Let A be an n by n nearly singular matrix with $\text{Rank}(A) \geq n-1$ and singular values $d_1 \geq \cdots \geq d_{n-1} > d_n$, where d_n can be small or zero. Consider the rank-1 modification of A:

$$\hat{A} = A + \alpha z w^T,$$

with $\|z\| = \|w\| = 1$. We give lower and upper bounds for the condition number of $\hat{A}$ in terms of $|\alpha|$ and the angles between z and w and the singular vectors of A corresponding to d_n.

On the Condition of Nearly Singular Matrices Under Rank-1 Perturbations

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1. Introduction

Let A be a $n \times n$ nearly singular matrix with Rank$(A) \geq n-1$ and singular value decomposition (SVD) given by

$$A = UDV^T = U \begin{pmatrix} d_1 & & & \\ & \ddots & & \\ & & d_{n-1} & \\ & & & d_n \end{pmatrix} V^T, \tag{1.1}$$

where $d_1 \geq \ldots \geq d_{n-1} > d_n \geq 0$ and d_n can be zero or very small. Let ϕ be the last column of V, and ψ the last column of U, so

$$A\phi = d_n \psi$$

and

$$A^T \psi = d_n \phi.$$

We shall use only the 2-norm in this paper.

Note that the condition number of A is

$$K(A) = \frac{d_1}{d_n},$$

which can be very large if d_n is small. Since the null space of A is at most one-dimensional, it is possible to remove the singularity by an appropriately chosen rank-1 modification. Consider the following modification to A:

$$\hat{A} = A + \alpha z w^T \tag{1.2},$$

with $\|z\| = \|w\| = 1$. Golub [4] and Bunch et.al. [1] studied the problem of updating the eigenvalues of $\hat{A}$ when A and $\hat{A}$ are symmetric. Thompson [6] gives the following interlocking property of the singular values of A and $\hat{A}$:

Theorem 1.1. (Thompson) *Let $c_1 \geq \cdots \geq c_n$ be the singular values of $\hat{A}$. Then,*

$$d_{i+1} \leq c_i \leq d_{i-1} \qquad i = 1, \cdots, n \tag{1.3}$$

under the convention $d_0 = +\infty, d_{n+1} = 0$. Conversely, given $c_1 \geq \cdots \geq c_n$ satisfying (1.3), there exists a rank-one modification to A with singular values $c_1, \cdots, c_n$.

Since the above theorem does not give any information about the extreme singular values of $\hat{A}$, we cannot infer any conclusion on the conditioning of the perturbed matrix. In fact, one can conclude from the results given in section 2, that the condition number of $\hat{A}$ can take any real value between $\frac{d_2}{d_{n-1}}$ and infinity. Ideally, one would like $K(\hat{A})$ to be as close to $\frac{d_2}{d_{n-1}}$ as possible. This, however, may be difficult to achieve because the optimal choice of α, z and w depends on a knowledge of the SVD of A. If the singular vectors ϕ and ψ are known (e.g. obtained by inverse iteration [5, 2]), then, by choosing $\alpha = d_{n-1} - d_n, z = \psi$ and $w = \phi$ we get $K(\hat{A}) = \frac{d_1}{d_{n-1}}$, which is close enough to the optimal bound, since we are assuming that A is at most rank-one deficient. But even this is not always possible in practice because ψ and ϕ may not be known explicitly and also one may not be free to choose z and w. In section 3, we give upper and lower bounds for the condition number of $\hat{A}$ in terms of $|\alpha|$ and the angle between ψ and z, and the angle between ϕ and w. Some examples of the use of these bounds are presented in section 4.

2. A priori Lower Bound

We shall show that $\frac{d_2}{d_{n-1}}$ is the optimal condition number that can be achieved with a rank-1 modification to A. We also give a particular rank-1 modification which gives the nearly optimal value $\frac{d_1}{d_{n-1}}$. Thompson's theorem will be used heavily in this and the next section.

Lemma 2.1. *Let A be an $n \times n$ matrix with SVD given by (1.1), and $\hat{A}$ defined as in (1.2). Then*

a) $K(\hat{A}) \geq \frac{d_2}{d_{n-1}}$ *for all α, z and w, and*

b) *there exist α_0, z_0 and w_0 such that $K(\hat{A}) = \frac{d_2}{d_{n-1}}$.*

Proof. a) By theorem 1.1 we have

$$c_1 \geq d_2$$

and

$$c_n \geq d_{n-1}$$

where $c_1 \geq \cdots \geq c_n$ are the singular values of $\hat{A}$. Therefore,

$$K(\hat{A}) = \frac{c_1}{c_n} \geq \frac{d_2}{d_{n-1}}.$$

b) The second part of Thompson's theorem shows that, given $c_1 \geq \cdots \geq c_n$ satisfying (1.3), there exists a rank-one modification to A with singular values $c_1, \cdots, c_n$. In particular, there exist α_0, z_0 and w_0 such that the singular values of $\hat{A} = A + \alpha_0 z_0 w_0^T$ are $c_1 = d_2, c_i = d_i$ for $i = 2, \cdots n-1$ and $c_n = d_{n-1}$. Thus, $K(\hat{A}) = \frac{d_2}{d_{n-1}}$.

∎

Lemma 2.2. *Let*

$$\hat{A} = A + (d_{n-1} - d_n)\psi\phi^T,$$

then

$$K(\hat{A}) = \frac{d_1}{d_{n-1}}.$$

Proof. The matrix $\hat{A}$ can be written as

$$\begin{aligned}\hat{A} &= U(D + (d_{n-1} - d_n)U^T\psi\phi^T V)V^T \\ &= U(D + (d_{n-1} - d_n)e_n e_n^T)V^T\end{aligned}$$

Therefore, the singular values of $\hat{A}$ are: $d_1, \cdots, d_{n-1}, d_{n-1}$. Thus,

$$K(\hat{A}) = \frac{d_1}{d_{n-1}}.$$

∎

3. Upper and Lower Bounds

In this section we give an estimate of the effect that a general rank-one perturbation has on the conditioning of a matrix A which is at most rank-one deficient. We will show that the improvement in the condition number depends on how close the vectors z and w are to the null spaces of A and A^T, respectively.

The following lemma will be needed later:

Lemma 3.1. *Let*

$$B = I + (\beta - 1)e_n e_n^T + uv^T$$

where e_n denotes the n-th unit vector, $0 \le \beta \le 1$ and u, v are arbitrary vectors in R^n. Then B has at most 3 singular values different from 1, namely s_1, s_2 and s_3 such that

$$s_1 \ge 1 \ge s_2 \ge s_3, \tag{3.1}$$

and moreover,

$$\frac{1}{\sqrt{|\det B|}} \le K(B) \le \frac{s_1^2}{|\det B|} \tag{3.2}$$

Proof. If we define $T = I + (\beta - 1)e_n e_n^T$, we can easily verify that

$$T = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \beta \end{pmatrix}.$$

Let $\sigma_i, i = 1, \ldots, n$ be the singular values of B. Then, by applying Theorem 1.1 to $B = T + uv^T$, we have

$$\begin{aligned} \sigma_1 &\ge 1 \\ 1 \ge \sigma_i &\ge 1 \qquad i = 2, \ldots, n-2 \\ 1 \ge \sigma_{n-1} &\ge \beta \\ 1 \ge \sigma_n &\ge 0, \end{aligned}$$

therefore, $\sigma_i = 1$ for $i = 2, \ldots, n-2$ and, by calling $s_1 = \sigma_1, s_2 = \sigma_{n-1}$ and $s_3 = \sigma_n$, we have (3.1). Suppose that $s_3 \ne 0$. Then, since

$$|\det B| = s_1 s_2 s_3, \tag{3.3}$$

we can derive the upper bound in (3.2) as follows:

$$K(B) = \frac{s_1}{s_3} = \frac{s_1^2 s_2}{|\det B|} \le \frac{s_1^2}{|\det B|}$$

On the other hand, given (3.3) we have

$$|\det B| \ge s_2 s_3 \ge s_3^2.$$

Thus,

$$K^2(B) = \frac{s_1^2}{s_3^2} \ge \frac{1}{|\det B|},$$

giving the lower bound in (3.2).

∎

Theorem 3.1. *Let A be an $n \times n$ matrix with SVD given by (1.1), ϕ the last column of V, and ψ the last column of U. Define $\hat{A}$ as in (1.2) and let θ_z be the angle between ψ and z, and θ_w the angle between ϕ and w, i.e.*

$$\cos\theta_z = \psi^T z$$

and

$$cos\theta_w = \phi^T w.$$

Then,

$$K_1 \leq K(\hat{A}) \leq K_2$$

where

$$K_1 = \frac{d_{n-1}^{\frac{3}{2}}}{d_1} \frac{1}{\left[|\alpha| \cdot |cos\theta_z cos\theta_w| + d_n \left(1 + \frac{|\alpha|}{d_{n-1}}|sin\theta_z sin\theta_w|\right)\right]^{1/2}}$$

and

$$K_2 = \frac{d_1 \left[1 + \frac{|\alpha|}{d_{n-1}}\right]^2}{|\alpha| \cdot |cos\theta_z cos\theta_w| - d_n \left(1 + \frac{|\alpha|}{d_{n-1}}|sin\theta_z sin\theta_w|\right)}$$

provided that the denominator in K_2 is positive.

Proof. The matrix $\hat{A}$ can be written as:

$$\hat{A} = UCV^T \tag{3.4}$$

where

$$C = D + \alpha pq^T$$

with $p = U^T z$ and $q = V^T w$. Since U and V are orthogonal matrices, we have $\|p\| = \|q\| = 1$ and $K(\hat{A}) = K(C)$. Define

$$D_1 = D + (d_{n-1} - d_n)e_n e_n^T = \begin{pmatrix} d_1 & & & \\ & \ddots & & \\ & & d_{n-1} & \\ & & & d_{n-1} \end{pmatrix},$$

where e_n denotes the n-th unit vector. Then,with

$$B \equiv \begin{pmatrix} I & 0 \\ 0 & \frac{d_n}{d_{n-1}} \end{pmatrix} + \alpha D_1^{-1} pq^T. \tag{3.5}$$

we have

$$C = D_1 B.$$

The matrix B can be written as a rank-two modification to the identity matrix as follows:

$$B = I + (-1 + \frac{d_n}{d_{n-1}})e_n e_n^T + \alpha(D_1^{-1}p)q^T. \tag{3.6}$$

By applying the identity:

$$\det(I + uv^T + xy^T) = 1 + v^T u + y^T x + (v^T u)(y^T x) - (v^T x)(y^T u)$$

(proven in [3]),we get:

$$\det B = \frac{d_n}{d_{n-1}} \left(1 + \alpha \sum_{i=1}^{n-1} \frac{q_i p_i}{d_i}\right) + \alpha \frac{q_n p_n}{d_{n-1}}, \tag{3.7}$$

where p_i and q_i denote the components of the vectors p and q. Since $p_n = \cos\theta_z$ and $q_n = \cos\theta_w$, we have

$$\sum_{i=1}^{n-1} p_i^2 = \|p\|^2 - p_n^2 = 1 - \cos^2\theta_z = \sin^2\theta_z.$$

Similarly,

$$\sum_{i=1}^{n-1} q_i^2 = \sin^2\theta_w.$$

Using the Cauchy-Schwartz inequality, we have

$$\left|\sum_{i=1}^{n-1} q_i p_i\right| \leq \sqrt{\sum_{i=1}^{n-1} p_i^2}\sqrt{\sum_{i=1}^{n-1} q_i^2} \leq |\sin\theta_z \cdot \sin\theta_w|.$$

Therefore, by applying the inequality $|a+b| \geq |a| - |b|$ to (3.7), we get:

$$|\det B| \geq \frac{|\alpha|}{d_{n-1}}|\cos\theta_z \cdot \cos\theta_w| - \frac{d_n}{d_{n-1}}\left(1 + \frac{|\alpha|}{d_{n-1}}|\sin\theta_z \cdot \sin\theta_w|\right) \tag{3.8}$$

and also

$$|\det B| \leq \frac{|\alpha|}{d_{n-1}}|\cos\theta_z \cdot \cos\theta_w| + \frac{d_n}{d_{n-1}}\left(1 + \frac{|\alpha|}{d_{n-1}}|\sin\theta_z \cdot \sin\theta_w|\right). \tag{3.9}$$

On the other hand, by Lemma 3.1 we have

$$\frac{s_1^2}{|\det B|} \geq K(B) \geq \frac{1}{\sqrt{|\det B|}}. \tag{3.10}$$

Since $s_1 = \|B\|$, from the definition of B we can easily derive that

$$s_1 \leq \left\|\begin{pmatrix} I & 0 \\ 0 & \frac{d_n}{d_{n-1}} \end{pmatrix}\right\| + |\alpha| \cdot \|D_1^{-1} p q^T\| \leq 1 + \frac{|\alpha|}{d_{n-1}}. \tag{3.11}$$

This and (3.8) can be applied to (3.10) to get the following upper bound:

$$K(\hat{A}) \leq K(D_1)K(B) \leq \frac{d_1}{d_{n-1}}\left(\frac{\left(1 + \frac{|\alpha|}{d_{n-1}}\right)^2}{\frac{|\alpha|}{d_{n-1}}|\cos\theta_z \cdot \cos\theta_w| - \frac{d_n}{d_{n-1}}\left(1 + \frac{|\alpha|}{d_{n-1}}|\sin\theta_z \sin\theta_w|\right)}\right) \equiv K_2,$$

provided that the right hand side in (3.8) is positive. We can also get the lower bound K_1 by applying (3.9) to (3.10) as follows:

$$K(\hat{A}) \geq \frac{K(B)}{K(D_1^{-1})} \geq \frac{d_{n-1}}{d_1}\left[\frac{|\alpha|}{d_{n-1}}|\cos\theta_z \cdot \cos\theta_w| + \frac{d_n}{d_{n-1}}\left(1 + \frac{|\alpha|}{d_{n-1}}|\sin\theta_z \cdot \sin\theta_w|\right)\right]^{-\frac{1}{2}} \equiv K_1.$$

∎

4. Examples

Theorem 3.1 suggests that, in order to make the upper bound K_2 as small as possible, one should choose z and w sufficiently aligned with ψ and ϕ respectively so that $|\alpha||\cos\theta_z \cos\theta_w|$ is large compared to

$$d_n \left(1 + \frac{|\alpha|}{d_{n-1}} |\sin\theta_z \sin\theta_w|\right).$$

As $d_n \rightarrow 0$, the upper bound K_2 takes the form

$$K_2 = \frac{d_1 \left[1 + \frac{|\alpha|}{d_{n-1}}\right]^2}{|\alpha| \cdot |\cos\theta_z \cos\theta_w| + O(d_n)}$$

In particular, when A is exactly singular (i.e. $d_n = 0$), and $\alpha = d_{n-1}$, the bound becomes

$$\mathcal{K}(\hat{A}) \leq \frac{d_1}{d_{n-1}} \cdot \frac{4}{|\cos\theta_z \cdot \cos\theta_w|}$$

Further, to make this bound as small as possible, one should choose z and w such that $|\cos\theta_z| = |\cos\theta_w| = 1$, so that

$$\mathcal{K}(\hat{A}) \leq 4 \cdot \frac{d_1}{d_{n-1}},$$

which is only a factor of 4 larger than the actual value of $\mathcal{K}(\hat{A})$ given by lemma 2.2.

On the other hand, if z and w are such that $|\cos\theta_z|$ and $|\cos\theta_w|$ are small, there may not be any improvement in the condition number of A. In particular, when z is orthogonal to ψ and w is orthogonal to ϕ, we have the following lower bound for $\mathcal{K}(\hat{A})$:

$$\mathcal{K}(\hat{A}) \geq \sqrt{\frac{d_{n-1}}{d_n}} \frac{d_{n-1}}{d_1} \frac{1}{\sqrt{1 + \frac{|\alpha|}{d_{n-1}}}}$$

which is $O(d_n^{-\frac{1}{2}})$, as $d_n \rightarrow 0$. In fact, we have not improved $\mathcal{K}(A)$ because in this case the smallest singular value of $\hat{A}$ is still equal to d_n.

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