On the Numerical Solution of Transmission Problems for the Laplace equation on Polygonal Domains

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1 Introduction

In classical potential theory, boundary value problems for Laplace's equation are reduced to second-kind boundary integral equations by representing the solutions to the differential equations by a combination of single-layer and double-layer potentials on the boundary of the domain. When the boundary of the domain is smooth, the kernels for the corresponding integral equations are smooth. Moreover, if the boundary data is smooth then the solutions are also smooth. This class of problems is well understood; the existence and uniqueness of solutions follow from Fredholm theory, and the integral equations can be solved numerically using standard tools.

However, when the region has corners the solutions to both the differential equation and the boundary integral equation are singular. The nature of the singularities has been the subject of extensive analysis (see [8], [13], [14], and [21] for representative examples). In particular, it is well-known that solutions are unique and exist in the L^2-sense (see [5], [20]) both for the differential and integral equations. Moreover, the leading singular terms of the solutions in the vicinity of corners are known for both the integral and the differential equation. More recently, in [18] the behaviour of solutions to the boundary integral equations arising from Dirichlet and Neumann boundary conditions in the vicinity of corners for polygonal domains was characterized in detail. In particular, it was found that the solution in the vicinity of corners can be represented by linear combinations of certain explicitly-known powers and powers multiplied by logarithms. The analysis has subsequently been extended to curved boundaries [17] as well as the Dirichlet and Neumann boundary value problems for the Helmholtz [19] and biharmonic [15] equations. In [12] these representations were used to construct efficient discretizations for the solution to Laplace's equation with Dirichlet and Neumann boundary conditions.

In this paper we give a detailed description of the behavior of the solutions to the integral equations arising in Laplace transmission problems with polygonal boundaries. Such problems arise naturally in electrostatics. We find that the solutions in the vicinity of corners are representable by certain series of implicitly-defined powers which are analytic functions of the material parameters. We then use these analytical results to construct highly accurate and efficient numerical algorithms for the solution to Laplace transmission problems on polygonal domains. We demonstrate the performance of this approach with a number of numerical examples.

The structure of the paper is as follows. In Section 2 we describe the necessary mathematical preliminaries. Section 3 contains the analytical results. Section 4 contains numerical results illustrating the performance of the algorithm.

2 Mathematical Preliminaries

2.1 Boundary Value Problems

Let Ω be the interior of a polygonal domain in $\mathbb{R}^2$ and $\gamma : [0, L] \to \mathbb{R}^2$ a counterclockwise arc length parametrization of its boundary. Let $\nu(t)$ be the inward-pointing normal to γ at $t \in [0, L]$, and let Γ denote the boundary of Ω. For boundary data $f, g : [0, L] \to \mathbb{R}$ and real numbers λ_1, λ_2, λ_3 and, λ_4, we consider the following problem.

Laplace transmission problem:

$$\nabla^2 \phi(x) = 0 \qquad x \in \mathbb{R}^2 \setminus \Gamma, \tag{1}$$

$$\lambda_1 \lim_{\substack{x \to \gamma(t) \\ x \in \Omega}} \phi(x) - \lambda_2 \lim_{\substack{x \to \gamma(t) \\ x \in \mathbb{R}^2 \setminus \overline{\Omega}}} \phi(x) = f(t) \qquad t \in [0, L], \tag{2}$$

$$\lambda_3 \lim_{\substack{x \to \gamma(t) \\ x \in \Omega}} \frac{\partial \phi(x)}{\partial \nu(t)} - \lambda_4 \lim_{\substack{x \to \gamma(t) \\ x \in \mathbb{R}^2 \setminus \overline{\Omega}}} \frac{\partial \phi(x)}{\partial \nu(t)} = g(t) \qquad t \in [0, L]. \tag{3}$$

Let ϕ_1 denote the solution of (1) in the interior and ϕ_2 denote the solution in $\mathbb{R}^2 \setminus \overline{\Omega}$.

Remark 2.1. *In this paper we assume that $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$. A similar analysis applies to the case where some, or all, of the coefficients are negative or zero.*

Remark 2.2. *For ease of exposition we restrict our discussion to regions with polygonal boundaries. The analysis and algorithm outlined in this paper extend in a straightforward manner to multiply connected domains and domains with curved boundaries.*

2.2 Single- and double-layer potentials

In classical potential theory, boundary value problems are solved by representing the solution of the differential equation inside the region as a potential induced by charges and dipoles on the boundary. Let $\psi^0_{x_0}(x)$ denote the potential of a unit charge at $x_0 \in \mathbb{R}^2$ and let $\psi^1_{x_0,h}(x)$ denote the potential of a unit dipole at $x_0 \in \mathbb{R}^2$ oriented in the direction h. Specifically, $\psi^0_{x_0}, \psi^1_{x_0,h} : \mathbb{R}^2 \setminus x_0 \to \mathbb{R}$ are given by the following formulas

$$\psi^0_{x_0}(x) = \log(\|x - x_0\|), \tag{4}$$

$$\psi^1_{x_0,h}(x) = \frac{\langle h, x_0 - x\rangle}{\|x_0 - x\|^2}. \tag{5}$$

where $\|\cdot\|$ denotes the standard Euclidean distance and $\langle\cdot,\cdot\rangle$ denotes the inner product.

The potential due to a charge distribution ρ on the boundary Γ is referred to as a single-layer potential and is given by

$$\phi^0_\rho(x) = \int_0^L \psi^0_{\gamma(t)}(x)\rho(t)\,\mathrm{d}t, \tag{6}$$

for any $x \in \mathbb{R}^2 \setminus \Gamma$. Similarly, the potential due to a dipole distribution ρ on the boundary is referred to as a double-layer potential and is given by

$$\phi^1_\rho(x) = \int_0^L \psi^1_{\gamma(t),\nu(t)}(x)\rho(t)\,\mathrm{d}t, \tag{7}$$

for any $x \in \mathbb{R}^2 \setminus \Gamma$.

The following theorem describes the behaviour of (6) and (7) as x approaches the boundary Γ.

Theorem 2.1. *Suppose the point x approaches a point $x_0 = \gamma(t_0)$ from the inside along* *a path such that*

$$-1+\alpha < \frac{x-x_0}{\|x-x_0\|}\cdot\gamma'(t_0) < 1-\alpha \tag{8}$$

for some $\alpha > 0$*. Then*

$$\lim_{x\to x_0}\phi^0_\rho(x) = \phi^0_\rho(x_0) \tag{9}$$

$$\lim_{x\to x_0}\phi^1_\rho(x) = \phi^1_\rho(x_0) - \pi\rho(x_0) \tag{10}$$

$$\lim_{x\to x_0}\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0}\phi^0_\rho(x+\tau\nu(t_0)) = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0}\phi^0_\rho(x_0+\tau\nu(t_0)) + \pi\rho(x_0). \tag{11}$$

Similarly, if x *approaches a point* $x_0=\gamma(t_0)$ *from the outside then*

$$\lim_{x\to x_0}\phi^0_\rho(x) = \phi^0_\rho(x_0) \tag{12}$$

$$\lim_{x\to x_0}\phi^1_\rho(x) = \phi^1_\rho(x_0) + \pi\rho(x_0) \tag{13}$$

$$\lim_{x\to x_0}\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0}\phi^0_\rho(x+\tau\nu(t_0)) = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0}\phi^0_\rho(x_0+\tau\nu(t_0)) - \pi\rho(x_0). \tag{14}$$

Definition 2.1. *Define the operator* $\phi^* : L^2([0,L]) \to L^2([0,L])$ *by*

$$\phi^*_\rho(t) = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0}\phi^0_\rho(t+\tau\nu(t)) \tag{15}$$

The following lemma gives a representation of the normal derivative of the single-layer potential ϕ^*_ρ in terms of the function ψ^1.

Lemma 2.2. *Let* Ω *be the interior of a polygonal domain in* $\mathbb{R}^2$ *and* $\gamma : [0,L] \to \mathbb{R}^2$ *a counterclockwise arc length parametrization of its boundary. Let* $\nu(t)$ *be the inward-pointing normal to* γ *at* $t \in [0,L]$*, and let* Γ *denote the boundary of* Ω*. Then, for* $\rho \in L^2([0,L])$

$$\phi^*_\rho(t) = \int_0^L \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\,\rho(s)\,\mathrm{d}s. \tag{16}$$

The following theorem is proved in [7] and describes the behaviour of the normal derivative of a double-layer potential near the boundary.

Theorem 2.3. *Let $0 < t_0 < L$ and suppose that the curve γ and the density ρ are smooth on the interval $(t_0 - \delta, t_0 + \delta)$ for some $\delta > 0$. Suppose further that the point x approaches a point $x_0 = \gamma(t_0)$ along a path such that*

$$-1 + \alpha < \frac{x - x_0}{\|x - x_0\|} \cdot \gamma'(t_0) < 1 - \alpha \tag{17}$$

for some $\alpha > 0$. Then

$$\lim_{\substack{x \to x_0 \\ x \in \Omega}} \frac{\mathrm{d}}{\mathrm{d}\tau} \phi_\rho^1(x + \tau\nu(t_0))\bigg|_{\tau=0}, \tag{18}$$

and

$$\lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^2 \setminus \overline{\Omega}}} \frac{\mathrm{d}}{\mathrm{d}\tau} \phi_\rho^1(x + \tau\nu(t_0))\bigg|_{\tau=0}, \tag{19}$$

are well-defined and equal.

2.3 Properties of the double-layer potential

The following lemma establishes the regularity of the function $\psi^1_{x_0,h}(x)$ when $x_0, x \in \Gamma$ and h is normal to Γ. It can be found in [1], for example.

Lemma 2.4. *Let $\gamma : [0, L] \to \mathbb{R}^2$ be a curve parametrized by arclength and $\nu(t)$ be the normal vector to $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, $0 < t < L$, satisfying*

$$\nu(t) = (-\gamma_2'(t), \gamma_1'(t)). \tag{20}$$

Suppose that for some integer $k \geq 2$, γ is C^k in a neighbourhood of a point s, $0 < s < L$.

Then

$$\psi^1_{\gamma(s),\nu(s)}(\gamma(t)), \tag{21}$$

$$\psi^1_{\gamma(t),\nu(t)}(\gamma(s)), \tag{22}$$

are C^{k-2} functions of t in a neighborhood of s and

$$\lim_{t\to s}\psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = \lim_{t\to s}\psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = -\frac{1}{2}\kappa(s), \tag{23}$$

where $\kappa : [0, L] \to \mathbb{R}$ is the signed curvature of γ. Similarly, if γ is analytic in a neighborhood of a point s, where $0 < s < L$, then (21) and (22) are analytic functions of t in a neighborhood of s.

The following lemmas describes the behaviour of ψ^1 in the vicinity of a corner.

Lemma 2.5. *Under the same assumptions as the previous lemma, if γ has a corner with interior angle $\pi\alpha$ at t_0 then*

$$\lim_{\substack{s\to t_0^+\\ t\to t_0^-}}\psi^1_{\gamma(s),\nu(s)}(\gamma(t)), \tag{24}$$

$$\lim_{\substack{s\to t_0^+\\ t\to t_0^-}}\psi^1_{\gamma(t),\nu(t)}(\gamma(s)), \tag{25}$$

do not exist. In particular, along $s - t_0 = -(t - t_0) = h$,

$$\psi^1_{\gamma(t_0+h),\nu(t_0+h)}(\gamma(t_0 - h)) = \left(\frac{\cos\pi\alpha}{2\sin\pi\alpha}\right)\frac{1}{h} + O(1) \tag{26}$$

if γ is smooth in a neighbourhood of the corner.

Lemma 2.6. *Suppose Γ is an open wedge with side lengths one and an interior angle of $\pi\alpha$, where $0 < \alpha < 2$. Let $\gamma : [-1, 1] \to \mathbb{R}^2$ be an arclength parametrization of Γ, and*

$\nu(t)$ be the inward-pointing normal to Γ. Then

$$\psi^1_{\gamma(t),\nu(t)}(\gamma(s)) = \frac{-|s| \sin \pi\alpha}{s^2 + t^2 + 2st \cos \pi\alpha}, \tag{27}$$

if $s < 0$, $t > 0$ or $s > 0$, $t < 0$. For all other values of s and t, $\psi^1_{\gamma(t),\nu(t)}(\gamma(s)) = 0$.

Lemma 2.7. *Let Ω be a polygonal domain with boundary curve Γ. Let $\gamma : [0, L] \to \mathbb{R}^2$ be a counter-clockwise arc length parametrization of Γ and $\nu(t)$ be the inward-pointing normal vector to Γ at $\gamma(t)$. Let Γ_* be a subset of Γ corresponding to a wedge of sidelength $\delta > 0$, and without loss of generality assume that $\gamma : [0, 2\delta] \to \Gamma_*$. Then*

$$\int_{2\delta}^{L} \psi^1_{\gamma(t),\nu(t)}(s)\, \rho(s)\, \mathrm{d}s, \tag{28}$$

and

$$\int_{2\delta}^{L} \psi^1_{\gamma(s),\nu(s)}(t)\, \rho(s)\, \mathrm{d}s, \tag{29}$$

are smooth for all $0 < t < 2\delta$ and $\rho \in L^2([0, L])$.

We conclude this section with the following definitions which will be used in the remainder of the paper.

Definition 2.2. *For a given boundary Γ, we define the kernel $K : \Gamma \times \Gamma \to \mathbb{R}$ by the formula*

$$K(x, y) = \psi^1_{y,\nu(y)}(x). \tag{30}$$

Here by a slight abuse of notation we denote the normal derivative to Γ at a point $y \in \mathbb{R}^2$ by $\nu(y)$, instead of $\nu(\gamma^{-1}(y))$. Similarly, we define the adjoint kernel $K^* : \Gamma \times \Gamma \to \mathbb{R}$ by the formula

$$K^*(x, y) = \psi^1_{x,\nu(x)}(y). \tag{31}$$

2.4 Boundary conditions in terms of potentials

In order to solve the system (1) we represent the interior and exterior solutions each as the sum of a single-layer potential and a double-layer potential;

$$\phi_1(x) = \phi^0_{\rho_1}(x) + \phi^1_{\rho_3}(x), \tag{32}$$

$$\phi_2(x) = \phi^0_{\rho_2}(x) + \phi^1_{\rho_4}(x), \tag{33}$$

where ρ_1 and ρ_2 are charge densities on the inside and outside surface of the boundary, respectively, and ρ_3 and ρ_4 are dipole densities on the inside and outside surface of the boundary, respectively. We are then free to choose two additional relations between the densities ρ_1, ρ_2, ρ_3 and, ρ_4. As in [16], for example, we choose the additional constraints so that the singular integrals in (32) and (33) cancel. Equivalently we set

$$\lambda_1\rho_1 = \lambda_2\rho_2, \tag{34}$$

$$\lambda_3\rho_3 = \lambda_4\rho_4. \tag{35}$$

Let $\rho_s = \rho_1/\lambda_2$ and $\rho_d = \rho_3/\lambda_4$. The following lemma reduces the boundary value problem (1) to a system of integral equations.

Lemma 2.8. *Suppose ρ_s and ρ_d satisfy the integral equations*

$$-\pi\,(\lambda_1\lambda_4 + \lambda_2\lambda_3)\,\rho_d(t) + (\lambda_1\lambda_4 - \lambda_2\lambda_3)\,\phi^1_{\rho_d}(t) = f(t), \tag{36}$$

$$\pi\,(\lambda_1\lambda_4 + \lambda_2\lambda_3)\,\rho_s(t) - (\lambda_1\lambda_4 - \lambda_2\lambda_3)\,\phi^*_{\rho_s}(t) = g(t). \tag{37}$$

Then ϕ_1 and ϕ_2 given by (32) and (33), respectively, satisfy (1)-(3) with $\rho_1 = \lambda_2\rho_s$, $\rho_2 = \lambda_1\rho_s$, $\rho_3 = \lambda_4\rho_d$, and $\rho_4 = \lambda_3\rho_d$. Moreover, if ϕ satisfies (1)-(3) then there exist ρ_s and ρ_d such that ϕ is given by (32) inside Γ and by (33) outside.

Remark 2.3. *For simply-connected domains with the choice of auxiliary conditions given by (34) and (35), the equations for the single-layer densities and double-layer densities completely decouple. For multiply-connected domains, though the single- and double-layer* *densities are coupled, it does not affect the structure of the singularities near corners.*

For ease of exposition we introduce the constant β, defined by

$$\beta = \frac{\lambda_1\lambda_4 - \lambda_2\lambda_3}{\lambda_1\lambda_4 + \lambda_2\lambda_3}, \tag{38}$$

and define the rescaled boundary conditions $\tilde{f}, \tilde{g} : [0, L] \to \mathbb{R}$ by

$$\tilde{f}(t) = \frac{1}{\lambda_1\lambda_4 + \lambda_2\lambda_3} f(t) \tag{39}$$

$$\tilde{g}(t) = -\frac{1}{\lambda_1\lambda_4 + \lambda_2\lambda_3} g(t), \tag{40}$$

in which case (36) and (37) can be re-written as

$$-\pi\rho_d(t) + \beta\phi^1_{\rho_d}(t) = \tilde{f}(t), \tag{41}$$

$$-\pi\rho_s(t) + \beta\phi^*_{\rho_s}(t) = \tilde{g}(t). \tag{42}$$

Note that $|\beta| < 1$ if $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$.

2.5 Analytical Formulas

The following lemmas, proved in [18], provide the values of certain integrals which will be used in this paper.

Lemma 2.9. *Suppose that* $0 < \alpha < 2$ *is a real number and* μ *is a complex number such that* $\mathrm{Re}\mu > -1$, *and* $\mu \neq 0, 1, 2, 3, \ldots$ *Then*

$$\int_0^1 \frac{s \sin \pi\alpha}{t^2 + s^2 - 2st\cos\pi\alpha} t^\mu \, \mathrm{d}t = \pi s^\mu \frac{\sin \pi\mu(1-\alpha)}{\sin \pi\mu} + \sum_{k=0}^{\infty} \frac{\sin{(k+1)\pi\alpha}}{\mu - k - 1} s^{k+1}, \tag{43}$$

for all $0 < s < 1$.

Lemma 2.10. *Suppose that* $0 < \alpha < 2$ *is a real number and* m *is a non-negative integer.*

Then

$$\int_0^1 \frac{s \sin \pi\alpha}{t^2 + s^2 - 2st \cos \pi\alpha} t^m \, \mathrm{d}t = s^m \pi(1-\alpha) \cos m\pi\alpha - s^m \log(s) \sin m\pi\alpha + \sum_{\substack{k \geq 0 \\ k \neq m-1}}^{\infty} \frac{\sin{(k+1)\pi\alpha}}{m-k-1} s^{k+1}, \tag{44}$$

for all $0 < s < 1$.

The following lemmas establish properties of a certain collection of implicitly-defined functions.

Lemma 2.11. *For a fixed* $-1 < \delta < 1$, *consider the equation*

$$\beta \sin \delta z = \sin z. \tag{45}$$

Then for each $j = 0, 1, 2, \dots,$ *there exists an open region* $R_j \subset \mathbb{C}$ *containing the interval on the real axis* $(-1, 1)$, *and a function* $z_j(\beta)$ *which is holomorphic on* R_j *and satisfies the following equations*

$$\beta \sin \delta z_j(\beta) = \sin z_j(\beta), \tag{46}$$

$$z_j(0) = \pi j. \tag{47}$$

Moreover, these solutions are complete. Namely, if there exists $(z_0, \beta_0) \in \mathbb{R}^+ \times (-1, 1)$ *satisfying (45) then* $z_0 = z_j(\beta_0)$ *for a unique* $j \geq 0$.

Proof. Note that

$$\frac{\mathrm{d}z_j}{d\beta} = \frac{\sin(\delta z)}{1 - \delta^2\beta^2 - (1-\delta^2)\sin^2(z)} \left[\cos(z) + \delta\beta\cos(\delta z)\right], \tag{48}$$

and so is bounded for all $-1 < \beta < 1$. By the analytic implicit function theorem, there exists an open region in $\mathbb{C}$ containing $(-1, 1)$ on which the function z_j can be uniquely defined. ∎

Lemma 2.12. *Fix* $-1 < \delta < 1$. *If*

$$z_j(\beta_0) = k\pi \tag{49}$$

for some integer k *and* $\beta_0 \in (-1, 1)$, $\beta_0 \neq 0$, *then* $\delta k\pi$ *is an integer,* $k = j$ *and*

$$z_j(\beta) = j\pi \tag{50}$$

for all $\beta \in (-1, 1)$.

Proof. If $z_j(\beta_0) = k\pi$, $\beta_0 \neq 0$ then

$$\sin(\delta k\pi) = 0, \tag{51}$$

and hence $z(\beta) = k\pi$ is a solution to (45) for all $\beta \in [-1, 1]$. The uniqueness guaranteed by the implicit function implies that $z(\beta) = z_j(\beta)$ which completes the proof. ∎

Lemma 2.13. *Suppose that* $\delta\pi j$ *is not an integer. Then* R_j *contains* $[-1, 1]$. *Moreover, let* σ_j^+ *be the* j*th smallest element of*

$$\left\{ \frac{k}{1 - (-1)^k \delta} \;\middle|\; k = 0, 1, 2, \ldots \right\} \tag{52}$$

and let σ_j^- *be the* j*th smallest element of*

$$\left\{ \frac{k}{1 + (-1)^k \delta} \;\middle|\; k = 0, 1, 2, \ldots \right\}. \tag{53}$$

Then

$$\lim_{\beta \to -1} z_j(\beta) = \sigma_j^-, \tag{54}$$

$$\lim_{\beta \to 1} z_j(\beta) = \sigma_j^+. \tag{55}$$

Proof. The proof is an immediate consequence of the previous lemma and the analytic implicit function theorem. ■

Lemma 2.14. *For $n \geq 1$, suppose that $(1-\delta)j\pi$ is not an integer for any $j \leq n$ and let A be the $n \times n$ matrix with entries defined by*

$$A_{i,j}(\beta) = \begin{cases} \pi\beta \frac{\sin(1-\delta)i\pi}{z_j(\beta)-i\pi} & \beta \neq 0, i \neq 0, \text{ or } j \neq 0 \\ -\pi\left(1-\beta\delta\right) & \beta \neq 0,\ i=j=0, \\ -\pi\delta_{i,j} & \beta = 0. \end{cases} \tag{56}$$

Then $\det A(\beta)$ *is a holomorphic function of* β *in an open set containing* $[-1,1]$.

Proof. First, we observe that

$$z_j(\beta) - i\pi \neq 0 \tag{57}$$

if $\beta \neq 0$ or $i \neq j$. Moreover

$$\left.\frac{\mathrm{d}z_j(\beta)}{\mathrm{d}\beta}\right|_{\beta=0} = (-1)^j \sin(\pi j\delta) \neq 0. \tag{58}$$

and thus

$$\frac{1}{z_j(\beta) - i\pi} \tag{59}$$

has a simple pole at $\beta = 0$ if $i = j$. If $j \neq i$ then it is holomorphic on R_j. Hence

$$\pi\beta \frac{\sin(1-\delta)j\pi}{z_j(\beta) - i\pi} \tag{60}$$

is holomorphic on $R_j \setminus \{0\}$ and has a removable singularity at $\beta = 0$. The holomorphicity of A_{ij} on R_j follows by l'Hôpital's rule.

Letting R_n^* be the intersection of all R_j, $j \leq n$, we observe that $[-1, 1]$ is in the interior of R_n^* from which the result follows. ■

Lemma 2.15. *For $n \geq 1$, suppose that $(1-\delta)j\pi$ is not an integer for any $j \leq n$ and let A be the $n \times n$ matrix defined in the previous lemma. Then $\det(A) \neq 0$ for all $\beta \in [-1, 1]$.*

Proof. If $\beta = 0$ then the result follows immediately from the definition of A. For $\beta \neq 0$ let C be the $n \times n$ matrix with entries given by

$$C_{i,j}(\beta) = \pi\beta \frac{1}{z_j(\beta) - i\pi} \tag{61}$$

and B the $n \times n$ diagonal matrix with entries

$$B_{j,j} = \sin(1-\delta)\pi j, \tag{62}$$

and note that $A = BC$. Since C is a square Cauchy matrix it is invertible. Moreover, B is diagonal and by assumption, since $(1-\delta)z_j(\beta) \neq k\pi$ for any integer k, it follows that $\det B \neq 0$. ■

Definition 2.3. *For $j = 0, 1, \ldots$ and $-1 < \delta < 1$, define $w_j(\beta)$ via the formula*

$$w_j(\beta; \delta) = z_j(\beta; -\delta). \tag{63}$$

In particular, note that

$$\beta\sin(\delta w_j(\beta)) = -\sin(w_j(\beta)), \quad -1<\beta<1, \tag{64}$$

$$w_j(0) = \pi j. \tag{65}$$

Definition 2.4. *For $j = 0, 1, \ldots$ and $-1 < \delta < 1$, define the $n \times n$ matrix $B(\beta;\delta)$ via the formula*

$$B_{ij}(\beta;\delta) = A_{ij}(\beta;-\delta) \tag{66}$$

where the entries of A_{ij} are given by (56).

The following lemma establishes the convergence of a family of series.

Lemma 2.16. *Suppose that n is a positive integer and $0 < z_j < (n+1)\pi$ and $0 < \alpha < 2$. Define the function $\phi_j(z;n)$ by*

$$\phi_{j,n}(s) = \sum_{k=n+1}^{\infty} \frac{\sin k\pi\alpha}{\frac{z_j}{\pi} - k} s^{k-n}. \tag{67}$$

Then $\phi_{j,n}$ is bounded for all $0 \le s \le 1$.

Proof. We begin by observing that

$$\sum_{k=n+1}^{\infty} \frac{\sin k\pi\alpha}{\frac{z_j}{\pi} - k} s^{k-n} \tag{68}$$

converges for all $|s| < 1$ and thus $\phi_{j,n}$ is continuous on $[0, 1)$.

Next, define the sequences $w_1, w_2, \ldots$ and $q_1, q_2, \ldots$ by

$$w_k = -\sin k\pi\alpha, \tag{69}$$

$$q_k = \frac{1}{k - \frac{z_j}{\pi}} s^k. \tag{70}$$

Observe that for all $N \geq n+1$,

$$\left|\sum_{k=n+1}^{N} w_k\right| = \frac{1}{2}\left|\sum_{k=n+1}^{N} e^{ik\pi\alpha} - e^{-ik\pi\alpha}\right| \tag{71}$$

$$\leq \frac{1}{2}\left|\frac{1-e^{i(N-n)\pi\alpha}}{1-e^{i\pi\alpha}} - \frac{1-e^{-i(N-n)\pi\alpha}}{1-e^{-i\pi\alpha}}\right| \tag{72}$$

$$\leq \frac{1}{2\sin\frac{\pi\alpha}{2}} < \infty. \tag{73}$$

Moreover, note that $q_k > q_{k+1} > 0$. Therefore, by the Dirichlet test,

$$\sum_{k=n+1}^{\infty} \frac{\sin k\pi\alpha}{\frac{z_j}{\pi} - k} s^{k-n} \tag{74}$$

converges for $s = 1$. By Abel's theorem it follows that $\phi_{j,n}$ is continuous on $[0,1]$ and thus is bounded. ■

3 Analytical results

In the following two sections we characterize the behaviour of solutions to (36) and (37) on a wedge, provided the prescribed boundary data is analytic on either side of the corner, though possibly discontinuous across it. This analysis immediately applies to general polygonal domains since Lemma 2.7 guarantees that the potential due to the remainder of the boundary is analytic when restricted to a sufficiently small neighborhood of a corner. Finally, we note that in light of Remark 2.3 it suffices to consider (41) and (42) independently.

In the following we take Γ to be a wedge with unit side lengths and interior angle $\pi\alpha$, where $0 < \alpha < 2$ and α is irrational. Additionally, let $\gamma : [-1,1] \to \Gamma$ be an arclength parametrization of Γ and $\nu(t)$ denote the inward facing normal.

3.1 Series representation of the solutions of the double-layer equation

Theorem 3.1. *Suppose that f is representable by a convergent Taylor series on the intervals $[-1,0]$ and $[0,1]$, so that*

$$f(t) = \sum_{k=0}^{\infty} c_k t^k + \sum_{k=0}^{\infty} d_k \operatorname{sgn}(t)\,|t|^k. \tag{75}$$

Then, for all $n \geq 0$, there exist sequences $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ such that if

$$\rho_d(t) = \sum_{k=0}^{n} a_k |t|^{w_j(\beta)/\pi} + \sum_{k=0}^{n} b_k \operatorname{sgn}(t)\,|t|^{z_j(\beta)/\pi} \tag{76}$$

then

$$-\pi\rho_d(s) + \beta \int_{-1}^{1} \psi_{\gamma(t),\nu(t)}(\gamma(s))\rho_d(t)\,\mathrm{d}t = f(t) + s^n\phi(s) \tag{77}$$

for $-1 \leq s \leq 1$. Here $\phi : [-1,1] \to \mathbb{R}$ is a bounded function, and $z_j(\beta; 1-\alpha)$ and $w_j(\beta; 1-\alpha)$ are the functions defined in Lemma 2.11 and Definition 2.3, respectively.

Proof. If $\beta = 0$ then the result follows immediately. Now suppose $\beta \neq 0$. Applying the results of Lemma 2.9, if $\mu > -1$, $\mu \neq 0, 1, 2, \ldots$ then

$$\int_{-1}^{1} K(s,t)|t|^{\mu}\,\mathrm{d}t = -\pi|s|^{\mu}\frac{\sin \pi\mu(1-\alpha)}{\sin \pi\mu} - \sum_{k=0}^{\infty} \frac{\sin\,(k+1)\pi\alpha}{\mu - k - 1}|s|^{k+1} \tag{78}$$

and

$$\int_{-1}^{1} K(s,t)\operatorname{sgn}(t)\,|t|^{\mu}\,\mathrm{d}t = \pi \operatorname{sgn}(s)|s|^{\mu}\frac{\sin \pi\mu(1-\alpha)}{\sin \pi\mu} + \sum_{k=0}^{\infty} \frac{\sin\,(k+1)\pi\alpha}{\mu - k - 1}\operatorname{sgn}(s)|s|^{k+1} \tag{79}$$

from which it follows that

$$-\pi|s|^{w_j/\pi} + \beta \int_{-1}^{1} K(s,t)|t|^{w_j/\pi}\,\mathrm{d}t = \\ -\left[\pi + \beta\pi(1-\alpha)\right]\delta_{j,0} - \beta \sum_{k=1}^{\infty} \frac{\sin k\pi\alpha}{\frac{w_j}{\pi} - k}|s|^k, \tag{80}$$

and

$$-\pi\,\mathrm{sgn}(s)|s|^{z_j/\pi} + \beta \int_{-1}^{1} K(s,t)\,\mathrm{sgn}(t)|t|^{z_j/\pi}\,\mathrm{d}t = \\ -\pi\left[1 - (1-\alpha)\beta\right]\delta_{j,0} + \beta \sum_{k=1}^{\infty} \frac{\sin k\pi\alpha}{\frac{z_j}{\pi} - k}\,\mathrm{sgn}(s)|s|^k. \tag{81}$$

Recalling the definition of the matrices A and B given in (56) and (66), respectively, it follows that

$$-\pi\rho_d(s) + \beta \int_{-1}^{1} K(s,t)\rho_d(t)\,\mathrm{d}t = \sum_{k=0}^{n} \mathrm{sgn}(s)|s|^k \sum_{j=0}^{n} A_{kj}b_j - \sum_{k=0}^{n} |s|^k \sum_{j=0}^{n} B_{kj}a_j \tag{82}$$

$$+ \sum_{k=n+1}^{\infty} \sum_{j=0}^{n} b_j \frac{\sin k\pi\alpha}{\frac{z_j}{\pi} - k}\,\mathrm{sgn}(s)|s|^k - \sum_{k=n+1}^{\infty} \sum_{j=0}^{n} a_j \frac{\sin k\pi\alpha}{\frac{w_j}{\pi} - k}|s|^k. \tag{83}$$

By Lemma 2.15, since A and B are invertible, there exist sequences $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ such that

$$\sum_{j=0}^{n} A_{kj}b_j = d_k, \tag{84}$$

and

$$\sum_{j=0}^{n} B_{kj}a_j = -c_k. \tag{85}$$

Moreover, by Lemma 2.16

$$\phi(s) = \sum_{j=0}^{n} b_j \sum_{k=n+1}^{\infty} \frac{\sin k\pi\alpha}{\frac{z_j}{\pi} - k} \operatorname{sgn}(s)|s|^{k-n} - \sum_{j=0}^{n} a_j \sum_{k=n+1}^{\infty} \frac{\sin k\pi\alpha}{\frac{w_j}{\pi} - k} |s|^{k-n}, \tag{86}$$

is a bounded function for all s, which completes the proof.

■

3.2 Series representation of the solutions of the single-layer equation

Theorem 3.2. *Suppose that f is representable by a convergent Taylor series on the intervals $[-1, 0]$ and $[0, 1]$, so that*

$$g(t) = \sum_{k=0}^{\infty} c_{k+1} t^k + \sum_{k=0}^{\infty} d_{k+1} \operatorname{sgn}(t)\, |t|^k. \tag{87}$$

Then, for all $n \geq 1$, there exist sequences $a_1, \ldots, a_{n+1}$ and $b_1, \ldots, b_{n+1}$ such that if

$$\rho_s(t) = \sum_{k=1}^{n+1} a_k \operatorname{sgn}(t)|t|^{w_j(\beta)/\pi - 1} + \sum_{k=1}^{n+1} b_k\, |t|^{z_j(\beta)/\pi - 1} \tag{88}$$

then

$$-\pi \rho_s(s) + \beta \int_{-1}^{1} \psi_{\gamma(s),\nu(s)}(\gamma(t)) \rho_s(t)\, \mathrm{d}t = g(t) + s^n \phi(s) \tag{89}$$

for $-1 \leq s \leq 1$. Here $\phi : [-1, 1] \to \mathbb{R}$ is a bounded function, and $z_j(\beta; 1-\alpha)$ and $w_j(\beta; 1-\alpha)$ are the functions defined in Lemma 2.11 and Definition 2.3, respectively.

Proof. If $\beta = 0$ then the result follows immediately. Now suppose $\beta \neq 0$. Applying the results of Lemma 2.9, if $\mu > -1,\ \mu \neq 0, 1, 2, \ldots$ then

$$\int_{-1}^{1} K^*(s,t)|t|^{\mu-1}\,\mathrm{d}t = -\pi|s|^{\mu-1}\frac{\sin\pi\mu(1-\alpha)}{\sin\pi\mu} - \sum_{k=0}^{\infty}\frac{\sin{(k+1)\pi\alpha}}{\mu-k-1}|s|^{k} \tag{90}$$

and

$$\int_{-1}^{1} K^*(s,t)\,\mathrm{sgn}(t)\,|t|^{\mu-1}\,\mathrm{d}t = \pi\,\mathrm{sgn}(s)|s|^{\mu-1}\frac{\sin\pi\mu(1-\alpha)}{\sin\pi\mu} + \sum_{k=0}^{\infty}\frac{\sin{(k+1)\pi\alpha}}{\mu-k-1}\,\mathrm{sgn}(s)|s|^{k} \tag{91}$$

from which it follows that

$$-\pi|s|^{w_j/\pi-1} + \beta\int_{-1}^{1} K^*(s,t)|t|^{w_j/\pi-1}\,\mathrm{d}t = -\beta\sum_{k=1}^{\infty}\frac{\sin k\pi\alpha}{\frac{w_j}{\pi}-k}|s|^{k-1}, \tag{92}$$

and

$$-\pi\,\mathrm{sgn}(s)|s|^{z_j/\pi-1} + \beta\int_{-1}^{1} K^*(s,t)\,\mathrm{sgn}(t)|t|^{z_j/\pi-1}\,\mathrm{d}t = \beta\sum_{k=1}^{\infty}\frac{\sin k\pi\alpha}{\frac{z_j}{\pi}-k}\,\mathrm{sgn}(s)|s|^{k-1}. \tag{93}$$

Recalling the definition of the matrices A and B given in (56) and (66), respectively, it follows that

$$-\pi\rho_s(s) + \beta\int_{-1}^{1} K^*(s,t)\rho_s(t)\,\mathrm{d}t = \sum_{k=1}^{n+1}\mathrm{sgn}(s)|s|^{k}\sum_{j=1}^{n+1}A_{kj}b_j - \sum_{k=1}^{n+1}|s|^{k}\sum_{j=1}^{n+1}B_{kj}a_j \tag{94}$$

$$\sum_{k=n+2}^{\infty}\sum_{j=1}^{n+1}b_j\frac{\sin k\pi\alpha}{\frac{z_j}{\pi}-k}\,\mathrm{sgn}(s)|s|^{k-1} - \sum_{k=n+2}^{\infty}\sum_{j=1}^{n+1}a_j\frac{\sin k\pi\alpha}{\frac{w_j}{\pi}-k}|s|^{k-1}. \tag{95}$$

By Lemma 2.15, since A and B are invertible, there exist sequences $a_1, \dots, a_{n+1}$ and $b_1, \dots, b_{n+1}$ such that

$$\sum_{j=1}^{n+1} A_{kj} b_j = d_k, \tag{96}$$

and

$$\sum_{j=1}^{n+1} B_{kj} a_j = -c_k. \tag{97}$$

Moreover, by Lemma 2.16

$$\phi(s) = \sum_{j=1}^{n+1} b_j \sum_{k=n+2}^{\infty} \frac{\sin k\pi\alpha}{\frac{z_j}{\pi} - k} \operatorname{sgn}(s)|s|^{k-n} - \sum_{j=1}^{n+1} \sum_{k=n+2}^{\infty} a_j \frac{\sin k\pi\alpha}{\frac{w_j}{\pi} - k} |s|^{k-n}, \tag{98}$$

is a bounded function for all s, which completes the proof.

■

4 Numerical results

We solve the integral equations corresponding to the Laplace transmission problem on polygonal domains using a modified version of the algorithms described in [12]. In order to avoid recomputation of the quadratures for different values of β we modify the procedure described in [12] to construct quadratures which integrate ranges of powers multiplied by the kernel for ranges of angles instead of the set of singular powers corresponding to that particular angle. As described in [12], the discretization for the integral equations involving the single-layer potential are obtained by taking the adjoint of the discretized operator for the double-layer potential. To illustrate the performance of the algorithm we solve the exterior and interior Dirichlet problem on the domains shown in Figures 1-3. After discretization, the resulting linear systems were solved using standard techniques. The algorithm was implemented in Fortran 77 and the experiments were run on a 2.7 GHz Apple laptop with 8 GB RAM.

To demonstrate the accuracy of the algorithm for the Laplace transmission problem, we choose our boundary data f and g to be the result of an incident charge placed inside or outside the region. We solve the linear system to obtain the potential and use (6) and (7) to construct the solution away from the boundary. In both cases an analytic solution exists and is used to determine the accuracy of our solution. Specifically, the potential is evaluated analytically and numerically at a few arbitrary points and the maximum of the difference is calculated. The results are summarized in Table 1.

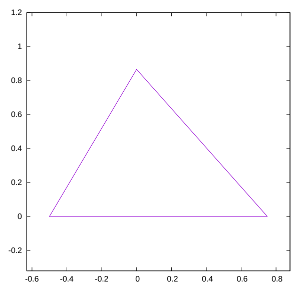

Figure 1: Γ_1 - a triangle in $\mathbb{R}^2$.

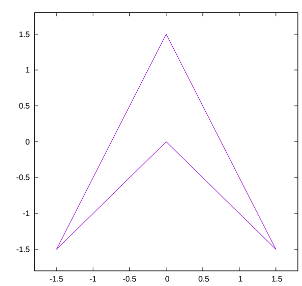

Figure 2: Γ_2 - a chevron in $\mathbb{R}^2$.

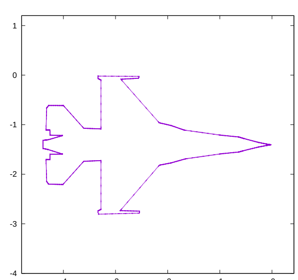

Figure 3: Γ_5 - top view of a falcon in $\mathbb{R}^2$.

5 Conclusions and extensions

In this paper we analyze the solution to boundary integral equations related to Laplace transmission problems on polygonal domains. In particular, if the boundary data is

Curve	Boundary conditions	Maximum error	Number of nodes	Total run time	Condition number
Γ_1	0.93651	$1.7764 \cdot 10^{-15}$	378	0.41724	35.307
Γ_2	-0.93651	$6.0091 \cdot 10^{-13}$	1528	16.352	32.115
Γ_3	-0.93651	$2.7496 \cdot 10^{-11}$	7242	1666.7	50.402

Table 1: Numerical results for the Laplace transmission problem.

smooth on either side of the corner then the solution to the boundary integral equations in the vicinity of the corner are represented by a series of non-integer powers and non-integer powers multiplied by logarithms. The resulting singular powers are analytic functions of the material parameters. Using a modification of the algorithm described in [12] the representations of the solutions near the corners can be used to create efficient and accurate discretizations for solving Laplace transmission problems on simply-connected and multiply-connected polygonal domains.

5.1 Multiple junctions

Here we consider only the case of Laplace transmission problems in regions with corners. This analysis can easily be extended to tilings in which more than two materials meet at a point. A detailed description of this analysis will be described in a future paper.

5.2 The Helmholtz transmission problem on domains with corners

In this paper we consider the solution to transmission problems for Laplace's equation on polygonal domains. The method used here extends naturally to the Helmholtz equation and the biharmonic equation on polygonal domains. Papers detailing the analysis and corresponding numerical algorithms to solve transmission problems for the Helmholtz and biharmonic equations on polygonal domains are currently in preparation.

5.3 Generalization to three dimensions

The generalization of the apparatus of this paper to three dimensional polyhedra is fairly straightforward, but the detailed analysis has not been carried out. This line of research is being vigorously pursued.

References

[1] K. E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, 1997.

[2] J. Bremer, *A fast direct solver for the integral equations of scattering theory on planar curves with corners*, J. Comput. Phys., 231 (2012), pp. 1879–1899.

[3] ——, *On the Nyström discretization of integral equations on planar curves with corners*, Appl. Comput. Harmon. Anal., 32 (2012), pp. 45–64.

[4] J. Bremer, V. Rokhlin, and I. Sammis, *Universal quadratures for boundary integral equations on two-dimensional domains with corners*, J. Comput. Phys., 229 (2010), pp. 8259–8280.

[5] R. Coifman, P. Jones, and S. Semmes, *Two elementary Proofs of the L^2 Boundedness of Cauchy Integrals on Lipschitz Curves.*, J. Am. Math. Soc., 2 (1989), pp. 553–564.

[6] A. Gillman, S. Hao, and P. G. Martinsson, *A simplified technique for the efficient and highly accurate discretization of boundary integral equations in 2d on domains with corners*, J. Comput. Phys., 256 (2014), pp. 214–219.

[7] J. Giroire and J. Nedelec, *Numerical solution of an exterior Neumann problem using a double layer potential*, Math. Comp., 32 (1978), pp. 973–990.

[8] P. Grisvard, *Elliptic problems in nonsmooth domains*, Boston: Pitman Advanced Pub. Program, 1985.

[9] S. Hao, A. H. Barnett, P. G. Martinsson, and P. Young, *High-order accurate Nyström discretization of integral equations with weakly singular kernels on smooth curves in the plane*, Adv. Comput. Math., 40 (2014).

[10] J. Helsing, *A fast and stable solver for singular integral equations on piecewise smooth curves*, SIAM J. Sci. Comput., 33 (2011), pp. 153–174.

[11] J. Helsing and K. Perfekt, *On the polarizability and capacitance of the cube*, Appl. Comput. Harmon. Anal., 34 (2013), pp. 445–468.

[12] J. Hoskins, V. Rokhlin, and K. Serkh, *On the Numerical Solution of Elliptic Partial Differential Equations on Polygonal Domains*, tech. rep., Yale University, 2017.

[13] R. S. Lehman, *Development of the Mapping Function at an Analytic Corner*, Pacific Journal of Mathematics, 7 (1957), pp. 1437–1449.

[14] V. G. Maz'ya and S. M. Nikol'skii, *Analysis IV: Linear and Boundary Integral Equations (Encycopedia of Mathematical Sciences)*, New York: Springer-Verlag, 1991.

[15] M. Rachh and K. Serkh, *On the solution of Stokes equation on regions with corners*, arXiv:1711.04072, (2017).

[16] V. Rokhlin, *Solution of acoustic scattering problems by means of second kind integral equations*, Wave Motion, 5 (1983), pp. 257–272.

[17] K. Serkh, *On the solution of elliptic partial differential equations on regions with corners III: curved boundaries.* Manuscript in preparation.

[18] ——, *On the solution of elliptic partial differential equations on regions with corners II: Detailed analysis*, Appl. Comput. Harmon. Anal., 5 (2017).

[19] K. Serkh and V. Rokhlin, *On the solution of the Helmholtz equation on regions with corners*, PNAS, 113 (2016), pp. 9171–9176.

[20] G. Verchota, *Layer potentials and regularity for the dirichlet problem for laplace's equation in lipschitz domains*, J. Funct. Anal, 59 (1984), pp. 572–611.

[21] V. G. Zargaryan, S. S.and Maz'ya, *The Asymptotic Form of the Solutions of the Integral Equations of Potential Theory in the Neighborhood of the Corner Points of a Contour*, PMM U.S.S.R., 48 (1984), pp. 120–124.