ABSTRACT

The technique introduced by McDaniel-Lee for the handling of the fluid/fluid interface boundary under range-dependent environments is extended to handle the horizontal fluid/elastic interface boundary. Representative wave equations of the parabolic type are considered in both fluid and elastic media. The required interface conditions, (1) continuity of vertical components of displacement, (2) continuity of vertical components of stress, and (3) horizontal components of stress vanish on the interface, are satisfied with this numerical treatment. A complete theoretical development is presented along with a test example to demonstrate its validity.

A Numerical Treatment of the Fluid/Elastic Interface under Range – dependent Environments

E. C. Shang[1] and Ding Lee[2]

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[1]Present Address: NOAA/ERL Wave Propagation Lab., Boulder, CO 80303.
[2]Present Address: Naval Underwater Systems Center, New London,CT 06320.

I. INTRODUCTION

Modern computational techniques improved much of the efficiency of range-dependent wave propagation models. The efficiency as well as usefulness of the range-dependent model can be further enhanced if a capability can be incorporated to handle the fluid/elastic interface. This paper introduces a numerical treatment of the horizontal fluid/elastic interface boundary. Since the numerical treatment introduced by McDaniel-Lee [1] to handle the fluid/fluid interface in 1982, interest was on the rise in searching for efficient methods to handle the fluid/elastic interface. Some contributions were made in the seismology science to treat the fluid/elastic interface. R. Stephen [2] developed finite difference methods to treat this problem dealing with a hyperbolic wave equation. J. T. Kuo and Y. C. Teng [3] applied the finite element as well as finite difference schemes extensively to solve the same problem as above but dealing with an elliptic wave equation. The above techniques, though workable, are not simple to adapt into any existing range-dependent model without requiring excessive efforts; moreover, these techniques are by no means simple. The numerical treatment we introduced in this paper is based on the standard Parabolic Equation (PE) in the fluid medium introduced by Tappert [4] and on the coupled parabolic equations in the elastic medium derived by McCoy [5]. The fluid/elastic interface requires three conditions to be satisfied on the interface, i.e., (1) the continuity of vertical components of displacement, (2) the continuity of vertical components of stress, and (3) horizontal components of stress must vanish on the interface. These conditions were derived to be consistent with the PE representation in both fluid and elastic media. McDaniel-Lee's technique was modified to treat these conditions numerically. This modification allows the existing implicite finite difference (IFD) [6] marching scheme to be applied systematically. A test problem with known solution, given by Ewing and Press [7], is used to examine the validity of this development.

II. BACKGROUND SUMMARY

Since the new treatment to the fluid/elastic interface boundary is an extension of the McDaniel-Lee technique, it is desirable that the McDaniel-Lee's treatment to the fluid/fluid interface boundary be briefly reviewed.

We use u(r,z) to indicate the wave field, the pressure, in a 2-dimensional medium, depth and range. Thus, u(r,z) satisfies the parabolic wave equation

$$u_r = a(k_o,r,z)u + b(k_o,r,z)\, u_{zz} , \qquad (2.1)$$

where $a(k_o,r,z) = ik_o(n^2(r,z) - 1)/2$

and

$$b(k_o,r,z) = i/(2k_o),$$

where k_o is the reference wavenumber, $n(r,z)$ stands for the index of refraction which is defined as a ratio of the reference sound speed c_o and the sound speed $c(r,z)$.

At the ocean bottom, the change of sound speed and density form an interface (see Figure 1). From one medium

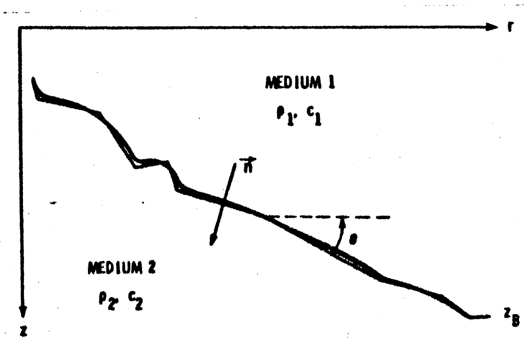

Figure 1: Interface Boundary

to the next, at each interface, the interface conditions must be satisfied, i.e., the pressure and normal components of particle velocity are continuous at the interface.

The standard PE, Eq. (2.1), does not contain the density. In order to satisfy the interface conditions, McDaniel-Lee developed a special equation with density variations to represent the wave field on the interface. It turned out that this special equation is again a PE.

In developing this interface PE, McDaniel-Lee applied the Taylor series expansion to the points near the interface and then match the fields on the interface which is denoted by z_B. A clear configuration is given in Figure 2.

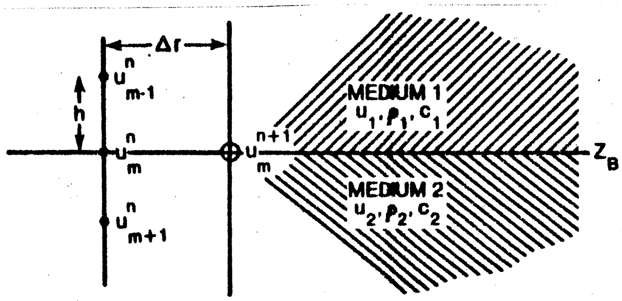

Figure 2: The Interface Between Two Media

In carrying out the matching process, let us describe the interface mathematically below.

The continuity of pressure requires that

$$u(r,z_B) = u_2(r,z_B), \tag{2.2}$$

the continuity of normal component of particle velocity requires that

$$\rho_2 \frac{\partial u_1}{\partial z}\bigg|_{z_B} = \rho_1 \frac{\partial u_2}{\partial z}\bigg|_{z_B} . \tag{2.3}$$

Note that the interface is assumed horizontal.

In every medium the field $u(r,z)$ satisfies the PE, (2.1). Therefore, in medium 1, $u_1(r,z)$ satisfies

$$(u_1)_r = a_1(k_o,r,z)\, u_1 + b_1(k_o,r,z)\, (u_1)_{zz} , \tag{2.4}$$

where $a_1(k_o,r,z) = ik_o(n_1^2(r,z) - 1)/2$ (2.5)

and

$$b_1(k_o,r,z) = i/(2k_o). \tag{2.6}$$

Using the first three terms of the Taylor expansion for $(u_1)_{m-1}$ upon $(u_1)_m$ and solve for $(u_1)_{zz}$, then substitute them into Eq. (2.4), we find

$$\frac{\partial u_1}{\partial z} = \frac{h}{2b_1}\,(u_1)_r - \frac{h}{2b_1}\,a_1u_1 + \frac{1}{h}\,(u_1 - (u_1)_{m-1}) \tag{2.7}$$

where $h = \Delta z$.

Similarly, in medium 2 use a three-term Taylor expansion for $(u_2)_{m+1}$ upon $(u_2)_m$ and follow the same procedures as carried out in medium 1, we find

$$\frac{\partial u_2}{\partial z} = -\frac{h}{2b_2}\,(u_2)_r + \frac{h}{2b_2}\,a_2u_2 + \frac{1}{h}\,((u_2)_{m+1} - u_2). \tag{2.8}$$

The first interface condition (2.2) allows one to write $u_1 = u_2 = u$ on the interface. Then, multiply both sides of (2.7) by ρ_2 and multiply both sides of (2.8) by ρ_1; then the second interface condition (2.3) allows the above results to be equal. After simplification, the McDaniel-Lee horizontal interface wave field is obtained between two fluid media to be

$$u_r = \left(\frac{1}{b_1} + \frac{\rho_1}{\rho_2}\,\frac{1}{b_2}\right)^{-1}\left[\left(\frac{a_1}{b_1} + \frac{\rho_1}{\rho_2}\,\frac{a_2}{b_2}\right)u + \frac{2}{h^2}\left(\frac{\rho_1}{\rho_2}\left((u_2)_{m+1} - u\right) - \left(u-(u_1)_{m-1}\right)\right)\right] \quad (2.9)$$

Note that the density is assumed to remain constant in each medium.

III. FIELD REPRESENTATIONS IN FLUID/ELASTIC MEDIUM

Dealing with the fluid/elastic interface, two media are involved, i.e., a fluid medium and an elastic medium. The field representation in the fluid medium by the parabolic approximation is a scalar PE while in the elastic medium the representative wave equation is a vector PE. A standard derivation of the scalar PE in fluid medium can be found in references 4 [Tappert] and 8 [Lee-Siegmann]. The vector PE has been derived by a few authors, though their derivations, and method of derivation differ from one another. These derivations are

(1) Lander and Claerbout's equation [9], derived using dilatation and rotation,

(2) Hudson's equation [10], derived using displacement, and

(3) McCoy's equation [5], derived using dilational and shear potentials.

In this paper we use potential representation to deal with the fluid/elastic interface, thus, it is appropriate that we adopt the vector PE developed by McCoy.

In the fluid medium the parabolic wave equation was expressed in the 2-dimensional cylindrical coordinates where we use $\phi_1(r,z)$ to indicate the potential; in the elastic medium we use $\phi_2(r,z)$ and $\psi_2(r,z)$ to represent the potentials there. In the above notation, the subscript "1" is used to indicate the fluid medium, and the subscript "2" to indicate the elastic medium. We use D_1 to indicate the displacement in the fluid medium and D_2 to indicate the displacement in the elastic medium. Their relationships with the potential are given by

$$D_1\ (u_1,w_1) = \text{grad}\ \phi_1(r,z), \tag{3.1}$$

$$D_2\ (u_2,w_2) = \text{grad}\ \phi_2\ (r,z) + \text{rot}\ \psi_2(r,z). \tag{3.2}$$

Equivalently we can write the above as

$$\begin{cases} u_1 = \dfrac{\partial\phi_1}{\partial r} \\ w_1 = \dfrac{\partial\phi_1}{\partial z} \end{cases} \tag{3.3}$$

$$\begin{cases} u_2 = \dfrac{\partial \phi_2}{\partial r} - \dfrac{\partial \psi_2}{\partial z} \\ \\ w_2 = \dfrac{\partial \phi_2}{\partial z} + \dfrac{\partial \psi_2}{\partial r} + \dfrac{1}{r}\psi_2 \end{cases} \tag{3.4}$$

In the fluid medium, the potential $\phi_1(r,z)$ satisfies the Helmholtz equation below.

$$\nabla^2\phi_1(r,z) + k_1^2\,(r,z)\phi_1 = 0. \tag{3.5}$$

Following the derivation of the PE in the fluid medium by Lee-Siegmann, the wave field $\phi_1(r,z)$ obeys the following decomposition

$$\phi(r,z) = A_1(r,z)\; H_0^{(1)}(k_0 r) \sim A_1(r,z)\sqrt{\frac{2}{\pi k_0 r}}\; e^{i\left(k_0 r - \frac{\pi}{4}\right)}, \tag{3.6a}$$

where k_0 is the reference wavenumber and $H_0^{(1)}(k_0 r)$ is the zeroth order Hankel function of the first kind. In the foregoing development, all Hankel functions, in their asymptotic expansion form, have a common multiplicative constant $\sqrt{\frac{2}{\pi}}\, e^{-i\frac{\pi}{4}}$; this can be ignored for simple calculation. This treatment allows Eq. (3.6)a to be written as

$$\phi(r,z) \cong A(r,z)\sqrt{\frac{1}{k_0 r}}\; e^{ik_0 r}\;. \tag{3.6b}$$

Assuming that $A_1(r,z)$ is weakly dependent in range, the representative PE in the fluid medium is obtained, by the parabolic approximation

$$\frac{\partial A_1}{\partial r} = a_1 A_1 + b_1 \frac{\partial^2 A_1}{\partial z^2} \tag{3.7}$$

where $a_1 = \frac{i}{2k_o}(k_1^2\,(r,z) - k_o^2)$ (3.8)

and

$$b_1 = \frac{i}{2k_o} \; . \tag{3.9}$$

In an inhomogeneous elastic medium, $\phi_2(r,z)$ and $\psi_2(r,z)$ no longer satisfy the two independent Helmholtz equations because of the coupling effects. Following the derivation of McCoy, the propagation of time-harmonic stress waves through an inhomogeneous, linear elastic solid medium which is locally isotropic, is governed by the equation

$$(\bar{\lambda}_2 + \bar{\mu}_2)\nabla\,\nabla \bullet D_2 + \bar{\mu}_2 \nabla^2\, D_2 + \bar{\rho}\omega^2 D_2 + \mathbf{F}_2 = 0 \tag{3.10}$$

where λ_2, μ_2 are Lame parameters and ρ_2 is the density such that

$$\lambda_2 = \overline{\lambda_2}\,[1 + \varepsilon_r(r,z)], \tag{3.11}$$

$$\mu_2 = \overline{\mu_2}\,[1 + \varepsilon_\mu(r,z)], \tag{3.12}$$

$$\rho_2 = \overline{\rho_2}\,[1 + \varepsilon_\rho(r,z)], \tag{3.13}$$

a

and

$$\mathbf{F}_2 = \nabla \bullet [\bar{\lambda}_2 \varepsilon_\lambda (\nabla \bullet D_2) \mathbf{e} + \bar{\mu}_2 \varepsilon_\mu (\nabla D_2 + D_2 \nabla)] + \bar{\rho}_2 \varepsilon_\rho \omega^2 D_2, \tag{3.14}$$

where **e** is a unit vector.

In the above equation, the upper bar indicates the spatial average and ε with a subscript indicates the perturbation with respect to that subscript.

Define

$$\bar{k}_D = \frac{\omega^2}{\bar{c}_D^2} = \frac{\bar{\rho}_2 \omega^2}{\bar{\lambda}_2 + 2\bar{\mu}_2}, \tag{3.15}$$

and

$$\bar{k}_s = \frac{\omega^2}{\bar{c}_s^2} = \frac{\bar{\rho}_2 \omega^2}{\bar{\mu}_2}, \tag{3.16}$$

assuming that

$$\phi_2(r,z) = A_2(r,z)\, H_o^{(1)}(\bar{k}_D\, r) \sim A_2(r,z) \sqrt{\frac{1}{\bar{k}_D r}}\, e^{i \bar{k}_D r} \tag{3.17}$$

and

$$\psi_2(r,z) = B_2(r,z)\, H_o^{(1)}(\bar{k}_s r) \sim B_2(r,z) \sqrt{\frac{1}{\bar{k}_s r}}\, e^{i \bar{k}_s r}. \tag{3.18}$$

With the assumptions above, $A_2(r,z)$ and $B_2(r,z)$ satisfy the following PE's, respectively:

$$\frac{\partial A_2}{\partial r} = a_2 A_2 + b_2 \frac{\partial^2 A_2}{\partial z^2} + c_2 \frac{\partial B_2}{\partial z} , \tag{3.19}$$

and

$$\frac{\partial B_2}{\partial r} = a_2' B_2 + b_2' \frac{\partial^2 B_2}{\partial z^2} + c_2' \frac{\partial A_2}{\partial z} , \tag{3.20}$$

where c_2 and c_2' are coupling coefficients whose definitions along with other symbols are given below:

$$a_2 = \frac{i}{2\bar{k}_D} [k_D^{\,2}(r,z) - \bar{k}_D^{\,2}], \tag{3.21}$$

$$b_2 = \frac{i}{2\bar{k}_D} , \tag{3.22}$$

$$c_2 = \frac{-1}{2\bar{k}_D} [i\Delta\bar{k}\ \bar{\varepsilon}_{\mu\rho}], \tag{3.23}$$

$$a_2' = \frac{1}{2\bar{k}_s} [k_s^2(r,z) - \bar{k}_s^2], \tag{3.24}$$

$$b_2' = \frac{i}{2\bar{k}_s} , \tag{3.25}$$

$$c_2' = \frac{-1}{2\bar{k}_s} \left[i\ \Delta\bar{k}\ \bar{\varepsilon}_{\mu\rho}^{\,*} \right], \tag{3.26}$$

$$\bar{\varepsilon}_{\mu\rho} = \frac{1}{\Delta r} \int_{n\Delta r}^{(n+1)\Delta r} \varepsilon_{\mu\rho}(x,z)\, e^{i\,\bar{\Delta k}\,x}\, dx, \tag{3.27}$$

and

$$\varepsilon_{\mu\rho} = 2\left(\bar{c}_S/\bar{c}_0\right)\varepsilon_\mu - \varepsilon_\rho \tag{3.28}$$

IV. FLUID/ELASTIC INTERFACE CONDITIONS

In this section, we derive the fluid/elastic interface conditions associated with the representative PE's. Furthermore, from these interface conditions, a system of three equations which relate the fluid potential A_1 to the elastic potentials A_2 and B_2 and their derivatives on the fluid/elastic interface will be derived.

We begin by discussing the fluid/elastic interface conditions using the expression of displacement. These conditions, in cylindrical coordinates, are

1. Continuity of vertical components of displacement:

$$w_1 = w_2. \tag{4.1}$$

2. Continuity of vertical components of stress:

$$\lambda_1\left(\frac{\partial u_1}{\partial r} + \frac{1}{r} u_1 + \frac{\partial w_1}{\partial z}\right) = \lambda_2 \left(\frac{\partial u_2}{\partial r} + \frac{1}{r} u_2 + \frac{\partial w_2}{\partial z}\right) + 2\mu_2 \left(\frac{\partial w_2}{\partial z}\right) \quad (4.2)$$

3. The horizontal components of stress must vanish on the interface:

$$\mu_2 \left(\frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial r}\right) = 0. \quad (4.3)$$

In terms of potentials, the equivalent interface conditions to (4.1), (4.2), and (4.3) are

1. Continuity of vertical components of displacement:

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z} + \frac{\partial \psi_2}{\partial r} + \frac{1}{r} \psi_2. \quad (4.4)$$

2. Continuity of vertical components of stress:

$$\lambda_1\left(\frac{\partial^2 \phi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_1}{\partial r} + \frac{\partial^2 \phi_1}{\partial z^2}\right) = \lambda_2 \left(\frac{\partial^2 \phi_2}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_2}{\partial r} + \frac{\partial^2 \phi_2}{\partial z^2}\right) + 2\mu_2 \left(\frac{\partial^2 \phi_2}{\partial z^2} + \frac{\partial^2 \psi_2}{\partial r \partial z} + \frac{1}{r} \frac{\partial \psi_2}{\partial z}\right) \quad (4.5)$$

3. The horizontal component of stress must vanish on the interface:

$$\mu_2 \left(2 \frac{\partial^2 \phi_2}{\partial r \partial z} - \frac{\partial^2 \psi_2}{\partial z^2} + \frac{\partial^2 \psi_2}{\partial r^2} + \frac{1}{r} \frac{\partial \psi_2}{\partial r} - \frac{1}{r^2} \psi_2\right) = 0. \quad (4.6)$$

Making use of Eq. (3.6)a for ϕ_1, (3.17) for ϕ_2, (3.18) for ψ_2, and the PE's (3.7), (3.19), and (3.20), the corresponding interface conditions for the PE's are obtained below.

1. Continuity of vertical components of displacement:

$$\frac{\partial A_1}{\partial z} = \sqrt{\frac{k_o}{\bar{k}_D}}\ \frac{\partial A_2}{\partial z}\ e^{i\Delta_D r} + \sqrt{\frac{k_o}{\bar{k}_s}} \left(\frac{2B_2}{\partial r} + i\ \bar{k}_s\ B_2\right) e^{i\Delta_s r} \qquad (4.7)$$

2. Continuity of vertical components of stress:

$$-\rho_1 \omega^2 A_1 = \left[\lambda_2\left(\frac{\partial^2 A_2}{\partial z^2} + 2i\bar{k}_D\ \frac{\partial A_2}{\partial r} - \bar{k}_D{}^2 A_2\right) + 2\mu_2\ \frac{\partial^2 A_2}{\partial z^2}\right]\sqrt{\frac{k_o}{\bar{k}_D}}\ e^{i\Delta_D r}$$

$$+ 2\mu_2\left(\frac{\partial^2 B_2}{\partial r \partial z} + i\bar{k}_s\ \frac{\partial B_2}{\partial z}\right)\sqrt{\frac{k_o}{\bar{k}_s}}\ e^{i\Delta_s r} \qquad (4.8)$$

3. The horizontal components of stress must vanish on the interface:

$$2\ \frac{\partial^2 A_2}{\partial z \partial r} + 2i\bar{k}_D\ \frac{\partial A_2}{\partial z} = \left(\frac{\partial^2 B_2}{\partial z^2} - 2i\bar{k}_s\ \frac{\partial B_2}{\partial r} - (ik_s)^2 B_2\right)\sqrt{\frac{\bar{k}_D}{\bar{k}_s}}\ e^{i\Delta_{sD} r} \qquad (4.9)$$

where $\Delta_D = k_D - k_o$, $\Delta_s = k_s - k_o$.

From this point on, since we deal with the potentials and their partial derivatives on the interface boundary and at the present range level, for economy in writing we drop the superscript n and the subscript j, e.g., A_2 means $(A_2)_j^n$ unless otherwise specified.

Following McDaniel-Lee's technique in fluid medium, we use the Taylor expansion for $(A_1)_{j-1}$ upon $(A_1)_j$, solve for $\frac{\partial^2 A_1}{\partial z^2}$ and substituting into

Eq. (3.7), we find

$$\frac{\partial A_1}{\partial z} = \frac{h}{2b_1} \frac{\partial A_1}{\partial r} - \frac{h}{2b_1} a_1 A_1 + \frac{1}{h} (A_1 - (A_1)_{j-1}) \tag{4.13}$$

where h, is defined as the depth increment Δz. Later we will use k to represent the range increment Δr.

If we substitute (4.13) into (4.7), we obtain

$$\frac{h}{2b_1} \frac{\partial A_1}{\partial r} - \frac{h}{2b_1} a_1 A_1 + \frac{1}{h} (A_1 - (A_1)_{j-1}) = K_D \frac{\partial A_2}{\partial z} + K_s \left(\frac{\partial B_2}{\partial r} + i\bar{k}_s B_2 \right), \tag{4.14}$$

where

$$K_D = \sqrt{\frac{k_o}{\bar{k}_D}} \, e^{i\Delta_D r}, \tag{4.15}$$

and

$$K_s = \sqrt{\frac{k_o}{\bar{k}_s}} \, e^{i\Delta_s r}. \tag{4.16}$$

Using the finite difference for the partial derivatives in (4.14), we obtain

$$\frac{h}{2b_1} \frac{1}{k} \left(A_1^{n+1} - A_1 \right) - \frac{h}{2b_1} a_1 A_1 + \frac{1}{h} (A_1 - (A_1)_{j-1}) = K_D \frac{1}{h} \left((A_2)_{j+1} - A_2 \right)$$
$$+ K_s \left(\frac{1}{k} [B_2^{n+1} - B_2] + i \, \bar{k}_s \, B_2 \right). \tag{4.17}$$

A simplification of (4.17) gives

$$p_{11}\, A_1^{n+1} + p_{12}\, A_2^{n+1} + p_{13}\, B_2^{n+1} = \text{RHS } 1 \qquad (4.18)$$

where

$$p_{11} = \frac{h}{2b_1}\,\frac{1}{k}\,, \qquad (4.19)$$

$$p_{12} = 0, \qquad (4.20)$$

$$p_{13} = -K_s/k, \qquad (4.21)$$

and

$$\text{RHS1} = \left[\frac{h}{2b_1}\left(\frac{1}{k} + a_1\right) - \frac{1}{h}\right] A_1 + \frac{1}{h}(A_1)_{j-1} + \frac{K_D}{h}\left((A_2)_{j+1} - A_2\right)$$

$$+ K_s\,(i\bar{k}_s - \frac{1}{k})\,\bar{B}_2. \qquad (4.22)$$

The next two equations to be derived into the system are based on Eqs. (4.8) and (4.9). In those two equations, two terms are involved, namely, $\frac{\partial^2 A_2}{\partial r \partial z}$ and $\frac{\partial^2 B_2}{\partial r \partial z}$. We first try to develop explicit expressions for these two partial derivatives.

Applying the finite difference to the partial derivatives in Eq. (3.20) gives

$$(B_2)_{j+1}^{n+1} = (B_2)_{j+1} + k\, a_2'\, (B_2)_{j+1} + k\, b_2'\left(\frac{\partial^2 B_2}{\partial z^2}\right)_{j+1}$$

$$+ k\, c_2' \left(\frac{\partial A_2}{\partial z}\right)_{j+1}$$

$$= (1 + k\, a_2')\ (B_2)_{j+1} + k\, b_2' \left(\frac{\partial^2 B_2}{\partial z^2}\right)_{j+1} + k\, c_2' \left(\frac{\partial A_2}{\partial z}\right)_{j+1} \tag{4.23}$$

$$\frac{\partial^2 B_2}{\partial r \partial z} = \frac{\partial}{\partial r}\left(\frac{\partial B_2}{\partial z}\right) = \frac{\partial}{\partial r}\ \frac{1}{h}\ ((B_2)_{j+1} - B_2) = \frac{1}{h}\left(\frac{(B_2)_{j+1}^{n+1} - (B_2)_{j+1}}{k} - \frac{(B_2)_j^n - B_2}{k}\right)$$

$$= \frac{1}{hk}\Bigg\{(1 + k\, a_2')\ (B_2)_{j+1} + k\, b_2' \left(\frac{\partial^2 B_2}{\partial z^2}\right)_{j+1}$$

$$+ k\, c_2' \left(\frac{\partial A_2}{\partial z}\right)_{j+1} - (B_2)_{j+1} - \left((B_2)_j^{n+1} - B_2\right)\Bigg\}$$

$$= \frac{1}{hk}\Bigg\{k\, a_2'\ (B_2)_{j+1} + k\, b_2' \left(\frac{\partial^2 B_2}{\partial z^2}\right)_{j+1} + k\, c_2' \left(\frac{\partial A_2}{\partial z}\right)_{j+1}$$

$$- \left((B_2)_j^{n+1} - B_2\right)\Bigg\}. \tag{4.24}$$

Applying the finite difference to the partial derivatives in Eq. (3.19), we obtain

$$(A_2)_{j+1}^{n+1} = (A_2)_{j+1} + k\, a_2 (A_2)_{j+1} + k\, b_2 \left(\frac{\partial^2 A_2}{\partial z^2}\right)_{j+1} + k\, c_2 \left(\frac{\partial B_2}{\partial z}\right)_{j+1} \tag{4.25}$$

$$= (1 + k\, a_2)\ (A_2)_{j+1} + k\, b_2 \left(\frac{\partial^2 A_2}{\partial z^2}\right)_{j+1} k\, c_2 \left(\frac{\partial B_2}{\partial z}\right)_{j+1}.$$

$$\frac{\partial^2 A_2}{\partial r \partial z} = \frac{\partial}{\partial r}\left(\frac{\partial A_2}{\partial z}\right) = \frac{1}{h}\frac{\partial}{\partial r}\left((A_2)_{j+1} - A_2\right)$$

$$= \frac{1}{h}\left(\frac{(A_2)_{j+1}^{n+1} - (A_2)_{j+1}}{k} - \frac{(A_2)_j^{n+1} - A_2}{k}\right)$$

$$= \frac{1}{hk}\left\{k\, a_2 (A_2)_{j+1} + k\, b_2 \left(\frac{\partial^2 A_2}{\partial z^2}\right)_{j+1} + k\, c_2 \left(\frac{\partial B_2}{\partial z}\right)_{j+1} - \left((A_2)_j^{n+1} - A_2\right)\right\}. \tag{4.26}$$

Eq. (4.8) can be written as

$$-\rho_1 \omega^2 A_1 = \left[(\lambda_2 + 2\mu_2) \frac{\partial^2 A_2}{\partial z^2} + \lambda_2\, 2i\bar{k}_D \frac{\partial A_2}{\partial r} - \lambda_2 \bar{k}_D^2 A_2\right] K_D + 2 \frac{\rho_2 \omega^2}{\bar{k}_s}\left(\frac{\partial^2 B_2}{\partial z^2} + i\, \bar{k}_s \frac{\partial B_2}{\partial z}\right) K_s. \tag{4.27}$$

Substituting (4.24) into (4.27), we obtain

$$-\rho_1 \omega^2 A_1 = \left[\frac{\rho_2 \omega^2}{\bar{k}_D} \frac{\partial^2 A_2}{\partial z^2} + \lambda_2\, 2\, i\bar{k}_D \frac{\partial A_2}{\partial r} - \lambda_2 \bar{k}_D^2 A_2\right] K_D + 2 \frac{\rho_2 \omega^2}{\bar{k}_s^2}\left(\frac{1}{hk}\left\{k a_2' (B_2)_{j+1} + k\, b_2' \left(\frac{\partial^2 B_2}{\partial z^2}\right)_{j+1} + k\, c_2' \left(\frac{\partial A_2}{\partial z}\right)_{j+1} - \left((B_2)_j^{n+1} - B_2\right)\right\} + i\, \bar{k}_s \frac{\partial B_2}{\partial z}\right) K_s. \tag{4.28}$$

Using the finite difference for all partial derivatives in (4.28), we obtain

$$-\rho_1\omega^2 A_1 = \left[\frac{\rho_2\omega^2}{\bar{k}_D}\ \frac{1}{h^2}\ ((A_2)_{j+2} - 2(A_2)_{j+1} + A_2) + \lambda_2\ 2i\bar{k}_D\ \frac{1}{k}\left(A_2^{n+1} - A_2\right)\right.$$

$$\left. - \lambda_2\ \bar{k}_D^2\ A_2\right] K_D + 2\ \frac{\rho_2\omega^2}{\bar{k}_s^{\ 2}}\ \left(\frac{1}{hk}\left\{k\ a_2'\ (B_2)_{j+1}\right.\right.$$

$$+ k\ b_2'\ \frac{1}{h^2}\left[(B_2)_{j+2} - 2(B_2)_{j+1} + B_2\right] + k\ c_2'\ \frac{1}{h}\left((A_2)_{j+1} - A_2\right)$$

$$\left.\left. - \left((B_2)_j^{n+1} - B_2\right)\right\} + i\ \bar{k}_s\ \frac{1}{h}\ [(B_2)_{j+1} - B_2]\right) K_s. \tag{4.29}$$

A simplification of (4.29) gives

$$p_{21}\ A_1^{n+1} + p_{22}\ A_2^{n+1} + p_{23}\ B_2^{n+1} = \text{RHS2} \tag{4.30}$$

where

$$p_{21} = 0, \tag{4.31}$$

$$p_{22} = \lambda_2\ 2i\bar{k}_D\ \frac{1}{k}\ K_D, \tag{4.32}$$

$$p_{23} = -2\ \frac{\rho_2\omega^2}{\bar{k}_s^{\ 2}\ hk}\ K_s, \tag{4.33}$$

and

$$\text{RHS2} = -\rho_1\omega^2 A_1 - \frac{\rho_2\omega^2}{h^2}\ \frac{K_D}{\bar{k}_D^{\ 2}}\ (A_2)_{j+2}$$

$$+ \frac{\rho_2\omega^2}{h}\left\{\frac{2K_D}{\bar{k}_D h} + \frac{2K_s k\, c_2'}{\bar{k}_s}\right\}(A_2)_{j+1}$$

$$+ \left\{\frac{-\rho_2\omega^2}{\bar{k}_D\, h^2} + \lambda_2\; 2i\bar{k}_D\,\frac{1}{k} + \lambda_2\;\bar{k}_D^2\right\}K_D A_2$$

$$- 2\,\frac{\rho_2\omega^2}{\bar{k}_s^{\;2}}\,\frac{1}{h}\,\frac{1}{k}\;\; k\, b_2'\,\frac{1}{h^2}\; K_s\;(B_2)_{j+2}$$

$$- 2\,\frac{\rho_2\omega^2}{\bar{k}_s^{\;2}}\left(\frac{1}{hk}\;\; k\, a_2' - 2\, k\, b_2'\,\frac{1}{h^2} + i\bar{k}_s\,\frac{1}{h}\right)\; K_s\;(B_2)_{j+1}$$

$$- 2\,\frac{\rho_2\omega^2}{\bar{k}_s^{\;2}}\left(\frac{b_2'}{h^3} + \frac{1}{(kh)} - i\,\bar{k}_s\,\frac{1}{h}\right)K_s\, B_2 - \rho_1\omega^2\, A_1\;. \qquad (4.34)$$

Next, substitute (4.26) into (4.9), we obtain

$$\frac{2}{hk}\left\{k\; a_2\;(A_2)_{j+1} + k\; b_2\left(\frac{\partial^2 A_2}{\partial z^2}\right)_{j+1} + k\; c_2\left(\frac{\partial B_2}{\partial z}\right)_{j+1}\;\left((A_2)^{n+1} - A_2\right)\right\}$$

$$+ 2i\bar{k}_D\,\frac{\partial A_2}{\partial z} = \left(\frac{\partial^2 B_2}{\partial z^2} - 2i\bar{k}_s\,\frac{\partial B_2}{\partial r} - (i\bar{k}_s)^2 B_2\right)K_{DS}, \qquad (4.35)$$

where $K_{DS} = \sqrt{\dfrac{\bar{k}_D}{\bar{k}_s}}\; e^{\,i\Delta_{SD} r}$, and $\qquad (4.36)$

$$\Delta_{SD} = \Delta_S - \Delta_D\;.$$

Using the finite difference for all partial derivatives in (4.35), we obtain

$$\frac{2}{hk}\left\{k\ a_2(A_2)_{j+1}\ k\ b_2\ \frac{1}{h^2}\left[(A_2)_{j+2} - 2(A_2)_{j+1} + A_2\right] + k\ \frac{c_2}{h}\left[(B_2)_{j+1} - B_2\right]\right.$$

$$\left. - \left((A_2)^{n+1} - A_2\right)\right\} + 2i\bar{k}_D\ \frac{1}{h}\left((A_2)_{j+1} - A_2\right)$$

$$= \frac{1}{h^2}\ [(B_2)_{j+2} - 2(B_2)_{j+1} + B_2] - 2i\bar{k}_s\ \frac{1}{k}\ [B_2^{n+1} - B_2]$$

$$- (i\bar{k}_s)^2\ B_2)\ K_{DS}. \qquad (4.37)$$

A simplification of (4.37) gives

$$p_{31}\ A_1^{n+1} + p_{32}\ A_2^{n+1} + p_{33}\ B_2^{n+1} = RHS3 \qquad (4.38)$$

where

$$p_{31} = 0, \qquad (4.39)$$

$$p_{32} = -2/(hk), \qquad (4.40)$$

$$p_{33} = 2ik_s\ (1/k)\ K_{DS}, \qquad (4.41)$$

and

$$RHS3 = -\frac{2}{h^3}\ b_2\ (A_2)_{j+2}$$

$$-\left[\frac{2}{hk}\left\{ka_2 - 2k\ b_2\ \frac{1}{h^2}\right\} + 2i\bar{k}_D\ \frac{1}{h}\right]\ (A_2)_{j+1}$$

$$+\left(-\frac{2}{hk}\ k\ b_2\ \frac{1}{h^2}\ \frac{2}{hk} + 2i\bar{k}_D\ \frac{1}{h}\right)A_2 + \frac{1}{h^2}\ K_{DS}(B_2)_{j+2}$$

$$- \left\{ k \frac{c_2}{h} + \frac{2}{h^2} K_{DS} \right\} (B_2)_{j+1}$$

$$+ \left\{ \left(\frac{1}{h^2} + 2i\bar{k}_s \frac{1}{k} - i\bar{k}_s^{\,2} \right) K_{DS} + k \frac{c_2}{h} \right\} B_2. \tag{4.42}$$

Equations (4.18), (4.30), and (4.38) constitute a system of 3 equations among which unknowns A_1^{n+1}, A_2^{n+2}, and B_2^{n+1} are to be solved. From this system we see that A_1^{n+1} only appears in Eq. (4.18). Actually, we need only to solve a system of 2 equations, i.e., Eqs. (4.30) and (4.38). This system can be written as

$$\begin{bmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} A_2^{n+1} \\ B_2^{n+1} \end{bmatrix} = \begin{bmatrix} \text{RHS2} \\ \text{RHS3} \end{bmatrix}. \tag{4.43}$$

After system (4.43) is solved, we substitute B_2^{n+1} into Eq. (4.18) to obtain A_1^{n+1} by means of

$$A_1^{n+1} = (\text{RHS 1} - p_{13} \; B_2^{n+1} \;)/\; p_{11}. \tag{4.44}$$

To examine the existence and uniqueness of the solution of system (4.43), we evaluate the determinent of (4.43),

$$\text{determinant of (4.43)} = \left(\lambda_2 \; 2i\bar{k}_D \frac{1}{k} \right) \left(2i\bar{k}_s \frac{1}{k} K_{DS} \right)$$

$$- \left(- \frac{2}{hk} \right) \left(-2 \frac{\rho_2 \omega^2}{\bar{k}_s hk} \right)$$

$$= -4\lambda_2 \frac{\bar{k}_D \bar{k}_s}{k^2} K_{DS} - 4 \frac{1}{h^2k^2} Ks \frac{\rho^2\omega^2}{\bar{k}_s} \neq 0$$

because K_{DS} is complex while other quantities are real; therefore, the determinent is not singular and the solution exists and is unique.

V. A NEW COMPUTATIONAL APPROACH

Following McDaniel-Lee [1], a uniform partition in the z-direction is assumed. The $h = \Delta z$ is used for the depth increment and the integer index j is the interface boundary. As before, the superscript indicates the range level, and the subscript indicates the depth level. It is also understood that if both the superscript and the subscript are dropped, it denotes the field at $(n\Delta r, j\Delta z)$, i.e., $A = A_j^n$.

Our approach can be described by the following statement:

> Solving the representative parabolic wave equations in the different media by means of an implicit finite difference (IFD) marching scheme, applying the field values on the interface as boundary information from fluid and elastic media.

We proceed to discuss the meaning of the above statement. Solving the representative parabolic wave equation using an implicit finite difference scheme in a marching process requires the initial field values at range r_o

plus the surface and bottom boundary information. The surface boundary point at the present level is denoted by $(A_1)_o^n$ and at the advanced level by $(A_1)_o^{n+1}$ the bottom boundary point at the present level is denoted by $(A_1)_j^n$ and at the advanced level by $(A_1)_j^{n+1}$. The IFD scheme predicts the wave field at the advanced level, $r + \Delta r$, regardless whether the medium is fluid or elastic. If the medium is fluid, there is only one PE to solve; if the medium is elastic, there is a system of 2 PE's to solve, in this case the field is a vector containing components A_2 and B_2, each component is a subvector. Dealing with the solution of the entire problem, the surface remains unchanged, but an interface boundary comes into existence. This interface boundary separates a fluid medium and an elastic medium. One crosses the interface boundary, the density and the sound speeds change. In the elastic medium two sound speeds occur, the speed of P-wave c_D and the speed of S-wave c_S. Initial field values at r_o are used along with surface points $(A_1)_o^n$, $(A_1)_o^{n+1}$ and interface boundary points $(A_1)_j^n$, $(A_1)_j^{n+1}$ to predict the wave field $(r_o + \Delta r)$ in the fluid medium. The $(A_2)_j^n$, $(A_2)_j^{n+1}$, $(B_2)_j^{n+1}$ are used as surface points along with initial field values at the same range level. Beyond interface boundary, the boundary points $(A_2)_{bottom}^n$, $(A^2)_{bottom}^{n+1}$, $(B_2)_{bottom}^n$, $(B_2)_{bottom}^{n+1}$ are generated artificially. This setup allows the same IFD procedure to solve a system of PE's in the elastic medium in the same manner as in the fluid medium. The key treatment is to determine the interface boundary values $(A_1)_j$, $(A_2)_j$, and $(B_2)_j$ which are related by the system

(4.43) and Eq. (4.44). With these interface boundary values, the IFD can march in range to predict the wave field in the fluid at the next range. This is where the McDaniel-Lee interface treatment is extended to handle the fluid/elastic horizontal interface. Note that $(A_1)_j$ is used as a bottom boundary point to solve the PE in the fluid medium while $(A_2)_j$ and $(B_2)_j$ are used as two "surface" points to solve the system of two parabolic equations in the elastic medium.

For better understanding, the diagram below explains our description and will help clear up our early statements.

Let $(\vec{A}_1)_j^n$, $(\vec{A}_1)_j^{n+1}$ be 2 vectors associated with range levels n and n+1 respectively. These 2 vectors contain only the nonzero components at the interface boundaries, i.e., $(A_1)_j^n$ and $(A_1)_j^{n+1}$.

Then the matrix representation for the fluid/elastic interface problem can be expressed by

Fluid: $$D^{n+1} \quad \vec{A}_1^{n+1} = D^n \quad \vec{A}_1^n + (\vec{A}_1)_j^n + (\vec{A}_1)_j^{n+1}$$

---interface

Elastic: $$\begin{pmatrix} E & E_1 \\ \hline F_1 & F \end{pmatrix}^{n+1} \begin{pmatrix} \vec{A}_2^{n+1} \\ \hline \vec{B}_2^{n+1} \end{pmatrix} = \begin{pmatrix} E & E_1 \\ \hline F_1 & F \end{pmatrix}^{n} \begin{pmatrix} \vec{A}_2^{n} \\ \hline \vec{B}_2^{n} \end{pmatrix} + \begin{pmatrix} (\vec{A}_2)_j^n \\ \hline (\vec{B}_2)_j^n \end{pmatrix} + \begin{pmatrix} (\vec{A}_2)_j^{n+1} \\ \hline (\vec{B}_2)_j^{n+1} \end{pmatrix}$$

where $(\vec{A}_2)_j$ and $(\vec{B}_2)_j$ are 2 vectors having the same structure as $(\vec{A}_1)_j$ except the nonzero elements are the first components and D, E, F are tri-diagonal matrices, and E_1, F_1 are sparse matrices whose non-zero elements appear on the diagonal and lower diagonal.

Note that the points which influence the fluid/elastic interface boundary are $(A_1)_j$, $(A_2)_j$, and $(B_2)_j$ at all range levels. It is very clear that their relationships are defined by the system (4.43) and the Eq. (4.44). Once these boundary values are obtained, the IFD scheme can march forward in range.

VI. A NUMERICAL EXAMPLE

This section examines the validity of our approach and an example whose solutions [6] are known is used as a demonstration.

First of all, in the case where the elastic waves are absent which implies that μ_2 is zero. Under such an environment only one PE is needed to represent the wave propagation in the fluid. Thus, the equation (2.1) is the representative equation. Furthermore, Eq. (4.6) becomes zero identically on both sides. Eq. (4.4) reduces to

$$\frac{\partial A_1}{\partial z} = \frac{\partial A_2}{\partial z} \quad , \tag{6.1}$$

and Eq. (4.5) reduces to

$$\rho_2 A_1 = \rho_1 A_2 \quad . \tag{6.2}$$

Eqs. (6.1) and (6.2) are equivalent interface conditions to (2.2) and (2.3) in the fluid medium. Then all Taylor expansions can be applied following the McDaniel-Lee technique.

Next we use an example discussed by Ewing et al [7] to computationally demonstrate the validity. This example neglects the branch line integrals and is feasible for large values of range; thus, it is very suitable for our PE's (far-field). The solutions ϕ_1, ϕ_2, ψ_2 are given for Helmholtz equations. For consistency we derive the corresponding solutions to the PE.

The exact mathematical expressions for ϕ_1, ϕ_2, and ψ_2 are

$$\phi_1 = \frac{2\pi}{H} \sqrt{\frac{2}{\pi r}} \sum_n \frac{I}{\sqrt{k_n}} e^{i(\omega t - k_n r - \frac{\pi}{4})} \Phi_1(k_n) \sin(\xi_n d) \sin(\xi_n z)$$

$$0 \leq z \leq H = z_j, \ d \doteq \text{source depth} \tag{6.3}$$

$$\phi_2 = \frac{2\pi}{H} \sqrt{\frac{2}{\pi r}} \sum_n \frac{I}{\sqrt{k_n}} e^{i(\omega t - k_n r - \frac{\pi}{4})} \Phi_2(k_n) \sin(\xi_n d) e^{-\eta(z-H)}$$

$$z \geq H \tag{6.4}$$

$$\psi_2 = \frac{2\pi}{H} \sqrt{\frac{2}{\pi r}} \sum_n \frac{I}{\sqrt{k_n}} e^{i(\omega t - k_n r - \frac{\pi}{4})} \Psi_2(k_n) \sin(\xi_n d) e^{-\zeta(z-H)}$$

$$z \geq H \ . \tag{6.5}$$

where

$$\Phi_1(k_n) = \frac{-\frac{\rho_1}{\rho_2}\,\frac{c^4}{\beta_2^4}\,\frac{\eta_n}{\xi_n}\,k_n H}{\sqrt{\frac{c^2}{v_1^2} - I}\;\{\cdot\}\;\cos(\xi_n H)} , \tag{6.6}$$

$$\Phi_2(k_n) = \frac{-\frac{\rho_1}{\rho_2}\,\frac{c^2}{\beta_2^2}\,(2 - \frac{c^2}{\beta_2^2})\,k_n H}{\sqrt{\frac{c^2}{v_1^2} - I}\;\{\cdot\}} , \tag{6.7}$$

$$\Psi_2(k_n) = -\frac{\frac{\rho_1}{\rho_2}\,\frac{c^2}{\beta_2^2}\,\frac{\eta_n}{\xi_n}}{\{\cdot\}} , \tag{6.8}$$

$$\{\cdot\} = \frac{\rho_1}{\rho_2}\,\frac{c^4}{\beta_2^4}\left\{\frac{\sin(\xi_n H)}{\sqrt{\frac{c^2}{v_1^2} - 1}\;\sqrt{1 - \frac{c^2}{\alpha_2^2}}}\left[I + \frac{1 - c^2/\alpha_2^2}{\frac{c^2}{c_1^2} - 1}\right] - \left[\frac{k_n H\sqrt{1 - c^2/\alpha_2^2}}{(c^2/v_1^2 - 1)}\,\sec(\xi_n H)\right]\right\}$$

$$- 4\,\frac{\sqrt{1 - c^2/\beta_2^2}}{\sqrt{1 - c^2/\alpha_2^2}} + \frac{1 - c^2/\alpha_2^2}{1 - c^2/\beta_2^2} + 2\sqrt{1 - \frac{c^2}{\alpha_2^2}}\,\sqrt{1 - \frac{c^2}{\beta_2^2}}$$

$$- 2(2 - \frac{c^2}{\beta_2^2})\Bigg\}\cos(\xi_n H) , \tag{6.9}$$

and $$\xi_n = k_n \sqrt{\frac{c^2}{v_1^2} - 1} \quad , \tag{6.10}$$

$$\eta_n = k_n \sqrt{1 - \frac{c^2}{\alpha_2^2}} \quad , \tag{6.11}$$

$$\zeta_n = k_n \sqrt{1 - c^2/\beta_2^2} \quad , \tag{6.12}$$

where n is a subscript which indicates that the quantity is to be evaluated at $k = k_n$, where k_n are the roots of

$$\frac{\rho_1}{\rho_2} \frac{\omega^4}{\beta_2^4} \frac{\eta}{\xi} \tan(\xi H) - 4k^2 \left[\zeta - (2k^2 - \frac{\omega^2}{\beta_2^2})^2 \right] = 0 \ . \tag{6.13}$$

For computational simplicity, we use one mode, i.e., n = 1. After separating the $H_0^{(1)}(k_* r)$ ($k_* = k_0$, k_s, or k_D) and the time-harmonic, the corresponding PE's A_1, A_2, and B_2 to Eqs. (6.3), (6.4), and (6.5) become

$$A_1 = \frac{2\pi}{z_j} \sqrt{\frac{k_0}{k_1}} \, e^{i(k_n - k_0)r} \, \Phi_1(k_1) \sin(\xi_n d) \sin(\xi_n z) \quad , \tag{6.14}$$

$$A_2 = \frac{2\pi}{H} \sqrt{\frac{k_D}{k_1}} \, e^{i(k_n - k_D)r} \, \Phi_2(k_1) \sin(\xi_n d) e^{-\eta(z-H)} \tag{6.15}$$

$$B_2 = 2\pi \sqrt{\frac{k_s}{k_1}} \, e^{i(k_n - k_s)r} \, \Psi_2(k_1) \sin(\xi_n d) e^{-\zeta(z-H)} \tag{6.16}$$

and

$\{\bullet\}$ has the same definition as (6.9) but has the subscript n = 1.

Next select the compressional wave velocity v_1 in a fluid of 1500 m/s with compressibility $\lambda_1 = v_1^2$ and a water density of $\rho_1 = 1.0$. c is the phase velocity such that

$$f = c\, k_n/2\pi \longrightarrow \frac{2\pi f}{k_n} = c$$

We follow the case I of Press-Ewing's to select α_2, β_2, and v_1 such that

$$\alpha_2 > \beta_2 \geq c \geq v_1 \tag{6.17}$$

We make the following choices:

$f = 68.03$ Hz

$H = z_j = 100$m

$d = z_s = 25$m

$v_1 = 1500$m/s

$\beta_2 = 1530.0$ m/s

$\alpha_2 = 1725.0$ m/s

$c_1 = 1507.5$ m/s

$k_1 = 0.283546$

$\lambda_1 = (1500)^2$

$\lambda_2 = 2.5 * (1725.0)^2 - 2*2.5*(1530.0)^2$

$\mu_2 = 2.5*(1530.0)^2$

$\rho_1 = 1.0$ g/cm^3

$\rho_2 = 1.97$ g/cm^3

$k_o = 0.284963$

$k_D = 0.247794$

$k_s = 0.279376$

$(A_1)_j^{n+1}$ is calculated by means of formula (4.44) where RHS1 is defined by formula (4.22), p_{13} is defined by formula (4.21), p_{11} is defined by formula (4.19), and B_2^{n+1} is calculated by formula (6.16).

As illustrated in the previous section, our main effort is to determine $(A_1)^{n+1}$ and use it as a fluid/elastic boundary information. $(A_1)_j^{n+1}$ is obtained in such a way that it is related to the information of elastic potentials $(A_2)_j^{n+1}$ and $(B_2)_j^{n+1}$ on the fluid/elastic interface. System (4.43) was developed to relate these points. In this test example, we use an accurate $(A_2)_j$ and $(B_2)_j$ at every range from a known solution as the accurate boundary interface values. These accurate values are applied at every step when solving system (4.43). Results are tabulated describing the comparison of computed field values against the known solution in dB. Accuracy was carried up to 2 significant digits using the VAX 11/780 computer.

TABLE OF RESULTS

Range (m) / Depth (m)	1500	2000	
30	(-0.366E-02,-0.590E-02) (-0.366E-02,-0.590E-02)	(-0.669E-02,-0.198E-02) (-0.662E-02,-0.209E-02)	Computed Exact
60	(0.480E-02,-0.785E-02) (0.482E-02,-0.777E-02)	(-0.874E-02,-0.270E-02) (-0.872E-02,-0.276E-02)	
90	(-0.276E-02,-0.428E-02) (-0.269E-02,-0.433E-02)	(-0.464E-02,-0.152E-02) (-0.487E-02,-0.154E-02)	

From this selected test example, it is clearly seen that the numerical results, produced by this model, agree satisfactorily with the exact solution. The results not only demonstrate the validity of this model, but also show the correct computational procedure following the IFD procedure. This also serves as an early indication that this model can be readily incorporated into the IFD code.

VII. CONCLUSIONS

A mathematical model has been developed by means of the parabolic approximation method for handling the fluid/elastic interface. The complete mathematical development plus the numerical example proved the validity of the model. As it stands now, even though this model is accurate, it is limited to narrow angle propagation only. However, an important feature of this model is that it can handle a range-dependent index of refraction in the elastic medium. Moreover, another attractive feature is that this model is readily adaptable into the existing IFD code without requiring excessive effort.

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REFERENCES

1. S. T. McDaniel and D. Lee, "A Finite-Difference Treatment of Interface Conditions for the Parabolic Wave Equation: The Horizontal Interface," J. Acoust. Soc. Am., 71(4), 855-858 (1982)

2. R. A. Stephen, "A Comparison of Finite Difference and Reflectivity Seisomograms for Marine Models," Geophy. J. R. Astr. Soc., 72, 39-58 (1983)

3. Yu-Chiung Teng and John T. Kuo, "A Finite Element Algorithm for Solving the Transient Problem in a Fluid-Solid Coupled Medium," in COMPUTATIONAL ACOUSTICS-Volume 1: WAVE PROPAGATION, ED., D. Lee, R. L. Sternberg, and M. H. Schultz, North-Holland, Amsterdam (1987)

4. F. D. Tappert, "The Parabolic Approximation Method," in Wave Propagation and Underwater Acoustics, eds. J. B. Keller and J. S. Papadakis, Lecture Notes in Physics, Vol. 70, Springer-Verlag, Heidelberg (1977).

5. J. J. McCoy, "A Parabolic Theory of Stress Wave Propagation through Inhomogeneous Linarly Elastic Solid," Tr. ASME, 462-468 (1977)

6. D. Lee and S. T. McDaniel, Ocean Acoustic Propagation by Finite Difference Methods, Pergamon Press (1988).

7. W. M. Ewing, W. S. Jarditzky, and F. Press, Elastic Waves in Layered Media, McGraw-Hill Book Co., Inc., New York (1957)

8. D. Lee and W. L. Siegmann, "A Mathematical Model for the 3-Dimensional Ocean Acoustic Propagation," J. Math. Model, Vol. 7, 143-162 (1986)

9. T. Lander and J. F. Claerbout, "Numerical Calculations of Elastic Waves in Laterally Inhomogeneous Media," J. Geophys. Res., 77, 1476 (1972)

10. J. A. Hudson, "A Parabolic Approximation for Elastic Waves," Wave Motion 2, 207 (1980).

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      SUBROUTINE UFIELD
C     ****************************************************************
C     *** USER STARTING FIELD
C     *** USER WRITES THIS SUBROUTINE IF GAUSSIAN FIELD NOT DESIRED
C     *** UFIELD IS CALLED IF INPUT PARAMETER ISF IS NOT ZERO
C     ****************************************************************
C     *** UFIELD SUBROUTINE SUPPLIES:
C         U   - COMPLEX STARTING FIELD
C     ****************************************************************
      PARAMETER MXLYR=101,MXN=10000,MXSVP=101,MXTRK=101,NIU=1,
     C          NOU=2,NPU=6
      COMPLEX ACOFX,ACOFY,BCOF,BOTX,BOTY,BTA,HNK,HNKL,SURX,SURY,TEMP,
     C        U,X,Y
C
      REAL*4 K0, K1, KS, KD, MU2, LAMNDA2
      COMPLEX PHI1(MXN), PHI2(MXN), PSI2(MXN), CARG, B1, A1,
     C        T12, T21, T22, T32, T33, CHI, A1JM,
     C        KDD, KSS, CARH, CARI
      COMMON /USTFLD/ K0, K1, KS, KD, SIGMA1, CPHI1, CPHI2, CPSI2,
     C       B1, DELTAS, DELTAD, A1JM
      EQUIVALENCE (H, ZLYR(1)), (D, ZS), (Z, ZI)
C
      COMMON /IFDCOM/ACOFX,ACOFY,ALPHA,BCOF,BETA(MXLYR),BOTX,BOTY,
     C       BTA(MXN),C0,CSVP(MXSVP),DR,DR1,DZ,FRQ,IHNK,ISF,ITYPEB,
     C       ITYPES,IXSVP(MXLYR),KSVP,N,N1,NLYR,NSVP,NWSVP,R12(MXN),RA,
     C       RHO(MXLYR),RSVP,SURX,SURY,THETA,TRACK(MXTRK,2),U(MXN),
     C       X(MXN),XK0,Y(MXN),ZA,ZLYR(MXLYR),ZP,ZS,ZSVP(MXSVP)
      DATA PI/3.141592654/,DEG/57.29578/
C
C     PUT THESE VALUES IN TEMPORARILY
C
      DATA C/1507.5/, V1/1500.0/, ALPHA2/1725.00/, BETA2/1530.00/,
     D     RHO1/1.0/, RHO2/1.97/, IND/0/
C
C
      SEC(ANG) = 1.0 / COS(ANG)
C
C     *** STARTING FIELD - EWING & PRESS
C
      AR = RA
      IF (IND .NE. 0) AR = RA - DR
      OMEGA = 2.0 * PI * FRQ
      K0 = OMEGA / C0
      K1 = OMEGA / C
      KS = OMEGA / BETA2
      KD = OMEGA / ALPHA2
C     WRITE (NPU, 1) 'K0: ',K0,'K1: ',K1,'KS: ',KS,'KD: ',KD
1     FORMAT (2X,A4,E12.6,3X,A4,E12.6,3X,A4,E12.6,3X,A4,E12.6)
      CARG = CMPLX(0.0, (K1 - K0) * AR)
      CARH = CMPLX(0.0, (K1 - KD) * AR)
      CARI = CMPLX(0.0, (K1 - KS) * AR)
      B1 = CMPLX(0.0, 1.0 / (2.0 * K0))
C     WRITE (NPU, 2) 'CARG: ',CARG,'B1: ',B1
2     FORMAT (2X,A6,'(',E12.6,2X,E12.6,')',5X,A4,'(',E12.6,2X,E12.6,
     F        ')')
      DELTAS = (KS - K0) * AR
      DELTAD = (KD - K0) * AR
      DELTADS = (KD - KS) * AR
C     WRITE (NPU, 3) 'DELTAS: ',DELTAS,'DELTAD: ',DELTAD,'DELTADS: ',
```

```
C     W                  DELTADS
3      FORMAT (2X,A8,E12.6,3X,A8,E12.6,3X,A9,E12.6)
       MU2 = RHO2 / BETA2 ** 2
       LAMNDA2 = RHO2 * (ALPHA2 ** 2 - 2.0 * BETA2 ** 2)
C      WRITE (NPU, 4) 'MU2: ',MU2,'LAMNDA2: ',LAMNDA2
4      FORMAT (2X,A5,E12.6,3X,A9,E12.6)
C
       ARG1 = C ** 2 / V1 ** 2 - 1.0
       ARG2 = 1.0 - C ** 2 / ALPHA2 ** 2
       ARG3 = 1.0 - C ** 2 / BETA2 ** 2
       ARG4 = 2.0 - C ** 2 / BETA2 ** 2
C      WRITE (NPU,5) 'ARG1: ',ARG1,'ARG2: ',ARG2,'ARG3: ',ARG3,
C     W              'ARG4: ',ARG4
5      FORMAT (2X,A6,E12.6,3X,A6,E12.6,3X,A6,E12.6,3X,A6,E12.6)
C
       SIGMA1 = K1 * SQRT (ARG1)
       ETA1 = K1 * SQRT (ARG2)
       ZETA1 = K1 * SQRT (ARG3)
C      WRITE (NPU,6) 'SIGMA1: ',SIGMA1,'ETA1: ',ETA1,'ZETA1: ',ZETA1
6      FORMAT (2X,A8,E12.6,3X,A6,E12.6,3X,A7,E12.6)
C
       BRACE = (RHO1 / RHO2) * (C ** 4 / BETA2 ** 4)
     B         * (SIN (SIGMA1 * H) / (SQRT (ARG1) * SQRT (ARG2))
     B         * (1.0 + ARG2 / ARG1) - ((K1 * H * SQRT (ARG2)) / ARG1
     B         * SEC (SIGMA1 * H))) - 4.0 * (SQRT (ARG3) / SQRT (ARG2)
     B         + SQRT (ARG2) / SQRT (ARG3) + 2.0 * SQRT (ARG2)
     B         * SQRT (ARG3) - 2.0 * ARG4) * COS (SIGMA1 * H)
C
       CPHI1 = -((RHO1 / RHO2) * (C **4 / BETA2 ** 4) * (ETA1 / SIGMA1)
     C         * K1 * H) / (1.0 * BRACE * COS(SIGMA1 * H))
       CPHI2 = -((RHO1 / RHO2) * (C ** 2 / BETA2 ** 2) * ARG4 * K1 * H)
     C         / (SQRT(ARG1) * BRACE)
       CPSI2 = -((RHO1 / RHO2) * (C ** 2 / BETA2 ** 2) * (ETA1 / SIGMA1))
     C         / BRACE
C      WRITE (NPU,7) 'BRACE: ',BRACE,'CPHI1: ',CPHI1
7      FORMAT (2X,A7,E12.6,5X,A7,E12.6)
C
       DO 10 I=1,N
       ZI=I*DZ
       IF ((Z .GE. 0.0) .OR. (Z .LE. H)) THEN
          PHI1(I) = ((2.0 * PI) / H) * CEXP(CARG) * CPHI1
     U              * SIN(SIGMA1 * D) * SIN(SIGMA1 * Z)
     U              * CSQRT(CMPLX(K0 / K1, 0.0))
       END IF
       IF (Z .GE. H) THEN
          PHI2(I) = ((2.0 * PI) / H) * CEXP(CARH) * CPHI2
     U              * SIN(SIGMA1 * D) * EXP(-ETA1 * (Z - H))
     U              * CSQRT(CMPLX(KD / K1, 0.0))
          PSI2(I) = (2.0 * PI) * CEXP(CARI) * CPSI2
     U               *CMPLX(0.0,-K1)
     U              * SIN(SIGMA1 * D) * EXP(-ZETA1 * (Z - H))
     U              * CSQRT(CMPLX(KS / K1, 0.0))
       END IF
       U(I) = PHI1(I)
10     CONTINUE
       PHI2(N+1) = ((2.0*PI)/H)*CEXP(CARH)*CPHI2
     U             *SIN(SIGMA1*D)*EXP(-ETA1*((N+1)*DZ-H))
     U             *CSQRT(CMPLX(KD/K1,0.0))
```

```
      PSI2(N+1) = (2.0*PI)*CEXP(CARI)*CPSI2*CMPLX(0.0,-K1)
     U              *SIN(SIGMA1*D)*EXP(-ZETA1*((N+1)*DZ-H))
     U              *CSQRT(CMPLX(KS/K1,0.0))
      KDD = CSQRT(CMPLX(K0 / KD, 0.0)) * CEXP(CMPLX(0.0, DELTAD))
      KSS = CSQRT(CMPLX(K0 / KS, 0.0)) * CEXP(CMPLX(0.0, DELTAS))
      T33 = CMPLX(0.0, DR / (K0 * DZ ** 2))
C     WRITE (NPU, 8) 'T33: ',T33
8     FORMAT (2X,A5,'(',E12.6,2X,E12.6,')')
      T32 = (CMPLX(1.0, 0.0) - T33) * PHI1(N)
      T32 = T32 + T33 * PHI1(N-1)
C     WRITE (NPU, 8) 'T32: ',T32
      T12 = CMPLX(0.0, KS)
      T12 = KSS * T33 * DZ * PSI2(N) * (T12 - CMPLX(1.0 / DR, 0.0))
C     WRITE (NPU, 8) 'T12: ',T12
      CHI = T12 - T32 / T33
C     WRITE (NPU, 8) 'CHI: ',CHI
      T21 = KSS * T33 * CEXP(CMPLX(0.0, (K1 - KS) * DR))
     T      * PSI2(N) * DZ / DR
      IF (IND .EQ. 0) T21 = KSS * T33 * PSI2(N) * DZ / DR
C     WRITE (NPU, 8) 'T21: ',T21
      T22 = KDD * T33 * (PHI2(N+1) - PHI2(N))
C     WRITE (NPU, 8) 'T22: ',T22
9     FORMAT (2X,A6,'(',E12.6,2X,E12.6,')')
      A1 = T32 + T22 + T21 + T12
      IF (IND .GT. 0) GO TO 15
      U(N) = A1
      IND = 1
      RETURN
15    A1JM = A1
      RETURN
      END
```

```
      SUBROUTINE BCON
C     ******************************************************************
C     *** USER PREPARED BOTTOM CONDITION SUBROUTINE
C         BCON IS CALLED IF INPUT PARAMETER ITYPEB = 1
C         SEE MAIN PROGRAM FOR DEFINITIONS
C     ******************************************************************
C     *** SUBROUTINE RETURNS:
C         BOTY,BOTX
C     ******************************************************************
C
      PARAMETER MXLYR=101,MXN=10000,MXSVP=101,MXTRK=101,NIU=1,
     C          NOU=2,NPU=6
      COMPLEX ACOFX,ACOFY,BCOF,BOTX,BOTY,BTA,HNK,HNKL,SURX,SURY,TEMP,
     C        U,X,Y
      REAL*4 K0, K1, KS, KD
      COMPLEX CARG, B1, CKS, CKD, P11, P13, A1JN, A1JM, A2JN, A2JP1N,
     C        B2JN, B2JNP1, RHS1, CARH, CARI
      COMMON /USTFLD/ K0, K1, KS, KD, SIGMA1, CPHI1, CPHI2, CPSI2, B1,
     C                DELTAS, DELTAD, A1JM
      COMMON /IFDCOM/ACOFX,ACOFY,ALPHA,BCOF,BETA(MXLYR),BOTX,BOTY,
     C       BTA(MXN),C0,CSVP(MXSVP),DR,DR1,DZ,FRQ,IHNK,ISF,ITYPEB,
     C       ITYPES,IXSVP(MXLYR),KSVP,N,N1,NLYR,NSVP,NWSVP,R12(MXN),RA,
     C       RHO(MXLYR),RSVP,SURX,SURY,THETA,TRACK(MXTRK,2),U(MXN),
     C       X(MXN),XK0,Y(MXN),ZA,ZLYR(MXLYR),ZP,ZS,ZSVP(MXSVP)
      EQUIVALENCE (H, ZLYR(1)), (D, ZS), (ZJ, ZLYR(1))
      DATA PI/3.141592654/,DEG/57.29578/
C
      IF(THETA) 50,100,150
C
C     *** THETA LESS THAN 0.0. BOTTOM SLOPES UP.
50    CONTINUE
      BOTY=U(N)
C     BOTX=..............
      RETURN
C
C     *** THETA = 0.0. BOTTOM IS FLAT.
100   CONTINUE
      BOTY = U(N)
      CALL UFIELD
      BOTX = A1JM
      RETURN
C
C     *** THETA GREATER THAN 0.0, BOTTOM SLOPES DOWN.
150   CONTINUE
C     BOTY=..............
C     BOTX=..............
      RETURN
      END
```