We introduce a new version of the combined field integral equation (CFIE) for the solution of electromagnetic scattering problems in three dimensions. Unlike the conventional CFIE, the version reported here is well-conditioned. While we use a standard magnetic field integral operator, we precondition the electric field integral operator, converting it into a second-kind integral operator; the resulting CFIE is an integral equation of the second kind that has no spurious resonances. We also report numerical results showing that the new formulation stabilizes the number of iterations needed to solve the CFIE on closed surfaces. This is in contrast to the conventional CFIE, where the number of iterations grows as the discretization is refined.

Well-Conditioned Boundary Integral Equations for Three-Dimensional Electromagnetic Scattering§

H. Contopanagos*, B. Dembart†, M. Epton†, J.J. Ottusch*,
V. Rokhlin‡, J. Visher*, S. Wandzura*
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§This research was supported in part by the Defense Advanced Research Projects Agency under Contract No. MDA972-95-C-0021. *HRL Laboratories, LLC, 3011 Malibu Canyon Rd. Malibu, CA 90265. †Boeing, P.O. Box 3707, Seattle, WA 98124. ‡Yale University, Department of Computer Science, New Haven, CT 06520

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1 Introduction

Recent progress in the construction of "fast" methods for the solution of the boundary integral equations of scattering theory [1] has vastly increased the size of tractable problems [2, 3]; it has also increased the need for well-conditioned boundary integral formulations. There are two principal reasons for this:

- Since we have sparse decompositions of the integral operators of scattering theory, but not their inverses, we employ iterative solvers. Well-conditioned systems of equations can be solved with few iterations.
- Using a fine discretization to resolve source variations or geometric detail on a subwavelength scale results in an ill-conditioned linear equation. This is sometimes called the "low frequency" problem in computational electromagnetics.

Only second-kind integral equations (see Appendix), or objects with similar spectral behavior (such as appropriately preconditioned differential equations) can be solved with fully controlled approximation error. The correct operators are the sum of a constant (or at least well-conditioned and easily invertible) operator and a compact operator.

Boundary integral operators of scattering typically violate this requirement in one of three ways:

- The spectrum may accumulate at zero. A typical example is the first-kind integral equation for the scalar Dirichlet problem (used for $2d$ electromagnetic scattering calculations in TM polarization),
- the operator may have an unbounded spectrum, such as a pseudodifferential or hypersingular operator,
- the operator may have small eigenvalues associated with resonances, often unphysical; the latter are often referred to as "spurious resonances" (see, for example, [4]).

For electromagnetic scattering from perfectly electrically conducting (PEC) surfaces, the standard boundary integral equations are the electric field integral equation (EFIE)

$$-\mathbf{n} \times \mathbf{E}^i = T\mathbf{J} \tag{1}$$

and the magnetic field integral equation (MFIE)

$$Z\mathbf{n} \times \mathbf{H}^i = \left(\frac{1}{2} + K\right)\mathbf{J}, \tag{2}$$

where the integral operators T and K are defined (as in [5]) by*

$$\begin{aligned} T\mathbf{J} &\equiv T(k) \\ &\equiv ik\mathbf{n}(\mathbf{x}) \times \int_S ds' \left\{ G(k,\mathbf{x},\mathbf{x}')\,\mathbf{J}(\mathbf{x}) + \frac{1}{k^2}\nabla\left[\nabla G(k,\mathbf{x},\mathbf{x}')\cdot\mathbf{J}(\mathbf{x}')\right]\right\} \quad (3) \\ K\mathbf{J} &\equiv K(k)\mathbf{J} \equiv -\mathbf{n}(\mathbf{x}) \times \int_S ds'\,\nabla G(k,\mathbf{x},\mathbf{x}') \times \mathbf{J}(\mathbf{x}'), \quad (4) \end{aligned}$$

where ∇ denotes differentiation with respect to $\mathbf{x}$, and $\mathbf{n}(\mathbf{x})$ is the unit normal to the surface at $\mathbf{x}$.

The MFIE is a second-kind integral equation. Unfortunately, this equation is suitable for an unacceptably small class of electromagnetic problems. It is inapplicable to open surfaces, becomes ill-conditioned in the presence of geometric singularities, and suffers from spurious resonances. The EFIE has both a compact piece and a hypersingular piece (coming from the double gradient term). One can eliminate the spurious resonances of the MFIE by adding the EFIE to form a combined field integral equation (CFIE) [6]. The cost of doing so is the introduction of the EFIE's hypersingular piece, which spoils the conditioning for fine discretizations (or low frequencies).

Adams and Brown [7, 8] and Kolm and Rokhlin [9] recently observed that a hypersingular integral operator and a first-kind integral operator are ideal preconditioners for each other, in the sense that the composition of the two has the spectral characteristics of a second-kind integral operator. In this letter, we show how the same approach can be employed to analytically precondition the EFIE. In fact (as was implicit in a result of Hsiao and Kleinman [5]), the electric field integral operator T preconditions itself.

Two issues raised in [8] are important for the successful application of this idea to closed bodies. First, only the local (or short distance) behavior of the preconditioner is important for asymptotic conditioning. Thus, one can precondition the EFIE by multiplying it by an electric field integral operator corresponding to an arbitrary wavenumber, real or complex; if the wavenumber has a positive imaginary part, one avoids the introduction of any additional resonances. (Obviously, if the EFIE preconditioner reproduced the MFIE resonances, then the CFIE would also have them.) Second, one must take care that the discretization of the product of preconditioner and preconditioned operators preserves the correct spectral properties.

In this letter we describe well-conditioned formulations for both open and closed surfaces. We also present numerical results for closed surfaces which demonstrate the advantages of the new CFIE formulation over the conventional CFIE.

*The other terms follow the usual conventions: $\mathbf{J} \equiv Z\mathbf{n}\times\mathbf{H}$ is the unknown surface current, $\mathbf{E}^i$ and $\mathbf{H}^i$ are the incident electric and magnetic fields, respectively, $Z = \sqrt{\mu/\epsilon}$ is the wave impedance, and $G(k,\mathbf{x},\mathbf{x}') = \exp(ikr)/4\pi r$ is the $3d$ Helmholtz kernel with $r = |\mathbf{x}-\mathbf{x}'|$ being the distance separating field and source points. Harmonic time dependence $e^{-i\omega t}$ is assumed.

2 Preconditioning the EFIE operator

References [8] and [9] consider integral operators constructed from the kernel for the Laplace and Helmholtz equations in $2d$. They observe that the product of a first-kind operator, constructed from an undifferentiated kernel, and a hypersingular operator, constructed from a twice differentiated kernel, has the desirable spectral characteristics of a second-kind operator. Since the EFIE integral operator T has both of these, one might expect that the composition of two such operators $T^2 \equiv T \circ T$ would include a constant operator and a compact operator. One might also worry that the product of hypersingular components would produce another hypersingular operator. It is easy to see, however, that the rotation operation $\mathbf{n}\times$ in the definition (3) of T, which annihilates the component of the surface vector field normal to the surface, also ensures that the product of the two hypersingular operators is identically equal to zero. Indeed, applying the hypersingular component of the second T operator to an arbitrary tangential surface vector function $\mathbf{f}(\mathbf{x}')$ produces a surface gradient function

$$\mathbf{n}\times\nabla\phi(\mathbf{x}) = \frac{i}{k}(\mathbf{n}\times\nabla)\int_S ds'\ \nabla G(k,\mathbf{x},\mathbf{x}')\cdot\mathbf{f}(\mathbf{x}')\ , \tag{5}$$

which the hypersingular component of the first T operator, in turn, annihilates (for closed surfaces) by virtue of the identity

$$\nabla_S\cdot[\mathbf{n}\times\nabla\phi(\mathbf{x})] = 0, \tag{6}$$

with ∇_S denoting the surface gradient operator on S; identity (6), the surface analog of the $3d$ identity $\nabla\cdot[\nabla\times\phi(\mathbf{x})] = 0$, can be found, for example, in [10], and is valid for any sufficiently smooth function ϕ on S. It follows immediately from (3), (5), and (6) that T^2 behaves as a second-kind integral operator.

In this letter we investigate in detail the spectral properties of the EFIE and MFIE integral operators and combinations thereof for the PEC sphere, a simple $3d$ target for which the spectral properties of these operators are known analytically. A complete set of basis functions on the surface of a sphere of radius a is given by the vector spherical harmonics [11]

$$\mathbf{X}_{lm}(\theta,\varphi) \equiv \frac{a}{i\sqrt{l(l+1)}}\mathbf{n}\times\nabla Y_{lm}(\theta,\varphi)\,, \tag{7}$$

$$\mathbf{U}_{lm}(\theta,\varphi) \equiv \mathbf{n}\times\mathbf{X}_{lm}(\theta,\varphi)\,, \tag{8}$$

defined here in terms of the scalar spherical harmonics $Y_{lm}(\theta,\varphi)$.

The result of applying T and $K^+ \equiv \left(K+\frac{1}{2}\right)$ to each basis function is† [5]

$$T(k)\left\{\begin{array}{c}\mathbf{X}_{lm}\\ \mathbf{U}_{lm}\end{array}\right\} = \left\{\begin{array}{c}-\mathbb{J}_l(ka)\,\mathbb{H}_l(ka)\,\mathbf{U}_{lm}\\ \mathbb{J}_l'(ka)\,\mathbb{H}_l'(ka)\,\mathbf{X}_{lm}\end{array}\right\} \tag{9}$$

†The MFIE eigenvalues in [5] contain a sign error which is corrected here.

and

$$K^{+}(k)\left\{\begin{array}{c}\mathbf{X}_{lm}\\ \mathbf{U}_{lm}\end{array}\right\}=\left\{\begin{array}{c}i\mathbb{J}_l'(ka)\,\mathbb{H}_l(ka)\,\mathbf{X}_{lm}\\ -i\mathbb{J}_l(ka)\,\mathbb{H}_l'(ka)\,\mathbf{U}_{lm}\end{array}\right\}, \tag{10}$$

where $\mathbb{J}_l$ and $\mathbb{H}_l$ are Riccati-Bessel and (first-kind) Riccati-Hankel functions of order l, and k is the wavenumber associated with the kernel of each integral operator. The Riccati-Bessel and Riccati-Hankel functions are defined [12] in terms of spherical Bessel and Hankel functions $j_l(x)$ and $h_l^{(1)}(x)$ by

$$\mathbb{J}_l(x)\equiv x j_l(x), \tag{11}$$

$$\mathbb{H}_l(x)\equiv x h_l^{(1)}(x). \tag{12}$$

Although our chosen basis functions $\mathbf{X}_{lm}$ and $\mathbf{U}_{lm}$ are not eigenfunctions of the operator $T(k)$, they are eigenfunctions of $T^2(k)\equiv T(k)\circ T(k)$:

$$T^2(k)\left\{\begin{array}{c}\mathbf{X}_{lm}\\ \mathbf{U}_{lm}\end{array}\right\}=-\mathbb{J}_l(ka)\,\mathbb{H}_l(ka)\,\mathbb{J}_l'(ka)\,\mathbb{H}_l'(ka)\left\{\begin{array}{c}\mathbf{X}_{lm}\\ \mathbf{U}_{lm}\end{array}\right\}. \tag{13}$$

The operator $T^2(k)$ has a bounded spectrum, since, in the limit of large l, its eigenvalues accumulate at $-\frac{1}{4}$ (a result which follows from the asymptotic properties of j_l and $h_l^{(1)}$ given, for example, in [12]). However, as is evident from (10) and (13), the operator $T^2(k)$ also shares resonances (at the zeros of $\mathbb{J}_l'(ka)$ for the $\mathbf{X}_{lm}$ modes, and at the zeros of $\mathbb{J}_l(ka)$ for the $\mathbf{U}_{lm}$ modes) with the MFIE operator $K^{+}(k)$. This fact is also evident from the identity

$$T^2(k)=K^2(k)-\frac{1}{4}=K^{-}(k)\circ K^{+}(k), \tag{14}$$

(where $K^{-}\equiv K-\frac{1}{2}$) derived in [5]. Therefore, although $T^2(k)$ is a second-kind integral operator, it is not a suitable component of a resonance-free combined field integral equation for closed bodies.

As stated earlier, boundedness of the spectrum of the product of two EFIE operators (of the form (3)) is assured if they have the same short-distance behavior, a condition that does not require the two operators to share the same wavenumber (propagation constant). If we choose EFIE operators with different wavenumbers, $T(k_1)$ and $T(k_2)$, we can simultaneously obtain a bounded product and avoid MFIE resonances.

The following analysis indicates that ik is a particularly good choice for the wavenumber in the preconditioning operator (assuming that the wavenumber k is real). The eigensystem for $T(ik)\circ T(k)$ on a sphere is

$$T(ik)\circ T(k)\left\{\begin{array}{c}\mathbf{X}_{lm}\\ \mathbf{U}_{lm}\end{array}\right\}=-\left\{\begin{array}{c}\mathbb{J}_l'(ika)\,\mathbb{H}_l'(ika)\,\mathbb{J}_l(ka)\,\mathbb{H}_l(ka)\,\mathbf{X}_{lm}\\ \mathbb{J}_l(ika)\,\mathbb{H}_l(ika)\,\mathbb{J}_l'(ka)\,\mathbb{H}_l'(ka)\,\mathbf{U}_{lm}\end{array}\right\}. \tag{15}$$

It is straightforward to show (given the properties [12] of j_l and $h_l^{(1)}$) that the eigenvalues of $T(ik)\circ T(k)$ accumulate at $\frac{i}{4}$ and $-\frac{i}{4}$ for the $\mathbf{X}_{lm}$ and $\mathbf{U}_{lm}$

eigenmodes, respectively, and that $T(ik) \circ T(k)$ does not share any resonances with the MFIE operator $K^{+}(k)$.

Since $T(ik) \circ T(k)$ is a second-kind integral operator (in the sense described in the Appendix) and does not share any resonances with $K^{+}(k)$, we are finally in a position to write a well-conditioned CFIE operator. The simplest form of such an operator is

$$T(ik) \circ T(k) + \alpha K^{+}(k), \tag{16}$$

where α is a constant to be chosen. In creating this CFIE operator we have preconditioned the EFIE part before adding to it the MFIE part (which is already a second-kind integral operator). The same applies to the excitation side of the equation. The resulting CFIE is

$$-T(ik)\left(\mathbf{n} \times \mathbf{E}^{i}\right) + \alpha Z \mathbf{n} \times \mathbf{H}^{i} = \left[T(ik) \circ T(k) + \alpha K^{+}(k)\right] \mathbf{J}. \tag{17}$$

The eigensystem for the CFIE operator (16) is

$$\begin{aligned} &\left[T(ik) \circ T(k) + \alpha K^{+}(k)\right] \left\{ \begin{array}{c} \mathbf{X}_{lm} \\ \mathbf{U}_{lm} \end{array} \right\} \\ &= -\left\{ \begin{array}{c} \left[\mathbb{J}_l'(ika)\, \mathbb{H}_l'(ika)\, \mathbb{J}_l(ka) - i\alpha \mathbb{J}_l'(ka)\right] \mathbb{H}_l(ka)\, \mathbf{X}_{lm} \\ \left[\mathbb{J}_l(ika)\, \mathbb{H}_l(ika)\, \mathbb{J}_l'(ka) + i\alpha \mathbb{J}_l(ka)\right] \mathbb{H}_l'(ka)\, \mathbf{U}_{lm} \end{array} \right\}. \end{aligned} \tag{18}$$

If one chooses $\alpha = \pm 1$ then, as a function of the argument ka, these eigenvalues have no zeros. For $\alpha = +1$, they circle the origin of the complex plane.

Other well-conditioned CFIE operators can be devised, for example, by preconditioning the MFIE part before combining it with the preconditioned EFIE part. We have investigated two forms:

$$T(ik) \circ T(k) + \alpha K^{+}(ik) \circ K^{+}(k) \tag{19}$$

and

$$T(ik) \circ T(k) + \alpha \mathbf{n} \times K^{+}(ik) \circ \mathbf{n} \times K^{+}(k). \tag{20}$$

Our experience shows the numerical behavior of all three CFIE formulations to be similar.

We have proven the CFIE operators in (16), (19) and (20) to be second-kind and resonance-free for spheres. However, given that the asymptotic behavior of the eigenvalues on a smooth surface stems from the short distance behavior of the kernel, we argue (following the theorems proved in [9]) that the asymptotic behavior of the various operators on spheres should also obtain for any closed surface that can be obtained by smooth deformation of a sphere. The numerical results presented in Section 4 support this argument. We also present results for a cube, which, like many targets of practical interest, has geometric singularities. These results suggest that the new CFIE formulations should be well conditioned for a wide class of closed surfaces.

3 A Different Form of the Preconditioned EFIE Operator

There are several ways to produce a Nyström discretization of the product operator $T(k_1) \circ T(k_2)$. The simplest and most straightforward approach, multiplying the discretized representations of the individual operators, can lead to numerical difficulties. The reason is that it is relatively difficult to make the discretized representations of the hypersingular part of each operator sufficiently accurate (especially for high-spatial-frequency eigenmodes) to numerically effect the cancellation that obtains analytically.

Effective discretizations of $T(k_1) \circ T(k_2)$ can be obtained either by discretizing the product operator directly or by reformulating the product operator to eliminate the product of hypersingular operators. We have not implemented the first method because of the added complexity it entails. We have implemented the second approach using a reformulated product operator that eliminates all instances of hypersingular operators. A short derivation of the reformulated equation is given below.

The first step toward obtaining a more useful form of the product operator $T(k_1) \circ T(k_2)$ is to separate each integral operator into its singular and hypersingular components. Introducing the abbreviations

$$T_1 = T(k_1), \tag{21}$$

$$T_2 = T(k_2), \tag{22}$$

we write

$$T_1 = ik_1 T_1^S + \frac{i}{k_1} T_1^H, \tag{23}$$

$$T_2 = ik_2 T_2^S + \frac{i}{k_2} T_2^H, \tag{24}$$

where

$$T_m^S \mathbf{J} \equiv \mathbf{n}(\mathbf{x}) \times \int_S ds' \; G(k_m, \mathbf{x}, \mathbf{x}') \, \mathbf{J}(\mathbf{x}'), \tag{25}$$

$$T_m^H \mathbf{J} \equiv (\mathbf{n}(\mathbf{x}) \times \nabla) \int_S ds' \; \nabla G(k_m, \mathbf{x}, \mathbf{x}') \cdot \mathbf{J}(\mathbf{x}'). \tag{26}$$

The product operator $T_1 \circ T_2$ can be expanded into four terms. Each of the two cross terms, $T_1^S \circ T_2^H$ and $T_1^H \circ T_2^S$, can be transformed (by Stokes's theorem) into the product of new, single-gradient integral operators on S plus a line integral around the boundary of S. The term formed by the product of hypersingular integral operators, $T_1^H \circ T_2^H$, reduces to a line integral. The result is further simplified by noticing that two of the three line integrals, when applied to $\mathbf{J}$, can be combined into a single term whose argument is identical to the incident electric field $\mathbf{E}^i$ by virtue of (1).

The next step is to reformulate the excitation side of the equation, taking advantage of the fact that the incident wave obeys Maxwell's equations. By applying Stokes's theorem, we rewrite the term $T_1^H\left[\mathbf{n}(\mathbf{x}')\times\mathbf{E}^i(\mathbf{x}')\right]$ as the sum of a single-gradient integral operator on $\nabla'\times\mathbf{E}^i(\mathbf{x}')$ and a line integral that exactly cancels the line integral involving $\mathbf{E}^i$ on the other side of the equation. A further simplification follows from Faraday's Law, $\nabla\times\mathbf{E}=i\omega\mu\mathbf{H}$.

The final result for the analytically preconditioned EFIE with reformulated integral operator product is

$$\begin{aligned}&-ik_1T_1^S\left(\mathbf{n}\times\mathbf{E}^i\right)-Z\frac{k_2}{k_1}T_1^\alpha\left(\mathbf{n}\cdot\mathbf{H}^i\right)\\&=\left(\frac{k_2}{k_1}T_1^\alpha\circ T_2^T+\frac{k_1}{k_2}T_1^\beta\circ T_2^L-k_1k_2T_1^S\circ T_2^S-\frac{k_1}{k_2}T_1^E\circ T_2^L\right)\mathbf{J},\end{aligned}\tag{27}$$

where the various integral operators are defined by

$$T_m^\alpha\phi\equiv\mathbf{n}(\mathbf{x})\times\int_S ds'\;\nabla G\left(k_m,\mathbf{x},\mathbf{x}'\right)\phi\left(\mathbf{x}'\right),\tag{28}$$

$$T_m^\beta\phi\equiv\mathbf{n}(\mathbf{x})\times\int_S ds'\;\mathbf{n}(\mathbf{x}')\times\nabla'G\left(k_m,\mathbf{x},\mathbf{x}'\right)\phi\left(\mathbf{x}'\right),\tag{29}$$

$$T_m^L\mathbf{f}\equiv\int_S ds'\;\nabla G\left(k_m,\mathbf{x},\mathbf{x}'\right)\cdot\mathbf{f}\left(\mathbf{x}'\right),\tag{30}$$

$$T_m^T\mathbf{f}\equiv\mathbf{n}(\mathbf{x})\cdot\int_S ds'\;\nabla G\left(k_m,\mathbf{x},\mathbf{x}'\right)\times\mathbf{f}\left(\mathbf{x}'\right),\tag{31}$$

$$T_m^S\mathbf{f}\equiv\mathbf{n}(\mathbf{x})\times\int_S ds'\;G\left(k_m,\mathbf{x},\mathbf{x}'\right)\mathbf{f}\left(\mathbf{x}'\right),\tag{32}$$

$$T_m^E\phi\equiv\mathbf{n}(\mathbf{x})\times\oint_{\partial S} d\mathbf{l}'\;G\left(k_m,\mathbf{x},\mathbf{x}'\right)\phi\left(\mathbf{x}'\right),\tag{33}$$

with $m=1,2$. Note that T_m^α, T_m^β, and T_m^E map scalar functions to surface vector functions, whereas T_m^L and T_m^T do the reverse. The operator on the right hand side of (27) maps surface vector functions into surface vector functions.

In the remainder of this section we discuss closed surfaces and observe that $T_1\circ T_2$ behaves like a second-kind integral operator. For open surfaces, the situation is somewhat more complicated in that additional analytical machinery is required to convert (27) into a second-kind integral operator. We have performed such analyses for the $2d$ and $3d$ scalar cases, and will report these results in the future.

If S is a closed surface, the term $T_1^E\circ T_2^L\mathbf{J}$ vanishes, and (27) simplifies to

$$-ik_1T_1^S\left(\mathbf{n}\times\mathbf{E}^i\right)-Z\frac{k_2}{k_1}T_1^\alpha\left(\mathbf{n}\cdot\mathbf{H}^i\right)=S\left(k_1,k_2\right)\mathbf{J},\tag{34}$$

where

$$S_{12}\equiv S\left(k_1,k_2\right)\equiv\frac{k_2}{k_1}T_1^\alpha\circ T_2^T+\frac{k_1}{k_2}T_1^\beta\circ T_2^L-k_1k_2T_1^S\circ T_2^S.\tag{35}$$

We note several features of S_{12}.

First, all of the individual integral operators that comprise S_{12} involve kernels with one or no gradients on the Helmholtz Green's function G. All such integral operators are bounded.

Second, the eigenvalues of the integral operator S_{12} do not accumulate at the origin. We will demonstrate this by examining its three components $T_1^{\alpha} \circ T_2^T$, $T_1^{\beta} \circ T_2^L$, and $T_1^S \circ T_2^S$. The operator $T_1^{\alpha} \circ T_2^T$ is a second-kind integral operator for the transverse (divergence-free) component of $\mathbf{J}$, and is identically zero for the longitudinal (irrotational) component of $\mathbf{J}$. Likewise, the operator $T_1^{\beta} \circ T_2^L$ is a second-kind operator for the longitudinal component of $\mathbf{J}$, and is identically zero for the transverse component of $\mathbf{J}$. Since any surface current distribution can be decomposed into longitudinal and transverse components [11], the sum $\frac{k_2}{k_1} T_1^{\alpha} \circ T_2^T + \frac{k_1}{k_2} T_1^{\beta} \circ T_2^L$ is a second-kind integral operator; subtracting $k_1 k_2 T_1^S \circ T_2^S$, a compact operator, does not change this result. As observed in Section 2, we can avoid resonance sharing by setting $k_1 = ik$ and $k_2 = k$. In this case, the eigenvalues of S_{12} accumulate at two points, $\pm \frac{i}{4}$, rather that at $-\frac{1}{4}$.

Third, the spectrum of S_{12}, after discretization, is bounded and includes accumulation points at the expected locations. However, an accurate discretization will have zero (or very small) eigenvalues wherever the EFIE operator $T\left(k_2\right)$ has a resonance. Thus, it has to be combined with an appropriate discretization of the MFIE operator, to obtain an effective discretization of the CFIE.

Finally, it should be noted that (34) is manifestly insusceptible to the "low-frequency" problem that plagues the EFIE. Since the well-conditioned behavior of S_{12} comes from the composite operators $\frac{k_2}{k_1} T_1^{\alpha} \circ T_2^T$ and $\frac{k_1}{k_2} T_1^{\beta} \circ T_2^L$, both of whose prefactors have modulus unity (assuming $|k_1| = |k_2| = k$), and since the term $k_1 k_2 T_1^S \circ T_2^S$ tends to zero as $k \to 0$, the full operator S_{12} remains well conditioned in the limit of low frequency.

In summary, although the operators $T_1 \circ T_2$ and S_{12} have identical spectral properties for closed bodies, it is easier to construct an accurate Nyström discretization for S_{12} because it is composed of less singular integral operators. Matrix representations of S_{12} have bounded spectra, but also suffer from spurious resonances inherited from the EFIE operator $T\left(k_2\right)$. These resonances can be eliminated by combining S_{12} with $K^{+}(k_2)$ (or the modified MFIE operators in (19) and (20)). The result is a well-conditioned system of linear algebraic equations.

4 Numerical Results

In this section we compare the numerical performance of the conventional CFIE (referred to below as CFIE1)

$$-\mathbf{n} \times \left(\mathbf{n} \times \mathbf{E}^i\right) + Z\mathbf{n} \times \mathbf{H}^i = \left[\mathbf{n} \times T\left(k\right) + K^{+}(k)\right] \mathbf{J} \tag{36}$$

with the preconditioned CFIE (CFIE2)

$$\begin{aligned} &kT^S(ik)\,\mathbf{n}\times\mathbf{E}^i + iZT^\alpha(ik)\,\mathbf{n}\cdot\mathbf{H}^i - Z\mathbf{n}\times\mathbf{H}^i \\ &= \left\{ i\left[-T^\alpha(ik)\circ T^T(k) + T^\beta(ik)\circ T^L(k) - k^2 T^S(ik)\circ T^S(k) \right] - K^+(k) \right\}\mathbf{J} \end{aligned} \tag{37}$$

produced by combining (17) (with $\alpha = -1$) and (34) (with $k_1 = ik$ and $k_2 = k$). We discretized the individual operators in these equations using a high-order Nyström scheme [13]. In all cases, the wave impedance Z was set to unity.

We present three examples. The first example shows how the condition number of each operator, defined as the ratio of the largest to smallest singular values, depends on the fineness of discretization. Table 1 lists the condition number (CN) of the matrix representing each CFIE operator as the size of the sphere decreases. In all cases, the same discretization was used, created by placing a 6-point quadrature rule on each of the 80 nearly identical patches that cover the sphere, for a total of 960 unknowns. As the sphere radius decreases, the condition number for the CFIE2 integral operator stabilizes at about 2, whereas the condition number of the CFIE1 integral operator continues to grow in inverse proportion to the radius.

radius(λ)	CFIE1	CFIE2
1	4.2	3.04
1/4	15	2.68
1/16	59	2.04
1/64	230	1.99
1/256	940	1.97
1/1024	3800	1.97
1/4096	15000	1.97

Table 1: Condition number of CFIE matrices for shrinking PEC spheres

The second test compares iterative solver performance for the new CFIE and the conventional CFIE. The target geometry consists of two PEC spheres, one with a radius of $\lambda/2$, the other set at a resonant radius, namely, the first zero of $\mathbb{J}_1'(2\pi r/\lambda)$ or $r \approx 0.43667457\ \lambda$. The spheres are separated by a $\lambda/100$ gap. We subdivided the patches near the gap by a factor of about 10 to adequately resolve the currents, which vary rapidly there. Table 2 compares iteration counts and radar cross section (RCS) errors for several discretizations. The iterations columns list the maximum number of iterations a conjugate gradient squared (CGS) routine required to reach a residual error of 10^{-3}. A solution computed from a substantially more refined discretization provided an accuracy reference. The stated error is the root mean squared (RMS) value of the difference between the monostatic $\phi\phi$ RCS of the comparison solution and the reference solution at 181 angles. For identical discretizations, the two methods had about the same error. The data show a dramatic difference, however, in the iteration count behavior of the two methods in response to discretization refinements.

unknowns	patches	CFIE1		CFIE2	
		iterations	error	iterations	error
1496	748	60	0.46	9	0.40
4488	748	126	0.18	11	0.18
996	498	44	0.61	9	0.48
2988	498	103	0.23	12	0.16
5976	498	163	0.016	11	0.029

Table 2: Iteration count and solution error vs. discretization for two PEC spheres.

The third test also compares iterative solver performance for the two CFIE formulations. In this case the target is a cube of size 1λ. We present numerical results for five different discretizations, the first of which was derived from a mesh (i.e., a set of patches) obtained by dividing each face into four squares. The second mesh was constructed from the first one by subdividing each square into four smaller squares. The third mesh was constructed from the second by subdividing edge-touching patches in half along a line parallel to the edge; patches adjacent to two edges (i.e., corner patches) were divided into quarters. Meshes for the fourth and fifth discretizations were constructed by recursively applying the procedure by which the third mesh was constructed from the second. This process, known as patch tapering, is useful for resolving the source singularities that arise in the vicinity of geometric singularities. It also puts stress on the conventional CFIE because points near edges get close together. Table 3 lists the maximum and average number of iterations the CGS routine needed to obtain solutions for 92 independent excitations to a residual error of 10^{-3}. The total number of unknowns is the result of using a 9-point quadrature rule on each square or rectangular patch. The iteration count for CFIE2 grows very slowly with increasing taper depth, whereas for CFIE1 it increases steadily, in accordance with expectations.

unknowns	taper depth	CFIE1		CFIE2	
		max	ave	max	ave
432	0	12	6.5	10	4.3
1728	1	18	9.9	11	4.9
3888	2	26	14	11	4.9
6912	3	41	23	11	5.5
10800	4	58	36	13	5.7

Table 3: Iteration count vs. taper depth for 1λ PEC cube.

5 Conclusions and Generalizations

The classical electric field integral operator is its own perfect preconditioner, in the sense that applying it to both sides of the EFIE converts the latter into a second-kind integral equation. When the preconditioned electric field integral operator is used as a component of the CFIE, the latter is also converted into a second-kind integral equation. Furthermore, if the preconditioning electric field operator corresponds to a complex wavenumber, the resulting CFIE has no spurious resonances.

In this paper, we describe in some detail an improved CFIE for electromagnetic scattering from perfectly conducting closed surfaces, leading to a significant improvement in the performance of iterative solvers; incorporating the approach into the existing "fast" solvers is completely straightforward. The results presented here admit generalizations in several directions. The extensions discussed below are currently under investigation, and will be reported at a later date.

The approach of this paper can be applied, with minor modifications, to surface scattering with more general boundary conditions. The extension to an interface between two dielectrics, for example, is straightforward; the resulting operators have condition numbers that are in fact somewhat lower than in the case described here. While structures consisting of several dielectrics do not appear to present serious difficulties, places where several different dielectrics come in contact with each other require separate analytical treatment.

The approach of this paper has to be modified only slightly in order to obtain second kind integral equations describing electromagnetic scattering from open perfectly conducting surfaces. In this environment, the CFIE is replaced with an appropriately preconditioned EFIE, and the edge of the surface requires separate treatment. The result is a pair of coupled integral equations, one on the surface itself, and the other on the edge of the surface (which is, obviously, a curve in R^3). At this time, the theory has been constructed for the scalar case when the boundary of the surface is a sufficiently smooth curve; the analysis of open surfaces whose boundaries have corners is in progress.

A Appendix

The standard definition of a second-kind integral operator is an operator of the form

$$\lambda I + K, \tag{38}$$

where λ is a constant, I is the identity, and K is a compact operator. In scattering theory, one encounters operators of the form

$$\lambda_1 P_1 + \lambda_2 P_2 + K, \tag{39}$$

where λ_1 and λ_2 are constants and P_1 and P_2 are orthogonal projection operators such that

$$P_1 + P_2 = I. \tag{40}$$

Operators of the form (39) possess most of the desirable properties of second-kind integral operators. In a mild abuse of terminology, we refer to such expressions as second-kind integral operators throughout this letter.

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