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ERROR BOUNDS FOR BIVARIATE CUBIC INTERPOLATION

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In this note we improve the asymptotic aspect of the error bounds recently given by C. A. Hall in [4] for an interpolation scheme using piecewise bivariate cubic polynomials which was suggested by G. Birkhoff. We make essential use of new "Peano kernel" type result of J. H. Bramble and S. R. Hilbert, c.f. [3]. The results of this note are useful in giving sharp a priori error bounds for the Rayleigh-Ritz-Galerkin method for approximating the solution of boundary value problems for elliptic partial differential equations. Throughout this note, K will denote a positive constant not necessarily the same at each occurrence.

Let R be any right triangular polygon in the x - y plane, i.e., R is the union of right triangles, $R \equiv \bigcup_{s=1}^{k} T_s$, such that $T_s \cap T_m$, $1 \leq s, \ m \leq k$, is either void or a side of T_s and a side of T_m. We are interested in interpolating smooth real-valued functions on R by means of continuous, piecewise bivariate cubic polynomials, $p(x,y)$, i.e.,

$$p(x,y) \in S_R \equiv \left\{ \begin{array}{l} \{p(x,y) \,|\, \text{for each } 1 \leq s \leq k, \text{ there exist real constants} \\ a^s_{ij} \text{ such that } p(x,y) = \sum\limits_{0 \leq i+j \leq 3} a^s_{ij} x^i y^j \text{ for all} \\ (x,y) \in T_s, \text{ and } p(x,y) \in C^0(R)\}. \end{array} \right.$$

Moreover, if the original function vanishes on the boundary of R, we may want to interpolate to belong to $S_R^0 \equiv \{p \in S_R | \; p(x,y) = 0$ for all (x,y) in the boundary of $R\}$.

We start by considering a single right triangle., Δ, with vertices at (0,0), (a,0), and (0,b) and $P \equiv S_\Delta$, the set of all bivariate cubic polynomials. Clearly the dimension of P as a vector space is 10.

We define an interpolation mapping I_Δ from $C^2(\Delta)$ to P by

(1) $(D^{i,j} I_\Delta f)(0,0) \equiv \left(\frac{\partial^{i+j}}{\partial x^i \partial x^j} I_\Delta f\right)(0,0) = D^{i,j} f(0,0),$

$0 \leq i, j \leq 1,$

(2) $(D^{i,j} I_\Delta f)(0,b) = D^{i,j} f(0,b), \quad 0 \leq i+j \leq 1,$

and

(3) $(D^{i,j} I_\Delta f)(a,0) = D^{i,j} f(a,0), \; 0 \leq i+j \leq 1,$

for all $f \in C^2(\Delta)$.

We have

Theorem 1. The interpolation mapping I_Δ is well-defined, i.e., $I_\Delta f$ exists and is unique for all $f \in C^2(\Delta)$.

Corollary. $I_\Delta p = p$ for all $p \in P$.

Now we define a mapping I of $C^2(R)$ into S_R as follows: if

$f(x,y) \in C^2(R)$, $If(x,y) \equiv s(x,y)$,

where

$$(4) \qquad s(x,y) \equiv I_{T_i}(f_i)(x,y) \quad \text{for all} \quad (x,y) \in T_i, \quad 1 \le i \le k,$$

and f_i denotes the restriction of f to T_i. As corollaries of Theorem 1, we have

Theorem 2. I is well-defined from $C^2(R)$ to S_R and $I(s) = s$ for all $s \in S_R$.

Proof. Clearly the restriction of $I(f)(x,y)$ to T_i is in P for all $1 \le i \le k$. The continuity of $I(f)$ follows from the proof of Theorem 1. Q.E.D.

Theorem 3. I is well-defined from

$$C_0^2(R) \equiv \{f \in C^2(R) \mid f(x,y) = 0 \quad \text{for all} \quad (x,y) \in \partial R\}$$

to S_R^0 and $I(s) = s$ for all $s \in S_R^0$.

After introducing some additional terminology, we discuss error bounds for the preceding interpolation scheme. If j is a non-negative integer and $1 \leq p \leq \infty$, we define the Sobolev norm

$$||f||_{W^{j,p}(R)} \equiv \left\{ \sum_{0 \leq k+\ell \leq j} \int_R |D^{k,\ell} f(x,y)|^p \, dxdy \right\}^{1/p}$$

for all $f \in C^\infty(R)$.

Moreover, we let $W^{j,p}(R)$ denote the completion of $C^\infty(R)$ with respect to $|| \cdot ||_{W^{j,p}(R)}$ and $W_0^{j,p}(R)$ denote the completion of $C_0^\infty(R)$ with respect to $|| \cdot ||_{W^{j,p}(R)}$.

A collection, C, of right triangular polygons, R, is said to be <u>regular</u> if and only if there exists and $\varepsilon > 0$ such that

$\varepsilon \leq \inf_{R \in C} \inf_{1 \leq i \leq k_R} h_i/H_i$, where H_i and h_i denote the lengths of longest and shortest sides of the triangle T_i, $1 \leq i \leq k_R$.

We shall write $H_R \equiv \max_{1 \leq i \leq k_R} H_i$.

Theorem 4. Let $\mathcal{C}$ be a regular collection of right triangular polygons. If $f \in W^{4,p}(R)$, (resp. $W_0^{4,p}(R)$), for $R \in \mathcal{C}$, where $p > 1$, then $\mathcal{I}f \in S_R$, (resp. S_R^0), is well-defined and there exists a positive constant, K, such that for $j = 0,1$ and all $R \in \mathcal{C}$

$$(5) \qquad \|f - \mathcal{I}f\|_{W^{j,q}(R)} \le K(H_R)^{4-j}\left(\sum_{m+j=4} \|D^{m,j} f\|^p_{W^{0,p}(R)}\right)^{1/p}$$

for all $q \le p$, and

$$(6) \qquad \|f - \mathcal{I}f\|_{W^{j,q}(R)} \le K(H_R)^{4-j-(2/p)+(2/q)}\left(\sum_{m+j=4} \|D^{m,j} f\|^p_{W^{0,p}(R)}\right)^{1/p},$$

for all $q \ge p$.

Proof. We consider only the case of $j = 0$, since the proof for the case of $j = 1$ is essentially identical. By the Sobolev Imbedding Theorem, $F \in C^2(R)$ and hence the interpolation mapping $\mathcal{I}$ is well-defined. Let Δ denote the standard right triangle with vertices at $(0,0)$, $(1,0)$, and $(0,1)$.

Clearly, there exists a positive constant, K, such that

$$|f(x,y) - I_\Delta f(x,y)| \leq K \sup_{(x,y)\varepsilon\Delta} \sum_{0\leq m,j\leq 1} |D^{m,j} f(x,y)|$$

for all $(x,y)\ \varepsilon\ \Delta$ and all $f\ \varepsilon\ C^2(\Delta)$. Moreover, since $I_\Delta p = p$ for all $p\ \varepsilon\ P$, we may apply a Peano Kernal type result, Corollary to Theorem 2 of [3], which states that if (I-F) is a linear functional on $C^t(\Delta)$ such that there exists a positive constant C such that

$$|(I-F)(u)| \leq C \sup_{(x,y)\ \varepsilon\ \Delta} \sum_{m+j\leq t} |D^{m,j}\ u(x,y)|$$

and $(I-F)(p) = 0$ for all polynomials, $p(x,y)$, of degree $k > t \geq 0$, then for $p > 2/(R-t)$ there exists a positive constant K such that

$$|F(u)| \leq K \sum_{m+j=k} |D^{m,j}\ u|_{W^{0,p}(\Delta)},$$

and conclude that there exists a positive constant, again denoted by K, such that for all $p > 1$

$$|f(x,y) - I_\Delta f(x,y)| \le K \sum_{m+j=4} ||D^{m,j} f||_{W^{0,p}(\Delta)} .$$

By a standard argument, involving a change of the independent variables, cf. [1] and [3], we have, using the regularity of C,

$$(7) \quad |f(x,y) - If(x,y)| \le K(H_i)^{4-(2/p)} \sum_{m+j=4} ||D^{m,j}f||_{W^{0,p}(T_i)} ,$$

for all $(x,y) \in T_i$, $f \in W^{4,p}(R)$, $1 \le i \le k_R$, and all $R \in C$.

To prove (5) we note that by inequality (7), if $q \le p$,

$$||f - If||^q_{W^{0,q}(R)} = \sum_{i=1}^{k_R} ||f - If||^q_{W^{0,q}(T_i)} \le \sum_{i=1}^{k_R} ||f - If||^p_{W^{0,p}(T_i)} \le$$

$$(1/2)H_R^2 \sum_{i=1}^{k_R} ||f - If||^p_{W^{0,\infty}(T_i)} \le K(H_R)^{4p} \sum_{i=1}^{k_R} \left[\sum_{m+j=4} ||D^{m,j}f||_{W^{0,p}(T_i)}\right]^p \le$$

$$K(H_R)^{4p} \sum_{i=1}^{k_R} \sum_{m+j=4} ||D^{m,j}f||^p_{W^{0,p}(T_i)} = K(H_R)^{4p} \sum_{m+j=4} ||D^{m,j}f||^p_{W^{0,p}(R)} ,$$

where we have used Jensen's inequality to obtain the last inequality.

To prove (6), write

$$v_i \equiv ||f - If||_{W^{0,q}(T_i)} \quad \text{and} \quad w_i \equiv \sum_{m+j=4} ||D^{m,j}f||_{W^{0,p}(T_i)}$$

for all $1 \le i \le k_R$. By (7), $v_i \le K(H_i)^{4-(2/p)+(2/q)} w_i$, $1 \le i \le k_R$.

Hence by Jensen's and Hölder's inequalities, we have

$$||f - If||_{W^{0,q}(R)} = \left[\sum_{i=1}^{k_R} v_i^q \right]^{1/q} \leq K(H_R)^{4-(2/p)+(2/q)} \left[\sum_{i=1}^{k_R} w_i^q \right]^{1/q}$$

$$\leq K(H_R)^{4-(2/p)+(2/q)} \left[\sum_{i=1}^{k_R} w_i^p \right]^{1/p}$$

$$\leq K(H_R)^{4-(2/p)+(2/q)} \left[\sum_{m+j=4} ||D^{m,j}f||^p_{W^{0,p}(R)} \right]^{1/p} \qquad \text{QED.}$$

By making minor changes in the proof of Theorem 3, it is possible to obtain the following result.

<u>Theorem 5</u>. Let C be a regular collection of right triangular polygons. If $f \in W^{3,p}(R)$, (resp. $W_0^{3,p}(R)$), for all $R \in C$, where $p > 2$, then $If \in S_R$, (resp. S_R^0), is well-defined and there exists a positive constant, K, such that for $j = 0, 1$ and all $R \in C$

$$(8) \qquad ||f - If||_{W^{j,q}(R)} \leq K(H_R)^{3-j} \left[\sum_{m+j=3} ||D^{m,j}f||^p_{W^{0,p}(R)} \right]^{1/p},$$

for all $q \leq p$, and

$$(9)\quad ||f - If||_{W^{j,q}(R)} \leq K(H_R)^{3-j-(2/p)+(2/q)} \left(\sum_{m+j=3} ||D^{m,j}f||^p_{W^{0,p}(R)} \right)^{1/p},$$

for all $q \geq p$.

We now turn to the application of Theorems 4 and 5 to obtaining error bounds for the Rayleigh-Ritz-Galerkin method for approximating the solutions of elliptic partial differential equations. In particular, we let $\overline{\Omega}$ be a closed convex polygon in the plane, $\Omega \equiv \overline{\Omega} - \partial\overline{\Omega}$, and consider the problem of approximating the solution of

$$(10)\quad -D^{1,0}(p(x,y)D^{1,0}u) - D^{0,1}(q(x,y)D^{0,1}u) + r(x,y)u = f(x,y),$$

for all $(x,y) \in \Omega$,

$$(11)\qquad u(x,y) = 0, \text{ for all } (x,y) \in \partial\overline{\Omega},$$

where $p(x,y)$ and $q(x,y)$ are positive, real-valued $C^1(\overline{\Omega})$-functions, $r(x,y)$ is a nonnegative, real-valued, $C(\overline{\Omega})$ function, and $f(x,y)$ is a real-valued function in $W^{0,2}(\Omega)$, by the Rayleigh-Ritz-Galerkin method. That is, if S is a finite dimensional subspace os $W_0^{1,2}(\Omega)$, we must determine $u_S \in S$ such that

$$(12)\quad \int_\Omega p(x,y)D^{1,0}u_S D^{1,0}\phi \, dx\, dy + \int_\Omega q(x,y)D^{0,1}u_S D^{0,1}\phi \, dx\, dy$$

$$+ \int_\Omega r(x,y)u_S\phi \, dx\, dy = \int_\Omega f(x,y)\phi \, dx\, dy, \quad \text{for all } \phi \in S.$$

Using the results of [2] and [5] and Theorems 4 and 5, we may establish the following error bound for the Rayleigh-Ritz-Galerkin method. The reader is referred to [5] for the precise details of the proof.

Theorem 6. Let C be a regular collection of right triangular polygonal partitions, R, of $\bar{\Omega}$ and for each $R \in C$, let S_R^0 denote the finite dimensional space of piecewise, bivariate cubic polynomials with respect to R which vanish on the boundary of Ω. Under the above hypotheses, problem (10)-(11) has a unique solution, u, $u \in W^{2,2}(\Omega)$, and if u_R denotes the Rayleigh-Ritz-Galerkin approximation in S_R^0 then there exists a positive constant, K, such that

$$(13) \quad ||u - u_R||_{W^{j,2}(\Omega)} \leq K \, H_R^{p-j} \, ||u||_{W^{p,2}(\Omega)}, \quad 0 \leq j \leq 1,$$

for all $R \in C$ and all $u \in W^{p,2}(\Omega)$, where $2 \leq p \leq 4$.'

We remark that the exponent of H in (13) is "best possible" for the class of solutions under consideration.

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