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And It's Algebraic Survey

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STABILITY OF HIGH ORDER DIFFERENCE EQUATIONS AND ITS ALGEBRAIC SURVEY

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Abstract

In the present paper difference methods for Cauchy problem of p-order differential equation in Banach space are investigated. Problems about Lax's equivalence theorem, stability and convergence are discussed. Emphases will be on finding some algebraic surveys of stability.

1.INTRODUCTION

In the present paper stability and convergence of the initial-value problems of difference equations approximating p-order differential equations are investigated. After suggesting the conception of p-order stability, an equivalence theorem which is similar to Lax's theorem in [1] is proven. Finally we spent most of the investigation to find some interesting surveys of stability. Many authors worked on these problems before, but for $p > 1$ only a few results were given [2]. People might think that a high order differential

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equation (system) could be reduced to first order system with more unknowns, so it seems unnecessary to consider high order problems. But as a matter of fact it is not always successful to do that, and in many cases one prefer the original to any motivated system. So it is worthy to investigate the stability of the high order problems. Considering these facts we ask: Are the concepts of stability and convergence in $p > 1$ the same as in $p=1$? We shall point out the diference between variety p by introducing the conception of 'p-order stability ' as the generalization of the ordinary one. As a result we shall state some algebraic surveys in practical uses.

We consider initial value problems of p-order differential equation in Banach space B as follows:

$$A_0 D_t^p u(t) + A_1 D_t^{p-1} u(t) + \dots + A_p u(t) = 0, \quad 0 < t \leq T, \tag{1.1}$$

$$(u(0), D_t u(0), \dots, D_t^{p-1} u(0)) = (u_0, u_1, \dots u_{p-1}), \tag{1.2}$$

where A_0, $A_1, \dots, A_p$ are linear operators (might be unbounded), their common domain of definition D(A) is dense in B; u_0, $u_1, \dots, u_{p-1}$ ε B; $u(t)$ is the single-parameter 'curve' , $D_t^r u(t)$, $r=1,2,\dots,p$ are the notations of time derivative [7], as follows:

$$D_t^r u(t) = \lim_{\delta t \to 0} (1/\delta t^r) \sum_{k=0}^{r} (-1)^{r-k} C_r^k u(t + k\delta t) \quad r=1,2,\dots p.$$

where $C_r^k = r!/[k!(r-k)!]$.

Definition 1. The classical solution of problem (1.1)(1.2) is such a single- parameter curve u(t) that has first p-order differential derivatives belonging to D(A) and satisfying (1.1), and $D_t^r u(t) \to u_r$, $r=0,1,\dots,p-1$ as $t \to 0$. Denote the resolvent operator as $E^o(t)$, then $u(t) = E^o(t)(u_0, u_1, \dots, u_{p-1})$

$=E_0^o u_0+E_1^o u_1+\ldots+E_{p-1}^o u_{p-1}$, $E_r^o(t)$, $r=0,1,\ldots,p-1$ are linear operators in B.

Definition 2. If (1) D(A) is dense in B; (2) $|E_o(t)|=\max\{|E_r^o(t)|\}<K$, then problem (1.1),(1.2) is well posed.

Now we suppose that the difference approximation of (1.1),(1.2) is as follows:

$$C_q u^{n+q}=C_{q-1}u^{n+q-1}+\ldots+C_1 u^{q+1}+C_0 u^q, \tag{1.3}$$

$$q\geq p\ ,\quad 0<(n-q)\Delta t\leq T,$$

where Δt is the length of t-step, $C_i=C_i(\Delta t)$, $i=0,1,\ldots,q$ are known beforehand.

For brevity we introduce the auxiliary Banach space $\bar{B}$. As an element of $\bar{B}$ $\bar{u}$ is a q-component vector:

$$\bar{u}=\begin{vmatrix} u \\ v \\ . \\ . \end{vmatrix}\ ,\qquad u,v,\ldots\ \varepsilon\ B$$

And denote

$$\bar{u}^n=\begin{vmatrix} u^{n+q-1} \\ u^{n+q-2} \\ . \\ . \\ u^n \end{vmatrix}\ ,\quad \bar{u}(n\Delta t)=\begin{vmatrix} u(\overline{n-q-1}\Delta t) \\ u(\overline{n-q-2}\Delta t) \\ . \\ . \\ u(n\Delta t) \end{vmatrix} \tag{1.4}$$

The norm of $\bar{u}$ might be defined as $|\bar{u}|=\{|u|^2+|v|^2+\ldots\}^{1/2}$ or

$$\max\{|u|,|v|,\ldots\},$$

or any norm such that the following inequality of equivalence holds:

$$m|\bar{u}|\leq\max\{|u|,|v|,\ldots\}\leq M|\bar{u}|,$$

where M,m are constants. Initial value can be determined through a linear transformation:

$$\bar{u}^0=\Delta t^{p-1}\bar{S}(\Delta t)(u_0,u_1,\dots,u_{p-1})'.$$

For example:

$$\bar{S}(\Delta t)=1/(\Delta t^{p-1})\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & \frac{\Delta t}{1!} & \frac{\Delta t^2}{2!} & \cdots & \frac{\Delta t^{p-1}}{(p-1)!} \\ \cdots\cdots & & & & \\ 1 & \frac{(q-1)\Delta t}{1!} & \frac{[(q-1)\Delta t]^2}{2!} & \cdots & \frac{[(q-1)\Delta t]^{p-1}}{(p-1)!} \end{vmatrix} \tag{1.5}$$

2. BASIC DEFINITIONS AND LAX'S THEOREM

We suppose that the scheme (1.3) is through direct substitution of the derivatives by some differences.

Definition 1. Let $u(t)$ be any single-parameter curve from $D(A)$, then we call

$$\gamma(t,\Delta t)=[C_q u(t+q\Delta t)-C_{q-1}u(t+\overline{q-1}\Delta t)-C_0 u(t)]/\Delta t^p-$$
$$-[A_0 D_t^p u(t)+\dots+A_p u(t)] \tag{2.1}$$

the approximate error of the difference equation (1.3) for $u(t)$.

Definition 2. Let $u(t)$ be any single-parameter curve from $D(A)$, then the approximate error of initial value is defined as follows:

$$\bar{\eta}(\Delta t)=(\bar{u}^0-\bar{u}(0))/\Delta t^{p-1}. \tag{2.2}$$

Definition 3. For a set of single-parameter curve $U=\{u(t)\in D(A)\}$, $\bar{D}(A)=B$, which contains the classical solutions, $|\gamma(t,\Delta t)|\to 0$ uniformly holds for all $t\in[0,T]$ as $\Delta t\to 0^+$; operator C_q has uniformly bounded inverse such that $|C_q^{-1}(\Delta t)|\le K=\text{const}$, then call the difference approximation (1.3) consistent.

Definition 4. As $\Delta t \to 0^+$ for any classical solution $u(t) \in U$ and its corresponding difference solution u^n that $|u^n - u(t)| \to 0$ can be derived from $|\eta(\Delta t)| \to 0$, then call the difference equation (1.3) convergent.

If we denote

$$\bar{C}(\Delta t)=\begin{vmatrix} C_q^{-1}C_{q-1} & C_q^{-1}C_{q-2} & \cdots\cdots & C_q^{-1}C_0 \\ I & & & \\ & I & & \\ & & \cdot & \\ & & & \cdot \\ & & & I \end{vmatrix} \tag{2.3}$$

then equation (1.3) will be

$$\bar{u}^{n+1}=\bar{C}(\Delta t)\bar{u}^n. \tag{2.4}$$

The convergence condition is equivalent to

$$|\bar{C}(\Delta t)^n\ \bar{u}^0-\bar{u}(t)| \to 0, \text{ if } |\bar{\eta}(\Delta t)| \to 0,\ n\Delta t \to t \text{ as } \Delta t \to 0.$$

Definition 5. For operator (2.3) there exists a positive constant $\tau > 0$ such that operator family $\{\Delta t^{p-1}C(\Delta t)^n |_{0 \le n\Delta t \le T}^{0 < \Delta t < \tau}\}$ be uniformly bounded, then call difference equation (1.3) p-order stable.

While p=1 definition 5 is the same as in [1]. Now we turn to following equivalence theorem.

Theorem 1. To any difference approximation of the well-posed problem (1.1) and (1.2) stability and convergence are equivalent.

Proof. C-->S: The only thing needs to be verified is that for every $\bar{u} \in \bar{B}$ we have $|\Delta t^{p-1}\bar{C}(\Delta t)^{n}\bar{u}| \leq K(u) < +\infty$ when $0 < \Delta t < \tau$, $0 \leq n\Delta t \leq T$, then from resonance theorem of operator p-order stability could be derived. If not, then there exists a sequence of $\{\Delta_j t\}_1^{\infty}$, $\Delta_j t \to 0$ as $j \to \infty$, such that $|\Delta t^{p-1}\bar{C}(\Delta_j t)^{n}| \to \infty$ where $0 < \Delta_j t < \tau$, $0 \leq n_j\Delta_j t \leq T$. Suppose $n_j\Delta_j t \to t \in [0,T]$ as $j \to \infty$, and denote $\bar{u}_j = \Delta_j t^{p-1}\bar{u} / |\Delta_j t^{p-1} C(\Delta_j t)^{n_j}\bar{u}|^{1/2}$. Now we have $|\bar{u}_j - \emptyset| = o([\Delta_j t^{p-1}]^{1/2})$, where $\emptyset$ is the zero element in $\bar{B}$. According to convergence theorem the solutions $C(\Delta_j t)^{n_j}\bar{u}_j$ of the problem (1.1),(1.2) should tend to the trivial solution $\emptyset$, but on the contrary $|C(\Delta_j t)^{n_j}\bar{u}_j| = |\Delta_j t^{p-1} C(\Delta_j t)^{n_j}\bar{u}|^{1/2} \to +\infty$. So we have completed the first part of this theorem.

S-->C: Let $u(t) \in U$ be a classical solution and U^n be a difference solution, then the error of the approximate solution $\varepsilon^n = u(n\Delta t) - u^n$ satsfies the following equation:

$$\bar{\varepsilon}^{n+1} = \bar{C}(\Delta t)\bar{\varepsilon}^{n} + \Delta t^{p}\bar{r}(t,\Delta t),$$
$$\bar{\varepsilon}^{0} = \Delta t^{p-1}\bar{\eta}(\Delta t) \tag{2.5}$$

where $\bar{r}(t,\Delta t) = (C_q^{-1} r(t,\Delta t), 0, \ldots, 0) \in \bar{B}$. Using the recurrence relation above, we obtain

$$\bar{\varepsilon}^{n} = \Delta t \sum_{k=o}^{n-1} \Delta t^{p-1} C(\Delta t)^{n-k-1} r(k\Delta t,\Delta t) + \\ + \Delta t^{p-1} C(\Delta t)^{n}\bar{\eta}(\Delta t)$$

and

$$|\bar{\varepsilon}| \leq TK \max|r(k\Delta t,\Delta t)| + K|\bar{\eta}(\Delta t)| \leq \\ \leq TKM|C_q^{-1}| \max |r(k\Delta t,\Delta t)| + K|\bar{\eta}(\Delta t)| \to 0 \text{ as } \Delta t \to 0.$$

Corollary. If the difference equation (1.3) approximating the p-order differential equation is p+α-order stable ($\alpha \geq 0$) and the errors of the approximation satisfy

$$|r(k\Delta t,\Delta t)|,|\bar{\eta}(\Delta t)| \leq K\rho(\Delta t)\Delta t^{\alpha}$$

$$\rho(\Delta t) \to 0 \text{ as } \Delta t \to 0,$$

then we obtain the error estimate:

$$|\varepsilon^{n}| \leq K\, \rho(\Delta t).$$

Theorem 2. For weakly nonlinear equations p-order stability of $\bar{C}_1(\Delta t)$ implies convergence, if the following conditions are satisfied:

(1)the right part $F(t,u)$ of (1.1) has uniformly bounded Frechet derivative $F'_u(t,u)$;

(2)the difference operator in (2.4) $\bar{C}(\Delta t)\bar{u} = \bar{C}_1(\Delta)u + \Delta t^{p}\bar{C}_2(\Delta t,u)$, and $\bar{C}_2(\Delta t,u)$ has uniformly bounded Frechet derivative $\bar{C}'_{2u}(\Delta t,u)$.

The proof of theorem 2 is easy if one replaces $\bar{C}(\Delta t)$ in (2.5) by $\bar{C}_1(\Delta t)$, $r(k\Delta t,\Delta t)$ by $\bar{C}_2(\Delta t,u(k\Delta t)) - \bar{C}_2(\Delta t,u^{k}) - r(k\Delta t,\Delta t)$ and uses

$$|\bar{C}_2(\Delta t,u(k\Delta t))-\bar{C}_2(\Delta t,u^{k})| \leq K|\varepsilon^{k}|,$$

then a estimation can be deduced:

$$|\varepsilon^{n}| \leq \Delta t\ K \sum_{k=o}^{n-1} |\varepsilon^{k}| + K\{ \sum_{k=o}^{n-1} |r(k\Delta t,\Delta t)|+|\bar{\eta}(\Delta t)|\}$$

or

$$|\varepsilon^{n}| \leq K\ e^{Kn\Delta t}\{ \sum_{k=o}^{n-1} |r(k\Delta t,\Delta t)|+|\bar{\eta}(\Delta t)|\}.$$

From this theorem 2 is obvious.

Corollary. If $\bar{C}(\Delta t)=\bar{C}_1(\Delta t)+\Delta t^{p}\bar{C}_2(\Delta t)$, and $\bar{C}_2(\Delta t)$ is bounded, then p-order stability of $\bar{C}_1(\Delta t)$ implies that of $\bar{C}(\Delta t)$.

3. CAUCHY PROBLEMS OF THE LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Consider the high order differential equation with constant coefficients as follows:

$$\{P_0(\frac{\partial}{\partial x})\frac{\partial^p}{\partial t^p}+P_1(\frac{\partial}{\partial x})\frac{\partial^{p-1}}{\partial t^{p-1}}+\ldots+P_p\}u(x,t)=0 \tag{3.1}$$

$$(1,\frac{\partial}{\partial t},\ldots,\frac{\partial^{p-1}}{\partial t^{p-1}})u(x,0) = (\phi_0,\phi_1,\ldots,\phi_{p-1}) \tag{3.2}$$

where $P_i(\frac{\partial}{\partial x})=P_i(\frac{\partial}{\partial x1},\frac{\partial}{\partial x2},\ldots,\frac{\partial}{\partial xd})$, $i=0,1,\ldots,p$ are the polynomials of its arguments $\frac{\partial}{\partial x1},\frac{\partial}{\partial x2},\ldots,\frac{\partial}{\partial xd}$; $u(x,t)$, $\phi_0,\phi_1,\ldots,\phi_{p-1}$ are the ℓ-dimensional periodic function vectors with period of unity. We use the following difference equation:

$$\sum_\beta B^q_\beta(\Delta t)\ u^{n+q}(x+\beta\Delta x) = \sum_{j=0}^{q-1}\sum_\beta B^j_\beta(\Delta t)\ u^{n+j}(x+\beta\Delta x) \tag{3.3}$$

where β is a d-dimensional vector$(\beta_1,\beta_2,\ldots,\beta_d)$, $\beta\Delta x=(\beta_1\Delta x_1,\ldots,\beta_d\Delta x_d)$, B^j_β, $j=0,1,\ldots q$ are ℓ-order square matrices. Let

$$u^n(x) = \sum_k v^n(k)e^{2\pi(k,x)i}, k=(k_1,k_2,\ldots,k_d), \tag{3.4}$$

$$\bar{u}^n(x)=(u^{n+q-1}(x),\ldots,u^n(x))',$$
$$\bar{u}^n(x) = \sum_k \bar{v}^n(k)e^{2\pi(k,x)i}, \tag{3.5}$$
$$k_j=0,\pm1,\pm2,\ldots.$$

Using these Fourier expressions to (3.3) we easily obtain

$$\bar{H}_1(\Delta t,k\Delta x)\bar{v}^{n+1}(k) = \bar{H}_0(\Delta t,k\Delta x)\bar{v}^n(k) \tag{3.6}$$

where $\bar{v}^n(k)=(v^{n+q-1}(x),\ldots,v^n(x))'$,

$$\bar{H}_1(\Delta t,k\Delta x)=\begin{vmatrix} Hq & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{vmatrix},$$

$$\bar{H}_0(\Delta t,k\Delta x)=\begin{vmatrix} Hq_{-1} & Hq_{-2} & \cdots & H_0 \\ I & & & \\ & I & & \\ & & \ddots & \\ & & I & 0 \end{vmatrix},$$

and $H_j=H_j(\Delta t,k\Delta x) = \sum_\beta B_\beta^j(\Delta t)e^{2\pi(k,\Delta x)i}$, $j=0,1,\ldots,q$. So we have

$$\bar{v}^{n+1}(k)=\bar{G}(\Delta t,k\Delta x)\bar{v}^n(k) \tag{3.7}$$

$$\bar{G}(\Delta t,k\Delta x)= \begin{vmatrix} Hq^{-1}Hq_{-1} & \cdots Hq^{-1}H_0 & & \\ I & 0 & & \\ 0 & I & & \\ & & \ddots & \\ & & & \ddots \\ & & I & 0 \end{vmatrix}.$$

The consistency conditions of the difference equation (3.3) are

$$\left|\frac{\Delta t^r}{r!}\{q^r H_q(\Delta t,k\Delta x)-\sum_{j=0}^{q-1} j^r H_j(\Delta t,k\Delta x)\}-P_r(2\pi ki)\right| \to 0,$$

$$r=0,1,\ldots,p,$$

$$\left|\frac{\Delta t^r}{r!}\{q^r H_q(\Delta t,k\Delta x)-\sum_{j=0}^{q-1} j^r H_j(\Delta t,k\Delta x)\}\right| \to 0,$$

$$r > p \text{, for fixed } k \text{ as } \Delta t\to 0. \tag{3.8}$$

The p-order stability condition of the difference equation (3.3) becomes the uniformly boundedness of the following matrix family:

$$\{\Delta t^{p-1}\bar{G}(\Delta t,k\Delta x)^n \,|\, 0 < \Delta t < \tau,\ 0\le n\Delta t\le T,\ k\}. \tag{3.9}$$

Simply denote $\theta=k\Delta x=(k_1\Delta x_1,k_2\Delta x_2,\dots,k_d\Delta x_d)=(\theta_1,\theta_2,\dots,\theta_d)$ because of the arbitrary Δx we can take all θ_i $i=1,2,\dots,d$ as continuous parameters in interval (0,1). So instead of (3.9) we turn to investigate matrix family:

$$\{\Delta t^{p-1}\bar{G}(\Delta t,\theta)^n|^{0<\Delta t<\tau}_{0\leq n\Delta t\leq T,\theta}\}. \tag{3.10}$$

We often pay more attention on matrix $\bar{G}(0,\theta)$ and its family

$$\{\Delta t^{p-1}\bar{G}(0,\theta)^n|^{0<\Delta t<\tau}_{0\leq n\Delta t\leq T,\theta}\}. \tag{3.11}$$

The uniformly boundedness of these matrices will be investigated in the remaining part of this paper. Suppose $G(\Delta t,\theta)$ be Lipsichtz continuous on $0\leq\Delta t\leq\tau$ and $0\leq\theta_i\leq 1$, $i=1,2,\dots d$.

Corollary. If $\bar{G}(\Delta t,k\Delta x)=\bar{G}_1(\Delta t,\theta)+\Delta t^p\bar{G}_2(\Delta t,\theta)$, and $\bar{G}_2(\Delta t,\theta)$ is uniformly bounded , then the uniformly boundedness of matrix family:

$$\{\Delta t^{p-1}\bar{G}_1(\Delta t,\theta)^n|^{0<\Delta t<\tau}_{0\leq n\Delta t\leq T,\theta}\}$$

implies the uniformly boundedness of matrix family (3.10).

Theorem (Necessary Condition).If matrix family (3,11) is uniformly bounded, then all the eigenvalues of matrix $G(0,\theta)$ must satisfy following two conditions:

(1) $|\lambda_j(0,\theta)|\leq 1$, for all j;

(2) to those $|\lambda_j(0,\theta)|=1$ the order of its subblock in the Jordan Canonical Form should not exceed p.

4.BASIC LEMMAS

In this section we collect the general properties about multi-parameter matrices with all proofs omitted, including the properties about eigenvalues. Let $G(\mu)=G(\mu_1,\mu_2,\dots,\mu_s)$ be s-parameter matrix with continuous elements.

Lemma 1. Let $G(\mu)$ be continuous , then its eigenvalues $\lambda_1(\mu)$, $\lambda_2(\mu),\dots$ are the continuous functions of μ.

Lemma 2. Let $G(\mu)$ be continuous , for $\mu=\mu_0$ $\lambda_j(\mu_0)$ and s_i be the eigenvalue and its multiplicity $j=1,2,\dots,r$, and $|\lambda_i(\mu_0)-\lambda_j(\mu_0)|\geq\delta>0$ when $i\neq j$, $i,j=1,2,\dots,r$ then there exists an arbitrary small ball δ_1 with the centre μ_0 and a non-degenerate matrix $S(\mu)$ such that 1) $|S(\mu)|$, $|S^{-1}(\mu)|$ < K-const; 2)

$$S(\mu)G(\mu)S^{-1}(\mu)=\begin{vmatrix} G_1(\mu) & & & \\ & G_2(\mu) & & \\ & & \cdots & \\ & & & G_r(\mu) \end{vmatrix}$$

$$G_i(\mu)=\begin{vmatrix} \lambda_{i1}(\mu) & & & * \\ & \lambda_{i2}(\mu) & & \\ & & \cdots & \\ & & & \lambda_{is_i}(\mu) \end{vmatrix}$$

where $\lambda_{ij}(\mu)\to\lambda_i(\mu_0)$; $j=1,2,\dots,s_i$; $i=1,2,\dots,r$ as $\mu\to\mu_0$, and the elements denoted as * are uniformly bounded.

Lemma 3. If multi-parameter matrix

$$G(\mu)=\begin{vmatrix} \lambda_1(\mu) & \cdots\cdots & & g_{1,q} \\ & \lambda_2(\mu)\cdots & & g_{2,q} \\ & & \cdots & \\ & & & \lambda_q(\mu) \end{vmatrix}_{q\times q}$$

and 1) all elements g are uniformly bounded; 2) $|\lambda_i(\mu)| \leq 1+K\,S\{A\}t, q \leq p$; 3) $|\lambda_i(\mu)| \leq \alpha < 1$, then the matrix family $\{\Delta t^{p-1}G(\mu)^n|\ 0\leq n\Delta t \leq T\}$ is uniformly bounded, where μ may depend on Δt.

Now we return to another important lemma . Consider a matrix $G(\mu)$ in a small neighbourhood of a fixed point, for instance $\mu_0=0$. Because of continuity we have $G(\mu)=G(0)+\rho(\mu)G_1(\mu)$, where $\rho(\mu)\to 0$, $|G_1(\mu)| \le K$ as $\mu\to 0$. On the other hand matrix $G(0)$ can be transformed into Jordan Canonical form, e.i. there is a constant matrix T such that

$$J=TG(0)T^{-1}=\begin{vmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \cdot & \\ 0 & & & J_r \end{vmatrix}, \quad J_i=\begin{vmatrix} \lambda_i & \varepsilon & & 0 \\ & \lambda_i & \varepsilon & \\ & & \vdots & \varepsilon \\ 0 & & & \lambda_i \end{vmatrix}$$

$$i=1,2,\dots,r.$$

To $G(\mu)$ we get

$$\tilde{J}(\mu)=TG(\mu)T^{-1}=J+\rho(\mu)C(\mu), \quad |C(\mu)|\le K \qquad (4.1)$$

Lemma 4. Suppose matrix J is of the form (4.1), for brevity,

$$J=\begin{vmatrix} J_1 & 0 \\ 0 & J_2 \end{vmatrix}, \qquad C(\mu)=\begin{vmatrix} C_{11}(\mu) & C_{12}(\mu) \\ C_{21}(\mu) & C_{22}(\mu) \end{vmatrix} \qquad (4.2)$$

where J_1 contains all the subblocks with the same eigenvalue λ_1 in $G(0)$; J_2 contains the rest. Then there exists a transformation $D(\mu)$ such that:

$$D(\mu)=\begin{vmatrix} I & 0 \\ X(\mu) & I \end{vmatrix}, \qquad D^{-1}(\mu)=\begin{vmatrix} I & 0 \\ -X(\mu) & I \end{vmatrix}, \qquad (4.3)$$

$$D(\mu)\tilde{J}(\mu)D^{-1}=$$

$$=J+\rho(\mu)\begin{vmatrix} C_1(\mu) & C_3(\mu) \\ 0 & C_2(\mu) \end{vmatrix};$$

further more there exists another trasformation $E(\mu)$ such that:

$$E(\mu)=\begin{vmatrix} I & Y(\mu) \\ 0 & I \end{vmatrix}, \qquad E^{-1}(\mu)=\begin{vmatrix} I & -Y(\mu) \\ 0 & I \end{vmatrix}, \tag{4.4}$$

$$E(\mu)D(\mu)\tilde{J}(\mu)D^{-1}(\mu)E^{-1}(\mu)=$$

$$=J+\rho(\mu)\begin{vmatrix} C_4(\mu) & 0 \\ 0 & C_5(\mu) \end{vmatrix}$$

where $C_4(\mu),C_5(\mu),X(\mu),Y(\mu)$ are uniformly bounded.

This lemma tells us that the parameter matrix can be divided into small blocks according to the different trend of eigenvalues, so that investigation of this matrix can be simplified.

Lemma 5. Let matrix $\tilde{J}(\mu)=J+\rho(\mu)C(\mu)$, J consists of Jordan subblocks of p-order with the same eigenvalue λ_0 at $\mu=0$, where $\rho(\mu)\text{--}>0$, $|C(\mu)|\leq K$ as $\mu\text{--}>0$. Then in the neighbourhood of $\mu=0$ we have the estimation of eiginvalues as follows:

$$1)\ |\lambda_i(\mu)-\lambda_0|\leq[\rho(\mu)]^{1/p};$$

$$2)\ |\lambda_i(\mu)-\lambda_j(\mu)|\leq 2K[\rho(\mu)]^{1/p}. \tag{4.5}$$

5. ALGEBRAIC SURVEY OF THE STABILITY

In this section we shall give some sufficient conditions for checking stebility of the difference equations. They are often very useful in practice.

THEOREM 1. Let $G(\Delta t,\theta)$ be continuous on region $0\leq\Delta t\leq\tau$, $0\leq\theta_i\leq 1$, $i=1,2,\ldots,d$; all the norms of the eiginvaluie not exceed $1+K\Delta t$; for any fixed θ_0

matrix $G(0,\theta_o)$ satisfies: 1) $|\lambda_i(0,\theta_o)|\leq 1$; 2) multiplicity of those eiginvalues $|\lambda_i(0,\theta_o)|=1$ should not exceed p, then matrices (3.10) are uniformly bounded.

THEOREM 2. (P=1) If matrix $G(\Delta t,\theta)$ fits the necessary condition, and 1) all the norms of the eiginvalues of $G(\Delta t,\theta)$ should not exceed $1+K\Delta t$; 2) for any neighbourhood of θ_o $G(\Delta t,\theta)=G(0,\theta_o)+\rho(\Delta t,\theta-\theta_o)C(\Delta t,\theta)$ and $\rho(\Delta t,\theta-\theta_o)\to 0$, $|C(\Delta t,\theta)|\leq K$ as $\Delta t\to 0,\theta\to\theta_o$ satisfy $|\lambda_i(\Delta t,\theta)-\lambda_j(\Delta t,\theta)|\geq K\rho(\Delta t,\theta-\theta_o)$ or $1-\max|\lambda_i(\Delta t,\theta)|\geq K\rho(\Delta t,\theta-\theta_o)$, then matrix family (3.10) is uniformly bounded.

THEOREM 3. (high order) If matrix $G(\Delta t,\theta)$ fits following conditions(A),(B) or (A),(B'), then matrix family(3.10) is uniformly bounded.

(A)The norm of any eiginvalue of $G(\Delta t,\theta)$ does not exceed $1+K\Delta t$, for any fixed θ_o $|G(\Delta t,\theta)-G(0,\theta_o)|=O(\rho(\Delta t,\theta-\theta_o))$, and $\rho(\Delta t,\theta-\theta_o)\to 0$, and $[\rho(\Delta t,\theta-\theta_o)]^{1/p}\geq K\Delta t$ as $\Delta t,\theta-\theta_o\to 0$.

(B) For any fixed θ_o $|\lambda(0,\theta_o)|\leq 1$; if r Jordan subblocks of p-order contain the same eigenvalue $\lambda(0,\theta_o)$, then the r x p corresponding eigenvalues can be sorted into r groups:

$$\{\lambda_{i,j}(\Delta t,\theta)\,|\,j=1,2,..,p\},\ i=1,2,...,r\ ,$$

to them the following inequality holds:

$$|\lambda_{i,j}(\Delta t,\theta)-\lambda_{i_1,j_1}(\Delta t,\theta)|\geq K[\rho(\Delta t,\theta-\theta_o)]^{1/p}\quad i\neq i_1; \tag{5.1}$$

(B') Instead of (5.1) in (B) we can impose

$$|\lambda_{i,j}(\Delta t,\theta)-\lambda(0,\theta_o)|\geq K[\rho(\Delta t,\theta-\theta_o)]^{1/p}, \tag{5.2}$$

and one of following two conditions to be satisfied:

$$\text{(a) } K_1\Delta t \geq [\rho(\Delta t, \theta-\theta_o)]^{1/p} \geq K_2\Delta t; \tag{5.3}$$

(b) all $\lambda_{i,j}(\Delta t,\theta)$ are within an angle of circumference with $\lambda(0,\theta_o)$ as its vertex.

The proofs of theorem 1,2 are omitted. We give some key steps of the proof about theorem 3 briefly. The whole idea is that the boundedness of (3.10) can be reduce to that of a special kind of matrix family $\{\Delta t^{p-1}J(\Delta t,\theta)^n|_{0\leq n\Delta t\leq T}^{0<\Delta t<\tau}\}$ in a small neighbourhood of a point $(0,\theta_o)$, then using matrix function formula we obtain

$$\Delta t^{p-1}J(\Delta t,\theta)^n=\Delta t^{p-1}\{f(\lambda_{11})I+(J-\lambda_{11}I)f(\lambda_{11},\lambda_{12})+\ldots+ \\ (J-\lambda_{11}I)\ldots(J-\lambda_{r,p-1}I)f(\lambda_{11},\ldots,\lambda_{r,p-1})\} \tag{5.4}$$

where difference quotients $f(\lambda_{11})=\lambda_{11}^n,\ldots,$and

$$f(\lambda_{11},\ldots,\lambda_{r,p-1})= \\ =\frac{1}{2\pi i}\oint_C\lambda^n d\lambda/[\prod_k\prod_j(\lambda-\lambda_{kj})],$$

and C is a closed contour with all $\lambda_{i,j}(\Delta t,\theta)$ in it , So we turn to estimation of every term in (5.4) , the most difficult one is

$$\Phi=(J-\lambda_{11}I)\ldots(J-\lambda_{r,p-1}I)\Delta t^{p-1}* \\ *\frac{1}{2\pi i}\oint_C\lambda^n d\lambda/[\prod_k\prod_j(\lambda-\lambda_{kj})].$$

Corollary. In theorem 3 we can combine (B),(B′) to handle $m+m_o$ multi-eigenvalue $\lambda(0,\theta_o)$ that m_o eigenvalues satisfy (B′) and the rest satisfy (B) then the conclusion still valid.

We leave the interesting applications of these theorems to the readers. Theorem 3 is the most useful in the hyperbolic equation and high order differential equations.

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