Embedding Three-Dimensional Meshes in Boolean Cubes by Graph Decomposition

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Embedding Three-Dimensional Meshes in Boolean Cubes by Graph Decomposition

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Abstract. This paper explores the embeddings of multidimensional meshes into minimal Boolean cubes by graph decomposition. The dilation and the congestion of the product graph $(G_1 \times G_2) \rightarrow (H_1 \times H_2)$ is the maximum of the dilation and congestion for the two embeddings $G_1 \rightarrow H_1$ and $G_2 \rightarrow H_2$. The graph decomposition technique can be used to form new embeddings based on existing embeddings while preserving the dilation and congestion. It can also improve the average dilation and average congestion of existing embeddings. For three-dimensional meshes we show that the graph decomposition technique, together with previously known techniques, yields dilation-two minimal-expansion embeddings of more than 96% of all three-dimensional meshes contained in a $512 \times 512 \times 512$ mesh. Previous embeddings have dilation 7 for all three-dimensional meshes. The graph decomposition technique is also used to generalize the embeddings to meshes with wraparound and many-to-one embeddings.

1 Introduction

Many linear algebra computations can be performed effectively on processor networks configured as two-dimensional meshes, with or without wraparound. Processor networks configured as two- or higher dimensional meshes are also effective for the solution of partial differential equations whenever regular grids are appropriate.

Embedding meshes in Boolean cubes by encoding the indices of each axis in a Gray code [22] yields a nearest neighbor embedding of adjacent nodes [16]. However, if the length of the axis is not a power of two, the Gray code embedding forces the number of processors allocated to an axis to be a power of two. For meshes of high dimension, this may yield a very poor processor utilization. Havel and Móravek [12] proved that any nearest-neighbor embedding must have the same processor utilization as that offered by the binary-reflected Gray code. Whenever the Gray code does not yield the maximum processor utilization, an increased utilization can only be achieved if some adjacent mesh nodes are assigned to Boolean cube nodes at a distance of at least two. The length of the path into which a mesh

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edge is mapped is the dilation of the edge, and the maximum dilation of any edge is the dilation of the embedding. The expansion of the embedding is the ratio of the number of cube nodes used for the embedding and the number of mesh nodes. For meshes that cannot be embedded with minimal expansion and dilation one, the best known lower bound for the dilation is two. The bound of dilation two is tight for two-dimensional meshes [4]. The best known upper bound for the dilation is 7 for three-dimensional meshes [6], and $4k+1$ for k-dimensional meshes, $k > 3$ [5].

We show how graph decomposition can be used to provide effective embeddings for many meshes, and how properties such as expansion, dilation and congestion are affected by graph decomposition. The graph decomposition technique in combination with previously known results [2], [4], [7], [14], [13] yields dilation-two minimum-expansion embeddings of 96% of all three-dimensional meshes $\ell_1 \times \ell_2 \times \ell_3$, such that $1 \leq \ell_1, \ell_2, \ell_3 \leq 512$. By using Gray code embedding, only 29% of the meshes achieve minimal expansion for the considered three-dimensional domain.

The outline of this paper is as follows. Notation and definitions are defined in the next section, and some of the previously known results are briefly reviewed in Section 3. In section 4, we prove theorems that transform embeddings of a mesh into one with lower dimensionality or simplier form using graph decomposition. The embeddings of three-dimensional meshes using graph decomposition is discussed in Section 5. We extend the results to embeddings of meshes with wraparound in Section 6, and many-to-one embedding in Section 7. Conclusion follows in Section 8.

2 Notation and definitions

$\mathcal{V}(G)$ is the vertex set and $\mathcal{E}(G)$ the edge set of a graph G. $Hamming(x, y)$ is the Hamming distance between x and y. $|S|$ is the cardinality of a set S. Let j_m be the mth bit of the binary representation of j with the least significant bit being the 0th bit. Let $\lceil x \rceil_2$ denote $2^{\lceil \log_2 x \rceil}$.

Definition 1 An embedding φ of a *guest* graph, G, into a *host* graph, H, is a one-to-one mapping from each node $\varphi(i)$ in G to a unique node in H, and from each edge (i,j) in G to a path in H starting at node $\varphi(i)$ and ending at node $\varphi(j)$. The *expansion* of the embedding φ is

$$exp_\varphi = \frac{|\mathcal{V}(H)|}{|\mathcal{V}(G)|}.$$

The expansion is a measure of processor utilization. Denote the path (an ordered set of edges) corresponding to the edge e by $\varphi(e)$.

Definition 2 The *dilation* of an edge $e \in \mathcal{E}(G)$ is the length of the path $\varphi(e)$:

$$dil_\varphi(e) = |\varphi(e)|.$$

The *dilation* of the embedding φ is

$$dil_\varphi = \max_{e \in \mathcal{E}(G)} dil_\varphi(e).$$

The *average dilation* of the embedding φ is

$$\frac{1}{|\mathcal{E}(G)|} \sum_{e \in \mathcal{E}(G)} dil_{\varphi}(e).$$

Definition 3 The *congestion* of an edge $e \in \mathcal{E}(H)$ is the number of edges in G with images including e:

$$cong_{\varphi}(e) = \sum_{e' \in \mathcal{E}(G)} |\{e\} \cap \varphi(e')|.$$

The *congestion* of the mapping φ is

$$cong_{\varphi} = \max_{e' \in \mathcal{E}(H)} cong_{\varphi}(e').$$

The *average congestion* is similarly defined.

3 Preliminaries

3.1 Dilation one embeddings

The following theorem due to Havel and Móravek [12] shows that for certain meshes, embedding with minimal expansion and dilation one is impossible.

Theorem 1 [12] *If an $\ell_1 \times \ell_2 \times \cdots \times \ell_k$ mesh is embedded in an n-cube with dilation one, then $n \geq \sum_{i=1}^{k} \lceil \log_2 \ell_i \rceil$.*

Theorem 1 was independently rediscovered in [3], [8], [14] and [11]. See [14] for the proof. From the theorem follows that the expansion is in the range of 1 to 2^k. The percentage of meshes for which Gray code embedding [3], [15], [16], [22] yields minimal-expansion embeddings decreases with the number of axes of the mesh. Determining the asymptotic expansion for Gray code embedding is transformed to the following probability problem. Let a_i, $i \geq 1$, be a variable uniformly distributed over an interval $(\frac{1}{2}, 1]$, and a_i and a_j be independent variables for all $i \neq j$. Then, the probability that $\prod_{i=1}^{k} a_i \in (1/2^{\beta+1}, 1/2^{\beta}]$ is the asymptotic fraction of embedding k-dimensional meshes using Gray code embedding with an expansion 2^{β}. For minimal expansion $\beta = 0$.

Let $\alpha \in (\frac{1}{2}, 1]$ and $f_k(\alpha)$ be the probability that $\alpha \leq \prod_{i=1}^{k} a_i \leq 1$. Then,

Theorem 2 [13] *The fraction of all k-dimensional meshes for which a binary-reflected Gray code embedding yields minimum expansion is $f_k(\frac{1}{2}) = 2^k(1 - \frac{1}{2}\sum_{i=0}^{k-1} \frac{\ln^i 2}{i!})$, asymptotically.*

Figure 1 shows $f_k(\frac{1}{2})$ as a function of the number of dimensions, k. $f_2(\frac{1}{2}) = 2(1 - \ln 2) \approx 0.61$ and $f_3(\frac{1}{2}) = 4(1 - \ln 2 - \frac{\ln^2 2}{2}) \approx 0.27$.

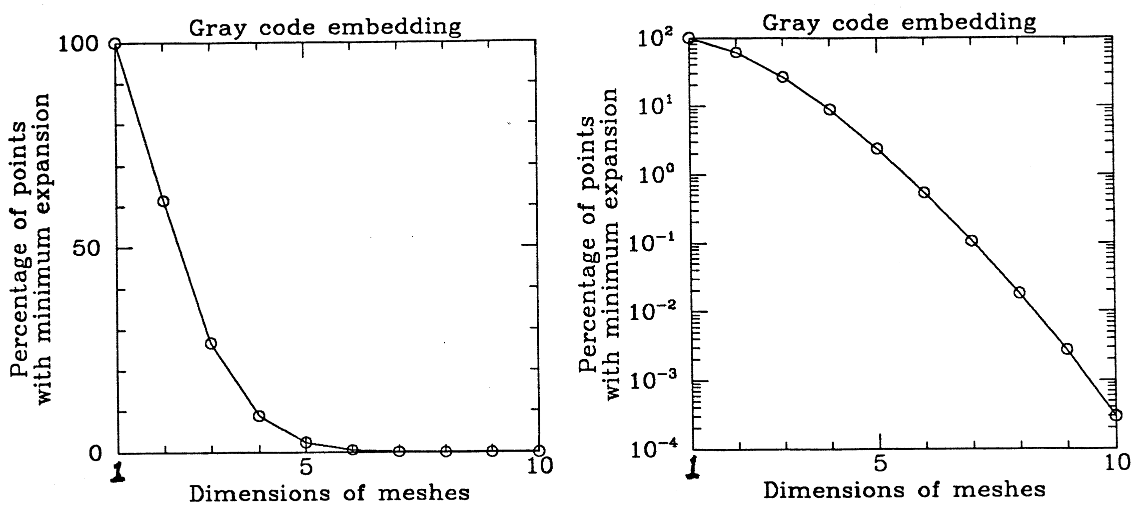

Figure 1: The asymptotic fraction of the domain $(\lceil \ell_i \rceil_2 / 2) < \ell_i \leq \lceil \ell_i \rceil_2$ for which minimum expansion is attained by Gray code embedding. The right plot has a logarithmic scale for the y-axis.

3.2 Reshaping techniques

Reshaping an $\ell_1 \times \ell_2 \times \cdots \times \ell_k$ mesh is the embedding of the mesh in an $\ell'_1 \times \ell'_2 \times \cdots \times \ell'_k$ mesh. The number of axes is preserved, but the length of the different axes are changed. We only consider reshaping an $\ell_1 \times \ell_2$ mesh into an $N_1 \times N_2$ mesh, where $N_1 = 2^{n_1}$ and $N_2 = 2^{n_2}$, such that $N_1 N_2 = \lceil \ell_1 \ell_2 \rceil_2$. Step embedding [1] and modified step embedding yield dilation three. Folding [19], line compression [1], and modified line compression [4] yield dilation two. Embeddings of a multidimensional mesh into another multidimensional mesh of different shape and cardinality are studied in [17] and [21]. By making the reshaped mesh having axes with lengths being powers of two, a Gray code embedding can be applied to the reshaped mesh [15]. See [13] for a detailed discussion.

3.3 Direct cube embeddings

In [14], we gave three dilation-two, minimal-expansion embeddings of two-dimensional meshes in Boolean cubes. The three meshes are of shapes 3×5, 7×9 and 11×11. The congestion is two as shown in [13]. By using these three embeddings, graph decomposition technique and Gray code embedding, all two-dimensional meshes with ≤ 64 nodes can be embedded into a minimal cube with dilation two and congestion two, with the exception of the embedding of the 3×21 mesh. Minimal expansion and dilation embeddings for all two-dimensional meshes in Boolean cubes was recently found by Chan [4]. However, the congestion is not known for Chan's embeddings. Moreover, Chan's embeddings tempt to have a larger average dilation than our direct embeddings especially for large meshes. Both the average dilation and average congestion for our direct embeddings are one asymptotically. In [13], we further gave two dilation-two, minimal-expansion embeddings of a $3 \times 3 \times 3$

mesh and a $3 \times 3 \times 7$ mesh in Boolean cubes. See [13] for further discussion and comparison of various reshaping techniques, Chan's methods and our direct embeddings.

4 Embedding by graph decomposition

4.1 The theorem

In this section we state and prove a few properties of product graphs, and the embedding characteristics of the product graph as a function of the embedding characteristics of the graphs forming the product graph.

Definition 4 The (Cartesian) *product* graph $G_1 \times G_2$ of a graph G_1 and a graph G_2 is defined as

$$\mathcal{V}(G_1 \times G_2) = \{[u_i, v_i] \mid \forall u_i \in \mathcal{V}(G_1), v_i \in \mathcal{V}(G_2)\}, \text{ and}$$

$$\mathcal{E}(G_1 \times G_2) = \{([u_i, v_i], [u_i, v_j]) \mid \forall u_i \in \mathcal{V}(G_1), (v_i, v_j) \in \mathcal{E}(G_2)\}$$

$$\bigcup \{([u_i, v_i], [u_j, v_i]) \mid \forall v_i \in \mathcal{V}(G_2), (u_i, u_j) \in \mathcal{E}(G_1)\}.$$

In the following, we will refer to the edges of the former set as G_2-type and of the latter set as G_1-type. $G_1 \times G_2$ can be derived by replacing each node of G_1 by G_2 and replacing each edge of G_1 by a set of edges connecting corresponding nodes of G_2. Note that the Cartesian product $\times$ is commutative and associative. Also, $|\mathcal{V}(G_1 \times G_2)| = |\mathcal{V}(G_1)| * |\mathcal{V}(G_2)|$ and $|\mathcal{E}(G_1 \times G_2)| = |\mathcal{V}(G_1)| * |\mathcal{E}(G_2)| + |\mathcal{V}(G_2)| * |\mathcal{E}(G_1)|$.

Theorem 3 *Let φ_i be an embedding function which maps a graph G_i into a graph H_i with expansion ε_i, dilation d_i, and congestion c_i, for $i = \{1, 2\}$. Then, there exists an embedding function φ that maps the graph $G = G_1 \times G_2$ into the graph $H = H_1 \times H_2$ with expansion $\varepsilon = \varepsilon_1\varepsilon_2$, dilation $d = \max(d_1, d_2)$ and congestion $c = \max(c_1, c_2)$.*

Proof: We prove the theorem by constructing an embedding function φ. Let $\mathcal{S}_1^{v_i} = \{([u_j, v_i], [u_k, v_i]) | \forall (u_j, u_k) \in \mathcal{E}(G_1)\}$ and $\mathcal{S}_2^{u_i} = \{([u_i, v_j], [u_i, v_k]) | \forall (v_j, v_k) \in \mathcal{E}(G_2)\}$. Clearly,

$$\mathcal{E}(G_1 \times G_2) = (\cup_{v_i \in \mathcal{V}(G_2)} \mathcal{S}_1^{v_i}) \bigcup (\cup_{u_i \in \mathcal{V}(G_1)} \mathcal{S}_2^{u_i}).$$

$\mathcal{S}_1^{v_i}$ is a copy of G_1 identified by node v_i in G_2. For the host graph H, we define $\mathcal{T}_1^{v_i}$ and $\mathcal{T}_2^{u_i}$ similarly. Hence,

$$\mathcal{E}(H_1 \times H_2) = (\cup_{v_i \in \mathcal{V}(H_2)} \mathcal{T}_1^{v_i}) \bigcup (\cup_{u_i \in \mathcal{V}(H_1)} \mathcal{T}_2^{u_i}).$$

An embedding function φ is derived from φ_1 and φ_2 by letting any edge $([u_j, v_i], [u_k, v_i]) \in \mathcal{S}_1^{v_i}$ corresponding to the edge $(u_j, u_k) \in \mathcal{E}(G_1)$ be mapped to the path

$$\{([\varphi_1(u_j), \varphi_2(v_i)], [w_1, \varphi_2(v_i)]), ([w_1, \varphi_2(v_i)], [w_2, \varphi_2(v_i)]), \cdots,$$

$$([w_{p-1}, \varphi_2(v_i)], [\varphi_1(u_k), \varphi_2(v_i)])\} \subseteq \mathcal{T}_1^{\varphi_2(v_i)}$$

in H, where the edge $(u_j, u_k) \in \mathcal{E}(G_1)$ is mapped to the path

$$\varphi_1((u_j, u_k)) = \{(\varphi_1(u_j), w_1), (w_1, w_2), \cdots, (w_{p-1}, \varphi_1(u_k))\}$$

in H_1. The mapping of edges in $\mathcal{S}_2^{v_i}$ are defined analogously. The dilation of any edge in $\mathcal{S}_1^{v_i}$ is the same as the dilation of the corresponding edge in $\mathcal{E}(G_1)$, and the dilation of any edge in $\mathcal{S}_2^{v_i}$ is the same as that of the corresponding edge in $\mathcal{E}(G_2)$. From the definition of φ it follows that any edge $e \in \mathcal{S}_1^{v_i}$, $\varphi(e) \subseteq \mathcal{T}_1^{\varphi_2(v_i)}$. From the definition of a product graph it follows that copies of H_1 (H_2) identified by different nodes in H_2 (H_1) are disjoint. Therefore, the congestion of all edges in $H_1 \times H_2$ is preserved. ▮

If the embedding function φ_i for $i = \{1, 2\}$ yields the average dilation $\bar{d}_i$, the average congestion $\bar{c}_i$ and

$$\alpha = \frac{|\mathcal{V}(G_1)| * |\mathcal{E}(G_2)|}{|\mathcal{E}(G_1 \times G_2)|} \text{ and } \beta = \frac{|\mathcal{V}(H_1)| * |\mathcal{E}(H_2)|}{|\mathcal{E}(H_1 \times H_2)|},$$

then, the embedding function φ has the average dilation $\bar{d} = \alpha\bar{d}_2 + (1-\alpha)\bar{d}_1$, and the average congestion $\bar{c} = \beta\bar{c}_2 + (1-\beta)\bar{c}_1$.

Corollary 1 *Let φ_i, $1 \leq i \leq r$, be embedding functions that maps graphs G_i into n_i-cubes with expansion ε_i, dilation d_i, and congestion c_i. Then, there exists an embedding function φ which maps a graph $G_1 \times G_2 \times \cdots \times G_r$ into a $\sum_{i=1}^r n_i$-cube with expansion $\varepsilon = \Pi_{i=1}^r \varepsilon_i$, dilation $d = \max_i d_i$, and congestion $c = \max_i c_i$.*

The fact that the dilation for the embedding of a product graph is the maximum dilation for the embedding of any graph used for the composition was observed in [20], [23], [18] and [10]. The corollary is used implicitly in [15] for the embedding of meshes by binary-reflected Gray codes, and in [7] and [14] for the embedding of two-dimensional meshes by a combination of direct embedding and Gray code embedding.

Corollary 2 *Let φ_i be an embedding function which maps an $\ell_{i1} \times \ell_{i2} \times \cdots \times \ell_{ik}$ mesh M_i into an n_i-cube with expansion ε_i, dilation d_i, and congestion c_i for $1 \leq i \leq r$. Then, there exists an embedding function φ which maps an $\ell_1 \times \ell_2 \times \cdots \times \ell_k$ mesh M into a $(\sum_{i=1}^r n_i)$-cube with expansion $\varepsilon = \prod_{i=1}^r \varepsilon_i$, dilation $d \leq \max_i d_i$, and congestion $c \leq \max_i c_i$, where $\ell_j = \prod_{i=1}^r \ell_{ij}$ for $1 \leq j \leq k$.*

Proof: It follows from Corollary 1 and the three facts below:

- The product graph of an $\ell_1 \times \ell_2 \times \cdots \times \ell_k$ mesh and an $\ell'_1 \times \ell'_2 \times \cdots \times \ell'_{k'}$ mesh is an $\ell_1 \times \ell_2 \times \cdots \times \ell_k \times \ell'_1 \times \ell'_2 \times \cdots \times \ell'_{k'}$ mesh.
- The product graph of an n_1-cube and an n_2-cube is an $(n_1 + n_2)$-cube.
- [21] An $\ell_1 \times \ell_2 \times \cdots \times \ell_k$ mesh is a subgraph of the mesh

$$(\ell_{11} \times \ell_{21} \times \cdots \times \ell_{r1}) \times (\ell_{12} \times \ell_{22} \times \cdots \times \ell_{r2}) \times \cdots$$

$$\times(\ell_{1k} \times \ell_{2k} \times \cdots \times \ell_{rk}), \quad \text{if } \prod_{i=1}^r \ell_{ij} = \ell_j, \ \forall 1 \leq j \leq k. \ ▮$$

Let M_i be an $\ell_{i1} \times \ell_{i2} \times \cdots \times \ell_{ik}$ mesh for $i = \{1, 2\}$ and $\ell_j = \ell_{1j}\ell_{2j}$ for $1 \leq j \leq k$. The embedding function φ for an $\ell_1 \times \ell_2 \times \cdots \times \ell_k$ mesh M being a subgraph of $M_1 \times M_2$ is defined in terms of the embedding functions φ_1 for M_1 and φ_2 for M_2. Let $z = (z_1, z_2, \cdots, z_k), 0 \leq z_i < \ell_i$ be a node in M, $x = (x_1, x_2, \cdots, x_k), 0 \leq x_i < \ell_{1i}$ be a node in M_1, $y = (y_1, y_2, \cdots, y_k), 0 \leq y_i < \ell_{2i}$ be a node in M_2, and $z_i = y_i\ell_{1i} + x_i, 1 \leq i \leq k$. The embedding of axis i of M consists of the embedding of ℓ_{2i} instances of axis ℓ_{1i}. Define

$$\tilde{\varphi}_1(y_1, y_2, \cdots, y_k, x_1, x_2, \cdots, x_k) = \varphi_1((x'_1, x'_2, \cdots, x'_k))$$

$$\text{where } x'_i = \begin{cases} x_i, & \text{if } y_i \text{ is even,} \\ \ell_{1i} - 1 - x_i, & \text{otherwise.} \end{cases}$$

The function $\tilde{\varphi}_1$ differs from the function φ_1 in that a reflection of the embedding of axis i of M_1 is performed for instances for which y_i is odd. The function φ is defined as follows:

$$\varphi((z_1, z_2, \cdots, z_k)) =$$
$$\varphi_2((y_1, y_2, \cdots, y_k))||\tilde{\varphi}_1(y_1, y_2, \cdots, y_k, x_1, x_2, \cdots, x_k),$$

where "||" is the concatenation operator. If $\ell_{1i} = 2^{n_i}$, then a binary-reflected Gray code G is used for each axis of M_1 and $\varphi_1((x_1, x_2, \cdots, x_k)) = G(x_1)||G(x_2)||\cdots||G(x_k)$. The embedding function φ takes the form:

$$\varphi((z_1, z_2, \cdots, z_k)) =$$
$$\varphi_2((y_1, y_2, \cdots, y_k))||\tilde{G}(y_1, x_1)||\tilde{G}(y_2, x_2)||\cdots||\tilde{G}(y_k, x_k),$$
$$\text{where } \tilde{G}(y_i, x_i) = \begin{cases} G(x_i), & \text{if } y_i \text{ is even,} \\ G(2^{n_i} - 1 - x_i), & \text{otherwise.} \end{cases}$$

An instance of axis i of mesh M_1 is traversed for every node along axis i of M_2. All edges along axis i of M_1 have dilation one for every i. With a dilation d embedding of mesh M_2 there exists at least one edge for some i that has dilation d for mesh M_2. By performing the embedding of axis i of the mesh M by traversing all edges along axis i of mesh M_1 for every edge of axis i of mesh M_2 the average dilation is minimized. Let $\bar{d_2}(i)$ be the average dilation of the edges of axis i in the mesh M_2, then the average dilation of the embedding of the mesh M is

$$1 + \sum_{i=1}^{k} \left\{ (\bar{d_2}(i) - 1)2^{(\sum_{j=1}^{k} n_j) - n_i}(\ell_{2i} - 1)(\prod_{j=1}^{k} \ell_{2j})/\ell_{2i} \right\} \Big/$$
$$\sum_{i=1}^{k} \left\{ (\ell_{2i}2^{n_i} - 1)(\prod_{j=1}^{k} \ell_{2j}2^{n_j})/(\ell_{2i}2^{n_i}) \right\} \approx 1 + \sum_{i=1}^{k} \frac{\bar{d_2}(i) - 1}{k2^{n_i}}.$$

The approximated term shows that the average dilation decreases as the length of axis i of mesh M_1 increases.

4.2 The strategy

The general strategy for mesh embedding by graph decomposition is the following:

1. If the number of nodes along any axis is a power of two then the embedding of all nodes along that axis is by a binary-reflected Gray code. For instance, the embedding of a $12 \times 16 \times 20 \times 32$ mesh is reduced to the problem of embedding a 12×20 and a 16×32 mesh.

2. For the axes with lengths not being powers of two, a decomposition is sought into meshes for which good embeddings are known, and the product of the expansions for the decomposed meshes is minimized. For instance, the embedding of a 12×20 mesh can be reduced to the embedding of a 3×5 and a 4×4 mesh. Embedding a $3 \times 25 \times 3$ mesh can be reduced to the embedding of two 3×5 meshes.

3. If the axes lengths are not powers of two, but can be increased slightly without increasing the size of the cube for a minimal expansion of the original mesh, then the mesh may be extended and the procedure just mentioned applied to the extended mesh. For instance, a $3 \times 3 \times 23$ mesh can be extended to a $3 \times 3 \times 25$ mesh, which is treated with the scheme above.

5 Embeddings of three-dimensional meshes

For three-dimensional meshes, we use these direct embeddings extended with the two-dimensional result in [4], and the graph decomposition technique. We achieve dilation-two minimal-expansion embeddings for 96% of the three-dimensional meshes contained within, or equal to, a $512 \times 512 \times 512$ mesh.

Performing a dilation-two embedding of a two-dimensional mesh defined by any pair of axes, and a Gray code embedding of the third axis results in one of the relative expansions

$$\frac{\lceil \ell_1\ell_2 \rceil_2 \lceil \ell_3 \rceil_2}{\lceil \ell_1\ell_2\ell_3 \rceil_2}, \frac{\lceil \ell_2\ell_3 \rceil_2 \lceil \ell_1 \rceil_2}{\lceil \ell_1\ell_2\ell_3 \rceil_2}, \text{ or } \frac{\lceil \ell_3\ell_1 \rceil_2 \lceil \ell_2 \rceil_2}{\lceil \ell_1\ell_2\ell_3 \rceil_2}.$$

The relative expansions are either equal to one or two. Note that more than one relative expansion may be one, such as for a $5 \times 10 \times 11$ mesh, or no relative expansion may be one, such as for the $6 \times 11 \times 7$ mesh. Choosing the two axes that have the lowest values of $\ell_1/\lceil \ell_1 \rceil_2$, $\ell_2/\lceil \ell_2 \rceil_2$, and $\ell_3/\lceil \ell_3 \rceil_2$, for the two-dimensional embedding results in the smallest relative expansion. For instance, for a $5 \times 6 \times 7$ mesh, the first two axes (of length five and six respectively) should be chosen for the two-dimensional embedding.

Another example where graph decomposition is effective is in the case of embedding a $21 \times 9 \times 5$ mesh. It can be embedded with minimal expansion by combining the $7 \times 9 \times 1$ direct embedding with the $3 \times 1 \times 5$ direct embedding. Another effective decomposition is the product of a $21 \times 3 \times 1$ mesh and a $1 \times 3 \times 5$ mesh.

The fraction of three-dimensional meshes, for which the decomposition technique combined with the two- and three-dimensional embedding techniques yield minimal-expansion embeddings with a dilation of at most two, is given in Figure 2. In the figure, $S_i(\varepsilon)$ is the cumulative percentage of meshes that have a relative expansion ε by applying the embedding methods with an index less than or equal to i below:

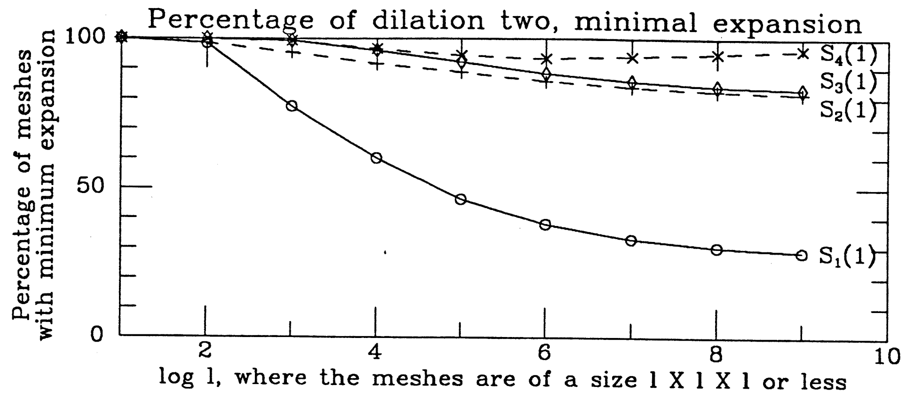

Figure 2: The cumulated percentage of the $\ell_1 \times \ell_2 \times \ell_3$ meshes where $1 \leq \ell_i \leq 2^n$ for $1 \leq n \leq 9$.

1. Apply Gray code embedding.
2. Apply the modified line compression technique [4] to any pair of axes and apply Gray code to the third axis.
3. Apply the $3 \times 3 \times 3$ or $3 \times 3 \times 7$ embedding combined with Gray code by Corollary 2.
4. For an $\ell_1 \times \ell_2 \times \ell_3$ mesh, find $\ell_2' \ell_2'' \geq \ell_2$ such that $\lceil \ell_1 \ell_2' \rceil_2 \lceil \ell_2'' \ell_3 \rceil_2 = \lceil \ell_1 \ell_2 \ell_3 \rceil_2$, Corollary 2 and [4]. The procedure is repeated for decomposing ℓ_1 and ℓ_3.

For a mesh of size less than or equal to $512 \times 512 \times 512$, the cumulated percentages grows as the sequence: 28.5%, 81.5%, 82.9%, 96.1%. Applying the method in [4] to any pair of axes, only allows about 81.5% of the meshes to achieve minimal expansion. Since the congestion for a product graph is the maximum congestion of any graph used for the composition, any three-dimensional mesh composed from any two-dimensional mesh with a congestion two mapping and Gray code have congestion two.

For the three-dimensional meshes of 128 nodes or less, the $5 \times 5 \times 5$ mesh is the only mesh for which we do not know of a minimal-expansion dilation-two embedding, if it exists. For three-dimensional meshes with up to 256 nodes, there are four additional meshes for which the same statement applies: $5 \times 7 \times 7$, $3 \times 9 \times 9$, $5 \times 5 \times 10$ and $3 \times 5 \times 17$.

6 Embeddings of wraparound meshes

Lemma 1 [21] *Let $\ell_i = \ell_i' \ell_i''$ and ℓ_i be even for all $1 \leq i \leq k$. Then, the $\ell_1 \times \ell_2 \times \cdots \times \ell_k$ wraparound mesh is a subgraph of the product graph of the $\ell_1' \times \ell_2' \times \cdots \times \ell_k'$ mesh and the $\ell_1'' \times \ell_2'' \times \cdots \times \ell_k''$ mesh (both without wraparound).*

Proof: Every $\ell_i' \times \ell_i''$ mesh for which $\ell_i' \ell_i''$ is even contains a ring of size $\ell_i' \ell_i''$ as a subgraph [21]. ∎

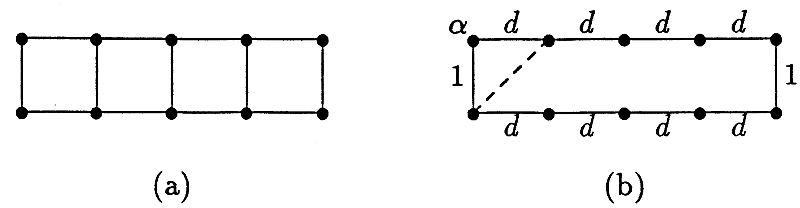

Figure 3: A linear array of size ℓ_i, ℓ_i odd, embedded in the product graph of a linear array of size $\lceil \ell_i/2 \rceil$ and a 1-cube. The $\lceil \ell_i/2 \rceil$ linear array has a dilation d embedding.

Lemma 2 *Let φ_1 be an embedding $G \to I$ and φ_2 be an embedding $I \to H$. Then, there exists an embedding function $\varphi : G \to H$ such that*

$$dil_{\varphi}(e) \leq \sum_{e_i \in \varphi_1(e)} dil_{\varphi_2}(e_i).$$

Lemma 3 *An $\ell_1 \times \ell_2 \times \cdots \times \ell_k$ wraparound mesh M can be embedded into a minimal hypercube with dilation $\leq d+1$, if there exists an embedding φ that maps the $\lceil \ell_1/2 \rceil \times \lceil \ell_2/2 \rceil \times \cdots \times \lceil \ell_k/2 \rceil$ mesh M_2 into a minimal hypercube with dilation d and $\lceil \prod_{i=1}^{k} \ell_i \rceil_2 = 2^k \lceil \prod_{i=1}^{k} \lceil \ell_i/2 \rceil \rceil_2$. The dilation is $\leq d$, if all ℓ_i's are even.*

Proof: Consider the embedding of a $2\lceil \ell_1/2 \rceil \times 2\lceil \ell_2/2 \rceil \times \cdots \times 2\lceil \ell_k/2 \rceil$ wraparound mesh $\tilde{M}$. By the assumptions of the lemma, Theorem 3, and Lemma 1 the wraparound mesh $\tilde{M}$ can be embedded into a minimal hypercube with dilation d. (The $2 \times 2 \times \cdots \times 2 = 2^k$ mesh is taken as the mesh M_1.)

We now embed the wraparound mesh M in the wraparound mesh $\tilde{M}$ by removing one hyperplane for each axis i of odd length. The edge in the mesh M connecting nodes on the two sides of the removed hyperplane is simulated by a length-two path through the removed hyperplane in $\tilde{M}$. In the Boolean cube embedding of $\tilde{M}$ the removed hyperplane connects to two neighboring hyperplanes through two sets of edges of dilation one and d, respectively. The edge of M which is a path of length two in $\tilde{M}$ has a dilation of edges of $d+1$ in the cube embedding, according to Lemma 2. ∎

Figure 3-(a) demonstrates the ith coordinate of the product graph of the mesh M_2 and the k-cube for which $\lceil \ell_i/2 \rceil = 5$. All the horizontal edges have dilation $\leq d$ due to the embedding φ. All the vertical edges have dilation one. It is easy to see from Figure 3-(b) that if $\ell_i = 9$, then the node α is removed and the dilations of the two edges incident to the removed node are $\leq d$ and one, respectively. So, the dilation for the new "logical edge" (the dashed edge in the figure) is $\leq d+1$.

Intuitively, the mesh $\tilde{M}$ is partitioned into 2^k submeshes of the form $\lceil \ell_1/2 \rceil \times \lceil \ell_2/2 \rceil \times \cdots \times \lceil \ell_k/2 \rceil$. The submeshes are labeled $\tilde{M}_i$, $0 \leq i < 2^k$, such that submesh i and submesh j are adjacent if $Hamming(i,j) = 1$. Submesh $i = (i_{k-1} i_{k-2} \cdots i_0)$ is reflected for axis r if $i_r = 1$ for all r's. After this reflection the same embedding function φ is applied to all submeshes for their embeddings in their respective cubes. Figure 4 shows the four

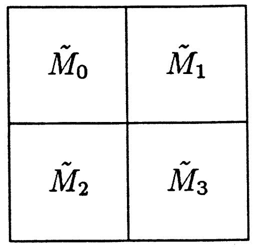

Figure 4: Partitioning for the embedding of an wraparound mesh.

submeshes for a two-dimensional case, in which the submeshes $\tilde{M}_1$ and $\tilde{M}_3$ are reflected horizontally and the submeshes $\tilde{M}_2$ and $\tilde{M}_3$ are reflected vertically before the embedding function is applied.

Clearly, if all the ℓ_i's are even, then the condition $\lceil\prod_{i=1}^k \ell_i\rceil_2 = 2^k\lceil\prod_{i=1}^k\lceil\ell_i/2\rceil\rceil_2$ is satisfied. If this condition holds, then the expansion remains minimal by using a mesh with wraparound of a slightly larger size (or of the same size) as an intermediate graph.

Lemma 4 *An $\ell_1 \times \ell_2 \times \cdots \times \ell_k$ wraparound mesh can be embedded into a minimal hypercube with dilation $\leq \max(d, 2)$, if there exists an embedding that maps the $\lceil\ell_1/4\rceil \times \lceil\ell_2/4\rceil \times \cdots \times \lceil\ell_k/4\rceil$ mesh M_2 into a minimal hypercube with dilation d and $\lceil\prod_{i=1}^k \ell_i\rceil_2 = 4^k\lceil\prod_{i=1}^k\lceil\ell_i/4\rceil\rceil_2$.*

Proof: Consider the embedding of a $4\lceil\ell_1/4\rceil \times 4\lceil\ell_2/4\rceil \times \cdots \times 4\lceil\ell_k/4\rceil$ wraparound mesh $\tilde{M}$. Apply an argument similar to the one in the proof of Lemma 3. (The $4\times4\times\cdots\times4 = 4^k$ mesh is taken as the mesh M_1.)

Figure 5-(a) and (c) shows one axis of the product graph of the mesh M_2 and the $2k$-cube with $\lceil\ell_i/4\rceil = 5$ and 4, respectively. All the horizontal edges have dilation $\leq d$, and all the vertical edges have dilation one. Figure 5-(b) and (d) show an embedded linear array of size $4\lceil\ell_i/4\rceil$ (by ignoring the dashed edges). Consider the case where $\ell_i \bmod 4 \neq 0$. We wish to show that by removing one, two and three nodes, respectively, the newly formed "logical edges" have a dilation of $\leq \max(d, 2)$. When $\ell_i \bmod 4 = 3$, remove node α. When $\ell_i \bmod 4 = 2$, remove nodes β and γ (but keep node α). When $\ell_i \bmod 4 = 1$, remove all the three nodes α, β and γ. The newly-formed "logical edges" are marked by the dashed edges in the figure. Clearly, all the dashed edges preserve the property of the dilation $\leq \max(d, 2)$.

Since the above proof requires that $\lceil\ell_i/4\rceil \geq 3$, it remains to be proved that if $\lceil\ell_i/4\rceil = 2$ or 1, the lemma still holds. Figure 5-(e) shows for $\ell_i = 5$, 6, 7 and 8. For $1 \leq \ell_i \leq 4$, the lemma can be derived easily. ∎

Note that there exist several ways to embed a ring for Figure 5-(a) and (b) that preserve the dilation of the edges. The selected embedding minimizes the average dilation.

Corollary 3 *Any two-dimensional wraparound mesh $\ell_1 \times \ell_2$ can be embedded into a minimal hypercube with dilation at most two, if $\lceil\ell_1\ell_2\rceil_2 = 16\lceil\lceil\ell_1/4\rceil\lceil\ell_2/4\rceil\rceil_2$ or both ℓ_1 and ℓ_2 are even. Any two-dimensional wraparound mesh $\ell_1 \times \ell_2$ can be embedded into a minimal hypercube with dilation at most three, if $\lceil\ell_1\ell_2\rceil_2 = 4\lceil\lceil\ell_1/2\rceil\lceil\ell_2/2\rceil\rceil_2$.*

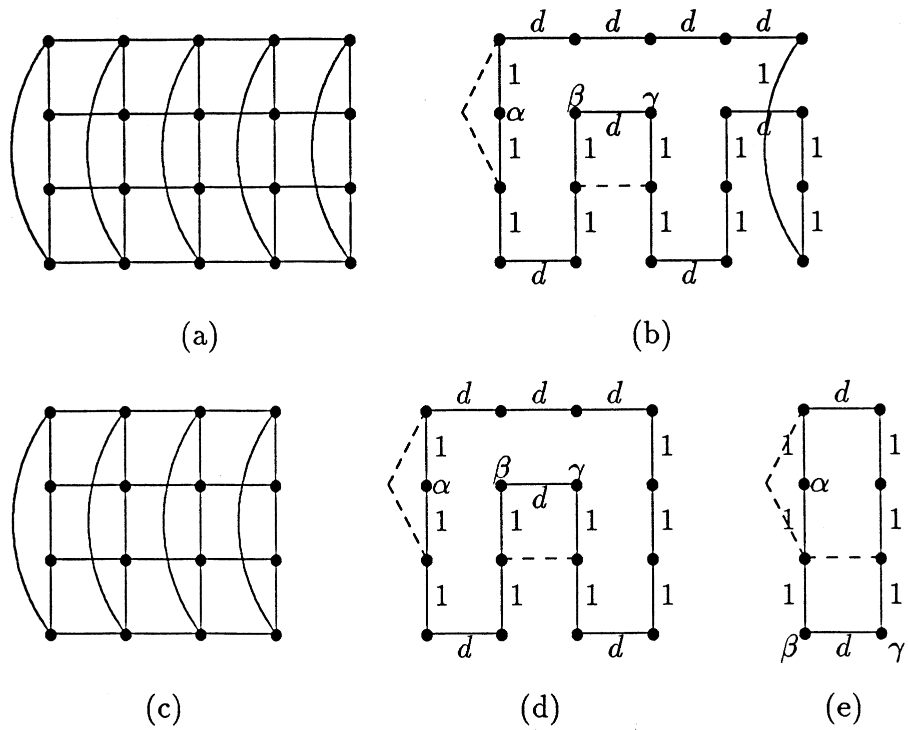

Figure 5: A linear array of size ℓ_i, embedded in the product graph of a linear array of size $\lceil \ell_i/4 \rceil$ and a 2-cube, where the latter linear array has a dilation d and the 2-cube has a dilation one embedding.

Proof: The former follows from [4], Lemmas 4 and 3. The latter follows from [4] and Lemma 3. ▮

7 Many-to-one embeddings

When embedding is performed in a many-to-one manner, we use *load-factor* instead of *expansion* to measure the processor utilization.

Definition 5 The *load-factor* of an embedding $\varphi : G \to H$ is the maximum number of nodes in the graph G which are mapped to the same node in the graph H, i.e.,

$$\max_{v' \in \mathcal{V}(H)} \left\{ \sum_{v \in \mathcal{V}(G)} |\{\varphi(v)\} \cap \{v'\}| \right\}.$$

Theorem 4 *Let φ_i be an embedding function which maps a graph G_i into a graph H_i with load-factor f_i, dilation d_i and congestion c_i for $i = \{1, 2\}$. Then, there exists an embedding function which maps the graph $G_1 \times G_2$ into the graph $H_1 \times H_2$ with load-factor $f = f_1 f_2$, dilation $d = \max(d_1, d_2)$ and congestion $c = \max(f_1 c_2, f_2 c_1)$. Further, for the graph $H_1 \times H_2$, the congestion of an H_1-type edge increases by at most a factor of f_2, and the congestion of an H_2-type edge increases by at most a factor of f_1.*

Proof: Consider a node $u_i \in H_1$ and a node $v_i \in H_2$. There are at most f_1 nodes in G_1 that are mapped to node u_i. Similarly, there are at most f_2 nodes in G_2 that are mapped to node v_i. The corresponding product node in $H_1 \times H_2$, $[u_i, v_i]$ contains at most $f_1 f_2$ nodes in $G_1 \times G_2$. The proof of dilation is similar to that in Theorem 3.

For the congestion, consider an edge in $H_1 \times H_2$. It is either an H_1-type edge or an H_2-type edge. For edges of the H_1-type, the congestion is $\leq f_1 c_2$ from definition of graph product. Similarly, edges of the H_2-type have a congestion $\leq f_2 c_1$. ▮

Lemma 5 *Let φ be an embedding function which maps an $\ell_1 \times \ell_2 \times \cdots \times \ell_k$ mesh M into an n-cube with load-factor f, dilation d, and congestion of the edges of the ith axis c_i. Then, there exists an embedding function $\tilde{\varphi}$ which maps an $\ell_1 \ell'_1 \times \ell_2 \ell'_2 \times \cdots \times \ell_k \ell'_k$ mesh $\tilde{M}$ into an n-cube with load-factor $\tilde{f} = f \prod_{i=1}^{k} \ell'_i$, dilation $\tilde{d} = d$, and congestion $\tilde{c} = \max_{i=1}^{k} \{(c_i \prod_{j=1}^{k} \ell'_j)/\ell'_i\}$.*

Proof: Consider the following two facts:

1. An $\ell_1 \ell'_1 \times \ell_2 \times \ell_3 \times \cdots \times \ell_k$ mesh M_1 can be embedded into an n-cube with load-factor $f\ell'_1$, dilation d, and congestion of the edges of the first axis c_1.

2. An $\ell'_2 \times \ell'_3 \times \cdots \times \ell'_k$ mesh M_2 can be embedded into an 0-cube (i.e., one-node cube) with load-factor $\prod_{i=2}^{k} \ell'_i$, dilation 0, and congestion 0.

From Theorem 4, the mesh $M_1 \times M_2$ can be embedded into an n-cube with load-factor $f \prod_{i=1}^{k} \ell'_i$, dilation d, and congestion for the edges of the first axis $c_1 \prod_{i=2}^{k} \ell'_i$. Since the mesh $\tilde{M}$ is a subgraph of the mesh $M_1 \times M_2$ [21,13], the load-factor, dilation and congestion for the edges of the first axis also hold for the mesh $\tilde{M}$. Congestion for the edges of the ith axis can be similarly derived to be bounded from above by $(c_i \prod_{j=1}^{k} \ell'_j)/\ell'_i$. ▮

The property of the load-factor in this lemma was also observed in [9], independently.

Corollary 4 *An $\ell_1 2^{n_1} \times \ell_2 2^{n_2} \times \cdots \times \ell_k 2^{n_k}$ mesh M can be embedded into an $(\sum_{i=1}^{k} n_i)$-cube with dilation one, congestion $(\prod_{i=1}^{k} \ell_i)/\min_i\{\ell_i\}$, and load-factor optimal.*

Proof: Simply apply Gray code embedding to φ in Lemma 5 in mapping a $2^{n_1} \times 2^{n_2} \times \cdots \times 2^{n_k}$ mesh into a $(\sum_{i=1}^{k} n_i)$-cube with load-factor one, dilation one and congestion one. ▮

Corollary 5 *An $\ell_1 \times \ell_2 \times \cdots \times \ell_k$ mesh M can be embedded into an n-cube with dilation one and load-factor optimal within a factor of two, if there exists $\ell'_i 2^{n_i} \geq \ell_i$ for all $1 \leq i \leq k$ such that $\lceil \prod_{i=1}^{k} \ell_i \rceil_2 = \lceil \prod_{i=1}^{k} \ell'_i 2^{n_i} \rceil_2$ and $\sum_{i=1}^{k} n_i \geq n$.*

Proof: Let $n' = \sum_{i=1}^{k} n_i$. Let mesh M' be of a form $\ell'_1 2^{n_1} \times \ell'_2 2^{n_2} \times \cdots \times \ell'_k 2^{n_k}$. By Corollary 4, the mesh M' can be mapped into an n'-cube with dilation one and load-factor optimal. Since the mesh M can be embedded into the mesh M' with expansion < 2 (by one-to-one mapping), the load-factor for mapping from the mesh M to the n'-cube is optimal within a factor of two. For $n' > n$, we simply fold the cube until it reaches the right size. Both the load-factor and optimal load-factor increase by a factor of $2^{n'-n}$. ▮

For example, a 19×19 mesh can be embedded in up to a 5-cube with dilation one and load-factor optimal within a factor of two. This is because the $3 \cdot 2^3 \times 5 \cdot 2^2$ mesh contains the 19×19 mesh and $\lceil 19 \cdot 19 \rceil_2 = \lceil 3 \cdot 2^3 \cdot 5 \cdot 2^2 \rceil_2$. The load-factor is 15 and the optimal load factor is $\lceil 19 \cdot 19 / 2^5 \rceil = 12$.

8 Summary

A graph embedded by graph decomposition has a dilation and congestion equal to the maximum dilation and congestion of the embedding of any of the graphs into which it is decomposed. By applying the graph decomposition technique and using the dilation-two embeddings for two-dimensional meshes [14], [4] and two dilation-two embeddings of three-dimensional meshes, we have attained dilation-two minimal-expansion embeddings into Boolean cubes for 96% of all three-dimensional meshes of a size less than, or equal to, $512 \times 512 \times 512$.

The decomposition technique can be applied to the embedding of meshes with an arbitrary number of dimensions. We conjecture that a majority of the the higher dimensional meshes can be embedded with dilation two using the existing two-, and three-dimensional mesh embeddings of dilation two.

The embeddings of wraparound meshes can be easily constructed out of the embeddings for meshes without wraparound using the graph decomposition technique. As a special case, for all two-dimensional wraparound meshes $\ell_1 \times \ell_2$, we have a minimal-expansion embedding with dilation two if $\lceil \ell_1 \ell_2 \rceil_2 = 16 \lceil \lceil \ell_1/4 \rceil \lceil \ell_2/4 \rceil \rceil_2$ or both ℓ_1 and ℓ_2 are even; and with dilation three if $\lceil \ell_1 \ell_2 \rceil_2 = 4 \lceil \lceil \ell_1/2 \rceil \lceil \ell_2/2 \rceil \rceil_2$ (where $\lceil x \rceil_2 = 2^{\lceil \log_2 x \rceil}$).

The embeddings of a large mesh into a smaller hypercube can be performed by decomposing the mesh into two smaller meshes, and applying a low-dilation embedding between processors and a high-dilation embedding within processors.

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