Diagonal Representation of Certain Matrices

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Abstract

An explicit expression is provided for the characteristic polynomial of a matrix M of the form

$$M = D - \begin{pmatrix} 0 & ab^{\mathrm{T}} \\ ba^{\mathrm{T}} & 0 \end{pmatrix}, \tag{1}$$

where D is a diagonal matrix, and a and b are column vectors. Also, an explicit expression is provided for the matrix of normalized eigenvectors of M, in terms of the roots of the characteristic polynomial (*i.e.*, in terms of the eigenvalues of M).

1 A Lemma, a Remark, and an Observation

The following lemma is verified by substituting into the left hand side of (7) the definitions of P in (6) and U in (9)–(16), and simplifying the result using (4). See [2] for similar results, and [3] and [1] for applications.

Lemma 1 *Suppose that m and n are positive integers, $a = (a_0, a_1, \ldots, a_{m-2}, a_{m-1})^{\mathrm{T}}$ and $b = (b_0, b_1, \ldots, b_{n-2}, b_{n-1})^{\mathrm{T}}$ are real vectors, and $d_0, d_1, \ldots, d_{m+n-2}, d_{m+n-1}$ and $\lambda_0, \lambda_1, \ldots, \lambda_{m+n-2}, \lambda_{m+n-1}$ are real numbers such that*

$$\lambda_j \neq d_k \tag{2}$$

for any j, k ($j, k = 0, 1, \ldots, m+n-2, m+n-1$),

$$\lambda_j \neq \lambda_k \tag{3}$$

when $j \neq k$, and

$$\left(\sum_{k=0}^{m-1} \frac{(a_k)^2}{d_k - \lambda_j} \right) \left(\sum_{k=0}^{n-1} \frac{(b_k)^2}{d_{m+k} - \lambda_j} \right) = 1 \tag{4}$$

(with $j = 0, 1, \ldots, m+n-2, m+n-1$).

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Suppose further that D is the diagonal $(m+n)\times(m+n)$ matrix defined by the formula

$$D = \left(\begin{array}{ccccc} d_0 & 0 & \cdots & \cdots & 0 \\ 0 & d_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & d_{m+n-2} & 0 \\ 0 & \cdots & \cdots & 0 & d_{m+n-1} \end{array} \right), \tag{5}$$

and P is the $(m+n)\times(m+n)$ matrix defined by the formula

$$P = \left(\begin{array}{cc} 0 & ab^{\mathrm{T}} \\ ba^{\mathrm{T}} & 0 \end{array} \right), \tag{6}$$

where 0 denotes matrices consisting entirely of zeroes.

Then,

$$(D-P)\,U = U\,\Lambda, \tag{7}$$

where Λ is the diagonal $(m+n)\times(m+n)$ matrix defined by the formula

$$\Lambda = \left(\begin{array}{ccccc} \lambda_0 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \lambda_{m+n-2} & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_{m+n-1} \end{array} \right), \tag{8}$$

and U is the orthogonal $(m+n)\times(m+n)$ matrix defined by the formula

$$U = \left(\begin{array}{c} AVR \\ BWS \end{array} \right). \tag{9}$$

In (9), A is the diagonal $m\times m$ matrix defined by the formula

$$A = \left(\begin{array}{ccccc} a_0 & 0 & \cdots & \cdots & 0 \\ 0 & a_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{m-2} & 0 \\ 0 & \cdots & \cdots & 0 & a_{m-1} \end{array} \right), \tag{10}$$

B is the diagonal $n\times n$ matrix defined by the formula

$$B = \left(\begin{array}{ccccc} b_0 & 0 & \cdots & \cdots & 0 \\ 0 & b_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & b_{n-2} & 0 \\ 0 & \cdots & \cdots & 0 & b_{n-1} \end{array} \right), \tag{11}$$

V is the $m \times (m+n)$ matrix with entry $V_{j,k}$ defined by the formula

$$V_{j,k} = \frac{1}{d_j - \lambda_k} \tag{12}$$

(with $j = 0, 1, \ldots, m-2, m-1$; $k = 0, 1, \ldots, m+n-2, m+n-1$), W is the $n \times (m+n)$ matrix with entry $W_{j,k}$ defined by the formula

$$W_{j,k} = \frac{1}{d_{m+j} - \lambda_k} \tag{13}$$

(with $j = 0, 1, \ldots, n-2, n-1$; $k = 0, 1, \ldots, m+n-2, m+n-1$), S is the diagonal $(m+n) \times (m+n)$ matrix with the diagonal entries $S_{0,0}$, $S_{1,1}$, $\ldots$, $S_{m+n-2,m+n-2}$, $S_{m+n-1,m+n-1}$ defined by the formula

$$S_{j,j} \;=\; 1 \Bigg/ \sqrt{\sum_{k=0}^{m-1} \left(\frac{a_k\, c_j}{d_k - \lambda_j} \right)^2 + \sum_{k=0}^{n-1} \left(\frac{b_k}{d_{m+k} - \lambda_j} \right)^2}, \tag{14}$$

and R is the diagonal $(m+n) \times (m+n)$ matrix with the diagonal entries $R_{0,0}$, $R_{1,1}$, $\ldots$, $R_{m+n-2,m+n-2}$, $R_{m+n-1,m+n-1}$ defined by the formula

$$R_{j,j} = c_j\, S_{j,j}. \tag{15}$$

In (14) and (15), c_0, c_1, $\ldots$, c_{m+n-2}, c_{m+n-1} are the real numbers defined by the formula

$$c_j = \sum_{k=0}^{n-1} \frac{(b_k)^2}{d_{m+k} - \lambda_j}. \tag{16}$$

Remark 2 The equation (4) is equivalent to the characteristic (secular) equation

$$\det |\lambda_j\, I - (D - P)| = 0 \tag{17}$$

for the eigenvalues λ_j (with $j = 0, 1, \ldots, m+n-2, m+n-1$) of the matrix $D - P$.

Observation 3 *The upper block AVR of the matrix U defined in (9) has the form of a diagonal matrix (A) times a matrix of inverse differences (V) times another diagonal matrix (R). The lower block BWS of the matrix U defined in (9) also has the form of a diagonal matrix (B) times a matrix of inverse differences (W) times another diagonal matrix (S). Therefore, there exists an algorithm which applies such an $N \times N$ matrix U (or its adjoint) to an arbitrary real vector of length N in $\mathcal{O}(N \log(1/\varepsilon))$ operations, where ε is the precision of computations (see [3]).*

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References

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