8*N + K
         RETURN
C ** ERROR:  Insufficient Storage for JU and U
 112     FLAG = 12*N + K
         RETURN
         END
```

```
C                          Appendix 3                          7/31/77
C
C         Subroutines for Solving Sparse Nonsymmetric Systems
C          of Linear Equations  (Compressed Pointer Storage)
C
C
C*** Subroutine CDRV
C*** Driver for subroutines for solving sparse nonsymmetric systems of
C       linear equations (compressed pointer storage)
C
        SUBROUTINE  CDRV
     *     (N, R,C,IC, IA,JA,A, B, Z, NSP,ISP,RSP,ESP, PATH, FLAG)
C
C    PARAMETERS
C    Class abbreviations are--
C       n - INTEGER variable
C       f - REAL variable
C       v - supplies a VALUE to the driver
C       r - returns a RESULT from the driver
C       i - used INTERNALly by the driver
C       a - ARRAY
C
C Class | Parameter
C ------+----------
C       |
C         The nonzero entries of the coefficient matrix M are stored
C    row-by-row in the array A.  To identify the individual nonzero
C    entries in each row, we need to know in which column each entry
C    lies.  The column indices which correspond to the nonzero entries
C    of M are stored in the array JA;  i.e., if  A(K) = M(I,J),  then
C    JA(K) = J.  In addition, we need to know where each row starts and
C    how long it is.  The index positions in JA and A where the rows of
C    M begin are stored in the array IA;  i.e., if M(I,J) is the first
C    nonzero entry (stored) in the I-th row and A(K) = M(I,J),  then
C    IA(I) = K.  Moreover, the index in JA and A of the first location
C    following the last element in the last row is stored in IA(N+1).
C    Thus, the number of entries in the I-th row is given by
C    IA(I+1) - IA(I),  the nonzero entries of the I-th row are stored
C    consecutively in
C            A(IA(I)),  A(IA(I)+1),  ..., A(IA(I+1)-1),
C    and the corresponding column indices are stored consecutively in
C            JA(IA(I)), JA(IA(I)+1), ..., JA(IA(I+1)-1).
C    For example, the 5 by 5 matrix
C                ( 1. 0. 2. 0. 0.)
C                ( 0. 3. 0. 0. 0.)
C            M = ( 0. 4. 5. 6. 0.)
C                ( 0. 0. 0. 7. 0.)
C                ( 0. 0. 0. 8. 9.)
C    would be stored as
C               | 1  2  3  4  5  6  7  8  9
C            ---+--------------------------
C            IA | 1  3  4  7  8 10
C            JA | 1  3  2  2  3  4  4  4  5
C             A | 1. 2. 3. 4. 5. 6. 7. 8. 9.         .
C
C nv    | N     - number of variables/equations.
C fva   | A     - nonzero entries of the coefficient matrix M, stored
C       |           by rows.
C       |           Size = number of nonzero entries in M.
C nva   | IA    - pointers to delimit the rows in A.
```

```
C        |            Size = N+1.
C nva    | JA    - column numbers corresponding to the elements of A.
C        |            Size = size of A.
C fva    | B     - right-hand side b;  B and Z can the same array.
C        |            Size = N.
C fra    | Z     - solution x;  B and Z can be the same array.
C        |            Size = N.
C
C        The rows and columns of the original matrix M can be
C    reordered (e.g., to reduce fillin or ensure numerical stability)
C    before calling the driver.  If no reordering is done, then set
C    R(I) = C(I) = IC(I) = I  for I=1,...,N.  The solution Z is returned
C    in the original order.
C        If the columns have been reordered (i.e.,  C(I).NE.I  for some
C    I), then the driver will call a subroutine (NROC) which rearranges
C    each row of JA and A, leaving the rows in the original order, but
C    placing the elements of each row in increasing order with respect
C    to the new ordering.  If  PATH.NE.1,  then NROC is assumed to have
C    been called already.
C
C nva    | R     - ordering of the rows of M.
C        |            Size = N.
C nva    | C     - ordering of the columns of M.
C        |            Size = N.
C nva    | IC    - inverse of the ordering of the columns of M;  i.e.,
C        |            IC(C(I)) = I  for I=1,...,N.
C        |            Size = N.
C
C        The solution of the system of linear equations is divided into
C    three stages --
C      NSFC -- The matrix M is processed symbolically to determine where
C                fillin will occur during the numeric factorization.
C      NNFC -- The matrix M is factored numerically into the product LDU
C                of a unit lower triangular matrix L, a diagonal matrix
C                D, and a unit upper triangular matrix U, and the system
C                Mx = b  is solved.
C      NNSC -- The linear system  Mx = b  is solved using the LDU
C                factorization from NNFC.
C    For several systems whose coefficient matrices have the same
C    nonzero structure, NSFC need be done only once (for the first
C    system);  then NNFC is done once for each additional system.  For
C    several systems with the same coefficient matrix, NSFC and NNFC
C    need be done only once (for the first system);  then NNSC is done
C    once for each additional right-hand side.
C
C nv     | PATH  - path specification;  values and their meanings are --
C        |           1  perform NROC, NSFC, and NNFC.
C        |           2  perform NNFC only  (NSFC is assumed to have been
C        |                done in a manner compatible with the storage
C        |                allocation used in the driver).
C        |           3  perform NNSC only  (NSFC and NNFC are assumed to
C        |                have been done in a manner compatible with the
C        |                storage allocation used in the driver).
```

```
C
C        Various errors are detected by the driver and the individual
C     subroutines.
C
C nr     | FLAG  - error flag;  values and their meanings are --
C        |              0     No Errors Detected
C        |              N+K   Null Row in A  --  Row = K
C        |             2N+K   Duplicate Entry in A  --  Row = K
C        |             3N+K   Insufficient Storage in NSFC  --  Row = K
C        |             4N+1   Insufficient Storage in NNFC
C        |             5N+K   Null Pivot  --  Row = K
C        |             6N+K   Insufficient Storage in NSFC  --  Row = K
C        |             7N+1   Insufficient Storage in NNFC
C        |             8N+K   Zero Pivot  --  Row = K
C        |            10N+1   Insufficient Storage in CDRV
C        |            11N+1   Illegal PATH Specification
C
C        Working storage is needed for the factored form of the matrix
C     M plus various temporary vectors.  The arrays ISP and RSP should be
C     the same;  integer storage is allocated from the beginning of ISP
C     and real storage from the end of RSP.
C
C nv     | NSP   - declared dimension of ISP and RSP;  NSP generally must
C        |           be larger than  8N+2 + 2K  (where  K = (number of
C        |           nonzero entries in M)).
C nvira  | ISP   - integer working storage divided up into various arrays
C        |           needed by the subroutines;  ISP and RSP should be
C        |           the same array.
C        |           Size = NSP.
C fvira  | RSP   - real working storage divided up into various arrays
C        |           needed by the subroutines;  ISP and RSP should be
C        |           the same array.
C        |           Size = NSP.
C nr     | ESP   - if sufficient storage was available to perform the
C        |           symbolic factorization (NSFC), then ESP is set to
C        |           the amount of excess storage provided (negative if
C        |           insufficient storage was available to perform the
C        |           numeric factorization (NNFC)).
C
      INTEGER  R(1), C(1), IC(1), IA(1), JA(1), ISP(1), ESP, PATH,
     *   FLAG, TMP, D, Q, U, RMN, ADD, UMAX
      REAL  A(1), B(1), Z(1), RSP(1)
C
      IF(PATH.LE.0 .OR. PATH.GT.3)    GO TO 111
C******  Initialize and divide up temporary storage  *******************
      FLAG =       0
      IL   =       1
      IJL  = IL  + N + 1
      IU   = IJL + N
      IJU  = IU  + N + 1
      IRL  = IJU + N
      JRL  = IRL + N
      JL   = JRL + N
      IRA  = NSP + 1 - N
      D    = IRA
      JRA  = D   - N
      TMP  = JRA
      Q    = TMP -(N + 1)
      JRU  = Q   - N
      IRU  = JRU - N
      IF(JL .GE. IRU)  GO TO 110
      IF(PATH .GT. 1)  GO TO 10
```

```
C
C******  Reorder A if necessary, call NSFC if flag is set  *************
        RMN = IRU - JL
        ADD = RMN/2
        JU = JL + ADD
        JLMAX = ADD
        JUMAX = RMN - ADD
        DO 5 II=1,N
          IF(C(II) .NE. II)  GO TO 6
   5      CONTINUE
        GO TO 7
C
   6    CALL  NROC (N, IC, IA, JA, A, ISP(IL), RSP(Q), ISP(IU), FLAG)
        IF(FLAG .NE. 0)  GO TO 100
C
   7    CALL  NSFC
     *       (N, R, IC, IA,JA, JLMAX,ISP(IL),ISP(JL),ISP(IJL), JUMAX,
     *        ISP(IU),ISP(JU),ISP(IJU), RSP(Q), RSP(IRA), RSP(JRA), Z,
     *        ISP(IRL),ISP(JRL), RSP(IRU),RSP(JRU), FLAG)
        IF(FLAG .NE. 0)  GO TO 100
C******  See if enough space remains, move JU next to JL  **************
  10    JLMAX = ISP(IJL+N-1)
        JUMAX = ISP(IJU+N-1)
        LMAX = ISP(IL+N) - 1
        UMAX = ISP(IU+N) - 1
        IF(PATH .GT. 1) GO TO 20
        NEED = JLMAX + JUMAX + LMAX + UMAX
        RMN = RMN + 3*N + 1
        ESP = RMN - NEED
        IF(NEED .GT. RMN)  GO TO 110
        JUOLD = JU - 1
        JU = JL + JLMAX - 1
        IF (JUMAX.LE.0)  GO TO 20
        DO 15 II=1,JUMAX
  15      ISP(JU+II) = ISP(JUOLD+II)
C******  Call remaining subroutines  ***********************************
  20    JU = JL + JLMAX
        L = JU + JUMAX
        U = L + LMAX
C
        IF(PATH .EQ. 3)  GO TO 30
        CALL NNFC
     *      (N, R, C, IC, IA,JA,A, LMAX,ISP(IL),ISP(JL),ISP(IJL),RSP(L),
     *       RSP(D), UMAX,ISP(IU),ISP(JU),ISP(IJU),RSP(U), Z, B, Z,
     *       RSP(TMP), ISP(IRL),ISP(JRL), FLAG)
        IF(FLAG .NE. 0)  GO TO 100
        RETURN
C
  30    CALL NNSC
     *      (N, R, C, ISP(IL),ISP(JL),ISP(IJL),RSP(L), RSP(D), ISP(IU),
     *       ISP(JU),ISP(IJU),RSP(U), Z, B, RSP(TMP))
        RETURN
C
C ** ERROR:  Error Detected in NROC, NSFC, NNFC, or NNSC
 100    RETURN
C ** ERROR:  Insufficient Storage
 110    FLAG = 10*N + 1
        RETURN
C ** ERROR:  Illegal PATH Specification
 111    FLAG = 11*N + 1
        RETURN
        END
```

```
C
C      -----------------------------------------------------------------
C
C               YALE SPARSE MATRIX PACKAGE - NONSYMMETRIC CODES
C                   SOLVING THE SYSTEM OF EQUATIONS Mx = b
C
C   I.    SUBROUTINE NAMES
C         Subroutine names and functions are --
C   (1)   NROC for reordering;
C   (2)   NSFC for symbolic factorization;
C   (3)   NNFC for numeric factorization and solution;
C   (4)   NNSC for solution.
C
C   II.   CALLING SEQUENCES
C         The coefficient matrix can be processed by an ordering routine
C   (e.g., to reduce fillin or ensure numerical stability) before using
C   the remaining subroutines.  If no reordering is done, then set
C   R(I) = C(I) = IC(I) = I  for I=1,...,N.  If an ordering subroutine
C   is used, then NROC should be used to reorder the coefficient matrix
C   The calling sequence is --
C       (        (matrix ordering))
C       (NROC    (matrix reordering))
C        NSFC    (symbolic factorization to determine where fillin will
C                   occur during numeric factorization)
C        NNFC    (numeric factorization into product LDU of unit lower
C                   triangular matrix L, diagonal matrix D, and unit
C                   upper triangular matrix U, and solution of linear
C                   system)
C        NNSC    (solution of linear system for additional right-hand
C                   side using LDU factorization from NNFC)
C   (If only one system of equations is to be solved, then the
C   subroutine TRK should be used.)
C
C   III. STORAGE OF SPARSE MATRICES
C         The nonzero entries of the coefficient matrix M are stored
C   row-by-row in the array A.  To identify the individual nonzero
C   entries in each row, we need to know in which column each entry
C   lies.  The column indices which correspond to the nonzero entries
C   of M are stored in the array JA;  i.e., if  A(K) = M(I,J),  then
C   JA(K) = J.  In addition, we need to know where each row starts and
C   how long it is.  The index positions in JA and A where the rows of
C   M begin are stored in the array IA;  i.e., if M(I,J) is the first
C   (leftmost) entry in the I-th row and  A(K) = M(I,J),  then
C   IA(I) = K.  Moreover, the index in JA and A of the first location
C   following the last element in the last row is stored in IA(N+1).
C   Thus, the number of entries in the I-th row is given by
C   IA(I+1) - IA(I),  the nonzero entries of the I-th row are stored
C   consecutively in
C           A(IA(I)),  A(IA(I)+1),  ..., A(IA(I+1)-1),
C   and the corresponding column indices are stored consecutively in
C           JA(IA(I)), JA(IA(I)+1), ..., JA(IA(I+1)-1).
C   For example, the 5 by 5 matrix
C               ( 1. 0. 2. 0. 0.)
C               ( 0. 3. 0. 0. 0.)
C           M = ( 0. 4. 5. 6. 0.)
C               ( 0. 0. 0. 7. 0.)
C               ( 0. 0. 0. 8. 9.)
C   would be stored as
C              | 1  2  3  4  5  6  7  8  9
C           ---+--------------------------
C           IA | 1  3  4  7  8 10
C           JA | 1  3  2  2  3  4  4  4  5
C            A | 1. 2. 3. 4. 5. 6. 7. 8. 9.
```

```
C        The strict upper (lower) triangular portion of the matrix
C     U (L) is stored in a similar fashion using the arrays  IU, JU, U
C     (IL, JL, L)  except that an additional array IJU (IJL) is used to
C     compress storage of JU (JL) by allowing some of the column (row)
C     indices to used for more than one row (column)  (n.b., L is stored
C     by columns).  IJU(K) (IJL(K)) points to the starting location in
C     JU (JL) of entries for the Kth row (column).  Compression in JU
C     (JL) occurs in two ways.  First, if a row (column) I was merged
C     into the current row (column) K, and the number of elements merged
C     in from (the tail portion of) row (column) I is the same as the
C     final length of row (column) K, then the Kth row (column) and the
C     tail of row (column) I are identical and IJU(K) (IJL(K)) may point
C     to the start of the tail.  Second, if some tail portion of the
C     (K-1)st row (column) is identical to the head of the Kth row
C     (column), then IJU(K) (IJL(K)) may point to the start of that tail
C     portion.  For example, the nonzero structure of the strict upper
C     triangular part of the matrix
C            d 0 x x x
C            0 d 0 x x
C            0 0 d x 0
C            0 0 0 d x
C            0 0 0 0 d
C     would be represented as
C                 | 1 2 3 4 5 6
C            ----+------------
C             IU | 1 4 6 7 8 8
C             JU | 3 4 5 4
C            IJU | 1 2 4 3
C     The diagonal entries of L and U are assumed to be equal to one and
C     are not stored.  The array D contains the reciprocals of the
C     diagonal entries of the matrix D.
C
C     IV.  ADDITIONAL STORAGE SAVINGS
C          In NSFC, R and IC can be the same array in the calling
C     sequence if no reordering of the coefficient matrix has been done.
C          In NNFC, Z and ROW can be the same array.  R, C and IC can all
C     be the same array if no reordering has been done.  If only the
C     rows have been reordered, then C and IC can be the same array.
C     If the row and column orderings are the same, then R and C can be
C     the same array.
C          In NNSC, R and C can be the same array if no reordering has
C     been done or if the row and column orderings are the same.  Z and B
C     can be the same array;  however, then B will be destroyed.
C
C     V.   PARAMETERS
C          Following is a list of parameters to the programs.  Names are
C     uniform among the various subroutines.  Class abbreviations are --
C        n - INTEGER variable
C        f - REAL variable
C        v - supplies a VALUE to a subroutine
C        r - returns a RESULT from a subroutine
C        i - used INTERNALly by a subroutine
C        a - ARRAY
C
C Class | Parameter
C ------+----------
C fva   | A      - nonzero entries of the coefficient matrix M, stored
C       |            by rows.
C       |            Size = number of nonzero entries in M.
```

```
C fva   | B     - right-hand side b.
C       |           Size = N.
C nva   | C     - ordering of the columns of M.
C       |           Size = N.
C fvra  | D     - reciprocals of the diagonal entries of the matrix D.
C       |           Size = N.
C nr    | FLAG  - error flag;  values and their meanings are --
C       |            0     No Errors Detected
C       |            N+K   Null Row in A  --  Row = K
C       |           2N+K   Duplicate Entry in A  --  Row = K
C       |           3N+K   Insufficient Storage for JL  --  Row = K
C       |           4N+1   Insufficient Storage for L
C       |           5N+K   Null Pivot  --  Row = K
C       |           6N+K   Insufficient Storage for JU  --  Row = K
C       |           7N+1   Insufficient Storage for U
C       |           8N+K   Zero Pivot  --  Row = K
C nva   | IA    - pointers to delimit the rows of A.
C       |           Size = N+1.
C nvra  | IJL   - pointers to the first element in each column in JL,
C       |           used to compress storage in JL.
C       |           Size = N.
C nvra  | IJU   - pointers to the first element in each row in JU, used
C       |           to compress storage in JU.
C       |           Size = N.
C nvra  | IL    - pointers to delimit the columns of L.
C       |           Size = N+1.
C nvra  | IU    - pointers to delimit the rows of U.
C       |           Size = N+1.
C nva   | JA    - column numbers corresponding to the elements of A.
C       |           Size = size of A.
C nvra  | JL    - row numbers corresponding to the elements of L.
C       |           Size = JLMAX.
C nv    | JLMAX - declared dimension of JL;  JLMAX must be larger than
C       |           the number of nonzeros in the strict lower triangle
C       |           of M plus fillin minus compression.
C nvra  | JU    - column numbers corresponding to the elements of U.
C       |           Size = JUMAX.
C nv    | JUMAX - declared dimension of JU;  JUMAX must be larger than
C       |           the number of nonzeros in the strict upper triangle
C       |           of M plus fillin minus compression.
C fvra  | L     - nonzero entries in the strict lower triangular portion
C       |           of the matrix L, stored by columns.
C       |           Size = LMAX.
C nv    | LMAX  - declared dimension of L;  LMAX must be larger than
C       |           the number of nonzeros in the strict lower triangle
C       |           of M plus fillin  (IL(N+1)-1 after NSFC).
C nv    | N     - number of variables/equations.
C nva   | R     - ordering of the rows of M.
C       |           Size = N.
C fvra  | U     - nonzero entries in the strict upper triangular portion
C       |           of the matrix U, stored by rows.
C       |           Size = UMAX.
C nv    | UMAX  - declared dimension of U;  UMAX must be larger than
C       |           the number of nonzeros in the strict upper triangle
C       |           of M plus fillin  (IU(N+1)-1 after NSFC).
C fra   | Z     - solution x.
C       |           Size = N.
C
C        -----------------------------------------------------------------
```

```
C
C*** Subroutine NROC
C*** Reorders rows of A, leaving row order unchanged
C
      SUBROUTINE NROC (N, IC, IA, JA, A, JAR, AR, P, FLAG)
C
C       Input parameters: N, IC, IA, JA, A
C       Output parameters: JA, A, FLAG
C
C       Parameters used internally:
C nia   | P     - at the Kth step, P is a linked list of the reordered
C       |           column indices of the Kth row of A;  P(N+1) points
C       |           to the first entry in the list.
C       |           Size = N+1.
C nia   | JAR   - at the Kth step,JAR contains the elements of the
C       |           reordered column indices of A.
C       |           Size = N.
C fia   | AR    - at the Kth step, AR contains the elements of the
C       |           reordered row of A.
C       |           Size = N.
C
      INTEGER  IC(1), IA(1), JA(1), JAR(1), P(1), FLAG
      REAL  A(1), AR(1)
C
C  ******  For each nonempty row  *******************************
      DO 5 K=1,N
        JMIN = IA(K)
        JMAX = IA(K+1) - 1
        IF(JMIN .GT. JMAX) GO TO 5
        P(N+1) = N + 1
C  ******  Insert each element in the list  *********************
        DO 3 J=JMIN,JMAX
          NEWJ = IC(JA(J))
          I = N + 1
   1      IF(P(I) .GE. NEWJ) GO TO 2
            I = P(I)
            GO TO 1
   2      IF(P(I) .EQ. NEWJ) GO TO 102
          P(NEWJ) = P(I)
          P(I) = NEWJ
          JAR(NEWJ) = JA(J)
          AR(NEWJ) = A(J)
   3      CONTINUE
C  ******  Replace old row in JA and A  *************************
        I = N + 1
        DO 4 J=JMIN,JMAX
          I = P(I)
          JA(J) = JAR(I)
   4      A(J) = AR(I)
   5    CONTINUE
      FLAG = 0
      RETURN
C
C ** ERROR:  Duplicate entry in A
 102  FLAG = N + K
      RETURN
      END
C
```

```
C       ------------------------------------------------------------
C
C*** Subroutine NSFC
C*** Symbolic LDU-factorization of nonsymmetric sparse matrix
C      (compressed pointer storage)
C
        SUBROUTINE  NSFC
     *      (N, R, IC, IA,JA, JLMAX,IL,JL,IJL, JUMAX,IU,JU,IJU, Q, IRA,
     *       JRA, IRAC, IRL,JRL, IRU,JRU, FLAG)
C
C       Input variables:  N, R, IC, IA, JA, JLMAX, JUMAX.
C       Output variables: IL, JL, IJL, IU, JU, IJU, FLAG.
C
C       Parameters used internally:
C nia   | Q     - Suppose  M'  is the result of reordering  M.  If
C       |           processing of the Ith row of  M'  (hence the Ith
C       |           row of  U) is being done,  Q(J)  is initially
C       |           nonzero if  M'(I,J) is nonzero (J.GE.I).  Since
C       |           values need not be stored, each entry points to the
C       |           next nonzero and  Q(N+1)  points to the first.  N+1
C       |           indicates the end of the list.  For example, if N=9
C       |           and the 5th row of  M'  is
C       |              0 x x 0 x 0 0 x 0
C       |           then  Q  will initially be
C       |              a a a a 8 a a 10 5          (a - arbitrary).
C       |           As the algorithm proceeds, other elements of  Q
C       |           are inserted in the list because of fillin.
C       |           Q  is used in an analogous manner to compute the
C       |           Ith column of  L.
C       |           Size = N+1.
C nia   | IRA,  - vectors used to find the columns of  M.  At the Kth
C nia   | JRA,      step of the factorization,  IRAC(K)  points to the
C nia   | IRAC      head of a linked list in  JRA  of row indices I
C       |           such that I .GE. K and  M(I,K)  is nonzero.  Zero
C       |           indicates the end of the list.  IRA(I)  (I.GE.K)
C       |           points to the smallest J such that J .GE. K and
C       |           M(I,J)  is nonzero.
C       |           Size of each = N.
C nia   | IRL,  - vectors used to find the rows of  L.  At the Kth step
C nia   | JRL       of the factorization,  JRL(K)  points to the head
C       |           of a linked list in  JRL  of column indices J
C       |           such J .LT. K and  L(K,J)  is nonzero.  Zero
C       |           indicates the end of the list.  IRL(J)  (J.LT.K)
C       |           points to the smallest I such that I .GE. K and
C       |           L(I,J)  is nonzero.
C       |           Size of each = N.
C nia   | IRU,  - vectors used in a manner analogous to  IRL and JRL
C nia   | JRU       to find the columns of  U.
C       |           Size of each = N.
C
C  Internal variables:
C    JLPTR - points to the last position used in  JL.
C    JUPTR - points to the last position used in  JU.
C    JMIN,JMAX - are the indices in  A or U  of the first and last
C                elements to be examined in a given row.
C                For example,  JMIN=IA(K), JMAX=IA(K+1)-1.
C
        INTEGER CEND, QM, REND, RK, VJ
        INTEGER IA(1), JA(1), IRA(1), JRA(1), IL(1), JL(1), IJL(1)
        INTEGER IU(1), JU(1), IJU(1), IRL(1), JRL(1), IRU(1), JRU(1)
        INTEGER R(1), IC(1), Q(1), IRAC(1), FLAG
```

```
C
C  ******  Initialize pointers  ******************************************
        NP1 = N + 1
        JLMIN = 1
        JLPTR = 0
        IL(1) = 1
        JUMIN = 1
        JUPTR = 0
        IU(1) = 1
        DO 1 K=1,N
          IRAC(K) = 0
          JRA(K) = 0
          JRL(K) = 0
   1      JRU(K) = 0
C  ******  Initialize column pointers for A  ****************************
        DO 2 K=1,N
          RK = R(K)
          IAK = IA(RK)
          IF (IAK .GE. IA(RK+1))  GO TO 101
          JAIAK = IC(JA(IAK))
          IF (JAIAK .GT. K)  GO TO 105
          JRA(K) = IRAC(JAIAK)
          IRAC(JAIAK) = K
   2      IRA(K) = IAK
C
C  ******  For each column of L and row of U  ***************************
        DO 41 K=1,N
C
C  ******  Initialize Q for computing Kth column of L  *****************
          Q(NP1) = NP1
          LUK = -1
C  ******  by filling in Kth column of A  ******************************
          VJ = IRAC(K)
          IF (VJ .EQ. 0)  GO TO 5
   3        QM = NP1
   4        M = QM
            QM =  Q(M)
            IF (QM .LT. VJ)  GO TO 4
            IF (QM .EQ. VJ)  GO TO 102
              LUK = LUK + 1
              Q(M) = VJ
              Q(VJ) = QM
              VJ = JRA(VJ)
              IF (VJ .NE. 0)  GO TO 3
C  ******  Link through JRU  ********************************************
   5      LASTID = 0
          LASTI = 0
          IJL(K) = JLPTR
          I = K
   6        I = JRU(I)
            IF (I .EQ. 0)  GO TO 10
            QM = NP1
            JMIN = IRL(I)
            JMAX = IJL(I) + IL(I+1) - IL(I) - 1
            LONG = JMAX - JMIN
            JTMP = JL(JMIN)
            IF (JTMP .NE. K)  LONG = LONG + 1
            IF (JTMP .EQ. K)  R(I) = -R(I)
            IF (LASTID .GE. LONG)  GO TO 7
              LASTI = I
              LASTID = LONG
   7        IF (LONG .LE. 0)  GO TO 6
```

```
C  ******  And merge the corresponding columns into the Kth column  ****
             DO 9 J=JMIN,JMAX
               VJ = JL(J)
   8           M = QM
               QM = Q(M)
               IF (QM .LT. VJ)  GO TO 8
               IF (QM .EQ. VJ)  GO TO 9
                 LUK = LUK + 1
                 Q(M) = VJ
                 Q(VJ) = QM
                 QM = VJ
   9           CONTINUE
               GO TO 6
C  ******  LASTI is the longest column merged into the Kth  ************
C  ******  See if it equals the entire Kth column  ********************
  10       QM = Q(NP1)
           IF (QM .NE. K)  GO TO 105
           IF (LUK .EQ. 0)  GO TO 17
           IF (LASTID .NE. LUK)  GO TO 11
C  ******  If so, JL can be compressed  ********************************
           IRLL = IRL(LASTI)
           IJL(K) = IRLL + 1
           IF (JL(IRLL) .NE. K)  IJL(K) = IJL(K) - 1
           GO TO 17
C  ******  If not, see if Kth column can overlap the previous one  *****
  11       IF (JLMIN .GT. JLPTR)  GO TO 15
           QM = Q(QM)
           DO 12 J=JLMIN,JLPTR
             IF (JL(J) - QM)  12, 13, 15
  12         CONTINUE
           GO TO 15
  13       IJL(K) = J
           DO 14 I=J,JLPTR
             IF (JL(I) .NE. QM)  GO TO 15
             QM = Q(QM)
             IF (QM .GT. N)  GO TO 17
  14         CONTINUE
           JLPTR = J - 1
C  ******  Move column indices from Q to JL, update vectors  ***********
  15       JLMIN = JLPTR + 1
           IJL(K) = JLMIN
           IF (LUK .EQ. 0)  GO TO 17
           JLPTR = JLPTR + LUK
           IF (JLPTR .GT. JLMAX)  GO TO 103
             QM = Q(NP1)
             DO 16 J=JLMIN,JLPTR
               QM = Q(QM)
  16           JL(J) = QM
  17       IRL(K) = IJL(K)
           IL(K+1) = IL(K) + LUK
C
C  ******  Initialize Q for computing Kth row of U  *******************
           Q(NP1) = NP1
           LUK = -1
```

```
C  ******  by filling in Kth row of reordered A  ***********************
          RK = R(K)
          JMIN = IRA(K)
          JMAX = IA(RK+1) - 1
          IF (JMIN .GT. JMAX)  GO TO 20
          DO 19 J=JMIN,JMAX
            VJ = IC(JA(J))
            QM = NP1
 18         M = QM
            QM = Q(M)
            IF (QM .LT. VJ)  GO TO 18
            IF (QM .EQ. VJ)  GO TO 102
              LUK = LUK + 1
              Q(M) = VJ
              Q(VJ) = QM
 19         CONTINUE
C  ******  Link through JRL,  ******************************************
 20       LASTID = 0
          LASTI = 0
          IJU(K) = JUPTR
          I = K
          I1 = JRL(K)
 21         I = I1
            IF (I .EQ. 0)  GO TO 26
            I1 = JRL(I)
            QM = NP1
            JMIN = IRU(I)
            JMAX = IJU(I) + IU(I+1) - IU(I) - 1
            LONG = JMAX - JMIN
            JTMP = JU(JMIN)
            IF (JTMP .EQ. K)  GO TO 22
C  ******  Update IRL and JRL, *****************************************
              LONG = LONG + 1
              CEND = IJL(I) + IL(I+1) - IL(I)
              IRL(I) = IRL(I) + 1
              IF (IRL(I) .GE. CEND)  GO TO 22
                J = JL(IRL(I))
                JRL(I) = JRL(J)
                JRL(J) = I
 22         IF (LASTID .GE. LONG)  GO TO 23
              LASTI = I
              LASTID = LONG
 23         IF (LONG .LE. 0)  GO TO 21
C  ******  And merge the corresponding rows into the Kth row  **********
            DO 25 J=JMIN,JMAX
              VJ = JU(J)
 24           M = QM
              QM = Q(M)
              IF (QM .LT. VJ)  GO TO 24
              IF (QM .EQ. VJ)  GO TO 25
                LUK = LUK + 1
                Q(M) = VJ
                Q(VJ) = QM
                QM = VJ
 25           CONTINUE
            GO TO 21
C  ******  Update JRL(K) and IRL(K)  ***********************************
 26       IF (IL(K+1) .LE. IL(K))  GO TO 27
            J = JL(IRL(K))
            JRL(K) = JRL(J)
            JRL(J) = K
```

```
C  ******  LASTI is the longest row merged into the Kth  ***************
C  ******  See if it equals the entire Kth row  ***********************
  27       QM = Q(NP1)
           IF (QM .NE. K)  GO TO 105
           IF (LUK .EQ. 0)  GO TO 34
           IF (LASTID .NE. LUK)  GO TO 28
C  ******  If so, JU can be compressed  ******************************
           IRUL = IRU(LASTI)
           IJU(K) = IRUL + 1
           IF (JU(IRUL) .NE. K)  IJU(K) = IJU(K) - 1
           GO TO 34
C  ******  If not, see if Kth row can overlap the previous one  ********
  28       IF (JUMIN .GT. JUPTR)  GO TO 32
           QM = Q(QM)
           DO 29 J=JUMIN,JUPTR
             IF (JU(J) - QM)  29, 30, 32
  29         CONTINUE
           GO TO 32
  30       IJU(K) = J
           DO 31 I=J,JUPTR
             IF (JU(I) .NE. QM)  GO TO 32
             QM = Q(QM)
             IF (QM .GT. N)  GO TO 34
  31         CONTINUE
           JUPTR = J - 1
C  ******  Move row indices from Q to JU, update vectors  **************
  32       JUMIN = JUPTR + 1
           IJU(K) = JUMIN
           IF (LUK .EQ. 0)  GO TO 34
           JUPTR = JUPTR + LUK
           IF (JUPTR .GT. JUMAX)  GO TO 106
             QM = Q(NP1)
             DO 33 J=JUMIN,JUPTR
               QM = Q(QM)
  33           JU(J) = QM
  34       IRU(K) = IJU(K)
           IU(K+1) = IU(K) + LUK
C
C  ******  Update IRU, JRU  *******************************************
           I = K
  35         I1 = JRU(I)
             IF (R(I) .LT. 0)  GO TO 36
             REND = IJU(I) + IU(I+1) - IU(I)
             IF (IRU(I) .GE. REND)  GO TO 37
               J = JU(IRU(I))
               JRU(I) = JRU(J)
               JRU(J) = I
               GO TO 37
  36         R(I) = -R(I)
  37         I = I1
             IF (I .EQ. 0)  GO TO 38
             IRU(I) = IRU(I) + 1
             GO TO 35
```

```
C
C  ******  Update IRA, JRA, IRAC  ***************************************
  38      I = IRAC(K)
          IF (I .EQ. 0)  GO TO 41
  39        I1 = JRA(I)
            IRA(I) = IRA(I) + 1
            IF (IRA(I) .GE. IA(R(I)+1))  GO TO 40
            IRAI = IRA(I)
            JAIRAI = IC(JA(IRAI))
            IF (JAIRAI .GT. I)  GO TO 40
            JRA(I) = IRAC(JAIRAI)
            IRAC(JAIRAI) = I
  40        I = I1
            IF (I .NE. 0)  GO TO 39
  41      CONTINUE
C
        IJL(N) = JLPTR
        IJU(N) = JUPTR
        FLAG = 0
        RETURN
C
C ** ERROR:  Null Row in A
 101    FLAG = N + RK
        RETURN
C ** ERROR:  Duplicate entry in A
 102    FLAG = 2*N + RK
        RETURN
C ** ERROR:  Insufficient Storage for JL
 103    FLAG = 3*N + K
        RETURN
C ** ERROR:  Null pivot
 105    FLAG = 5*N + K
        RETURN
C ** ERROR:  Insufficient Storage for JU
 106    FLAG = 6*N + K
        RETURN
        END
C
C       ----------------------------------------------------------------
C
C*** Subroutine NNFC
C*** Numerical LDU-factorization of sparse nonsymmetric matrix and
C      solution of system of linear equations (compressed pointer
C      storage)
C
        SUBROUTINE  NNFC
     *     (N, R, C, IC, IA,JA,A, LMAX,IL,JL,IJL,L, D, UMAX,IU,JU,IJU,
     *      U, Z, B, ROW, TMP, IRL,JRL, FLAG)
C
C       Input variables:   N, R, C, IC, IA, JA, A, B, IL, JL, IJL,
C                          LMAX, IU, JU, IJU, UMAX
C       Output variables:  Z, L, D, U, FLAG
```

```
C
C       Parameters used internally:
C nia    | IRL,   - vectors used to find the rows of  L.  At the Kth step
C nia    | JRL        of the factorization,  JRL(K)  points to the head
C        |            of a linked list in  JRL  of column indices J
C        |            such J .LT. K and  L(K,J)  is nonzero.  Zero
C        |            indicates the end of the list.  IRL(J)  (J.LT.K)
C        |            points to the smallest I such that I .GE. K and
C        |            L(I,J)  is nonzero.
C        |            Size of each = N.
C fia    | ROW    - holds intermediate values in calculation of  U and L.
C        |            Size = N.
C fia    | TMP    - holds new right-hand side  b'  for solution of the
C        |            equation Ux = b'.
C        |            Size = N.
C
C  Internal variables:
C    JMIN, JMAX - indices of the first and last positions in a row to
C      be examined.
C    SUM - used in calculating  TMP.
C
        INTEGER RK,UMAX
        REAL LKI
        INTEGER  R(1), C(1), IC(1), IA(1), JA(1), IL(1), JL(1), IJL(1)
        INTEGER  IU(1), JU(1), IJU(1), IRL(1), JRL(1), FLAG
        REAL  A(1), L(1), D(1), U(1), Z(1), B(1), ROW(1), TMP(1)
C
C  ******  Initialize pointers and test storage  **********************
        IF(IL(N+1)-1 .GT. LMAX) GO TO 104
        IF(IU(N+1)-1 .GT. UMAX) GO TO 107
        DO 1 K=1,N
          IRL(K) = IL(K)
          JRL(K) = 0
   1      CONTINUE
C
C  ******  For each row  **********************************************
        DO 19 K=1,N
C  ******  Reverse JRL and zero ROW where Kth row of L will fill in  ***
          ROW(K) = 0
          I1 = 0
          IF (JRL(K) .EQ. 0) GO TO 3
          I = JRL(K)
   2      I2 = JRL(I)
          JRL(I) = I1
          I1 = I
          ROW(I) = 0
          I = I2
          IF (I .NE. 0) GO TO 2
C  ******  Set ROW to zero where U will fill in  **********************
   3      JMIN = IJU(K)
          JMAX = JMIN + IU(K+1) - IU(K) - 1
          IF (JMIN .GT. JMAX) GO TO 5
          DO 4 J=JMIN,JMAX
   4        ROW(JU(J)) = 0
C  ******  Place Kth row of A in ROW  *********************************
   5      RK = R(K)
          JMIN = IA(RK)
          JMAX = IA(RK+1) - 1
          DO 6 J=JMIN,JMAX
            ROW(IC(JA(J))) = A(J)
   6        CONTINUE
```

```
C  ******  Initialize SUM, and link through JRL  ************************
           SUM = B(RK)
           I = I1
           IF (I .EQ. 0) GO TO 10
C  ******  Assign the Kth row of L and adjust ROW, SUM  ****************
    7        LKI = -ROW(I)
C  ******  If L is not required, then comment out the following line  **
             L(IRL(I)) = -LKI
             SUM = SUM + LKI * TMP(I)
             JMIN = IU(I)
             JMAX = IU(I+1) - 1
             IF (JMIN .GT. JMAX) GO TO 9
             MU = IJU(I) - JMIN
             DO 8 J=JMIN,JMAX
    8          ROW(JU(MU+J)) = ROW(JU(MU+J)) + LKI * U(J)
    9        I = JRL(I)
             IF (I .NE. 0) GO TO 7
C
C  ******  Assign Kth row of U and diagonal D, set TMP(K)  *************
   10      IF (ROW(K) .EQ. 0) GO TO 108
           DK = 1 / ROW(K)
           D(K) = DK
           TMP(K) = SUM * DK
           IF (K .EQ. N) GO TO 19
           JMIN = IU(K)
           JMAX = IU(K+1) - 1
           IF (JMIN .GT. JMAX)  GO TO 12
           MU = IJU(K) - JMIN
           DO 11 J=JMIN,JMAX
   11        U(J) = ROW(JU(MU+J)) * DK
   12      CONTINUE
C
C  ******  Update IRL and JRL, keeping JRL in decreasing order  ********
   13      I = I1
           IF (I .EQ. 0) GO TO 18
   14      IRL(I) = IRL(I) + 1
           I1 = JRL(I)
           IF (IRL(I) .GE. IL(I+1)) GO TO 17
           IJLB = IRL(I) - IL(I) + IJL(I)
           J = JL(IJLB)
   15      IF (I .GT. JRL(J)) GO TO 16
             J = JRL(J)
             GO TO 15
   16      JRL(I) = JRL(J)
           JRL(J) = I
   17      I = I1
           IF (I .NE. 0) GO TO 14
   18      IF (IRL(K) .GE. IL(K+1)) GO TO 19
           J = JL(IJL(K))
           JRL(K) = JRL(J)
           JRL(J) = K
   19      CONTINUE
```

```
C
C  ******  Solve  Ux = TMP  by back substitution  **********************
        K = N
        DO 22 I=1,N
          SUM =  TMP(K)
          JMIN = IU(K)
          JMAX = IU(K+1) - 1
          IF (JMIN .GT. JMAX)  GO TO 21
          MU = IJU(K) - JMIN
          DO 20 J=JMIN,JMAX
  20        SUM = SUM - U(J) * TMP(JU(MU+J))
  21      TMP(K) =  SUM
          Z(C(K)) =  SUM
  22      K = K-1
        FLAG = 0
        RETURN
C
C ** ERROR:  Insufficient Storage for L
 104    FLAG = 4*N + 1
        RETURN
C ** ERROR:  Insufficient Storage for U
 107    FLAG = 7*N + 1
        RETURN
C ** ERROR:  Zero Pivot
 108    FLAG = 8*N + K
        RETURN
        END
C
C       --------------------------------------------------------------
C
C*** Subroutine NNSC
C*** Numerical solution of sparse nonsymmetric system of linear
C      equations given LDU-factorization (compressed pointer storage)
C
        SUBROUTINE NNSC
     *     (N, R, C, IL, JL, IJL, L, D, IU, JU, IJU, U, Z, B, TMP)
C
C       Input variables:   N, R, C, IL, JL, IJL, L, D, IU, JU, IJU, U, B
C       Output variables:  Z
C
C       Parameters used internally:
C fia   | TMP   - temporary vector which gets result of solving  Ly = b.
C       |           Size = N.
C
C  Internal variables:
C    JMIN, JMAX - indices of the first and last positions in a row of
C      U or L  to be used.
C
       INTEGER R(1), C(1), IL(1), JL(1), IJL(1), IU(1), JU(1), IJU(1)
       REAL L(1), D(1), U(1), B(1), Z(1), TMP(1)
```

```
C
C ******  Set TMP to reordered B  ****************************************
      DO 1 K=1,N
 1      TMP(K) = B(R(K))
C ******  Solve Ly = b  by forward substitution  ***********************
      DO 3 K=1,N
        JMIN = IL(K)
        JMAX = IL(K+1) - 1
        TMPK = -D(K) * TMP(K)
        TMP(K) = -TMPK
        IF (JMIN .GT. JMAX) GO TO 3
        ML = IJL(K) - JMIN
        DO 2 J=JMIN,JMAX
 2        TMP(JL(ML+J)) = TMP(JL(ML+J)) + TMPK * L(J)
 3      CONTINUE
C ******  Solve Ux = y  by back substitution  **************************
      K = N
      DO 6 I=1,N
        SUM = -TMP(K)
        JMIN = IU(K)
        JMAX = IU(K+1) - 1
        IF (JMIN .GT. JMAX) GO TO 5
        MU = IJU(K) - JMIN
        DO 4 J=JMIN,JMAX
 4        SUM = SUM + U(J) * TMP(JU(MU+J))
 5      TMP(K) = -SUM
        Z(C(K)) = -SUM
        K = K - 1
 6      CONTINUE
      RETURN
      END
```

```
C                            Appendix 4                          7/31/77
C
C        Test Driver for Sparse Nonsymmetric Matrix Package
C
C
C*** Program NTST
C*** Test Driver for Nonsymmetric Codes in Yale Sparse Matrix Package
C
C  Variables:
C
C     NG    -  size of grid used to generate test problem.
C
C     N     -  number of variables and equations (= NG x NG).
C
C     IA    -  INTEGER one-dimensional array used to store row pointers
C              to JA and A;  DIMENSION = N+1.
C
C     JA    -  INTEGER one-dimensional array used to store column
C              indices of nonzero elements of M;  DIMENSION = number of
C              nonzero entries in M.
C
C     A     -  REAL one-dimensional array used to store nonzero elements
C              of M;  DIMENSION = number of nonzero entries in M.
C
C     X     -  REAL one-dimensional array used to store solution x;
C              DIMENSION = N.
C
C     B     -  REAL one-dimensional array used to store right-hand-side b
C              DIMENSION = N.
C
C     P     -  INTEGER one-dimensional array used to store permutation of
C              rows and columns for reordering linear system;
C              DIMENSION = N.
C
C     IP    -  INTEGER one-dimensional array used to store inverse of
C              permutation stored in P;  DIMENSION = N.
C
C     NSP   -  declared dimension of one-dimensional arrays ISP and RSP.
C
C     ISP   -  INTEGER one-dimensional array used as working storage
C              (equivalenced to RSP);  DIMENSION = NSP.
C
C     RSP   -  REAL one-dimensional array used as working storage
C              (equivalenced to ISP);  DIMENSION = NSP.
C
C     ESP   -  INTEGER amount of excess storage available
C
C
        INTEGER  IA(101), JA(500),  P(100), IP(100),  ISP(1500), ESP,
     *     CASE, PATH, FLAG,  APTR,VP,VQ,  X,XMIN,XMAX,  Y,YMIN,YMAX
        REAL  A(500), Z(100), B(100),  RSP(1500),  NAME(3)
        EQUIVALENCE  (ISP(1), RSP(1))
        DATA  NSP/1500/, EPS/1E-5/,
     *     NAME(1)/'N'/, NAME(2)/'T'/, NAME(3)/'C'/
C
        INDEX(I,J) = NG*I + J - NG
C
        NG = 3
        N = NG*NG
C
```

```
C  ******  CASE=1 => NDRV,  CASE=2 => TDRV,  CASE=3 => CDRV  ***********
        DO 5 CASE=1,3
C
C  ******  Set up matrix for five-point finite difference operator *****
        APTR = 1
        DO 2 I=1,NG
          DO 2 J=1,NG
            VP = INDEX (I, J)
            P(VP) = VP
            IP(VP) = VP
            IA(VP) = APTR
            SUM = 0
            XMIN = MAX0 ( 1, I-1)
            XMAX = MIN0 (NG, I+1)
            YMIN = MAX0 ( 1, J-1)
            YMAX = MIN0 (NG, J+1)
            DO 1 X=XMIN,XMAX
              DO 1 Y=YMIN,YMAX
                IF ((X-I) * (Y-J) .NE. 0)  GO TO 1
                  VQ = INDEX(X, Y)
                  JA(APTR) = VQ
                  A(APTR) = 8
                  IF (VP .LT. VQ)  A(APTR) = -1
                  IF (VP .GT. VQ)  A(APTR) = -2
                  SUM = SUM + A(APTR) * VQ
                  APTR = APTR + 1
   1            CONTINUE
            B(VP) = SUM
   2        CONTINUE
        IA(N+1) = APTR
        NZA = IA(N+1) - 1
C
C  ******  Output original array A  ************************************
        IF (CASE.EQ.1)  PRINT 1001,  NG,NG
1001    FORMAT (/' *** FIVE-POINT OPERATOR ON ',
     *             I1, ' BY ' I1, ' GRID ')
        IF (CASE.EQ.1)  PRINT 1002,  (IA(I),I=1,N), IA(N+1)
1002    FORMAT (/' COEFFICIENT MATRIX: '/
     *          /'    IA (INDICES OF FIRST ELEMENTS IN ROWS)'
     *          /(10I5))
        IF (CASE.EQ.1)  PRINT 1003,  (I,JA(I),A(I), I=1,NZA)
1003    FORMAT (/'             JA             A  '
     *          /'  I   COLUMN INDICES    MATRIX'
     *          /(I3, I10, F16.5))
        IF (CASE.EQ.1)  PRINT 1004,  (B(I), I=1,N)
1004    FORMAT (/' RIGHT HAND SIDE B: '
     *          /(5F10.5))
C
C  ******  Call ODRV  **************************************************
        FLAG = 0
        PATH = 1
        CALL  ODRV
     *     (N, IA,JA,A, P,IP, NSP,RSP, PATH, FLAG)
        IF (FLAG.NE.0)  GO TO 101
```

```
C
C  ******  Output ordering of variables/equations  ********************
       IF (CASE.EQ.1)  PRINT 1005,  (I,P(I),IP(I), I=1,N)
1005   FORMAT (/' ROW/COLUMN ORDERING FROM ODRV: '/
     *         /'               P                    IP        '
     *         /'  I  ROW/COL ORDERING    INVERSE ORDERING '
     *         /(I3, I10, I20))
C
C  ******  Call NDRV / TDRV / CDRV  ***********************************
       PATH = 1
       IF (CASE.EQ.1)  CALL  NDRV
     *    (N, P,P,IP, IA,JA,A, B, Z, NSP,ISP,RSP,ESP, PATH, FLAG)
       IF (CASE.EQ.2)  CALL  TDRV
     *    (N, P,IP,   IA,JA,A, B, Z, NSP,ISP,RSP,ESP,       FLAG)
       IF (CASE.EQ.3)  CALL  CDRV
     *    (N, P,P,IP, IA,JA,A, B, Z, NSP,ISP,RSP,ESP, PATH, FLAG)
       IF (FLAG.EQ.0)  GO TO 3
         PRINT 1006,  NAME(CASE),  FLAG
1006     FORMAT (/' ERROR IN ', A1, 'DRV:  FLAG = ', I5)
         GO TO 5
C
C  ******  Calculate error  *******************************************
   3   SUM = 0
       DO 4 I=1,N
   4     SUM = SUM + ((Z(I)-I)/I)**2
       RMS = SQRT(SUM/N)
C
C  ******  Output solution and error measure  *************************
       PRINT 1007,  NAME(CASE),  (Z(I), I=1,N)
1007   FORMAT (/' SOLUTION FROM ', A1, 'DRV: '
     *         /(5F10.5))
C
       IF (RMS.LE.EPS)  PRINT 1008,  RMS
1008   FORMAT (/' SOLUTION CORRECT:  RMS ERROR = ', 1PE8.2)
       IF (RMS.GT.EPS)  PRINT 1009,  RMS
1009   FORMAT (/' SOLUTION INCORRECT:  RMS ERROR = ', 1PE8.2)
C
       PRINT 1010,  ESP
1010   FORMAT (/' EXTRA STORAGE AVAILABLE = ', I4)
C
   5   CONTINUE
       STOP
C
C  ******  Error messages  ********************************************
 101   PRINT 1013, FLAG
1013   FORMAT (/' ERROR IN ODRV:  FLAG = ', I5)
       STOP
       END
```

Appendix 5

Sample Output From Test Driver

*** FIVE-POINT OPERATOR ON 3 BY 3 GRID

COEFFICIENT MATRIX:

IA (INDICES OF FIRST ELEMENTS IN ROWS)

1 4 8 11 15 20 24 27 31 34

I	JA COLUMN INDICES	A MATRIX
1	1	8.00000
2	2	-1.00000
3	4	-1.00000
4	1	-2.00000
5	2	8.00000
6	3	-1.00000
7	5	-1.00000
8	2	-2.00000
9	3	8.00000
10	6	-1.00000
11	1	-2.00000
12	4	8.00000
13	5	-1.00000
14	7	-1.00000
15	2	-2.00000
16	4	-2.00000
17	5	8.00000
18	6	-1.00000
19	8	-1.00000
20	3	-2.00000
21	5	-2.00000
22	6	8.00000
23	9	-1.00000
24	4	-2.00000
25	7	8.00000
26	8	-1.00000
27	5	-2.00000
28	7	-2.00000
29	8	8.00000
30	9	-1.00000
31	6	-2.00000
32	8	-2.00000
33	9	8.00000

```
RIGHT HAND SIDE B:
  2.00000   6.00000  14.00000  18.00000  14.00000
 23.00000  40.00000  31.00000  44.00000
```

ROW/COLUMN ORDERING FROM ODRV:

I	P ROW/COL ORDERING	IP INVERSE ORDERING
1	1	1
2	3	7
3	7	2
4	9	8
5	6	6
6	5	5
7	2	3
8	4	9
9	8	4

```
SOLUTION FROM NDRV:
  1.00000   2.00000   3.00000   4.00000   5.00000
  6.00000   7.00000   8.00000   9.00000

SOLUTION CORRECT:  RMS ERROR = 6.36E-09

EXTRA STORAGE AVAILABLE = 1384

SOLUTION FROM TDRV:
  1.00000   2.00000   3.00000   4.00000   5.00000
  6.00000   7.00000   8.00000   9.00000

SOLUTION CORRECT:  RMS ERROR = 6.36E-09

EXTRA STORAGE AVAILABLE = 1412

SOLUTION FROM CDRV:
  1.00000   2.00000   3.00000   4.00000   5.00000
  6.00000   7.00000   8.00000   9.00000

SOLUTION CORRECT:  RMS ERROR = 6.36E-09

EXTRA STORAGE AVAILABLE = 1364
```