Abstract The solution of linear systems having circulant coefficient matrices is considered in this paper. This kind of systems occur in many applications: prediction, time series analysis, spline approximation, difference solution of partial differential equations, etc. The methods presented here are more efficient than the Toeplitz type methods and are based on the fast Fourier transform as well as the circulant factorization of the "banded circulant" matrices.

On The Solution of Circulant Linear Systems

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1. Introduction

Many problems in mathematics and applied science lead to the solution of linear systems having circulant coefficient matrices, which are related to the periodicity of the problems. Examples are prediction [17], time series analysis[12], spline approximation[1], solution of certain partial differential equations[18], and possibly in many other applications.

Circulant matrices belong to the class of Toeplitz matrices. Linear systems with Toeplitz matrices can be solved with $O(n^2)$ arithmetic operations [2, 11, 15, 19]. If we use any of the methods for solving Toeplitz systems to solve linear systems having circulant coefficient matrices we will not be able to make full use of the properties of the circulant matrices, which are a) the orthonormal eigenvectors of all n-by-n circulant matrices are the columns of the Fourier matrix, the elements of which can be expressed with exponential or trigonometric functions, and b) their eigenvalues are the polynomials of the primitive n-th root of unity with the elements of the matrices as its coefficients[9]. Such nice properties make it simple to estimate the invertibility of the matrix, to compute their inverses(if they sxist), and to solve circulant linear systems. Furthermore, the fast Fourier transform may be employed to calculate the eigenvalues and to solve the systems since the components of the eigenvectors are exponential functions. The methods presented in this paper are based on the use of fast Fourier transform. The dominant work of the algorithms is in performing the fast Fourier transform, and therefore the algorithms solve circulant systems in $O(n \log_2 n)$ operations as opposed to the $O(n^2)$ arithmetic operations required by using other algorithms for solving linear systems having Toeplitz coefficient matrices.

In some applications, for example, in the finite difference solution of one dimensional elliptic equations subject to periodic boundary conditions[5, 18] and approximation of periodic functions using splines[1], "banded circulant" matrices are encountered. Under certain coditions such matrices can be factored as a product of tow simpler circulants, and the systems may then be solved by using the Sherman-Morrison formula, or its block version, the Woodbury formula[13]. This method is quite competitive with Gaussian elimination both in terms of arithmetic operations and storage requirements.

In the case where multi-dimensional problems are concerned the matrices of coefficients of the resulting linear systems are block circulant matrices. After some transformations and permutations we are led to a block diagonal matrix with circulant blocks on the diagonal. This reduces the problem to the solution of n circulant linear systems, which may be performed in parallel. An important example is the finite difference approximate solution of elliptic equation over a rectangle with periodic boundary conditions [5, 18].

In §2, we develop the Fourier-circulant method for solving circulant linear systems. In §3 the method for decomposing banded circulant matrices into two circulants and solving the system will be described. The methods presented in §2 and §3 will be extended to block circulant systems in §4. §5 applies the results of §2, §3 and §4 to elliptic equations, and finally, some numerical experiments will be given in §6.

In what follows, we employ the notation $diag(\delta_1, \delta_2, \ldots, \delta_n)$ to mean the n-by-n diagonal matrix with diagonal elements $\delta_1, \delta_2, \ldots, \delta_n$, A^* the complex conjugate transpose of the matrix A, and A^T the transpose of matrix A.

2. Fourier-Circulant Method

Consider the system of linear equations

$$Cx = b, \tag{2.1}$$

where C is a circulant matrix of order n, and x and b are n-vectors. An n-by-n matrix $C = (c_{ij})$ is a *circulant matrix* if $c_{ij} = c_{i+1,j+1}$, and the subscripts are taken modulo n. Thus a circulant matrix can be written

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix}.$$

It is clear that C contains at most n distinct elements, and therefore often denoted by

$$C = circ(c_0, c_1, c_2, \ldots, c_{n-1}). \tag{2.2}$$

It is well known[9] that if C is a circulant, then

$$C = F\, diag(\lambda_1, \lambda_2, \ldots, \lambda_n) F^*, \tag{2.3}$$

where F, called Fourier matrix, is n-by-n unitary matrix of the eigenvectors of matrix C with (i,j)-element $\left[n^{-\frac{1}{2}}\omega^{(i-1)(j-1)}\right]$, and

$$\lambda_k = \sum_{l=0}^{n-1} c_l [\omega^{k-1}]^l, \qquad k = 1, 2, \ldots, n \tag{2.4}$$

are the eigenvalues of C, and ω is the primitive n-th root of unity.

If C is nonsingular, then the inverse of C is also a circulant, and it is, from (2.3), given by

$$C^{-1} = F\,diag(\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1})F^*. \tag{2.5}$$

Thus the solution to (2.1) is

$$x = F\,diag(\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1})F^*b. \tag{2.6}$$

If C is singular, in which case some of the eigenvalues of C are zero, the inverse of C will not exist, but there will exist an unique matrix called the Moore-Penrose generalized inverse of C and denoted by C^+, which is geven by

$$C^+ = F\,diag(\lambda_1^+, \lambda_2^+, \ldots, \lambda_n^+)F^*, \tag{2.7}$$

where

$$\lambda_k^+ = \begin{cases} 0, & \text{if } \lambda_k = 0, \\ \lambda_k^{-1}, & \text{if } \lambda_k \neq 0, \end{cases} \qquad k = 1, 2, \ldots, n, \tag{2.8}$$

and the least squares solution of (2.1) is given by

$$\hat{x} = C^+b, \tag{2.9}$$

which is the solution to (2.1) with smallest 2-norm when the system is consistent[3, 14].

It is easily seen that premultiplying a vector by matrices F^* and F may be accomplished by the fast Fourier transform(FFT) and its inverse, respectively, and the eigenvalues of C can be calculated via FFT [8]. Thus the algorithm procceds as follows.

Algorithm CIRS (CIRculant Solver) solves the circulant system (2.1).

1. Transform $\tilde{b} = F^*b$ by FFT.
2. Compute the eigenvalues of C via FFT.
3. Calculate $\bar{b} = diag(\lambda_1^+, \lambda_2^+, \ldots, \lambda_n^+)\tilde{b}$.
4. Tramsform the vector $\bar{b}$ to obtain the solution vector $x = F\,\bar{b}$.

endalgorithm

The algorithm uses the fast Fourier tansform three times and is, therefore, an $O(n \log_2 n)$ algorithm.

3. Banded Circulant Systems

In the case where only c_0, c_1 and c_{n-1} are nonzero, the matrix, denoted by C_t, is *circulant tridiagonal*, and its eigenvalues can easily be computed. Assume that C_t is real, then the eigenvalues are given by

$$\lambda_k = c_0 + [c_1 + c_{n-1}]\cos(k-1)\frac{2\pi}{n} + i[c_1 - c_{n-1}]\sin(k-1)\frac{2\pi}{n}, \qquad k = 1, 2, \ldots, n, \tag{3.1}$$

where $i = \sqrt{-1}$. If the matrix is real and symmetric(denoted by C_{st}), i.e. $c_1 = c_{n-1}$, then the eigenvalues are real and given by

$$\lambda_k = c_0 + 2c_1\cos(k-1)\frac{2\pi}{n}, \qquad k = 1, 2, \ldots, n. \tag{3.2}$$

In these situations, the method presented in the previous section may be employed. Since the matrices C_t and C_{st} have the cyclic tridiagonal structure, we are able to take advantage of the fact that under certain conditions this special kind of matrices can be factored into the product of a circulant lower bidiagonal matrix and a circulant upper bidiagonal matrix, and the systems then may be solved by using the Sherman-Morrison formula[13]. In this section we will develop an algorithm for solving the systems

$$C_t x = b, \tag{3.3}$$

and

$$C_{st} x = b, \tag{3.4}$$

and then extend it to general banded circulant systems.

We assume, without loss of generality, that $c_0 > 0$. It was shown[6] that if the matrix C_t is strictly diagonal dominant, then there exist real numbers α, β and γ, such that $|\beta| < 1$, $|\gamma| < 1$, and

$$C_t = \alpha \hat{L}\hat{U}, \tag{3.5}$$

where

$$\hat{L} = \begin{pmatrix} 1 & & & & \beta \\ \beta & 1 & & & \\ & \ddots & \ddots & & \\ & & \beta & 1 & \\ & & & \beta & 1 \end{pmatrix},$$

$$\widehat{U} = \begin{pmatrix} 1 & \gamma & & & \\ & 1 & \gamma & & \\ & & \ddots & \ddots & \\ & & & 1 & \gamma \\ \gamma & & & & 1 \end{pmatrix}.$$

It is easy to verify that

(3.6)
$$\begin{cases} \alpha = \frac{1}{2}\left[c_0 + \sqrt{c_0^2 - 4c_1c_{n-1}}\right], \\ \beta = \frac{c_{n-1}}{\alpha}, \\ \gamma = \frac{c_1}{\alpha}. \end{cases}$$

To solve (3.3) we first solve

(3.7)
$$\widehat{L}h = \alpha^{-1}b,$$

and then

(3.8)
$$\widehat{U}x = h,$$

where h is an auxiliary vector and its components are denoted by h_j, $j = 1, 2, \ldots, n$.

Writing

$$\widehat{L} = L + \beta e_1 e_n^T$$

and

$$\widehat{U} = U + \gamma e_n e_1^T,$$

and using the Sherman-Morrison formula, (3.7) and (3.8) are reduced to the following bidiagonal systems

(3.9)
$$Lh = \alpha^{-1}\left\{b - \left[\varsigma \sum_{l=1}^{n} b_l(-\beta)^{n-l}\right] e_1\right\},$$

(3.10)
$$Ux = h - \left[\eta \sum_{l=1}^{n} h_l(-\gamma)^{l-1}\right] e_n,$$

where

$$L = \begin{pmatrix} 1 & & & & \\ \beta & 1 & & & \\ & \ddots & \ddots & & \\ & & \beta & 1 & \\ & & & \beta & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & \gamma & & & \\ & 1 & \gamma & & \\ & & \ddots & \ddots & \\ & & & 1 & \gamma \\ & & & & 1 \end{pmatrix},$$

e_1 and e_n are the first and the last column of the n-th order identity matrix, respectively, and

$$\varsigma = \beta/[1 - (-\beta)^n], \tag{3.11}$$

$$\eta = \gamma/[1 - (-\gamma)^n], \tag{3.12}$$

which may be solved by forward and backward sweeps, respectively, and the procedure is stable since $|\beta| < 1$ and $|\gamma| < 1$. The algorithm, thus, may be formulated as follows.

Algorithm CTS(Circulant Tridiagonal Solver) solves circulant tridiagonal system (3.3).

1. Compute the quantities α, β, and γ via (3.6).
2. Compute ς and η.
3. Solve equation (3.9) and (3.10).

endalgorithm

The algorithm solves equation (3.3) with $O(5n)$ arithmetic operations, which is more favorable than those proposed by Björck and Golub [4] requiring $O(9n)$ operations.

The method studied above may be extended to *"banded circulant" matrices.* Consider the symmetric "banded circulant" of the form

$$C_b = \begin{pmatrix} c_0 & c_1 & \cdots & c_p & & & & c_p & \cdots & c_1 \\ c_1 & \ddots & \ddots & & \ddots & & & & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & & & & c_p \\ c_p & & \ddots & \ddots & \ddots & & \ddots & & & \\ & \ddots & & \ddots & \ddots & \ddots & & \ddots & & \\ & & \ddots & & \ddots & \ddots & \ddots & & \ddots & \\ & & & \ddots & & \ddots & \ddots & \ddots & & c_p \\ c_p & & & & \ddots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & & & & \ddots & & \ddots & \ddots & c_1 \\ c_1 & \cdots & c_p & & & & c_p & \cdots & c_1 & c_0 \end{pmatrix}.$$

Assume that $p \ll n$, the order of the matrix C_b. If C_b is positive definite and elliptic[10], then there exists a real polynomial $l(z) = \beta_0 + \cdots + \beta_p z^p$, $\beta_0 > 0$, with no root inside the unit circle, such that the characteristic function $\phi(z)$ of A, defined by

$$\phi(z) = c_p z^p + \cdots + c_0 + \cdots + c_p z^{-p},$$

can be factored as $\phi(z) = l(z) \cdot l(1/z)$. There are several methods to compute this factor(see [10]).

We now assume that the factor $l(z)$ has been computed. It is easy to verify that

$$C_b = \tilde{L}\tilde{L}^T, \tag{3.13}$$

where

$$\tilde{L} = \begin{pmatrix} \beta_0 & & & & \beta_p & \cdots & \beta_1 \\ \vdots & \ddots & & & & \ddots & \vdots \\ \beta_p & & \ddots & & & & \beta_p \\ & \ddots & & \ddots & & & \\ & & \ddots & & \ddots & & \\ & & & \ddots & & \ddots & \\ & & & & \beta_p & \cdots & \beta_0 \end{pmatrix}.$$

The system

$$C_b x = d \tag{3.14}$$

can be solved by solving following two systems

$$\tilde{L} y = d \tag{3.15}$$

and

$$\tilde{L}^T x = y. \tag{3.16}$$

Let E_1 and E_p denote the n-by-p matrices consisting of the first p columns and the last p columns of the n-th order identity matrix, respectively, and

$$L = \begin{pmatrix} \beta_0 & & & & & \\ \vdots & \ddots & & & & \\ \beta_p & & \ddots & & & \\ & \ddots & & \ddots & & \\ & & \ddots & & \ddots & \\ & & & \beta_p & \cdots & \beta_0 \end{pmatrix}.$$

Then $\widetilde{L}$ can be written

$$\widetilde{L} = L + E_1 R E_p^T, \tag{3.17}$$

where

$$R = \begin{pmatrix} \beta_p & \cdots & \beta_1 \\ & \ddots & \vdots \\ & & \beta_p \end{pmatrix},$$

and using the Woodbury formula the inverse is given by

$$\widetilde{L}^{-1} = L^{-1} - L^{-1}E_1(R^{-1} + E_p^T L^{-1} E_1)^{-1} E_p^T L^{-1},$$

and thus

$$\begin{aligned} y &= L^{-1}d - L^{-1}E_1(R^{-1} + E_p^T L^{-1}E_1)^{-1}E_p^T L^{-1}d \\ &= h - Wg, \end{aligned} \tag{3.18}$$

where h, W and g are the solution of the following equations,respectively

$$Lh = d, \tag{3.19}$$

$$LW = E_1, \tag{3.20}$$

$$Bg = z, \tag{3.21}$$

and

$$z = (h_{n-p+1}, \ldots, h_n)^T$$

is the p-vector with the last p elements of the vector h as its components, and

$$B = R^{-1} + E_p^T L^{-1} E_1. \tag{3.22}$$

To compute the p-th order matrix B, we first solve the equation (3.20). Since W is the first p columns of L^{-1}, and L is a lower triangular Toeplitz matrix and so is its inverse, W is defined by the first column of L^{-1}, which is the solution of the equation

$$Lw = (1, 0, \ldots, 0)^T, \tag{3.23}$$

and can be computed with $O(np)$ operations.

Denote by $w_1, w_2, \ldots, w_n$ the components of vector w, then we have

$$(3.24)\qquad W = \begin{pmatrix} w_1 & & & \\ w_2 & w_1 & & \\ \vdots & \vdots & \ddots & \\ \vdots & \vdots & & w_1 \\ \vdots & \vdots & & \vdots \\ w_n & w_{n-1} & \cdots & w_{n-p+1} \end{pmatrix},$$

and

$$(3.25)\qquad E_p^T W = \begin{pmatrix} w_{n-p+1} & w_{n-p} & \cdots & w_{n-2p+2} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ w_n & w_{n-1} & \cdots & w_{n-p+1} \end{pmatrix},$$

which is a Toeplitz matrix too.

The matrix R^{-1} is an upper triangular Toeplitz matrix and can be calculated with $O(p^2)$ operations. Thus B is Toeplitz, so solving (3.21) will cost $O(p^2)$ operations. Having computed B and solved the equations (3.19), (3.20) and (3.21), the auxiliary vector y can be found, and then we can solve equation (3.16) in a similar way. Since

$$\tilde{L}^{-T} = L^{-T} - L^{-T} E_p B^{-T} E_1^T L^{-T},$$

the solution vector x is given by

$$(3.26)\qquad x = r - Vs,$$

where $r = (r_1, r_2, \ldots, r_n)^T$ is the solution of the equation

$$(3.27)\qquad L^T r = y,$$

$$(3.28)\qquad V = \begin{pmatrix} w_{n-p+1} & w_{n-p+2} & \cdots & w_n \\ w_{n-p} & w_{n-p+1} & \cdots & w_{n-1} \\ \vdots & \vdots & & \vdots \\ w_1 & w_2 & \cdots & w_{n-p+1} \\ & w_1 & & \vdots \\ & & \ddots & \vdots \\ & & & w_1 \end{pmatrix},$$

and s is the solution of the equation

$$B^T s = (r_1, r_2, \ldots, r_p)^T. \tag{3.29}$$

The asymptotic operation counts of the method would be $O(5pn)$ excluding the amount of work to calculate the factor $l(z)$. In most usual case, $p = 1$ or 2, and finding $l(z)$ does not cost much work. The algorithm may be summarized as follows.

Algorithm BCS (Banded Circulant Solver) solves banded circulant system (3.14). Assume that the parameters β_0, β_1, ..., β_p are precomputed.

1. Solve equation (3.19) for h by forward substitution.
2. Solve equation (3.23) and form W via (3.24).
3. Compute R^{-1} by backward substitution, and form matrix B.
4. Solve equation (3.21) for g using a Toeplitz type method.
5. Calculate the solution vector y of (3.15).
6. Solve equation (3.27) for r.
7. Form V via (3.28).
8. Solve (3.29) for s.
9. Compute the solution vector x via (3.26).

endalgorithm

4. Block Circulant Systems

A block matrix $M = circ(M_0, M_1, \ldots, M_{m-1})$, where each of the blocks M_j is itself an n-th order circulant, is called *block circulant with circulant blocks.* In this section we will consider the system of the form

$$Mx = b, \tag{4.1}$$

which arises in many applications. An important one is the solution of elliptic equation with periodic boundary conditions by finite difference methods. The method proposed here to solve (4.1) is based on the use of the fast Fourier transform, and on the fact that the coefficient matrices of the resulting linear systems are circulants, hence the methods described in section 2 and 3 may be employed directly.

Denote the eigenvalues of the n-by-n circulant matrices M_j by $\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_n^{(j)}$, then we have

$$M_j = F\, D_j F^* \tag{4.2}$$

where

$$D_j = diag(\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_n^{(j)}), \qquad j = 0, 1, \ldots, m-1. \tag{4.3}$$

It follows that the block circulant matrix M can be decomposed as

$$M = Q\, D\, Q^*, \tag{4.4}$$

where

$$Q = diag(F, F, \ldots, F) \tag{4.5}$$

is an unitary block diagonal matrix of order (mn), and

$$D = circ(D_0, D_1, \ldots, D_{m-1})$$

is a block circulant.

After certain exchanges of the rows and the columns of D we are led to

$$M = QPdiag(N_1, N_2, \ldots, N_n)PQ^*, \tag{4.6}$$

where

$$N_k = circ(\lambda_k^{(0)}, \lambda_k^{(1)}, \ldots, \lambda_k^{(m-1)}), \qquad k = 1, 2, \ldots, n \tag{4.7}$$

are m-by-m circulant matrices with the eigenvalues of M_j as their elements, and P is an (mn)-by-(mn) permutation matrix. It is easily seen from (4.6) that M is nonsingular if and only if the matrices N_k are nonsingular for all k. In this case we have

$$M^{-1} = QPdiag(N_1^{-1}, N_2^{-1}, \ldots, N_n^{-1})PQ^*, \tag{4.8}$$

and the solution to (4.1) is given by

$$x = QPdiag(N_1^{-1}, N_2^{-1}, \ldots, N_n^{-1})PQ^* b. \tag{4.9}$$

If at least one of the diagonal blocks is singular, then the inverse of M does not exist and the least squares solution of (4.1) is

$$\hat{x} = QPdiag(N_1^+, N_2^+, \ldots, N_n^+)PQ^*b, \tag{4.10}$$

which will be the solution to (4.1) if the system is consistent.

To use (4.9) and (4.10) to compute the solution of (4.1), we need to do some exchanges of the components of some vectors in addition to performing the FFT's and solving the circulant systems. These exchanges can easily be handled by partitioning the vector b and x into subvectors to conform with the block structure of the matrix M as

$$b = \begin{pmatrix} b^{(1)} \\ b^{(2)} \\ \vdots \\ b^{(m)} \end{pmatrix}, \qquad x = \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(m)} \end{pmatrix}, \tag{4.11}$$

where

$$\begin{cases} b^{(j)} = (b_{1j}, b_{2j}, \ldots, b_{nj})^T, \\ x^{(j)} = (x_{1j}, x_{2j}, \ldots, x_{nj})^T, \end{cases} \qquad j = 1, 2, \ldots, m \tag{4.12}$$

are subvectors of length n. Now let

$$\hat{b}^{(j)} = F^* b^{(j)}, \qquad j = 1, 2, \ldots, m, \tag{4.13}$$

and the components of $\hat{b}^{(j)}$ are denoted as in (4.12). Furthermore denote

$$\tilde{b} = \begin{pmatrix} \tilde{b}^{(1)} \\ \tilde{b}^{(2)} \\ \vdots \\ \tilde{b}^{(m)} \end{pmatrix}, \tag{4.14}$$

where

$$\tilde{b}^{(k)} = (\hat{b}_{k1}, \hat{b}_{k2}, \ldots, \hat{b}_{km})^T, \qquad k = 1, 2, \ldots, n, \tag{4.15}$$

then we have

$$\tilde{b} = PQ^*b.$$

Let

$$\tilde{x}^{(k)} = (\hat{x}_{k1}, \hat{x}_{k2}, \ldots, \hat{x}_{km})^T \tag{4.16}$$

denote the solution vectors of the systems

$$N_k y = \tilde{b}^{(k)}, \qquad k = 1, 2, \ldots, n, \tag{4.17}$$

then

$$x^{(i)} = F\hat{x}^{(j)}, \tag{4.18}$$

where

$$\hat{x}^{(j)} = (\hat{x}_{1j}, \hat{x}_{2j}, \ldots, \hat{x}_{nj})^T, \qquad j = 1, 2, \ldots, m \tag{4.19}$$

and we finaly obtain the solution of (4.1) by composing the vector x using the subvectors $x^{(j)}$ via (4.11). Thus we have the following algorithm.

Algorithm BLCS (BLock Circulant Solver) solves block circulant system (4.1).

1. Transform $b^{(j)} = F^* b^{(j)}$, $j = 1, 2, \ldots, m$.
2. Compute the eigenvalues of M_j, $j = 1, 2, \ldots, m$.
3. Solve equations (4.17) using algorithm CIRS for all k.
4. Transform $x^{(j)} = F\hat{x}^{(j)}$, $j = 1, 2, \ldots, m$.
5. Assemble the vectors $x^{(j)}$ via (4.11) to obtain the solution vector x.

endalgorithm

5. Applications to elliptic equations

We will now apply the Fourier-circulant method to certain elliptic equations. It is well known[5] that the approximation of Poisson's equation on a rectangle subject to periodic boundary conditions in both directions by the standard five-point difference scheme on a uniform mesh results in the block circulant linear system of the form

$$M_p u = b, \tag{5.1}$$

where

$$M_p = circ(A, -I, 0, \ldots, 0, -I),$$

and

$$A = circ(4, -1, 0, \ldots, 0, -1).$$

We partition the unknown vector u as in (4.11) and denote its components as in (4.12). Since A is symmetric circulant tridiagonal matrix its eigenvalues are, from (3.2),

$$\lambda_k = 4 - 2\cos(k-1)\frac{2\pi}{n}, \qquad k = 1, 2, \ldots, n.$$

Assume that n is even. We rewrite the eigenvalues, by reordering them, as follows

$$(5.2) \qquad \begin{cases} \lambda_1 = 2, \\ \lambda_2 = 6, \\ \lambda_{2\nu-1} = \lambda_{2\nu} = 4 - 2\cos(\nu-1)\frac{2\pi}{n}, \qquad \nu = 2, 3, \ldots, \frac{n}{2}. \end{cases}$$

The elements of the matrix of the corresponding eigenvectors(denoted by F also) are real and given by

$$(5.3) \qquad \begin{cases} f_{i,1} = \sqrt{\frac{1}{n}}, \\ f_{i,2} = (-1)^{i-1}\sqrt{\frac{1}{n}}, \\ f_{i,2\nu-1} = \sqrt{\frac{2}{n}}\cos\left[(\nu-1)(i-1)\frac{2\pi}{n}\right], \\ f_{i,2\nu} = \sqrt{\frac{2}{n}}\sin\left[(\nu-1)(i-1)\frac{2\pi}{n}\right], \qquad \nu = 2, 3, \ldots, \frac{n}{2}. \end{cases}$$

$$i = 1, 2, \ldots, n.$$

In this situation we have

$$(5.4) \qquad N_k = circ(\lambda_k, -1, 0, \ldots, 0, -1), \qquad k = 1, 2, \ldots, n.$$

All the diagonal blocks N_k except N_1,which is singular since it has an eigenvalue equal to zero, are nonsingular, and therefore M is singular. From (4.10) the least squares solution to (5.1) is given by

$$(5.5) \qquad u = QPdiag(N_1^+, N_2^{-1}, \ldots, N_n^{-1})PQ^Tb,$$

where the orthogonal matrix Q is as in (4.5). The vector u computed via (5.5) is the solution to (5.1) only if $e^Tb = 0$, where $e = (1, 1, \ldots, 1)^T$, since e is the only null vector of matrix M_p.

To compute

$$(5.6) \qquad \tilde{u}^{(1)} = N_1^+\tilde{b}^{(1)},$$

we use the algorithm CIRS. In this case the eigenvalues of N_1 are (Assume that m is also even.)

$$\begin{cases} \mu_1 = 0, \\ \mu_2 = 4, \\ \mu_{2\nu-1} = \mu_{2\nu} = 2 - 2\cos(\nu - 1)\dfrac{2\pi}{m}, \qquad \nu = 2, 3, \ldots, \dfrac{m}{2}. \end{cases}$$

and the elements of the orthogonal matrix of the corresponding eigenvectors are

$$\begin{cases} g_{i,1} = \sqrt{\dfrac{1}{m}}, \\ g_{i,2} = (-1)^{i-1}\sqrt{\dfrac{1}{m}}, \\ g_{i,2\nu-1} = \sqrt{\dfrac{2}{m}}\cos\left[(\nu - 1)(i - 1)\dfrac{2\pi}{m}\right], \\ g_{i,2\nu} = \sqrt{\dfrac{2}{m}}\sin\left[(\nu - 1)(i - 1)\dfrac{2\pi}{m}\right], \qquad \nu = 2, 3, \ldots, \dfrac{m}{2}, \end{cases}$$

$$i = 1, 2, \ldots, m.$$

If we denote this matrix by

$$G = (g_{ij})_{m\times m},$$

then (5.6) becomes

$$\tilde{u}^{(1)} = Gdiag(0, \mu_2^{-1}, \ldots, \mu_m^{-1})G^T\tilde{b}^{(1)}, \tag{5.7}$$

which can be computed by real transform and its inverse.

The other equations

$$N_k\tilde{u}^{(k)} = \tilde{b}^{(k)} \tag{5.8}$$

are nonsingular circulant tridiagonal, and the coefficient matrices have the forms

$$N_k = \begin{pmatrix} \lambda_k & -1 & & & -1 \\ -1 & \lambda_k & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & \lambda_k & -1 \\ -1 & & & -1 & \lambda_k \end{pmatrix}, \qquad k = 2, 3, \ldots, n.$$

Since the eigenvalues satisfy the relation $\lambda_k > 2$ for $k = 2, 3, \ldots, n$, the quadratic equations

$$\tau^2 - \lambda_k\tau + 1 = 0 \tag{5.9}$$

have real root τ_k for all k except $k = 1$, such that $0 < \tau_k < 1$. In fact we have

(5.10)
$$\tau_k = \frac{\lambda_k}{2} - \sqrt{\left(\frac{\lambda_k}{2}\right)^2 - 1}.$$

The matrices N_k can then be factored as

$$N_k = \widetilde{L}_k \widetilde{U}_k,$$

where

$$\widetilde{L}_k = \begin{pmatrix} 1 & & & & -\tau_k \\ -\tau_k & 1 & & & \\ & \ddots & \ddots & & \\ & & -\tau_k & 1 & \\ & & & -\tau_k & 1 \end{pmatrix},$$

and

$$\widetilde{U}_k = \begin{pmatrix} \tau_k^{-1} & -1 & & & \\ & \tau_k^{-1} & -1 & & \\ & & \ddots & \ddots & \\ & & & \tau_k^{-1} & -1 \\ -1 & & & & \tau_k^{-1} \end{pmatrix}.$$

On the analogy of what we did in §3, we can solve the equations (5.8) by solving the following two bidiagonal systems of order m

(5.11)
$$L_k y = \tilde{b}^{(k)} + \left[\frac{\tau_k}{1-\tau_k^m} \sum_{l=0}^{m-1} \hat{b}_{kl}\, \tau_k^{m-l-1}\right] e_1,$$

and

(5.12)
$$U_k \tilde{u}^{(k)} = y + \left[\frac{1}{1-\tau_k^m} \sum_{l=1}^{m} y_l\, \tau_k^l\right] e_m,$$

where

$$L_k = \begin{pmatrix} 1 & & & & \\ -\tau_k & 1 & & & \\ & \ddots & \ddots & & \\ & & -\tau_k & 1 & \\ & & & -\tau_k & 1 \end{pmatrix},$$

$$U_k = \begin{pmatrix} \tau_k^{-1} & -1 & & & \\ & \tau_k^{-1} & -1 & & \\ & & \ddots & \ddots & \\ & & & \tau_k^{-1} & -1 \\ & & & & \tau_k^{-1} \end{pmatrix},$$

$$k = 2, 3, \ldots, n,$$

and e_1 and e_m are the first and the last column of the m-th order identity matrix. Thus the algorithm procceds as follows.

Algorithm PPS (Periodic Poisson Solver) solves block circulant system (5.1).

1. Transform $\tilde{b}^{(j)} = F^* b^{(j)}$, $j = 1, 2, \ldots, m$.
2. Compute the eigenvalues of A via (5.2).
3. Calculate $\tilde{u}^{(1)}$ via (5.7).
4. Solve equation (5.8) for $k = 2, 3, \ldots, n$ using algorithm CTS.
5. Transform $x^{(j)} = F\hat{x}^{(j)}$, $j = 1, 2, \ldots, m$.
6. Assemble the vectors $x^{(j)}$ via (4.11) to obtain the solution vector x.

endalgorithm

The complexity of the algorithm is $O(mn \log_2 n)$. Although it is the same as cyclic reduction, the algorithm is more efficient than cyclic reduction since it is a FFT-based method(see [16])

Another kind of elliptic equations to which the method may be applied is the biharmonic equation

$$\nabla^4 u(x, y) = \psi(x, y), \qquad (x, y) \in \Omega \tag{5.13}$$

subject to periodic conditions.

When the region Ω is a rectangle, and the mesh lengths in both directions are the same, and the standard 13-point difference approximation of the biharmonic operator is used, the resulting system of linear equations is of the form

$$M_b u = b, \tag{5.14}$$

where

$$M_b = circ(C, B, I, 0, \ldots, 0, I, B),$$

and

$$C = circ(20, -8, 1, 0, \ldots, 0, 1, -8),$$

$$B = circ(-8, 2, 0, \ldots, 0, 2).$$

Since C and B are symmetric their eigenvalues are real and given by

$$\begin{cases} \lambda_1 = 6, \\ \lambda_2 = 38, \\ \lambda_{2\nu-1} = \lambda_{2\nu} = 20 - 16\cos(\nu - 1)\dfrac{2\pi}{n} + 2\cos 2(\nu - 1)\dfrac{2\pi}{n}, \qquad \nu = 2, 3, \ldots, \dfrac{n}{2}, \end{cases} \tag{5.15}$$

and

(5.16)
$$\begin{cases} \mu_1 = -4, \\ \mu_2 = -12, \\ \mu_{2\nu-1} = \mu_{2\nu} = -8 + 4\cos(\nu-1)\dfrac{2\pi}{m}, \qquad \nu = 2,3,\ldots,\dfrac{m}{2}, \end{cases}$$

respectively. Thus

(5.17)
$$N_k = circ(\lambda_k, \mu_k, 1, 0, \ldots, 0, 1, \mu_k), \qquad k = 1,2,\ldots,n.$$

The system

(5.18)
$$N_1 \tilde{u}^{(1)} = \tilde{b}^{(1)}$$

is singular, and can be solved by using the algorithm CIRS, and the others

(5.19)
$$N_k \tilde{u}^{(k)} = \tilde{b}^{(k)}, \qquad k = 2,3,\ldots,n$$

are circulant pentadiagonal, and can be solved by the method of §3. In this case, $p = 2$ and the factor $l(z)$, which is quadratic, may be found by solving the equations(see [7])

(5.20)
$$\rho^4 + \mu_k \rho^3 + \lambda_k \rho^2 + \mu_k \rho + 1 = 0.$$

Since $\mu_k^2 - 4\lambda_k + 8 = 0$ for $k = 2,3,\ldots,n$, the equations (5.20) have two real roots that each has multiplicity 2, and they are given by

(5.21)
$$\rho_1^{(k)} = \frac{1}{4}\left[-\mu_k + \sqrt{\mu_k^2 - 16}\right],$$

and

(5.22)
$$\rho_2^{(k)} = \frac{1}{4}\left[-\mu_k - \sqrt{\mu_k^2 - 16}\right].$$

It is easily seen that

$$\rho_1^{(k)} > 1, \qquad \rho_2^{(k)} < 1,$$

and

$$\rho_1^{(k)} = \frac{1}{\rho_2^{(k)}}$$

since $\mu_k < -4$ for $k = 2,3,\ldots,n$, and the factor $l_k(z)$ for the matrix N_k is found to be

$$l_k(z) = \beta_0^{(k)} + \beta_1^{(k)} z + \beta_2^{(k)} z^2,$$

where

(5.23)
$$\begin{cases} \beta_0^{(k)} = \rho_1^{(k)}, \\ \beta_1^{(k)} = -2, \\ \beta_2^{(k)} = \dfrac{1}{\rho_1^{(k)}}. \end{cases}$$

Thus we have the following $O(mn\log_2 n)$ algorithm.

Algorithm PBS (Periodic Biharmonic Solver) solves block circulant system (5.14).

1. Transform $\tilde{b}^{(j)} = F^* b^{(j)}$, $j = 1, 2, \ldots, m$.
2. Compute the eigenvalues of A and B via (5.15) and (5.16), respectively.
3. Solve equation (5.18) for $\tilde{u}^{(1)}$ using algorithm CIRS.
4. For $k = 2, 3, \ldots, n$,
   4.1. calculate $\beta_0^{(k)}$, $\beta_1^{(k)}$, $\beta_2^{(k)}$ via (5.21) and (5.23), and
   4.2. solve equations (5.19) for $\tilde{u}^{(k)}$ using algorithm BCS.
5. Transform $x^{(j)} = F\hat{x}^{(j)}$, $j = 1, 2, \ldots, m$.
6. assemble the vectors $x^{(j)}$ via (4.11) to obtain the solution vector x.

endalgorithm

6. Numerical experiments

The methods described in the previous section were tried on the FPS-164 of the Department of Computer Science, Yale University. The programs were written and timed in FORTRAN, and the FFT subroutines used are part of the NA Packege of the Research Center for Scientific Computation, Yale University.

In each test case, a periodic function was arbitrarily chosen and its values at the mesh points of an $N \times N$ grid, which are considered to be the "true" solution of (5.1) and (5.14), were calculated, and the corresponding right hand sides were computed with the use of (5.1) and (5.14), respectively. Approximate solutions to (5.1) and (5.14) were then generated by using the algorithms described in §5 as well as the cyclic reduction for Poisson's equation. The maximun errors of all tests are less than 10^{-10}, indicating that the algorithms presented in this paper are stable. The execution time for each case are summarized in Table 1. It should be noted that the Foureir-circulant method is more efficient than cyclic reduction.

N	Algorithm PPS	Cyclic reduction	Algorithm PBS
16	0.246×10^{-1}	0.816×10^{-1}	0.287×10^{-1}
32	0.696×10^{-1}	0.347×10^{0}	0.815×10^{-1}
64	0.258×10^{0}	0.155×10^{1}	0.297×10^{0}
128	0.915×10^{0}	0.691×10^{1}	0.105×10^{1}

Table 1: Execution times (millisecond) on FPS-164

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