Zernike polynomials are a basis of orthogonal polynomials on the unit disk that are a natural basis for representing smooth functions. They arise in a number of applications including optics and atmospheric sciences. In this paper, we provide a self-contained reference on Zernike polynomials, algorithms for evaluating them, and what appear to be new numerical schemes for quadrature and interpolation. We also introduce new properties of Zernike polynomials in higher dimensions. The quadrature rule and interpolation scheme use a tensor product of equispaced nodes in the angular direction and roots of certain Jacobi polynomials in the radial direction. An algorithm for finding the roots of these Jacobi polynomials is also described. The performance of the interpolation and quadrature schemes is illustrated through numerical experiments. Discussions of higher dimensional Zernike polynomials are included in appendices.

Zernike Polynomials: Evaluation, Quadrature, and Interpolation

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Contents

1 Introduction

Zernike polynomials are a family of orthogonal polynomials that are a natural basis for the approximation of smooth functions on the unit disk. Among other applications, they are widely used in optics and atmospheric sciences and are the natural basis for representing Generalized Prolate Spheroidal Functions (see [12]).

In this report, we provide a self-contained reference on Zernike polynomials, including tables of properties, an algorithm for their evaluation, and what appear to be new numerical schemes for quadrature and interpolation. We also introduce properties of Zernike polynomials in higher dimensions and several classes of numerical algorithms for Zernike polynomial discretization in $\mathbb{R}^n$. The quadrature and interpolation schemes provided use a tensor product of equispaced nodes in the angular direction and roots of certain Jacobi polynomials in the radial direction. An algorithm for the evaluation of these roots is also introduced.

The structure of this paper is as follows. In Section 2 we introduce several technical lemmas and provide basic mathematical background that will be used in subsequent sections. In Section 3 we provide a recurrence relation for the evaluation of Zernike polynomials. Section 4 describes a scheme for integrating Zernike polynomials over the unit disk. Section 5 contains an algorithm for the interpolation of Zernike polynomials. In Section 6 we give results of numerical experiments with the quadrature and interpolation schemes introduced in the preceding sections. In Appendix A, we describe properies of Zernike polynomials in $\mathbb{R}^n$. Appendix B contains a description of an algorithm for the evaluation of Zernike polynomials in $\mathbb{R}^n$. Appendix C includes an description of Spherical Harmonics in higher dimensions. In Appendix D, an overview is provided of the family of Jacobi polynomials whose roots are used in numerical algorithms for high-dimensional Zernike polynomial discretization. Appendix D also includes a description of an algorithm for computing their roots. Appendix E contains notational conventions for Zernike polynomials.

2 Mathematical Preliminaries

In this section, we introduce notation and several technical lemmas that will be used in subsequent sections.

For notational convenience and ease of generalizing to higher dimensions, we will be denoting by $S_N^\ell(\theta) : \mathbb{R} \to \mathbb{R}$, the function defined by the formula

$$S_N^\ell(\theta) = \begin{cases} (2\pi)^{-1/2} & \text{if } N = 0, \\ \sin(N\theta)/\sqrt{\pi} & \text{if } \ell = 0,\ N > 0, \\ \cos(N\theta)/\sqrt{\pi} & \text{if } \ell = 1,\ N > 0. \end{cases} \tag{1}$$

where $\ell \in \{0, 1\}$, and N is a non-negative integer. In accordance with standard practice, we will denoting by $\delta_{i,j}$ the function defined by the formula

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2}$$

The following lemma is a classical fact from elementary calculus.

Lemma 2.1. *For all $n \in \{1, 2, ...\}$ and for any integer $k \geq n+1$,*

$$\frac{1}{k}\sum_{i=1}^{k}\sin(n\theta_i) = \int_0^{2\pi}\sin(n\theta)d\theta = 0 \tag{3}$$

and

$$\frac{1}{k}\sum_{i=1}^{k}\cos(n\theta_i) = \int_0^{2\pi}\cos(n\theta)d\theta = 0 \tag{4}$$

where

$$\theta_i = i\frac{2\pi}{k} \tag{5}$$

for $i = 1, 2, ..., k$.

The following technical lemma will be used in Section 4.

Lemma 2.2. *For all $m \in \{0, 1, 2, ...\}$, the set of all points $(N, n, \ell) \in \mathbb{R}^3$ such that $\ell \in \{0, 1\}$, N, n are non-negative integers, and $N + 2n \leq 2m - 1$ contains exactly $2m^2 + 2m$ elements.*

Proof. Lemma 2.2 follows immediately from the fact that the set of all pairs of non-negative integers (N, n) satisfying $N + 2n \leq 2m - 1$ has $m^2 + m$ elements where m is a non-negative integer. ■

The following is a classical fact from elementary functional analysis. A proof can be found in, for example, [13].

Lemma 2.3. *Let $f_1, ..., f_{2n-1} : [a, b] \to \mathbb{R}$ be a set of orthonormal functions such that for all $k \in \{1, 2, ..., 2n-1\}$,*

$$\int_a^b f_k(x)dx = \sum_{i=1}^{n} f_k(x_i)\omega_i dx \tag{6}$$

where $x_i \in [a, b]$ and $\omega_i \in \mathbb{R}$. Let $\phi : [a, b] \to \mathbb{R}$ be defined by the formula

$$\phi(x) = a_1 f_1(x) + ... + a_{n-1} f_{n-1}(x). \tag{7}$$

Then,

$$a_k = \int_a^b \phi(x) f_k(x)dx = \sum_{i=1}^{n}\phi(x_i) f_k(x_i)\omega_i. \tag{8}$$

for all $k \in \{1, 2, ..., n-1\}$.

2.1 Jacobi Polynomials

In this section, we define Jacobi polynomials and summarize some of their properties. Jacobi Polynomials, denoted $P_n^{(\alpha,\beta)}$, are orthogonal polynomials on the interval $(-1,1)$ with respect to weight function

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}. \tag{9}$$

Specifically, for all non-negative integers n, m with $n \neq m$ and real numbers $\alpha, \beta > -1$,

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta} dx = 0 \tag{10}$$

The following lemma, provides a stable recurrence relation that can be used to evaluate a particular class of Jacobi Polynomials (see, for example, [1]).

Lemma 2.4. *For any integer $n \geq 1$ and $N \geq 0$,*

$$\begin{aligned} P_{n+1}^{(N,0)}(x) = &\frac{(2n+N+1)N^2 + (2n+N)(2n+N+1)(2n+N+2)x}{2(n+1)(n+N+1)(2n+N)} P_n^{(N,0)}(x) \\ &- \frac{2(n+N)(n)(2n+N+2)}{2(n+1)(n+N+1)(2n+N)} P_{n-1}^{(N,0)}(x), \end{aligned} \tag{11}$$

where

$$P_0^{(N,0)}(x) = 1 \tag{12}$$

and

$$P_1^{(N,0)}(x) = \frac{N + (N+2)x}{2}. \tag{13}$$

The Jacobi Polynomial $P_n^{(N,0)}$ is defined in (10).

The following lemma provides a stable recurrence relation that can be used to evaluate derivatives of a certain class of Jacobi Polynomials. It is readily obtained by differentiating (11) with respect to x,

Lemma 2.5. *For any integer $n \geq 1$ and $N \geq 0$,*

$$\begin{aligned} P_{n+1}^{(N,0)\prime}(x) = &\frac{(2n+N+1)N^2 + (2n+N)(2n+N+1)(2n+N+2)x}{2(n+1)(n+N+1)(2n+N)} P_n^{(N,0)\prime}(x) \\ &- \frac{2(n+N)(n)(2n+N+2)}{2(n+1)(n+N+1)(2n+N)} P_{n-1}^{(N,0)\prime}(x) \\ &+ \frac{(2n+N)(2n+N+1)(2n+N+2)}{2(n+1)(n+N+1)(2n+N)} P_n^{(N,0)}(x), \end{aligned} \tag{14}$$

where

$$P_0^{(N,0)\prime}(x) = 0 \tag{15}$$

and

$$P_1^{(N,0)\prime}(x) = \frac{(N+2)}{2}. \tag{16}$$

The Jacobi Polynomial $P_n^{(N,0)}$ is defined in (10) and $P_n^{(N,0)\prime}(x)$ denotes the derivative of $P_n^{(N,0)}(x)$ with respect to x.

The following lemma, which provides a differential equation for Jacobi polynomials, can be found in [1]

Lemma 2.6. *For any integer n,*

$$(1-x^2)P_n^{(k,0)\prime\prime}(x)+(-k-(k+2)x)P_n^{(k,0)\prime}(x)+n(n+k+1)P_n^{(k,0)}(x) = 0 \quad (17)$$

for all $x \in [0,1]$ where $P_n^{(N,0)}$ is defined in (10).

Remark 2.1. We will be denoting by $\widetilde{P}_n : [0,1] \to \mathbb{R}$ the shifted Jacobi polynomial defined for any non-negative integer n by the formula

$$\widetilde{P}_n(x) = \sqrt{2n+2}P_n^{(1,0)}(1-2x) \quad (18)$$

where $P_n^{(1,0)}$ is defined in (10). The roots of $\widetilde{P}_n$ will be used in Section 4 and Section 5 in the design of quadrature and interpolation schemes for Zernike polynomials.

It follows immediately from the combination of (10) and (18) that the polynomials $\widetilde{P}_n$ are orthogonal on $[0,1]$ with respect to weight function

$$w(x) = x. \quad (19)$$

That is, for any non-negative integers i, j,

$$\int_0^1 \widetilde{P}_i(r)\widetilde{P}_j(r)rdr = \delta_{i,j}. \quad (20)$$

2.2 Gaussian Quadratures

In this section, we introduce Gaussian Quadratures.

Definition 2.1. *A Gaussian Quadrature with respect to a set of functions $f_1, ..., f_{2n-1} : [a,b] \to \mathbb{R}$ and non-negative weight function $w : [a,b] \to \mathbb{R}$ is a set of n nodes, $x_1, ..., x_n \in [a,b]$, and n weights, $\omega_1, ..., \omega_n \in \mathbb{R}$, such that, for any integer $j \leq 2n-1$,*

$$\int_a^b f_j(x)w(x)dx = \sum_{i=0}^{n} \omega_i f_j(x_i). \quad (21)$$

The following is a well-known lemma from numerical analysis. A proof can be found in, for example, [13].

Lemma 2.7. *Suppose that $p_0, p_1, ... : [a,b] \to \mathbb{R}$ is a set of orthonormal polynomials with respect to some non-negative weight function $w : [a,b] \to \mathbb{R}$ such that polynomial p_i is of degree i. Then,*

i) Polynomial p_i has exactly i roots on $[a,b]$.

ii) For any non-negative integer n and for $i = 0, 1, ..., 2n-1$, we have

$$\int_a^b p_i(x)w(x)dx = \sum_{k=1}^{n} \omega_k p_i(x_k) \quad (22)$$

where $x_1, ..., x_n \in [a, b]$ are the n roots of p_n and where weights $\omega_1, ..., \omega_n \in \mathbb{R}$ solve the $n \times n$ system of linear equations

$$\sum_{k=1}^{n} \omega_k p_j(x_k) = \int_a^b w(x) p_j(x) dx \tag{23}$$

with $j = 0, 1, ..., n-1$.

iii) The weights, ω_i, satisfy the identity,

$$\omega_i = \left(\sum_{k=0}^{n-1} p_k(x_i)^2 \right)^{-1} \tag{24}$$

for $i = 1, 2, ..., n$.

2.3 Zernike Polynomials

In this section, we define Zernike Polynomials and describe some of their basic properties.

Zernike polynomials are a family of orthogonal polynomials defined on the unit ball in $\mathbb{R}^n$. In this paper, we primarily discuss Zernike polynomials in $\mathbb{R}^2$, however nearly all of the theory and numerical machinery in two dimensions generalizes naturally to higher dimensions. The mathematical properties of Zernike polynomials in $\mathbb{R}^n$ are included in Appendix A.

Zernike Polynomials are defined via the formula

$$Z_{N,n}^{\ell}(x) = R_{N,n}(r) S_N^{\ell}(\theta) \tag{25}$$

for all $x \in \mathbb{R}^2$ such that $\|x\| \leq 1$, (r, θ) is the representation of x in polar coordinates, N, n are non- negative integers, S_N^{ℓ} is defined in (1), and $R_{N,n}$ are polynomials of degree $N + 2n$ defined by the formula

$$R_{N,n}(x) = x^N \sum_{k=0}^{n} (-1)^k \binom{n + N + \frac{p}{2}}{k} \binom{n}{k} (x^2)^{n-k} (1 - x^2)^k, \tag{26}$$

for all $0 \leq x \leq 1$. Furthermore, for any non-negative integers N, n, m,

$$\int_0^1 R_{N,n}(x) R_{N,m}(x) x \, dx = \frac{\delta_{n,m}}{2(2n + N + 1)} \tag{27}$$

and

$$R_{N,n}(1) = 1. \tag{28}$$

We define the normalized polynomials $\overline{R}_{N,n}$ via the formula

$$\overline{R}_{N,n}(x) = \sqrt{2(2n + N + 1)} R_{N,n}(x), \tag{29}$$

so that

$$\int_0^1 \left(\overline{R}_{N,n}(x) \right)^2 x \, dx = 1, \tag{30}$$

where N and n are non-negative integers. We define the normalized Zernike polynomial, $\overline{Z}^{\ell}_{N,n}$, by the formula

$$\overline{Z}_{N,n}(x) = \overline{R}_{N,n}(r) S^{\ell}_{N}(\theta) \tag{31}$$

where $x \in \mathbb{R}^2$ satisfies $\|x\| \leq 1$, and N, n are non-negative integers. We observe that $\overline{Z}^{\ell}_{N,n}$ has L^2 norm of 1 on the unit disk.

In an abuse of notation, we use $Z^{\ell}_{N,n}(x)$ and $Z^{\ell}_{N,n}(r,\theta)$ interchangeably where (r,θ) is the polar coordinate representation of $x \in \mathbb{R}^2$.

3 Numerical Evaluation of Zernike Polynomials

In this section, we provide a stable recurrence relation (see Lemma 3.1) that can be used to evaluate Zernike Polynomials.

Lemma 3.1. *The polynomials $R_{N,n}$, defined in (26) satisfy the recurrence relation*

$$\begin{aligned} &R_{N,n+1}(x) = \\ &-\frac{((2n+N+1)N^2 + (2n+N)(2n+N+1)(2n+N+2)(1-2x^2))}{2(n+1)(n+N+1)(2n+N)} R_{N,n}(x) \\ &-\frac{2(n+N)(n)(2n+N+2)}{2(n+1)(n+N+1)(2n+N)} R_{N,n-1}(x) \end{aligned} \tag{32}$$

where $0 \leq x \leq 1$, N is a non-negative integer, n is a positive integer, and

$$R_{N,0}(x) = x^N \tag{33}$$

and

$$R_{N,1}(x) = -x^N \frac{N + (N+2)(1-2x^2)}{2}. \tag{34}$$

Proof. According to [1], for any non-negative integers n and N,

$$R_{N,n}(x) = (-1)^n x^N P_n^{(N,0)}(1-2x^2), \tag{35}$$

where $0 \leq x \leq 1$, N and n are nonnegative integers, and $P_n^{(N,0)}$ denotes a Jacobi polynomial (see (10)).

Identity (32) follows immediately from the combination of (35) and (11). ■

Remark 3.1. The algorithm for evaluating Zernike polynomials using the recurrence relation in Lemma 3.1 is known as Kintner's method (see [9] and, for example, [6]).

4 Quadrature for Zernike Polynomials

In this section, we provide a quadrature rule for Zernike Polynomials.

The following lemma follows immediately from applying Lemma 2.7 to the polynomials $\widetilde{P}_n$ defined in (18).

Lemma 4.1. *Let $\{r_1, ..., r_m\}$ be the m roots of $\widetilde{P}_m$ (see (18)) and $\{\omega_1, ..., \omega_m\}$ the m weights of the Gaussian quadrature (see (21)) for the polynomials $\widetilde{P}_0, \widetilde{P}_1, ..., \widetilde{P}_{2m-1}$ (see (18)). Then, for any polynomial q of degree at most $2m-1$,*

$$\int_0^1 q(x)x dx = \sum_{i=1}^{m} q(r_i)\omega_i. \tag{36}$$

The following theorem provides a quadrature rule for Zernike Polynomials.

Theorem 4.2. *Let $\{r_1, ..., r_m\}$ be the m roots of $\widetilde{P}_m$ (see (18)) and $\{\omega_1, ..., \omega_m\}$ the m weights of the Gaussian quadrature (see (21)) for the polynomials $\widetilde{P}_0, \widetilde{P}_1, ..., \widetilde{P}_{2m-2}$ (see (18)). Then, for all $\ell \in \{0, 1\}$ and for all $N, n \in \{0, 1, ...\}$ such that $N + 2n \le 2m - 1$,*

$$\int_D Z_{N,n}^{\ell}(x)dx = \sum_{i=1}^{m} R_{N,n}(r_i)\omega_i \sum_{j=1}^{2m} \frac{2\pi}{2m} S_N^{\ell}(\theta_j) \tag{37}$$

where $R_{N,n}$ is defined in (26), θ_j is defined by the formula

$$\theta_j = j\frac{2\pi}{2m} \tag{38}$$

for $j \in \{1, 2, ..., 2m\}$, and $D \subseteq \mathbb{R}^2$ denotes the unit disk. Furthermore, there are exactly $2m^2 + m$ Zernike Polynomials of degree at most $2m - 1$.

Proof. Applying a change of variables,

$$\int_D Z_{N,n}^{\ell}(x)dx = \int_0^1 \int_0^{2\pi} R_{N,n}(r)S_N^{\ell}(\theta)rdrd\theta, \tag{39}$$

where $Z_{N,n}^{\ell}$ is a Zernike polynomial (see (25)) and where $R_{N,n}$ is defined in (27). Changing the order of integration of (39), we obtain

$$\int_D Z_{N,n}^{\ell}(x)dx = \int_0^1 rR_{N,n}(r)dr \int_0^{2\pi} S_N^{\ell}(\theta)d\theta. \tag{40}$$

Applying Lemma 2.1 and Lemma 4.1 to (40), we obtain

$$\int_D Z_{N,n}^{\ell}(x)dx = \sum_{i=1}^{m} R_{N,n}(r_i)\omega_i \sum_{j=1}^{2m} \frac{2\pi}{2m} S_N^{\ell}(\theta_j) \tag{41}$$

for $N + 2n \le 2m - 1$. The fact that there are exactly $2m^2 + m$ Zernike polynomials of degree at most $2m-1$ follows immediately from the combination of Lemma 2.2 with the fact that there are exactly m Zernike polynomials of degree at most $2m - 1$ that are of the form $Z_{0,n}^{\ell}$. ■

Remark 4.1. It follows immediately from Lemma 4.2 that for all $m \in \{1, 2, ...\}$, placing m nodes in the radial direction and $2m$ nodes in the angular direction (as described in Lemma 4.2), integrates exactly the $2m^2 + m$ Zernike polynomials on the disk of degree at most $2m - 1$.

Remark 4.2. The n roots of $\widetilde{P}_n$ (see 20) can be found by using, for example, the algorithm described in Section 10.3.

Remark 4.3. For Zernike polynomial discretization in $\mathbb{R}^{k+1}$, roots of the polynomials $\widetilde{P}_n^k$ are used, where $\widetilde{P}_n^k$ is defined by the formula

$$\widetilde{P}_n^k(x) = \sqrt{k+2n+1}P_n^{(k,0)}(1-2x). \tag{42}$$

Properties of this class of Jacobi polynomials are provided in Appendix D in addition to an algorithm for finding their roots.

The following remark illustrates that the advantage of quadrature rule (37) is especially noticeable in higher dimensions.

Remark 4.4. Quadrature rule (37) integrates all Zernike polynomials up to order $2m-1$ using the m roots of $\widetilde{P}_m$ (see (20)) as nodes in the radial direction. Using Guass-Legendre nodes instead of roots of $\widetilde{P}_m$ would require using $m+1$ nodes in the radial direction.

The high-dimensional equivalent of quadrature rule (37) uses the roots of $\widetilde{P}_m^{p+1}$ (see (107)) as nodes in the radial direction. Using Gauss-Legendre nodes instead of these nodes would require using an extra $p+1$ nodes in the radial direction or approximately $(p+1)m^{p+1}$ extra nodes total.

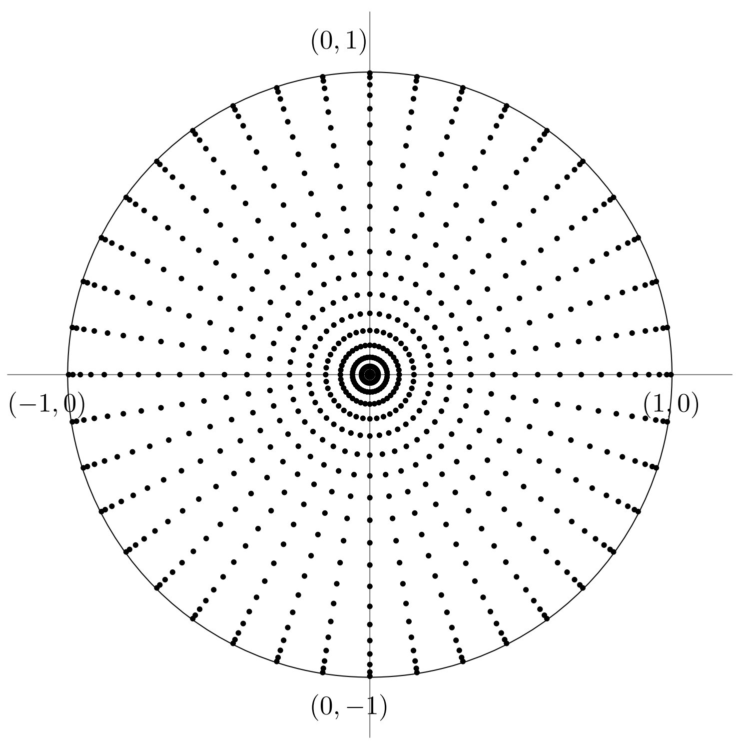

Figure 1: An illustration of locations of Zernike polynomial quadrature nodes with 20 radial nodes and 40 angular nodes.

The following remark shows that we can reduce the total number of nodes in quadrature rule (37) while still integrating the same number of functions.

node	θ
1	0.0000000000000000
2	0.1570796326794897
3	0.3141592653589793
4	0.4712388980384690
5	0.6283185307179586
6	0.7853981633974483
7	0.9424777960769379
8	1.0995574287564280
9	1.2566370614359170
10	1.4137166941154070
11	1.5707963267948970
12	1.7278759594743860
13	1.8849555921538760
14	2.0420352248333660
15	2.1991148575128550
16	2.3561944901923450
17	2.5132741228718340
18	2.6703537555513240
19	2.8274333882308140
20	2.9845130209103030
21	3.1415926535897930
22	3.2986722862692830
23	3.4557519189487720
24	3.6128315516282620
25	3.7699111843077520
26	3.9269908169872410
27	4.0840704496667310
28	4.2411500823462210
29	4.3982297150257100
30	4.5553093477052000
31	4.7123889803846900
32	4.8694686130641790
33	5.0265482457436690
34	5.1836278784231590
35	5.3407075111026480
36	5.4977871437821380
37	5.6548667764616280
38	5.8119464091411170
39	5.9690260418206070
40	6.1261056745000970

node	r
1	0.0083000442070672
2	0.0276430533525631
3	0.0575344576368137
4	0.0973041282065463
5	0.1460632469641095
6	0.2027224916634053
7	0.2660161417643405
8	0.3345303010944863
9	0.4067344665164935
10	0.4810157112964263
11	0.5557147130369888
12	0.6291628194156031
13	0.6997193231640498
14	0.7658081136864078
15	0.8259528873644578
16	0.8788101326763239
17	0.9231991629103781
18	0.9581285688822349
19	0.9828187818547442
20	0.9967238933309499

Table 1: Locations in the radial and angular directions of Zernike polynomial quadrature nodes with 40 angular nodes and 20 radial nodes.

Remark 4.5. Quadrature rule (37) integrates all Zernike polynomials of order up to $2m-1$ using a tensor product of $2m$ equispaced nodes in the angular direction and the m roots of $\widetilde{P}_m$ (see 18) in the radial direction. However, for large enough N and small enough j, $Z_{N,n}(r_j)$ is of magnitude smaller than machine precision, where r_j denotes the

j^{th} smallest root of $\widetilde{P}_m$. As a result, in order to integrate exactly $Z_{N,n}$ for large N, we can use fewer equispaced nodes in the angular direction at radius r_j.

5 Approximation of Zernike Polynomials

In this section, we describe an interpolation scheme for Zernike Polynomials.

We will denote by $r_1, ..., r_M$ the M roots of $\widetilde{P}_M$ (see 18).

Theorem 5.1. *Let M be a positive integer and $f : D \to \mathbb{R}$ be a linear combination of Zernike polynomials of degree at most $M-1$. That is,*

$$f(r,\theta) = \sum_{i,j} \alpha_{i,j}^{\ell} \overline{Z}_{i,j}^{\ell}(r,\theta) \tag{43}$$

where i, j are non-negative integers satisfying

$$i + 2j \le M - 1 \tag{44}$$

and where $\overline{Z}_{i,j}^{\ell}(r,\theta)$ is defined by (31) and S_i^{ℓ} is defined by (1). Then,

$$\alpha_{i,j}^{\ell} = \sum_{k=1}^{M} \left[\overline{R}_{i,j}(r_k)\omega_k \sum_{l=1}^{2M-1} \frac{2\pi}{2M-1} f(r_k,\theta_l) S_i^{\ell}(\theta_l) \right] \tag{45}$$

where $r_1, ..., r_M$ denote the M roots of $\widetilde{P}_M$ (see 18) and θ_l is defined by the formula

$$\theta_l = l \frac{2\pi}{2M-1} \tag{46}$$

for $l = 1, 2, ..., 2M-1$.

Proof. Clearly,

$$\alpha_{i,j}^{\ell} = \int_D f(r,\theta)\overline{Z}_{i,j}^{\ell} = \int_0^{2\pi}\int_0^1 f(r,\theta)\overline{R}_{i,j}(r) S_i^{\ell}(\theta) r dr d\theta. \tag{47}$$

Changing the order of integration of (47) and applying Lemma 2.1 and Lemma 2.3, we obtain

$$\begin{aligned} \alpha_{i,j}^{\ell} &= \int_0^1 \overline{R}_{i,j}(r) r \int_0^{2\pi} f(r,\theta) S_i^{\ell}(\theta) d\theta dr \\ &= \int_0^1 \overline{R}_{i,j}(r) r \sum_{l=1}^{2M-1} \frac{2\pi}{2M-1} f(r,\theta_l) S_i^{\ell}(\theta_l) dr. \end{aligned} \tag{48}$$

Applying Lemma 2.3 to (48), we obtain

$$\alpha_{i,j}^{\ell} = \sum_{k=1}^{M} \left[\overline{R}_{i,j}(r_k)\omega_k \sum_{l=1}^{2M-1} \frac{2\pi}{2M-1} f(r_k,\theta_l) S_{i,j}^{\ell}(\theta_l) \right]. \tag{49}$$

■

Remark 5.1. Suppose that $f : D \to \mathbb{R}$ is a linear combination of Zernike polynomials of degree at most $M-1$. It follows immediately from Theorem 5.1 and Theorem 4.2 that we can recover exactly the $M^2/2+M/2$ coefficients of the Zernike polynomial expanison of f by evaluation of f at $2M^2 - M$ points via (45).

Remark 5.2. Recovering the $M^2/2+M/2$ coefficients of a Zernike expansion of degree at most $M-1$ via (49) requires $O(M^3)$ operations by using a FFT to compute the sum

$$\sum_{l=1}^{2M-1} \frac{2\pi}{2M-1} f(r,\theta_l) S_{i,j}^{\ell}(\theta_l) \tag{50}$$

and then naively computing the sum

$$\alpha_{i,j}^{\ell} = \sum_{k=1}^{M} \overline{R}_{i,j}(r_k)\omega_k \sum_{l=1}^{2M-1} \frac{2\pi}{2M-1} f(r_k,\theta_l) S_{i,j}^{\ell}(\theta_l). \tag{51}$$

Remark 5.3. Sum (51) can be computed using an FMM (see, for example, [2]) which would reduce the evaluation of sum (49) to a computational cost of $O(M^2 \log(M))$.

Remark 5.4. Standard interpolation schemes on the unit disk often involve representing smooth functions as expansions in non-smooth functions such as

$$T_n(r) S_N^{\ell}(\theta) \tag{52}$$

where n and N are non-negative integers, T_n is a Chebyshev polynomial, and S_N^{ℓ} is defined in (1). Such interpolation schemes are amenable to the use of an FFT in both the angular and radial directions and thus have a computational cost of only $O(M^2 \log(M))$ for the interpolation of an M-degree Zernike expansion.

However, interpolation scheme (45) has three main advantages over such a scheme:
i) In order to represent a smooth function on the unit disk to full precision, a Zernike expansion requires approximately half as many terms as an expansion into functions of the form (52) (see Figure 3).
ii) Each function in the interpolated expansion is smooth on the disk.
iii) The expansion is amenable to filtering.

6 Numerical Experiments

The quadrature and interpolation formulas described in Sections 4 and 5 were implemented in Fortran 77. We used the Lahey/Fujitsu compiler on a 2.9 GHz Intel i7-3520M Lenovo laptop. All examples in this section were run in double precision arithmetic.

In each table in this section, the column labeled "nodes" denotes the number of nodes in both the radial and angular direction using quadrature rule (37). The column labeled "exact integral" denotes the true value of the integral being tested. This number is computed using adaptive gaussian quadrature in extended precision. The column labeled "integral via quadrature" denotes the integral approximation using quadrature rule (37).

We tested the performance of quadrature rule (37) in integrating three different functions over the unit disk. In Table 2 we approximated the integral over the unit disk of the function f_1 defined by the formula

$$f_1(x, y) = \frac{1}{1 + 25(x^2 + y^2)}. \tag{53}$$

In Table 3 we use quadrature rule (37) to approximate the integral over the unit disk of the function f_2 defined by the formula

$$f_2(r, \theta) = J_{100}(150r)\cos(100\theta)). \tag{54}$$

In Table 4, we use quadrature rule (37) to approximate the integral over the unit disk of the function f_3 defined by the formula

$$f_3(r, \theta) = P_8(x)P_{12}(y). \tag{55}$$

We tested the performance of interpolation scheme (43) on two functions defined on the unit disk.

In Figure 2 we plot the magnitude of the coefficients of the Zernike polynomials $R_{0,n}$ for $n = 0, 1, ..., 10$ using interpolation scheme (43) with 21 nodes in the radial direction and 41 in the angular direction on the function f_1 defined in (53). All coeficients of other terms were of magnitude smaller than 10^{-14}. In Table 5 we list the interpolated coefficients of the Zernike polynomial expansion of the function f_4 defined by the formula

$$f_4(x, y) = P_2(x)P_4(y) \tag{56}$$

where P_i is the ith degree Legendre polynomial. Listed are the coefficients using interpolation scheme (43) with 5 points in the radial direction and 9 points in the angular direction of Zernike polynomials

$$R_{N,n}\cos(N\theta) \tag{57}$$

where $N = 0, 1, ..., 8$ and $n = 0, 1, 2, 3, 4$. All other coefficients were of magnitude smaller than 10^{-14}. We interpolated the Bessel function

$$J_{10}(10r)cos(10\theta) \tag{58}$$

using interpolation scheme (43) and plot the resulting coefficients of the Zernike polynomials

$$R_{10,n}\cos(10\theta) \tag{59}$$

for $n = 0, ..., 16$ in Figure 3. All other coefficients were approximately 0 to machine precision. In Figure 3, we plot the coefficients of the Chebyshev expansion obtained via Chebyshev interpolation of the radial component of (58).

radial nodes	angular nodes	exact integral	integral via quadrature	relative error
5	10	0.4094244859413851	0.4097244673896003	0.732691×10^{-3}
10	20	0.4094244859413851	0.4094251051077367	0.151228×10^{-5}
15	30	0.4094244859413851	0.4094244870531256	0.271537×10^{-8}
20	40	0.4094244859413851	0.4094244859432513	0.455821×10^{-11}
25	50	0.4094244859413851	0.4094244859413883	0.791759×10^{-14}
30	60	0.4094244859413851	0.4094244859413848	0.630994×10^{-15}
35	70	0.4094244859413851	0.4094244859413850	0.142503×10^{-15}
40	80	0.4094244859413851	0.4094244859413858	0.181146×10^{-14}

Table 2: Quadratures for $f_1(x, y) = (1+25(x^2+y^2))^{-1}$ over the unit disk several different numbers of nodes

radial nodes	angular nodes	exact integral	integral via quadrature
5	10	0	$0.2670074163846569 \times 10^{-1}$
10	20	0	$0.2606355680939063 \times 10^{-2}$
15	30	0	$0.3119143925398078 \times 10^{-15}$
20	40	0	$0.0000000000000000 \times 10^{0}$
25	50	0	$0.3228321977714574 \times 10^{-1}$
30	60	0	$0.4945592102178045 \times 10^{-16}$
35	70	0	$0.1147861841710902 \times 10^{-16}$
40	80	0	$0.8148891073315595 \times 10^{-16}$
45	90	0	$-0.7432759692263743 \times 10^{-16}$
50	100	0	$0.3207999037057322 \times 10^{-1}$
55	110	0	$-0.1399753743762347 \times 10^{-15}$
60	120	0	$0.3075136040459932 \times 10^{-16}$
65	130	0	$-0.9458788981593222 \times 10^{-16}$
70	140	0	$0.2045957446273746 \times 10^{-17}$
75	150	0	$0.2416178317504225 \times 10^{-16}$

Table 3: Quadratures for $f_2(r,\theta) = J_{100}(150r)\cos(100\theta)$ using several different numbers of nodes

radial nodes	angular nodes	integral via quadrature	exact integral	relative error
5	10	$-0.8998055487754142 \times 10^{-2}$	$-0.1527947805159123 \times 10^{-2}$	-0.830191×10^{0}
10	20	$0.1655201967553289 \times 10^{-1}$	$-0.1527947805159123 \times 10^{-2}$	-0.109231×10^{1}
15	30	$-0.1527947805159138 \times 10^{-2}$	$-0.1527947805159123 \times 10^{-2}$	$-0.979221 \times 10^{-14}$
20	40	$-0.1527947805159132 \times 10^{-2}$	$-0.1527947805159123 \times 10^{-2}$	$-0.567665 \times 10^{-14}$
25	50	$-0.1527947805159108 \times 10^{-2}$	$-0.1527947805159123 \times 10^{-2}$	0.102180×10^{-13}
30	60	$-0.1527947805159144 \times 10^{-2}$	$-0.1527947805159123 \times 10^{-2}$	$-0.134820 \times 10^{-13}$
35	70	$-0.1527947805159128 \times 10^{-2}$	$-0.1527947805159123 \times 10^{-2}$	$-0.269641 \times 10^{-14}$
40	80	$-0.1527947805159155 \times 10^{-2}$	$-0.1527947805159123 \times 10^{-2}$	$-0.210036 \times 10^{-13}$

Table 4: Quadratures for $f_3(x, y) = P_8(x)P_{12}(y)$ (see (55)) using several different numbers of nodes

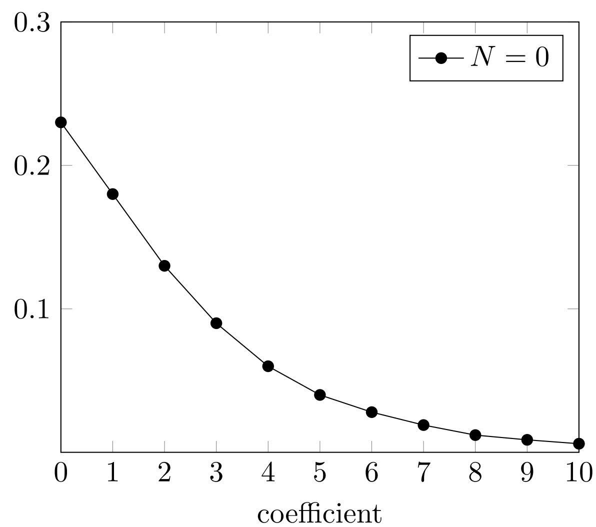

Figure 2: magnitudes of coefficients of interpolation of $f_1(x, y) = (1 + 25(x^2 + y^2))^{-1}$ for $N = 0$

N	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
0	0.02942	0.03297	-0.11998	0.01373	0.53776×10^{-16}
1	-0.48788×10^{-16}	0.76567×10^{-17}	0.99670×10^{-18}	0.22059×10^{-16}	-
2	0.02967	0.11495	-0.00647	-0.90206×10^{-16}	-
3	0.58217×10^{-16}	-0.73297×10^{-16}	0.19321×10^{-17}	-	-
4	0.04926	-0.03238	-0.13010×10^{-16}	-	-
5	0.77604×10^{-16}	0.10474×10^{-15}	-	-	-
6	0.09714	-0.11102×10^{-15}	-	-	-
7	-0.18100×10^{-16}	-	-	-	-
8	0.77241×10^{-16}	-	-	-	-

Table 5: Coefficients of the interpolation of the function $f_4(x, y) = P_2(x)P_4(y)$ into Zernike polynomials of degree at most 8. The entry corresponding to N, n is the coefficient of $R_{N,n} \cos(N\theta)$.

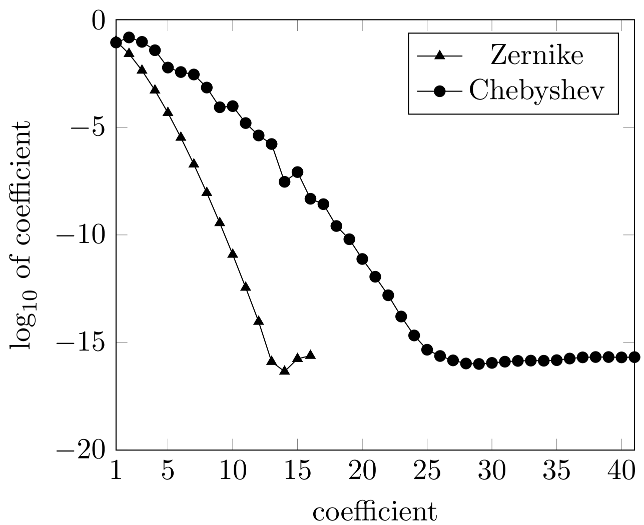

Figure 3: Coefficients of the Zernike expansion for $N = 10$ of $J_{10}(10r)\cos(10\theta)$ using Chebyshev and Zernike interpolation in the radial direction with 81 points in the angular direction and 41 points in the radial direction.

7 Appendix A: Mathematical Properties of Zernike Polynomials

In this appendix, we define Zernike polynomials in $\mathbb{R}^{p+2}$ and describe some of their basic properties. Zernike polynomials, denoted $Z_{N,n}^{\ell}$, are a sequence of orthogonal polynomials defined via the formula

$$Z_{N,n}^{\ell}(x) = R_{N,n}(\|x\|) S_N^{\ell}(x/\|x\|), \tag{60}$$

for all $x \in \mathbb{R}^{p+2}$ such that $\|x\| \le 1$, where N and n are nonnegative integers, S_N^{ℓ} are the orthonormal surface harmonics of degree N (see Appendix C), and $R_{N,n}$ are polynomials of degree $2n+N$ defined via the formula

$$R_{N,n}(x) = x^N \sum_{m=0}^{n} (-1)^m \binom{n+N+\frac{p}{2}}{m} \binom{n}{m} (x^2)^{n-m} (1-x^2)^m, \tag{61}$$

for all $0 \le x \le 1$. The polynomials $R_{N,n}$ satisfy the relation

$$R_{N,n}(1) = 1, \tag{62}$$

and are orthogonal with respect to the weight function $w(x) = x^{p+1}$, so that

$$\int_0^1 R_{N,n}(x) R_{N,m}(x) x^{p+1}\, dx = \frac{\delta_{n,m}}{2(2n+N+\frac{p}{2}+1)}, \tag{63}$$

where

$$\delta_{n,m} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \ne m. \end{cases} \tag{64}$$

We define the polynomials $\overline{R}_{N,n}$ via the formula

$$\overline{R}_{N,n}(x) = \sqrt{2(2n+N+p/2+1)}\, R_{N,n}(x), \tag{65}$$

so that

$$\int_0^1 \left(\overline{R}_{N,n}(x)\right)^2 x^{p+1}\, dx = 1, \tag{66}$$

where N and n are nonnegative integers. We define the normalized Zernike polynomial, $\overline{Z}_{N,n}^{\ell}$, by the formula

$$\overline{Z}_{N,n}(x) = \overline{R}_{N,n}(\|x\|) S_N^{\ell}(x/\|x\|) \tag{67}$$

for all $x \in \mathbb{R}^{p+2}$ such that $\|x\| \le 1$, where N and n are nonnegative integers, S_N^{ℓ} are the orthonormal surface harmonics of degree N (see Appendix C), and $R_{N,n}$ is defined in (61). We observe that $\overline{Z}_{N,n}^{\ell}$ has L^2 norm of 1 on the unit ball in $\mathbb{R}^{p+2}$.

In an abuse of notation, we refer to both the polynomials $Z_{N,n}^{\ell}$ and the radial polynomials $R_{N,n}$ as Zernike polynomials where the meaning is obvious.

Remark 7.1. When $p = -1$, the Zernike polynomials take the form

$$Z^1_{0,n}(x) = R_{0,n}(|x|) = P_{2n}(x), \tag{68}$$
$$Z^2_{1,n}(x) = \mathrm{sgn}(x) \cdot R_{1,n}(|x|) = P_{2n+1}(x), \tag{69}$$

for $-1 \leq x \leq 1$ and nonnegative integer n, where P_n denotes the Legendre polynomial of degree n and

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0, \end{cases} \tag{70}$$

for all real x.

Remark 7.2. When $p = 0$, the Zernike polynomials take the form

$$Z^1_{N,n}(x_1, x_2) = R_{N,n}(r)\cos(N\theta)/\sqrt{\pi}, \tag{71}$$
$$Z^2_{N,n}(x_1, x_2) = R_{N,n}(r)\sin(N\theta)/\sqrt{\pi}, \tag{72}$$

where $x_1 = r\cos(\theta)$, $x_2 = r\sin(\theta)$, and N and n are nonnegative integers.

7.1 Special Values

The following formulas are valid for all nonnegative integers N and n, and for all $0 \leq x \leq 1$.

$$R_{N,0}(x) = x^N, \tag{73}$$
$$R_{N,1}(x) = x^N\big((N + p/2 + 2)x^2 - (N + p/2 + 1)\big), \tag{74}$$
$$R_{N,n}(1) = 1, \tag{75}$$
$$R^{(k)}_{N,n}(0) = 0 \quad \text{for } k = 0, 1, \ldots, N-1, \tag{76}$$
$$R^{(N)}_{N,n}(0) = (-1)^n N! \binom{n + N + \frac{p}{2}}{n}. \tag{77}$$

7.2 Hypergeometric Function

The polynomials $R_{N,n}$ are related to the hypergeometric function ${}_2F_1$ (see [1]) by the formula

$$R_{N,n}(x) = (-1)^n \binom{n + N + \frac{p}{2}}{n} x^N {}_2F_1\Big(-n, n + N + \frac{p}{2} + 1; N + \frac{p}{2} + 1; x^2\Big), \tag{78}$$

where $0 \leq x \leq 1$, and N and n are nonnegative integers.

7.3 Interrelations

The polynomials $R_{N,n}$ are related to the Jacobi polynomials via the formula

$$R_{N,n}(x) = (-1)^n x^N P_n^{(N+\frac{p}{2},0)}(1 - 2x^2), \tag{79}$$

where $0 \le x \le 1$, N and n are nonnegative integers, and $P_n^{(\alpha,\beta)}$, $\alpha, \beta > -1$, denotes the Jacobi polynomials of degree n (see [1]).

When $p = -1$, the polynomials $R_{N,n}$ are related to the Legendre polynomials via the formulas

$$R_{0,n}(x) = P_{2n}(x), \tag{80}$$

$$R_{1,n}(x) = P_{2n+1}(x), \tag{81}$$

where $0 \le x \le 1$, n is a nonnegative integer, and P_n denotes the Legendre polynomial of degree n (see [1]).

7.4 Limit Relations

The asymptotic behavior of the Zernike polynomials near 0 as the index n tends to infinity is described by the formula

$$\lim_{n\to\infty} \frac{(-1)^n R_{N,n}\left(\frac{x}{2n}\right)}{(2n)^{p/2}} = \frac{J_{N+p/2}(x)}{x^{p/2}}, \tag{82}$$

where $0 \le x \le 1$, N is a nonnegative integer, and J_ν denotes the Bessel functions of the first kind (see [1]).

7.5 Zeros

The asymptotic behavior of the zeros of the polynomials $R_{N,n}$ as n tends to infinity is described by the following relation. Let $x_{N,m}^{(n)}$ be the mth positive zero of $R_{N,n}$, so that $0 < x_{N,1}^{(n)} < x_{N,2}^{(n)} < \ldots$. Likewise, let $j_{\nu,m}$ be the mth positive zero of J_ν, so that $0 < j_{\nu,1} < j_{\nu,2} < \ldots$, where J_ν denotes the Bessel functions of the first kind (see [1]). Then

$$\lim_{n\to\infty} 2n x_{N,m}^{(n)} = j_{N+p/2,m}, \tag{83}$$

for any nonnegative integer N.

7.6 Inequalities

The inequality

$$|R_{N,n}(x)| \le \binom{n+N+\frac{p}{2}}{n} \tag{84}$$

holds for $0 \le x \le 1$ and nonnegative integer N and n.

7.7 Integrals

The polynomials $R_{N,n}$ satify the relation

$$\int_0^1 \frac{J_{N+p/2}(xy)}{(xy)^{p/2}} R_{N,n}(y) y^{p+1}\, dy = \frac{(-1)^n J_{N+p/2+2n+1}(x)}{x^{p/2+1}}, \tag{85}$$

where $x \ge 0$, N and n are nonnegative integers, and J_ν denotes the Bessel functions of the first kind.

7.8 Generating Function

The generating function associated with the polynomials $R_{N,n}$ is given by the formula

$$\frac{\left(1+z-\sqrt{1+2z(1-2x^2)+z^2}\right)^{N+p/2}}{(2zx)^{N+p/2}x^{p/2}\sqrt{1+2z(1-2x^2)+z^2}} = \sum_{n=0}^{\infty} R_{N,n}(x)z^n, \tag{86}$$

where $0 \le x \le 1$ is real, z is a complex number such that $|z| \le 1$, and N is a nonnegative integer.

7.9 Differential Equation

The polynomials $R_{N,n}$ satisfy the differential equation

$$(1-x^2)y''(x) - 2xy'(x) + \left(\chi_{N,n} + \frac{\frac{1}{4} - (N+\frac{p}{2})^2}{x^2}\right)y(x) = 0, \tag{87}$$

where

$$\chi_{N,n} = (N + \tfrac{p}{2} + 2n + \tfrac{1}{2})(N + \tfrac{p}{2} + 2n + \tfrac{3}{2}), \tag{88}$$

and

$$y(x) = x^{p/2+1} R_{N,n}(x), \tag{89}$$

for all $0 < x < 1$ and nonnegative integers N and n.

7.10 Recurrence Relations

The polynomials $R_{N,n}$ satisfy the recurrence relation

$$\begin{aligned}
&2(n+1)(n+N+\tfrac{p}{2}+1)(2n+N+\tfrac{p}{2})R_{N,n+1}(x) \\
&\qquad = -\left((2n+N+\tfrac{p}{2}+1)(N+\tfrac{p}{2})^2 + (2n+N+\tfrac{p}{2})_3(1-2x^2)\right)R_{N,n}(x) \\
&\qquad\quad - 2n(n+N+\tfrac{p}{2})(2n+N+\tfrac{p}{2}+2)R_{N,n-1}(x),
\end{aligned} \tag{90}$$

where $0 \le x \le 1$, N is a nonnegative integer, n is a positive integer, and $(\cdot)_n$ is defined via the formula

$$(x)_n = x(x+1)(x+2)\dots(x+n-1), \tag{91}$$

for real x and nonnegative integer n. The polynomials $R_{N,n}$ also satisfy the recurrence relations

$$(2n+N+\tfrac{p}{2}+2)xR_{N+1,n}(x) = (n+N+\tfrac{p}{2}+1)R_{N,n}(x) + (n+1)R_{N,n+1}(x), \tag{92}$$

for nonnegative integers N and n, and

$$(2n+N+\tfrac{p}{2})xR_{N-1,n}(x) = (n+N+\tfrac{p}{2})R_{N,n}(x) + nR_{N,n-1}(x), \tag{93}$$

for integers $N \ge 1$ and $n \ge 0$, where $0 \le x \le 1$.

7.11 Differential Relations

The Zernike polynomials satisfy the differential relation given by the formula

$$
\begin{aligned}
&(2n+N+\tfrac{p}{2})x(1-x^2)\frac{d}{dx}R_{N,n}(x) \\
&\qquad = \big(N(2n+N+\tfrac{p}{2})+2n^2-(2n+N)(2n+N+\tfrac{p}{2})x^2\big)R_{N,n}(x) \\
&\qquad\qquad + 2n(n+N+\tfrac{p}{2})R_{N,n-1}(x),
\end{aligned}
\tag{94}
$$

where $0 < x < 1$, N is a nonnegative integer, and n is a positive integer.

8 Appendix B: Numerical Evaluation of Zernike Polynomials in $\mathbb{R}^{p+2}$

The main analytical tool of this section is Lemma 8.1 which provides a recurrence relation that can be used for the evaluation of radial Zernike Polynomials, $R_{N,n}$.

According to [1], radial Zernike polynomials, $R_{N,n}$, are related to Jacobi polynomials via the formula

$$
R_{N,n}(x) = (-1)^n x^N P_n^{(N+\frac{p}{2},0)}(1-2x^2), \tag{95}
$$

where $0 \le x \le 1$, N and n are nonnegative integers, and $P_n^{(\alpha,0)}$ is defined in (10).

The following lemma provides a relation that can be used to evaluate the polynomial $R_{N,n}$.

Lemma 8.1. *The polynomials $R_{N,n}$ satisfy the recurrence relation*

$$
\begin{aligned}
&2(n+1)(n+N+\tfrac{p}{2}+1)(2n+N+\tfrac{p}{2})R_{N,n+1}(x) \\
&\qquad = -\big((2n+N+\tfrac{p}{2}+1)(N+\tfrac{p}{2})^2+(2n+N+\tfrac{p}{2})_3(1-2x^2)\big)R_{N,n}(x) \\
&\qquad\qquad - 2n(n+N+\tfrac{p}{2})(2n+N+\tfrac{p}{2}+2)R_{N,n-1}(x),
\end{aligned}
\tag{96}
$$

where $0 \le x \le 1$, N is a nonnegative integer, n is a positive integer, and $(\cdot)_n$ is defined via the formula

$$
(x)_n = x(x+1)(x+2)\dots(x+n-1), \tag{97}
$$

for real x and nonnegative integer n.

Proof. It is well known that the Jacobi polynomial $P_n^{(\alpha,0)}(x)$ satisfies the recurrence relation

$$
a_{1n}P_{n+1}^{(\alpha,0)} = (a_{2n}+a_{3n}x)P_n^{(\alpha,0)}(x) - a_{4n}P_{n-1}^{(\alpha,0)}(x) \tag{98}
$$

where

$$
\begin{aligned}
a_{1n} &= 2(n+1)(n+\alpha+1)(2n+\alpha) \\
a_{2n} &= (2n+\alpha+1)\alpha^2 \\
a_{3n} &= (2n+\alpha)(2n+\alpha+1)(2n+\alpha+2) \\
a_{4n} &= 2(n+\alpha)(n)(2n+\alpha+2)
\end{aligned}
\tag{99}
$$

Identity (96) follows immediately from the combination of (98) and (99). ■

9 Appendix C: Spherical Harmonics in $\mathbb{R}^{p+2}$

Suppose that S^{p+1} denotes the unit sphere in $\mathbb{R}^{p+2}$. The spherical harmonics are a set of real-valued continuous functions on S^{p+1}, which are orthonormal and complete in $L^2(S^{p+1})$. The spherical harmonics of degree $N \geq 0$ are denoted by $S_N^1, S_N^2, \ldots, S_N^\ell, \ldots, S_N^{h(N)} \colon S^{p+1} \to \mathbb{R}$, where

$$h(N) = (2N+p)\frac{(N+p-1)!}{p!\, N!}, \tag{100}$$

for all nonnegative integers N.

The following theorem defines the spherical harmonics as the values of certain harmonic, homogeneous polynomials on the sphere (see, for example, [3]).

Theorem 9.1. *For each spherical harmonic S_N^ℓ, where $N \geq 0$ and $1 \leq \ell \leq h(N)$ are integers, there exists a polynomial $K_N^\ell \colon \mathbb{R}^{p+2} \to \mathbb{R}$ which is harmonic, i.e.*

$$\nabla^2 K_N^\ell(x) = 0, \tag{101}$$

for all $x \in \mathbb{R}^{p+2}$, and homogenous of degree N, i.e.

$$K_N^\ell(\lambda x) = \lambda^N K_N^\ell(x), \tag{102}$$

for all $x \in \mathbb{R}^{p+2}$ and $\lambda \in \mathbb{R}$, such that

$$S_N^\ell(\xi) = K_N^\ell(\xi), \tag{103}$$

for all $\xi \in S^{p+1}$.

The following theorem is proved in, for example, [3].

Theorem 9.2. *Suppose that N is a nonnegative integer. Then there are exactly*

$$(2N+p)\frac{(N+p-1)!}{p!\, N!} \tag{104}$$

linearly independent, harmonic, homogenous polynomials of degree N in $\mathbb{R}^{p+2}$.

The following theorem states that for any orthogonal matrix U, the function $S_N^\ell(U\xi)$ is expressible as a linear combination of $S_N^1(\xi), S_N^2(\xi), \ldots, S_N^{h(N)}(\xi)$ (see, for example, [3]).

Theorem 9.3. *Suppose that N is a nonnegative integer, and that $S_N^1, S_N^2, \ldots, S_N^{h(N)} \colon S^{p+1} \to \mathbb{R}$ are a complete set of orthonormal spherical harmonics of degree N. Suppose further that U is a real orthogonal matrix of dimension $p+2 \times p+2$. Then, for each integer $1 \leq \ell \leq h(N)$, there exists real numbers $v_{\ell,1}, v_{\ell,2}, \ldots, v_{\ell,h(N)}$ such that*

$$S_N^\ell(U\xi) = \sum_{k=1}^{h(N)} v_{\ell,k} S_N^k(\xi), \tag{105}$$

for all $\xi \in S^{p+1}$. Furthermore, if V is the $h(N) \times h(N)$ real matrix with elements $v_{i,j}$ for all $1 \leq i, j \leq h(N)$, then V is also orthogonal.

Remark 9.1. From Theorem (9.3), we observe that the space of linear combinations of functions S_N^ℓ is invariant under all rotations and reflections of S^{p+1}.

The following theorem states that if an integral operator acting on the space of functions $S^{p+1} \to \mathbb{R}$ has a kernel depending only on the inner product, then the spherical harmonics are eigenfunctions of that operator (see, for example, [3]).

Theorem 9.4 (Funk-Hecke)**.** *Suppose that $F\colon [-1,1] \to \mathbb{R}$ is a continuous function, and that $S_N\colon S^{p+1} \to \mathbb{R}$ is any spherical harmonic of degree N. Then*

$$\int_\Omega F(\langle \xi, \eta \rangle) S_N(\xi)\, d\Omega(\xi) = \lambda_N S_N(\eta), \tag{106}$$

for all $\eta \in S^{p+1}$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^{p+2}$, the integral is taken over the whole area of the hypersphere Ω, and λ_N depends only on the function F.

10 Appendix D: The Shifted Jacobi Polynomials $P_n^{(k,0)}(2x-1)$

In this section, we introduce a class of Jacobi polynomials that can be used as quadrature and interpolation nodes for Zernike polynomials in $\mathbb{R}^{p+2}$.

We define $\widetilde{P}_n^k(x)$ to be the shifted Jacobi polynomials on the interval $[0,1]$ defined by the formula

$$\widetilde{P}_n^k(x) = \sqrt{k+2n+1}P_n^{(k,0)}(1-2x) \tag{107}$$

where $k > -1$ is a real number and where $P_n^{(k,0)}$ is defined in (10). It follows immediately from (107) that $\widetilde{P}_n^k(x)$ are orthogonal with respect to weight function x^k. That is, for all non-negative integers n, the Jacobi polynomial $\widetilde{P}_n^k$ is a polynomial of degree n such that

$$\int_0^1 \widetilde{P}_i^k(x)\widetilde{P}_j^k(x)x^k dx = \delta_{i,j} \tag{108}$$

for all non-negative integers i, j where $k > -1$.
The following lemma, which follows immediately from the combination of Lemma 2.6 and (107), provides a differential equation satisfied by $\widetilde{P}_n^k$.

Lemma 10.1. *Let $k > -1$ be a real number and let n be a non-negative integer. Then, $\widetilde{P}_n^k$ satisfies the differential equation,*

$$r - r^2\widetilde{P}_n^{k\prime\prime}(r) + (k - rk + 1 - 2r)\widetilde{P}_n^{k\prime}(r) + n(n+k+1)\widetilde{P}_n^k(r) = 0. \tag{109}$$

for all $r \in (0,1)$.

The following recurrence for $\widetilde{P}_n^k$ follows readily from the combination of Lemma 107 and (11).

Lemma 10.2. *For all non-negative integers n and for all real numbers $k > -1$,*

$$\begin{aligned}\widetilde{P}^k_{n+1}(r) = & \frac{(2n+N+1)N^2 + (2n+N)(2n+N+1)(2n+N+2)(1-2r)}{2(n+1)(n+N+1)(2n+N)} \\ & \cdot \frac{\sqrt{2n+k+1}}{\sqrt{2(n+1)+k+1}} \widetilde{P}^k_n(r) \\ & - \frac{2(n+N)(n)(2n+N+2)}{2(n+1)(n+N+1)(2n+N)} \frac{\sqrt{2(n-1)+k+1}}{\sqrt{2(n+1)+k+1}} \widetilde{P}^k_{n-1}(r)\end{aligned} \tag{110}$$

10.1 Numerical Evaluation of the Shifted Jacobi Polynomials

The following observations provide a way to evaluate $\widetilde{P}^k_n$ and its derivatives.

Observation 10.1. Combining (11) with (107), we observe that $\widetilde{P}^k_n(x)$ can be evaluated by first evaluating $P^{(k,0)}_n(1-2x)$ via recurrence relation (11) and then multiplying the resulting number by

$$\sqrt{k+2n+1}. \tag{111}$$

Observation 10.2. Combining (14) with (107), we observe that the polynomial $\widetilde{P}^{k\prime}_n(x)$ (see (107)), can be evaluated by first evaluating $P^{(k,0)\prime}_n(1-2x)$ via recurrence relation (14) and then multiplying the resulting number by

$$-2\sqrt{k+2n+1}. \tag{112}$$

10.2 Prüfer Transform

In this section, we describe the Prüfer Transform, which will be used in Section 10.3. A more detailed description of the Prüfer Transform can be found in [7].

Lemma 10.3 (Prüfer Transform)**.** *Suppose that the function $\phi : [a,b] \to \mathbb{R}$ satisfies the differential equation*

$$\phi''(x) + \alpha(x)\phi'(x) + \beta(x)\phi(x) = 0, \tag{113}$$

where $\alpha, \beta : (a,b) \to \mathbb{R}$ are differential functions. Then,

$$\frac{d\theta}{dx} = -\sqrt{\beta(x)} - \left(\frac{\beta'(x)}{4\beta(x)} + \frac{\alpha(x)}{2}\right) sin(2\theta), \tag{114}$$

where the function $\theta : [a,b] \to \mathbb{R}$ is defined by the formula,

$$\frac{\phi'(x)}{\phi(x)} = \sqrt{\beta(x)} \tan(\theta(x)). \tag{115}$$

Proof. Introducing the notation

$$z(x) = \frac{\phi'(x)}{\phi(x)} \tag{116}$$

for all $x \in [a, b]$, and differentiating (116) with respect to x, we obtain the identity

$$\frac{\phi''}{\phi} = \frac{dz}{dx} + z^2(x). \tag{117}$$

Substituting (117) and (116) into (113), we obtain,

$$\frac{dz}{dx} = -(z^2(x) + \alpha(x)z(x) + \beta(x)). \tag{118}$$

Introducing the notation,

$$z(x) = \gamma(x)\tan(\theta(x)), \tag{119}$$

with θ, γ two unknown functions, we differentiate (119) and observe that,

$$\frac{dz}{dx} = \gamma(x)\frac{\theta'}{\cos^2(\theta)} + \gamma'(x)\tan(\theta(x)) \tag{120}$$

and squaring both sides of (119), we obtain

$$z(x)^2 = \tan^2(\theta(x))\gamma(x)^2. \tag{121}$$

Substituting (120) and (121) into (118) and choosing

$$\gamma(x) = \sqrt{\beta(x)} \tag{122}$$

we obtain

$$\frac{d\theta}{dx} = -\sqrt{\beta(x)} - \left(\frac{\beta'(x)}{4\beta(x)} + \frac{\alpha(x)}{2}\right) sin(2\theta). \tag{123}$$

■

Remark 10.3. The Prüfer Transform is often used in algorithms for finding the roots of oscillatory special functions. Suppose that $\phi : [a, b] \to \mathbb{R}$ is a special function satisfying differential equation (113). It turns out that in most cases, coefficient

$$\beta(x) \tag{124}$$

in (113) is significantly larger than

$$\frac{\beta'(x)}{4\beta(x)} + \frac{\alpha(x)}{2} \tag{125}$$

on the interval $[a, b]$, where α and β are defined in (113).

Under these conditions, the function θ (see (115)), is monotone and its derivative neither approaches infinity nor 0. Furthermore, finding the roots of ϕ is equivalent to finding $x \in [a, b]$ such that

$$\theta(x) = \pi/2 + k\pi \tag{126}$$

for some integer k. Consequently, we can find the roots of ϕ by solving well-behaved differential equation (123).

Remark 10.4. If for all $x \in [a, b]$, the function $\sqrt{\beta(x)}$ satisfies

$$\sqrt{\beta(x)} > \frac{\beta'(x)}{4\beta(x)} + \frac{\alpha(x)}{2}, \tag{127}$$

then, for all $x \in [a, b]$, we have $\frac{d\theta}{dx} < 0$ (see (114)) and we can view $x : [-\pi, \pi] \to \mathbb{R}$ as a function of θ where x satisfies the first order differential equation

$$\frac{dx}{d\theta} = \left(-\sqrt{\beta(x)} - \left(\frac{\beta'(x)}{4\beta(x)} + \frac{\alpha(x)}{2} \right) sin(2\theta) \right)^{-1}. \tag{128}$$

10.3 Roots of the Shifted Jacobi Polynomials

The primary purpose of this section is to describe an algorithm for finding the roots of the Jacobi polynomials $\widetilde{P}_n^k$. These roots will be used in Section 4 for the design of quadratures for Zernike Polynomials.
The following lemma follows immediately from applying the Prufer Transform (see Lemma 10.3) to (109).

Lemma 10.4. *For all non-negative integers n, real $k > -1$, and $r \in (0, 1)$,*

$$\frac{d\theta}{dr} = -\left(\frac{n(n+k+1)}{r - r^2} \right)^{1/2} - \left(\frac{1 - 2r + 2k - 2kr}{4(r - r^2)} \right) \sin(2\theta(r)). \tag{129}$$

where the function $\theta : (0, 1) \to \mathbb{R}$ is defined by the formula

$$\frac{\widetilde{P}_n^k(r)}{\widetilde{P}_n^{k\prime}(r)} = \left(\frac{n(n+k+1)}{r - r^2} \right)^{1/2} \tan(\theta(r)), \tag{130}$$

where $\widetilde{P}_n^k$ is defined in (108).

Remark 10.5. For any non-negative integer n,

$$\frac{d\theta}{dr} < 0 \tag{131}$$

for all $r \in (0, 1)$. Therefore, applying Remark 10.4 to (129), we can view r as a function of θ where r satisfies the differential equation

$$\frac{dr}{d\theta} = \left(-\left(\frac{n(n+k+1)}{r - r^2} \right)^{1/2} - \left(\frac{1 - 2r + 2k - 2kr}{4(r - r^2)} \right) \sin(2\theta(r)) \right)^{-1}. \tag{132}$$

Algorithm

In this section, we describe an algorithm for the evaluation of the n roots of $\widetilde{P}_n^k$. We denote the n roots of $\widetilde{P}_n^k$ by $r_1 < r_2 < ... < r_n$.

Step 1. Choose a point, $x_0 \in (0,1)$, that is greater than the largest root of $\widetilde{P}_n^k$. For example, for all $k \geq 1$, the following choice of x_0 will be sufficient:

$$x_0 = \begin{cases} 1 - 10^{-6} & \text{if } n < 10^3, \\ 1 - 10^{-8} & \text{if } 10^3 \leq n < 10^4, \\ 1 - 10^{-10} & \text{if } 10^4 \leq n < 10^5. \end{cases} \tag{133}$$

Step 2. Defining θ_0 by the formula

$$\theta_0 = \theta(x_0), \tag{134}$$

where θ is defined in (130), solve the ordinary differential equation $\frac{dr}{d\theta}$ (see (132)) on the interval $(\pi/2, \theta_0)$, with the initial condition $r(\theta_0) = x_0$. To solve the differential equation, it is sufficient to use, for example, second order Runge Kutta with 100 steps (independent of n). We denote by $\tilde{r}_n$ the approximation to $r(\pi/2)$ obtained by this process. It follows immediately from (126) that $\tilde{r}_n$ is an approximation to r_n, the largest root of $\widetilde{P}_n^k$.

Step 3. Use Newton's method with $\tilde{r}_n$ as an initial guess to find r_n to high precision. The polynomials $\widetilde{P}_n^k$ and $\widetilde{P}_n^{k\prime}$ can be evaluated via Observation 10.1 and Observation 10.2.

Step 4. With initial condition

$$x(\pi/2) = r_n, \tag{135}$$

solve differential equation $\frac{dr}{d\theta}$ (see (132)) on the interval

$$(-\pi/2, \pi/2) \tag{136}$$

using, for example, second order Runge Kuta with 100 steps. We denote by $\tilde{r}_{n-1}$ the approximation to

$$r(-\pi/2) \tag{137}$$

obtained by this process.

Step 5. Use Newton's method, with initial guess $\tilde{r}_{n-1}$, to find to high precision the second largest root, r_{n-1}.

Step 6. For $k = \{1, 2, ..., n-1\}$, repeat Step 4 on the interval

$$(-\pi/2 - k\pi, -\pi/2 - (k-1)\pi) \tag{138}$$

with intial condition

$$x(-\pi/2 - (k-1)\pi) = r_{n-k+1} \tag{139}$$

and repeat Step 5 on $\tilde{r}_{n-k}$.

11 Appendix E: Notational Conventions for Zernike Polynomials

In two dimensions, the Zernike polynomials are usually indexed by their azimuthal order and radial order. In this report, we use a slightly different indexing scheme, which leads to simpler formulas and generalizes easily to higher dimensions (see Section 2.3 for our definition of the Zernike polynomials $Z^{\ell}_{N,n}$ and the radial polynomials $R_{N,n}$). However, for the sake of completeness, we describe in this section the standard two dimensional indexing scheme, as well as other widely used notational conventions.

If $|m|$ denotes the azimuthal order and n the radial order, then the Zernike polynomials in standard two index notation (using asterisks to differentiate them from the polynomials $Z^{\ell}_{N,n}$ and $R_{N,n}$) are

$$\overset{*}{Z}{}^m_n(\rho,\theta) = \overset{*}{R}{}^{|m|}_n(\rho) \cdot \begin{cases} \sin(|m|\theta) & \text{if } m<0, \\ \cos(|m|\theta) & \text{if } m>0, \\ 1 & \text{if } m=0, \end{cases} \tag{140}$$

where

$$\overset{*}{R}{}^{|m|}_n(\rho) = \sum_{k=0}^{\frac{n-|m|}{2}} \frac{(-1)^k (n-k)!}{k!\left(\frac{n+|m|}{2}-k\right)!\left(\frac{n-|m|}{2}-k\right)!} \rho^{n-2k}, \tag{141}$$

for all $m = 0, \pm 1, \pm 2, \dots$ and $n = |m|, |m|+2, |m|+4, \dots$ (see Figure 140); they are normalized so that

$$\overset{*}{R}{}^{|m|}_n(1) = 1, \tag{142}$$

for all $m = 0, \pm 1, \pm 2, \dots$ and $n = |m|, |m|+2, |m|+4, \dots$. We note that

$$\overset{*}{R}{}^{|m|}_n(\rho) = R_{|m|,\frac{n-|m|}{2}}(\rho), \tag{143}$$

for all $m = 0, \pm 1, \pm 2, \dots$ and $n = |m|, |m|+2, |m|+4, \dots$, where R is defined by (26) (see Figure 6); equivalently,

$$R_{N,n}(\rho) = \overset{*}{R}{}^{N}_{N+2n}(\rho), \tag{144}$$

for all nonnegative integers N and n.

Remark 11.1. The quantity $n+|m|$ is sometimes referred to as the "spacial frequency" of the Zernike polynomial $\overset{*}{Z}{}^m_n(\rho,\theta)$. It roughly corresponds to the frequency of the polynomial on the disc, as opposed to the azimuthal frequency $|m|$ or the order of the polynomial n.

11.1 Zernike Fringe Polynomials

The Zernike Fringe Polynomials are the standard Zernike polynomials, normalized to have L^2 norm equal to π on the unit disc and ordered by their spacial frequency $n + |m|$ (see Table 6 and Figure 7). This ordering is sometimes called the "Air Force" or "University of Arizona" ordering.

11.2 ANSI Standard Zernike Polynomials

The ANSI Standard Zernike polynomials, also referred to as OSA Standard Zernike polynomials or Noll Zernike polynomials, are the standard Zernike polynomials, normalized to have L^2 norm π on the unit disc and ordered by n (the order of the polynomial on the disc; see Table 7 and Figure 8).

11.3 Wyant and Creath Notation

In [14], James Wyant and Katherine Creath observe that it is sometimes convienient to factor the radial polynomial $\overset{*}{R}{}^{|m|}_{2n-|m|}$ into

$$\overset{*}{R}{}^{|m|}_{2n-|m|}(\rho) = Q^{|m|}_n(\rho)\rho^{|m|}, \tag{145}$$

for all $m = 0, \pm1, \pm2, \ldots$ and $n = |m|, |m|+1, |m|+2, \ldots$, where the polynomial $Q^{|m|}_n$ is of order $2(n - |m|)$ (see Figure 4). Equivalently, the factorization can be written as

$$\overset{*}{R}{}^{|m|}_{n}(\rho) = Q^{|m|}_{\frac{n+|m|}{2}}(\rho)\rho^{|m|}, \tag{146}$$

for all $m = 0, \pm1, \pm2, \ldots$ and $n = |m|, |m|+2, |m|+4, \ldots$.

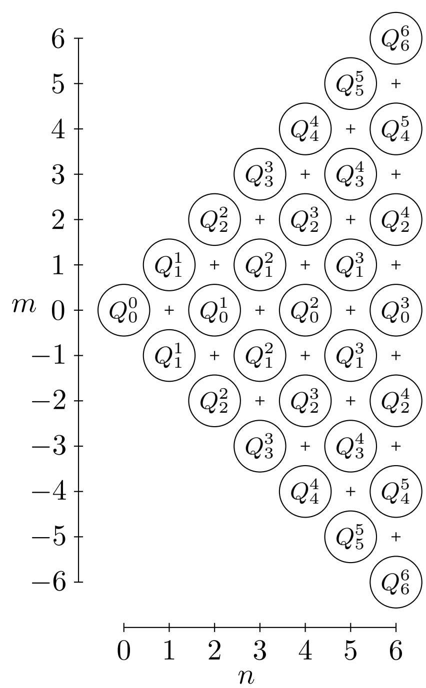

Figure 4: The Wyant and Creath polynomials $Q^{|m|}_{(n+|m|)/2}$

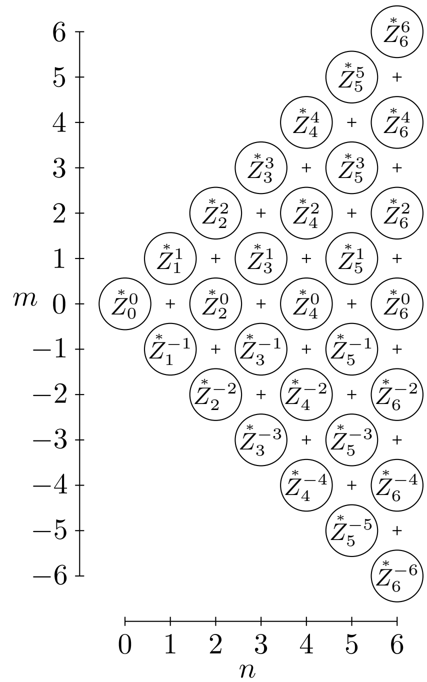

Figure 5: Zernike polynomials $\overset{*}{Z}{}_n^m$ in standard double index notation

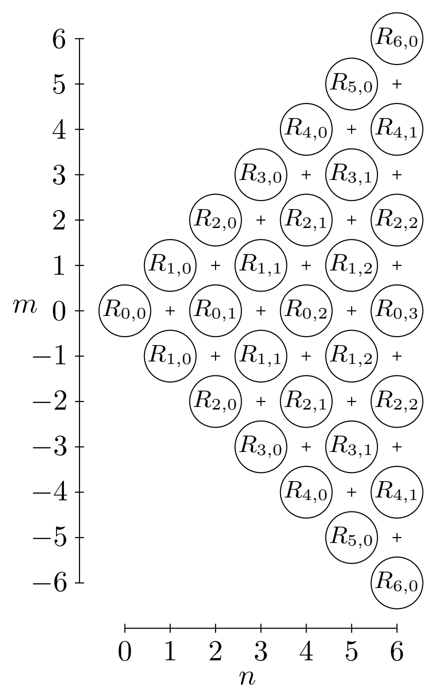

Figure 6: The polynomials $R_{|m|,(n-|m|)/2}$

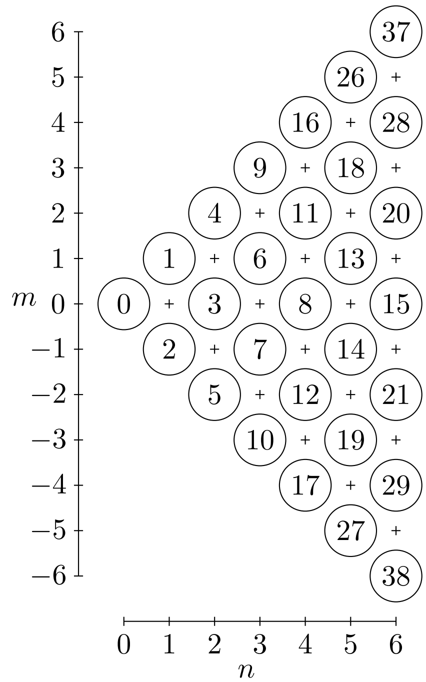

Figure 7: Fringe Zernike Polynomial Ordering

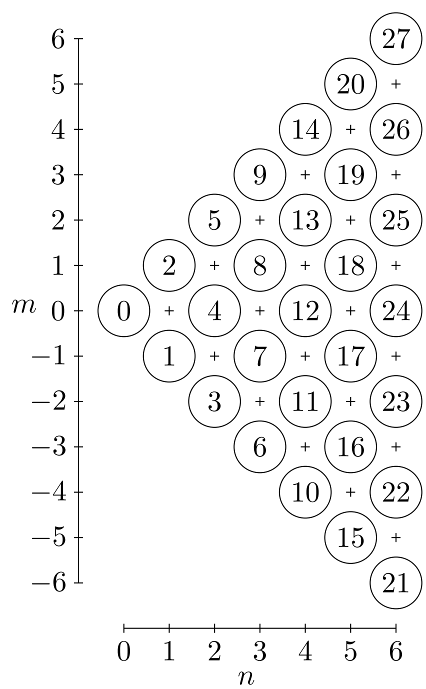

Figure 8: ANSI Standard Zernike Polynomial Ordering

index	n	m	spacial frequency	polynomial[◇]	name[†]
0	0	0	0	1	piston
1	1	1	2	$\overline{R}_{1,0}(\rho)\cos(\theta)$	tilt in x-direction
2	1	-1	2	$\overline{R}_{1,0}(\rho)\sin(\theta)$	tilt in y-direction
3	2	0	2	$\overline{R}_{0,1}(\rho)$	defocus (power)
4	2	2	4	$\overline{R}_{2,0}(\rho)\cos(2\theta)$	defocus + astigmatism 45°/135°
5	2	-2	4	$\overline{R}_{2,0}(\rho)\sin(2\theta)$	defocus + astigmatism 90°/180°
6	3	1	4	$\overline{R}_{1,1}(\rho)\cos(\theta)$	tilt + horiz. coma along x-axis
7	3	-1	4	$\overline{R}_{1,1}(\rho)\sin(\theta)$	tilt + vert. coma along y-axis
8	4	0	4	$\overline{R}_{0,2}(\rho)$	defocus + spherical aberration
9	3	3	6	$\overline{R}_{3,0}(\rho)\cos(3\theta)$	trefoil in x-direction
10	3	-3	6	$\overline{R}_{3,0}(\rho)\sin(3\theta)$	trefoil in y-direction
11	4	2	6	$\overline{R}_{2,1}(\rho)\cos(2\theta)$	
12	4	-2	6	$\overline{R}_{2,1}(\rho)\sin(2\theta)$	
13	5	1	6	$\overline{R}_{1,2}(\rho)\cos(\theta)$	
14	5	-1	6	$\overline{R}_{1,2}(\rho)\sin(\theta)$	
15	6	0	6	$\overline{R}_{0,3}(\rho)$	
16	4	4	8	$\overline{R}_{4,0}(\rho)\cos(4\theta)$	
17	4	-4	8	$\overline{R}_{4,0}(\rho)\sin(4\theta)$	
18	5	3	8	$\overline{R}_{3,1}(\rho)\cos(3\theta)$	
19	5	-3	8	$\overline{R}_{3,1}(\rho)\sin(3\theta)$	
20	6	2	8	$\overline{R}_{2,2}(\rho)\cos(2\theta)$	
21	6	-2	8	$\overline{R}_{2,2}(\rho)\sin(2\theta)$	
22	7	1	8	$\overline{R}_{1,3}(\rho)\cos(\theta)$	
23	7	-1	8	$\overline{R}_{1,3}(\rho)\sin(\theta)$	
24	8	0	8	$\overline{R}_{0,4}(\rho)$	

Table 6: **Zernike Fringe Polynomials.** This table lists the first 24 Zernike polynomials in what is sometimes called the “Fringe”, “Air Force”, or “University of Arizona” ordering (see, for example [11], p. 198, or [14], p. 31). They are often also denoted by $Z_\ell(\rho, \theta)$, where ℓ is the index.

◇ See formulas (26) and (29).

† See, for example, [8]. More complex aberrations are usually not named.

index	n	m	spacial frequency	polynomial$^\diamond$	name†
0	0	0	0	1	piston
1	1	-1	2	$\overline{R}_{1,0}(\rho)\sin(\theta)$	tilt in y-direction
2	1	1	2	$\overline{R}_{1,0}(\rho)\cos(\theta)$	tilt in x-direction
3	2	-2	4	$\overline{R}_{2,0}(\rho)\sin(2\theta)$	defocus + astigmatism 90°/180°
4	2	0	2	$\overline{R}_{0,1}(\rho)$	defocus (power)
5	2	2	4	$\overline{R}_{2,0}(\rho)\cos(2\theta)$	defocus + astigmatism 45°/135°
6	3	-3	6	$\overline{R}_{3,0}(\rho)\sin(3\theta)$	trefoil in y-direction
7	3	-1	4	$\overline{R}_{1,1}(\rho)\sin(\theta)$	tilt + vert. coma along y-axis
8	3	1	4	$\overline{R}_{1,1}(\rho)\cos(\theta)$	tilt + horiz. coma along x-axis
9	3	3	6	$\overline{R}_{3,0}(\rho)\cos(3\theta)$	trefoil in x-direction
10	4	-4	8	$\overline{R}_{4,0}(\rho)\sin(4\theta)$	
11	4	-2	6	$\overline{R}_{2,1}(\rho)\sin(2\theta)$	
12	4	0	4	$\overline{R}_{0,2}(\rho)$	
13	4	2	6	$\overline{R}_{2,1}(\rho)\cos(2\theta)$	
14	4	4	8	$\overline{R}_{4,0}(\rho)\cos(4\theta)$	
15	5	-5	10	$\overline{R}_{5,0}(\rho)\sin(5\theta)$	
16	5	-3	8	$\overline{R}_{3,1}(\rho)\sin(3\theta)$	
17	5	-1	6	$\overline{R}_{1,2}(\rho)\sin(\theta)$	
18	5	1	6	$\overline{R}_{1,2}(\rho)\cos(\theta)$	
19	5	3	8	$\overline{R}_{3,1}(\rho)\cos(3\theta)$	
20	5	5	10	$\overline{R}_{5,0}(\rho)\cos(5\theta)$	
21	6	-6	12	$\overline{R}_{6,0}(\rho)\sin(6\theta)$	
22	6	-4	10	$\overline{R}_{4,1}(\rho)\sin(4\theta)$	
23	6	-2	8	$\overline{R}_{2,2}(\rho)\sin(2\theta)$	
24	6	0	6	$\overline{R}_{0,3}(\rho)$	

Table 7: **ANSI Standard Zernike Polynomials.** This table lists the first 24 Zernike polynomials in the ANSI Standard ordering, also referred to as the “OSA Standard” or “Noll” ordering (see, for example [11], p. 201, or [10]). They are often also denoted by $Z_\ell(\rho, \theta)$, where ℓ is the index.

$\diamond$ See formulas (26) and (29).

† See, for example, [8]. More complex aberrations are usually not named.

References

[1] Abramowitz, Milton, and Irene A. Stegun, eds. *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables.* Washington: U.S. Govt. Print. Off., 1964.

[2] Alpert, Bradley K. and Vladimir Rokhlin. "A fast algorithm for the evaluation of Legendre expansions." *SIAM J. Sci. Stat. Comput.* 12.1 (1991): 158–179.

[3] Bateman, Harry. *Higher Transcendental Functions, Vol. 2.* Malabar, Florida: Robert E. Krieger Publishing Company, Inc., 1985.

[4] Born, Max, and Emil Wolf. *Principles of Optics.* 6th ed. (with corrections). New York: Pergamon Press Inc., 1980.

[5] Boyd, John P. and Fu Yu. "Comparing seven spectral methods for interpolation and for solving the Poisson equation in a disk: Zernike Polynomials, Logan-Shepp ridge polynomials, Chebyshev-Fourier Series, cylindrical Robert functions, Bessel-Fourier expansions, square-to-disk conformal mapping and radial basis functions" *J. Comput. Phys.* 230.4 (2011): 1408–1438.

[6] Chong, Chee-Way, P. Raveendran, and R. Mukundan. "A comparative analysis of algorithms for fast computation of Zernike moments." Pattern Recognition 36 (2003): 731–742.

[7] Glaser, Andreas, Xiangtao Liu, and Vladimir Rokhlin. "A Fast Algorithm for the Calculation of the Roots of Special Functions."*SIAM J. Sci. Comput.* 29.4 (2007): 1420–1438.

[8] Jagerman, Louis S. *Ophthalmologists, meet Zernike and Fourier!.* Victoria, BC, Canada: Trafford Publishing, 2007.

[9] Kintner, Eric C. "On the mathematical properties of the Zernike polynomials." *Optica Acta* 23.8 (1976): 679–680.

[10] *Ophthalmics - Corneal Topography Systems - Standard Terminology, Requirements.* ANSI Standard Z80.23-2008 (R2013).

[11] Radiant ZEMAX LLC. *ZEMAX: Optical Design Program User's Manual.* Redmond, WA: Author, 2011.

[12] Slepian, David. "Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty - IV: Extensions to Many Dimensions; Generalized Prolate Spheroidal Functions." *Bell Labs Technical Journal* 43.6 (1964): 3009-3057.

[13] Stoer, Josef and Roland Bulirsch. *Introduction to Numerical Analysis*, 2nd ed., Springer-Verlag, 1992.

[14] Wyant, James C., and Katherine Creath. "Basic wavefront aberration theory for optical metrology." Applied optics and optical engineering II (1992): 28–39.