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Ronald R. Coifman and Stefan Steinerberger

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RONALD R. COIFMAN AND STEFAN STEINERBERGER

ABSTRACT. We study a natural nonlinear analogue of a functional decomposition into Fourier series. Iterative Blaschke factorization allows one to formally write any $F \in \mathcal{H}^1$ as a series which successively unravels or unwinds the oscillation of the function

$$F = a_1 B_1 + a_2 B_1 B_2 + a_3 B_1 B_2 B_3 + \dots$$

where $a_i \in \mathbb{C}$ and B_i is a Blaschke product. Numerical experiments point towards rapid convergence of the formal series but the actual mechanism by which this is happening has yet to be explained. We derive a family of inequalities and use them to prove convergence: for example, we have convergence in L^2 for initial data F in the Dirichlet space \mathcal{D} . Furthermore, we present a numerically efficient way to expand a function without explicit calculations of the Blaschke zeroes. As the algorithm computes the mean of functions generated through successive holomorphic high pass filters, it relates naturally to Mallat's Scattering transform.

1. INTRODUCTION

1.1. Blaschke factorization. This paper is concerned with a natural nonlinear way for unraveling the oscillation of a function in the Hardy space $F \in \mathcal{H}^1$. Our starting point is a fundamental theorem in complex analysis (*Blaschke factorization*) stating that any $F \in \mathcal{H}^1$ can be decomposed as

$$F = B \cdot G,$$

where B is a Blaschke product, that is a function of the form

$$B(z) = z^m \prod_{i \in I} \frac{\bar{a}_i}{|a_i|} \frac{z - a_i}{1 - \bar{a}_i z},$$

where $m \in \mathbb{N}_0$ and $a_1, a_2, \dots \in \mathbb{D}$ are a finite or infinite number of zeroes inside the unit disk \mathbb{D} and $G \in \mathcal{H}^1$ has no roots in \mathbb{D} . For $|z| = 1$ we have $|B(z)| = 1$ which motivates the analogy

$$B \sim \text{frequency and } G \sim \text{amplitude}$$

for the function restricted to the boundary. However, the function G need not be constant: it can be any function that never vanishes inside the unit disk. Nonetheless, one would certainly hope that, since we have removed some of the 'frequency' intrinsic to F , it should be 'simpler'.

1.2. A formal series. There is a simple way of iterating: since G has no zeroes inside \mathbb{D} , its Blaschke factorization is the trivial one $G = 1 \cdot G$, however, the function $G(z) - G(0)$ certainly has at least one root inside \mathbb{D} and will therefore yield some nontrivial Blaschke factorization $G(z) - G(0) = B_1 G_1$. Altogether, this allows us to write

$$\begin{aligned} F &= B \cdot G \\ &= B \cdot (G(0) + (G(z) - G(0))) \\ &= B \cdot (G(0) + B_1 G_1) \\ &= G(0)B + BB_1 G_1. \end{aligned}$$

At least formally, an iterative application gives rise to what we call the *unwinding series*

$$F = a_1 B_1 + a_2 B_1 B_2 + a_3 B_1 B_2 B_3 + a_4 B_1 B_2 B_3 B_4 + \dots$$

This formal expansion first appeared in the PhD thesis of Michel Nahon [13]. Given a general function $F \in \mathcal{H}^1(\mathbb{D})$ it is not numerically feasible to actually compute the roots of the function; a crucial insight from [13] is that this is not necessary – one can numerically obtain the Blaschke

product in a stable way; the method is first mentioned in a paper of Guido and Mary Weiss [24] (see also [4]) and has been investigated with respect to stability by Nahon [13] and Letelier & Saito [11]. Thorough numerical investigation [13] indicates that the formal series

$$F = a_1 B_1 + a_2 B_1 B_2 + a_3 B_1 B_2 B_3 + a_4 B_1 B_2 B_3 B_4 + \dots$$

will converge to the actual function and, generically, this seems to happen at an exponential rate.

1.3. An example. We start with the Blaschke factorization of a function $F \in \mathcal{H}^\infty(\mathbb{D})$ given approximately by a modulated Gaussian on the boundary (obtained by projecting it onto holomorphic functions).

$$F(e^{i\theta}) = e^{-(\theta-\pi)^2} \cdot e^{10i\theta}.$$

The curves in the complex plane are parametrized by $B(e^{i\theta})$ and $G(e^{i\theta})$: indeed, a lot of the oscillation (and almost the entire phase) is transported from F to B leaving G significantly simpler than F . It also serves as a good example of the heuristic

$$B \sim \text{frequency and } G \sim \text{amplitude.}$$

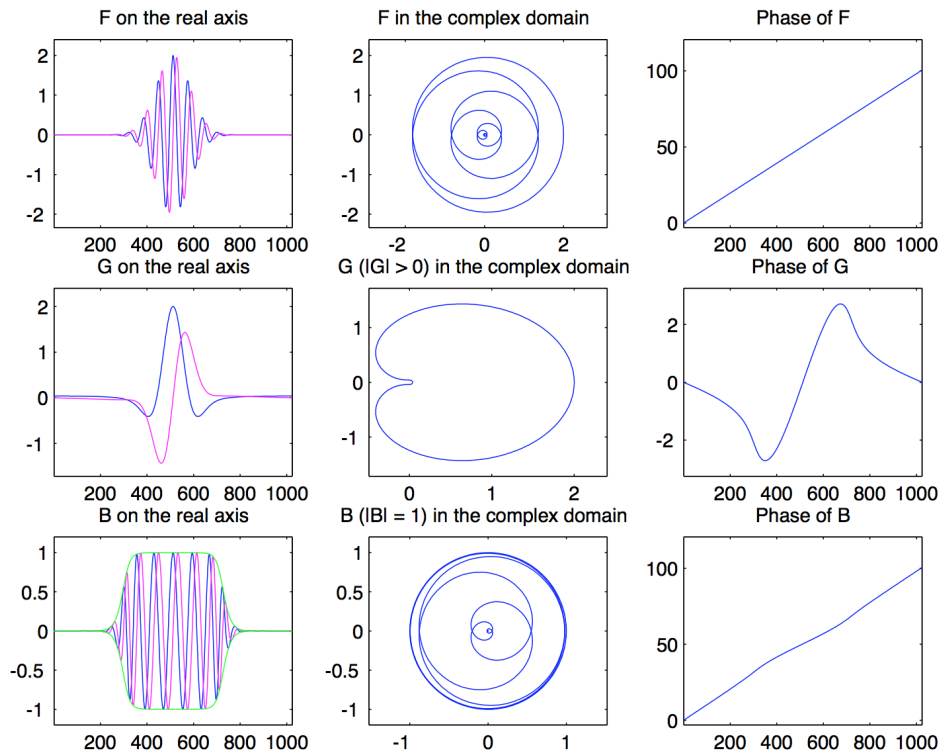


FIGURE 1. A picture taken from Michel Nahon's thesis [13]: the behavior of the Blaschke decomposition of F on $\partial\mathbb{D}$.

1.4. Related work. Blaschke products have long been used in the signal analysis. We refer to papers of Eisner & Pap [5], Feichtinger & Pap [7], Pap [15] and Picinbino [14] for some examples. The unwinding series is first studied in the PhD thesis of Michel Nahon [13]. Subsequently, a method for numerical stabilization in the case of $|F(e^{i\theta})|$ becoming small has been investigated by Letelier & Saito [11]. The unwinding series has been used by Healy [9, 10] in the study of the Doppler effect (see below for additional information). Most closely related to our paper is an

approach developed by Qian, Ho, Leong & Wang [18] (and elaborated in further papers by Qian and collaborators [16, 17, 19, 21]), which they call *adaptive Fourier transform*. The main idea is to use Blaschke products as a library and proceed by a projection pursuit approach, where at each step one projects onto the element in the library yielding the largest inner product with the function:

$$f_{n+1} = f_n - \langle f_n, B_n \rangle B_n$$

where B_n is chosen among all Blaschke products with n zeroes as the one yielding the largest inner product. Since, in particular, the functions z^n are elements of that library, this approach may be understood as a generalization of Fourier series – among their results is also an independent rediscovery of the Guido and Mary Weiss algorithm [17] and of the formal series [20]. There are also similarities with recent work of Mallat [12]. Mallat’s *scattering transform* is a translation-invariant operator, which is Lipschitz-continuous w.r.t. to C^1 -diffeomorphisms of the underlying space (for the Fourier transform even very smooth diffeomorphisms can have a large impact on high frequencies). The construction is based on an iterative application of wavelet transforms followed by restriction to the modulus. Our iterative application

$$G_n(z) = G_n(0) + (G_n(z) - G_n(0)) = G_n(0) + B_{n+1}(z)G_{n+1}(z)$$

uses the modulus of the corresponding functions while the coefficients are given as the mean. This yields a comparable level of stability: at least the leading coefficient is stable under both perturbations of the function and reparametrization of the torus.

2. STATEMENT OF RESULTS

2.1. Setup. Given a function $F \in \mathcal{H}^1(\mathbb{D})$, we define G_0 as the outer part in the Blaschke factorization

$$F = B_0 \cdot G_0$$

and then, iteratively, G_{n+1} as the outer part in the Blaschke factorization

$$G_n(z) - G_n(0) = B_{n+1}(z)G_{n+1}(z).$$

We are interested in ensuring that $\|G_n\|_X \rightarrow 0$ in some suitable space X and our main statements will be formulated that way. We remark that this formal series can be understood as the canonical extension of the Fourier series, which arises if one iteratively applies

$$G_n(z) - G_n(0) = z \cdot G_{n+1}(z),$$

which is in the same spirit but much simpler. The Blaschke series, in contrast to the Fourier series, proceeds by factoring out all zeroes inside \mathbb{D} – this gives a rise to a much larger library of functions and makes it seem intuitive that one should not only expect convergence but also faster convergence than for the Fourier series.

2.2. Algorithm and roots. We start (assuming for simplicity that there are no roots on the unit circle) with two basic observations. Recall that a general Blaschke factor has the form

$$B_1(z) = z^m \prod \frac{\bar{a}_i}{|a_i|} \frac{z - a_i}{1 - \bar{a}_i z}.$$

Suppose $F : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic, $F(0) \neq 0$ and F has the set of roots $R = \{r_1, r_2, \dots\}$. If $F = BG$, then the roots of G are simply

$$\begin{cases} r_i & \text{whenever } |r_i| > 1 \\ 1/\bar{r}_i & \text{whenever } 0 < |r_i| < 1. \end{cases}$$

Geometrically, this means that the roots of F outside the unit circle stay unchanged while roots inside the unit circle are inverted along the circle. We emphasize that in every step of the algorithm consists of studying not $G_n(z)$ but $G_n(z) - G_n(0)$, which will have a very different set of roots. However, one immediate easy consequence is the following.

Proposition. *Let $F : \mathbb{C} \rightarrow \mathbb{C}$ be given by a polynomial of degree n . Then the formal series converges and is exact after n steps.*

Proof. The algorithm is closed in the set of polynomials. Furthermore, since $G_k(z) - G_k(0)$ has at least one root in 0, the degree of the polynomial decreases by at least 1 in every step. \square

This argument is more algebraic than analytic and comes with the obvious limitation that it does not give any convergence speed (analogously, it is not surprising that a trigonometric polynomial can be written as finite Fourier series). An illustrative example is given by

$$F(z) = (z - (1 - \varepsilon))^{2n}.$$

The Blaschke factorization $F = BG$ is easy to write down and

$$G(z) = (1 - (1 - \varepsilon)z)^{2n} = (1 - \varepsilon)^{2n} \left(z - \frac{1}{(1 - \varepsilon)} \right)^{2n}.$$

By making ε sufficiently small, the functions F and G can be made as close to each other in any reasonable function space as we wish. These sort of examples immediately imply that it is not possible to construct a reasonable norm X with $\|G_{n+1}\|_X \leq (1 - \delta)\|G_n\|_X$ for some universal $\delta > 0$. Exponential convergence, which is observed in practice, will therefore either not always be the case or be the consequence of an underlying phenomenon ensuring that iterative Blaschke factorization cannot always stay close to set of functions behaving like these polynomials.

2.3. A general contraction property. This section presents our main convergence result. We first state the result in the most general form and comment on special cases of particular interest further below. Let $0 = \gamma_0 \leq \gamma_1 \leq \dots$ be an arbitrary monotonically increasing sequence of real numbers and let X be the subspace of $L^2(\mathbb{T})$ for which

$$\left\| \sum_{n \geq 0} a_n z^n \right\|_X^2 := \sum_{n \geq 0} \gamma_n |a_n|^2 < \infty.$$

We define a norm Y (merely a semi-norm whenever γ is not strictly increasing)

$$\left\| \sum_{n \geq 0} a_n z^n \right\|_Y^2 := \sum_{n \geq 0} (\gamma_{n+1} - \gamma_n) |a_n|^2.$$

Our main statement is that the Blaschke factorization acts nicely on these spaces. The first part of our statement is known (being ascribed to Digital Signal Processing in [17]) and can be equivalently phrased as follows: given a Blaschke decomposition $F = B \cdot G$ and assuming both functions are expanded into a Fourier series

$$F(z) = \sum_{n=0}^{\infty} f_n z^n \quad \text{and} \quad G(z) = \sum_{n=0}^{\infty} g_n z^n,$$

then, for every $N \in \mathbb{N}$

$$\sum_{n \geq N} |g_n|^2 \leq \sum_{n \geq N} |f_n|^2.$$

Phrased differently, inner outer factorization shifts the energy to lower frequencies in a strictly monotonous way. Our main tool will be a refinement of that inequality.

Theorem 1. *If $F \in \mathcal{H}^2$ has a Blaschke factorization $F = B \cdot G$, then*

$$\|G(e^{i\cdot})\|_X \leq \|F(e^{i\cdot})\|_X.$$

Moreover, if $F(\alpha) = 0$ for some $\alpha \in \mathbb{D}$, we even have

$$\|G(e^{i\cdot})\|_X^2 \leq \|F(e^{i\cdot})\|_X^2 - (1 - |\alpha|^2) \|G(e^{i\cdot})\|_Y^2.$$

The most important implication is convergence of the unwinding series in the space Y if the initial data lies in X . The argument is straight-forward: the construction of the unwinding series proceeds by setting

$$B_{n+1}G_{n+1} = G_n(z) - G_n(0)$$

and thus, by construction, the functions always have a root in $z_0 = 0$. Furthermore, adding and subtracting constants has no impact on $\|\cdot\|_X$ because $\gamma_0 = 0$ and therefore

$$\begin{aligned} \|G_{n+1}(e^{i\cdot})\|_Y^2 &\leq \|G_n(e^{i\cdot}) - G_n(0)\|_X^2 - \|G_{n+1}(e^{i\cdot})\|_X^2 \\ &= \|G_n(e^{i\cdot})\|_X^2 - \|G_{n+1}(e^{i\cdot})\|_X^2 \end{aligned}$$

Summing on both sides yields a telescoping series and thus

$$\sum_{n=2}^{\infty} \|G_n(e^{i\cdot})\|_Y^2 \leq \|G_1(e^{i\cdot})\|_X^2 = \|F(e^{i\cdot})\|_X^2,$$

which implies that $\|G_n(e^{i\cdot})\|_Y \rightarrow 0$. After n steps, we have the equation

$$F = a_1B_1 + a_2B_1B_2 + \cdots + a_{n-1}B_1 \cdots B_{n-1} + B_1B_2 \cdots B_nG_n$$

and exploiting that $|B_i| = 1$, we have that

$$\|f - (a_1B_1 + \cdots + a_{n-1}B_1 \cdots B_{n-1})\|_{L^2} = \|G_n\|_{L^2}.$$

This motivates putting special emphasis on the space X arising from $\gamma_n = n$ for which $Y = L^2$. This space is also known as the Dirichlet space \mathcal{D} and has special geometric significance and structure; for algebraic reasons we can get an even sharper inequality in that case (see below). Another natural (from a geometric perspective) space is given by $\gamma_n = n^2$, where $X = H^1$, $Y = \mathcal{D}$ and Theorem 1 can be alternatively proven using Green's formula (see below). All Sobolev spaces H^s with $s > 0$ are also special cases: the statement implies that for $F(e^{i\cdot}) \in H^s$ with $s > 0$, we have convergence in $H^{s-\frac{1}{2}}$. All these results have a completely analogous version on the upper half-space \mathbb{C}_+ with Blaschke-type products being defined on the real line \mathbb{R} ; even the proofs translate almost verbatim (see below).

2.4. A slight generalization. The algorithm is slightly more general than was presented up to now: indeed, at the n -th step, we could actually pick an arbitrary $\alpha_n \in \mathbb{D}$ and proceed via

$$B_{n+1}G_{n+1} = G_n(z) - G_n(\alpha_n).$$

Theorem 1 was formulated in a completely general way (for a general root α) and applies to this more general case as well (at the cost of introducing a factor $1 - |\alpha_n|^2$). Choosing a 'good' value for α_n requires balancing two competing factors: on the one hand, it seems natural that by selecting α_n so as to make $|G(\alpha_n)|$ large, we get a large coefficient which seems desirable. On the other hand $|G(\alpha_n)|$ is maximized for α_n on the boundary and whenever α_n is close to the boundary, we know that the Blaschke factorization gives a new factor G_{n+1} with the root α_n inverted by the unit disk: if α_n is the only root inside \mathbb{D} and close to the boundary, then G_{n+1} will not be very different from G_n . Furthermore, the entire process being highly nonlinear, it seems very difficult to predict the effect of picking a particular root after several iteration steps. We set $\alpha_n = 0$ throughout the rest of the paper but emphasize that in practical applications different choices might be useful.

2.5. A special case. Let us now explore the special cases with obvious geometric significance in greater detail. We identify functions $F \in \mathcal{H}^\infty(\mathbb{D})$ with maps $\gamma : \mathbb{T} \rightarrow \mathbb{R}^2$ via

$$\gamma_F(t) := F(e^{it}).$$

This is motivated by the fact that in the algorithm we obtain G_{n+1} not from $G_n(z)$ but from $G_n(z) - G_n(0)$ and it is therefore natural to study translation-invariant (i.e. geometric) quantities depending on G_n . Let us consider a particular example $F(z) = (z + 0.3 + i/3)(z - 0.2)(z - 1.5 - i/2)$ (taken essentially at random) and the Blaschke factorization $F(z) = B(z)G(z)$ (see Fig. 2 and Fig. 3). Since $G(z)$ has no roots in \mathbb{D} , the argument principle implies that $G(z)$ does not wind around 0. Note that furthermore

$$|F(e^{it})| = |G(e^{it})|$$

for all $t \in \mathbb{R}$. As suggested by the picture (and many others like it), one would expect that the

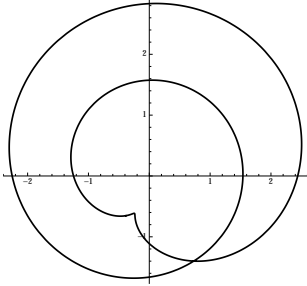


FIGURE 2. The curve γ_F .

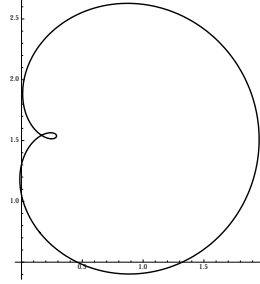


FIGURE 3. The curve γ_G .

length of the curve $\gamma_G(t)$ is smaller than that of $\gamma_F(t)$ but we have been unable to prove that; instead, we were able to obtain that result for the natural L^2 -version of length, sometimes called the *energy* of a curve

$$\text{energy}(\gamma) = \int_0^{2\pi} |\gamma'(t)|^2 dt.$$

By Hölder's inequality, we have that

$$\text{length}(\gamma)^2 = \left(\int_0^{2\pi} |\gamma'(t)| dt \right)^2 \leq 2\pi \left(\int_0^{2\pi} |\gamma'(t)|^2 dt \right) = 2\pi \text{energy}(\gamma).$$

Therefore, in particular, if the energy of a curve tends to 0, then so will the length. Algebraic simplifications allow us to quantify the decrease of the H^1 -norm of the boundary function in terms of its L^2 -norm weighted against the Poisson kernel of the roots (having the advantage of circumventing the use of the Dirichlet space).

Theorem 2. *Suppose $F \in \mathcal{H}^1(\mathbb{D})$ satisfies*

$$\int_0^{2\pi} |F'(e^{i\theta})|^2 d\theta < \infty.$$

If $\{\alpha_i : i \in I\}$ are the roots of F in \mathbb{D} and $F = B \cdot G$, then

$$\int_0^{2\pi} |G'(e^{i\theta})|^2 dt \leq \int_0^{2\pi} |F'(e^{i\theta})|^2 dt - \int_0^{2\pi} |G(e^{it})|^2 \sum_{i \in I} \frac{1 - |\alpha_i|^2}{|e^{it} - \alpha_i|^2} dt.$$

Exploiting an additional geometric argument based on random projections and the uncertainty principle, we were able to obtain the following estimate, which might be useful because it controls the error in L^∞ (again avoiding the Dirichlet space).

Corollary 1. *Suppose $F : \mathbb{D} \rightarrow \mathbb{C}$ satisfies*

$$\int_0^{2\pi} |F'(e^{i\theta})|^2 d\theta < \infty.$$

Then the formal series converges in L^∞ . Moreover,

$$|\{n \in \mathbb{N} : \|G_n(z) - G_n(0)\|_{L^\infty(\partial\mathbb{D})} \geq \varepsilon\}| \lesssim \frac{\left(\int_0^{2\pi} |F'(e^{it})|^2 dt \right)^2}{\varepsilon^4}.$$

2.6. Winding numbers and the Dirichlet space. The previously given examples (Fig. 2 and Fig. 3) show that the supremum of winding numbers, i.e. the quantity $\sup_{z \in \mathbb{C}} \text{wind}_\gamma(z)$ need not decrease under Blaschke factorization (as a matter of fact, it can increase, see Fig. 4 and Fig. 5). However, the examples strongly suggest that the *average* winding number, i.e. the quantity

$$\int_{\mathbb{C}} \text{wind}_\gamma(z) dz \quad \text{should decrease.}$$

This quantity can be regarded as *weighted* area, which is the area enclosed by the curve weighted with the winding number. It arises naturally when one applies Green's formula to compute the area surrounded by a simple, closed curve $\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2$ oriented counter-clockwise and written as $\gamma(t) = (x(t), y(t))$ via

$$\frac{1}{2} \int_0^{2\pi} x(t)\dot{y}(t) - \dot{x}(t)y(t) dt.$$

Applying the very same formula in the case of a non-simple closed curve naturally gives rise to

$$\frac{1}{2} \int_0^{2\pi} x(t)\dot{y}(t) - \dot{x}(t)y(t) dt = \int_{\mathbb{C}} \text{wind}_\gamma(z) dz.$$

This interpretation of the area formula dates back at least to a 1936 paper of Rado [22]. If F is holomorphic, then we have

$$\int_{\mathbb{C}} \text{wind}_{\gamma_F}(z) dz = \int_{\mathbb{D}} |F'(z)|^2 dz.$$

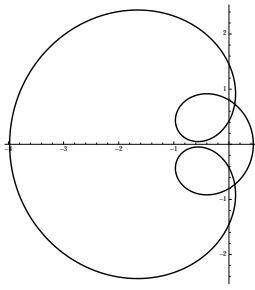


FIGURE 4. γ_F with $F(z) = (z - 0.8)(z + 1.21)$

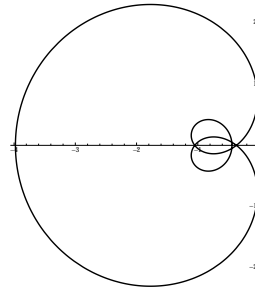


FIGURE 5. γ_G has larger maximal winding.

Writing that representation in Fourier space gives the so-called *area theorem* stating that if

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots, \quad \text{then} \quad \int_{\mathbb{C}} \text{wind}_{\gamma_F}(z) dz = \pi \sum_{n=1}^{\infty} n |a_n|^2.$$

The *Dirichlet space*

$$\mathcal{D} = \left\{ f : \mathbb{D} \rightarrow \mathbb{C} \mid f \text{ holomorphic and } \int_{\mathbb{D}} |f'(z)|^2 dz < \infty \right\}.$$

was first introduced by Beurling & Deny [1, 2]. When equipped with the inner product

$$\langle f, g \rangle_{\mathcal{D}} = \langle f, g \rangle_{\mathcal{H}^2} + \frac{1}{\pi} \int_{\mathbb{D}} f'(z) \overline{g'(z)} dz,$$

it becomes a Hilbert space. A monotonicity statement for Blaschke decomposition in that space is well-known and follows at once from Carleson's formula [3] (see also [6, Theorem 4.1.3]).

Corollary 2 (Special case of Carleson's formula). *Assume $F \in \mathcal{H}^\infty(\mathbb{D})$ with roots $\{\alpha_i : i \in I\}$ in \mathbb{D} and has the Blaschke factorization $F = B \cdot G$, then*

$$\int_{\mathbb{D}} |F'(z)|^2 dz = \int_{\mathbb{D}} |G'(z)|^2 dz + \frac{1}{2} \int_{\partial \mathbb{D}} |G|^2 \sum_{i \in I} \frac{1 - |a_i|^2}{|z - \alpha_i|^2}.$$

2.7. A curious stability property. When doing Blaschke factorization $F = BG$ numerically, we will introduce some roundoff errors; even though we never actually compute the roots of the functions, this roundoff error can be imagined as perturbing the roots a little bit. We have the following curious and purely algebraic *pointwise* stability statement.

Theorem 3. *Suppose $F_1, F_2 : \mathbb{C} \rightarrow \mathbb{C}$ are polynomials having the same roots outside of \mathbb{D} and the same number of roots inside \mathbb{D} . Then the Blaschke factorizations*

$$F_1 = B_1 G_1 \quad \text{and} \quad F_2 = B_2 G_2,$$

satisfy

$$|G_1(z) - G_2(z)| = |F_1(z) - F_2(z)| \quad \text{for all } z \in \partial\mathbb{D}.$$

2.8. An unwinding series on \mathbb{R} . The inner-outer factorization was the crucial ingredient to our entire approach; however, the same factorization can be achieved on the upper half-space and the entire notion of an unwinding series as well as all our statements transfer almost verbatim. This connection is established by the Cayley-transform, which is the conformal mapping from \mathbb{C}_+ to \mathbb{D} given by

$$C(z) = \frac{i - z}{i + z} \quad z \in \mathbb{C}_+.$$

Moreover, the linear map $T : \mathcal{H}^2(\mathbb{D}) \rightarrow \mathcal{H}^2(\mathbb{C}_+)$ given by

$$(Tf)(z) = \frac{1}{\sqrt{\pi}} \frac{1}{z + i} (f \circ C)(z)$$

is an isomorphism. Conversely, any function on \mathbb{C}_+ for which T^{-1} is defined allows us to apply the process on \mathbb{D} detailed above. However, one could also rephrase the entire process to \mathbb{C}_+ . The role of Blaschke products would then be played by functions indexed by $\lambda_1, \dots, \lambda_n \in \mathbb{C}_+$ of the form

$$\prod_{k=1}^n \frac{z - \lambda_k}{z - \bar{\lambda}_k}, \quad \text{where again } |B(z)| = 1 \text{ on } \mathbb{R}.$$

Let $\psi : [0, \infty] \rightarrow [0, \infty]$ be a monotonically increasing, differentiable function with $\psi(0) = 0$. The natural continuous extensions of the spaces are induced by the norms

$$\|f\|_X^2 := \int_0^\infty |\hat{f}(\xi)|^2 \psi(\xi) d\xi \quad \text{and} \quad \|f\|_Y^2 := \int_0^\infty |\hat{f}(\xi)|^2 \psi'(\xi) d\xi.$$

The main inequalities now have a slightly different form but can be proven the same way.

Theorem 4. *If F has roots $\lambda_1, \dots, \lambda_n \in \mathbb{C}_+$, then*

$$\left\| F \prod_{i=1}^n \frac{z - \bar{\lambda}_k}{z - \lambda_k} \right\|_X^2 \leq \|F\|_X^2.$$

For the removal of a single root $F(\lambda) = 0$, we have the stronger estimate

$$\left\| F \frac{z - \bar{\lambda}}{z - \lambda} \right\|_X^2 \leq \|F\|_X^2 - (2\Im(\lambda)) \left\| F \frac{z - \bar{\lambda}}{z - \lambda} \right\|_Y^2.$$

Moreover, in the Dirichlet space $\psi(\xi) = \xi$, we even have

$$\left\| F \prod_{i=1}^n \frac{z - \bar{\lambda}_k}{z - \lambda_k} \right\|_X^2 \leq \|F\|_X^2 - \int_{\mathbb{R}} |F(z)|^2 \sum_{i=1}^n \frac{2\Im(\alpha)}{|z - \lambda_k|^2} dz,$$

where the sum ranges over all roots of F on \mathbb{C}_+ .

3. COMPUTATION AND APPLICATION

In this section we provide a collection of known facts, additional background material, a way of computing the Blaschke factorization without ever having to compute the roots (dating back to a 1963 paper of Guido and Mary Weiss) and some sample applications.

3.1. Analytic signals. A classical way of using complex analysis when faced with a periodic, real signal $u(t) : [0, 2\pi] \rightarrow \mathbb{R}$ is to associate a natural imaginary part to the function. Already in 1946 Gabor [8] argued that

it has long been recognized that operations with the complex exponential $e^{j\omega t}$ [...] have distinct advantages over operations with sine or cosine functions.

and proposed to analyze the signal

$$f = u + iHu \quad \text{instead,}$$

where H is the Hilbert transform. Vakman [23] proved that requiring certain natural assumptions on the complexification process, this is the canonical complexification. A convenient fact for actual computation is that if

$$u(\theta) = \frac{a_0}{2} + \sum_{k \geq 1} a_k \cos k\theta + b_k \sin k\theta,$$

then

$$(Hu)(\theta) = \sum_{k \geq 1} a_k \sin k\theta - b_k \cos k\theta \quad \text{and} \quad (u + iHu)(\theta) = \frac{a_0}{2} + \sum_{k \geq 1} (a_k - ib_k)e^{ik\theta}.$$

3.2. The Guido and Mary Weiss algorithm. Let now $f(\theta)$ be a complex signal (possibly obtained from a real signal using the process above). Assume additionally that $f(\theta) \neq 0$. Note that any such $f(\theta)$ has only positive frequencies

$$f(\theta) = \sum_{k=0}^{\infty} a_k e^{ik\theta}$$

to which we may associate the function $F : \mathbb{D} \rightarrow \mathbb{C}$

$$F(z) = \sum_{k=0}^{\infty} a_k z^k,$$

which, assuming sufficient regularity, has f as boundary function. It is now our goal to construct the Blaschke decomposition of $F = B \cdot G$ without computing the roots of the function.

The algorithm proceeds as follows.

- (1) Compute the function $g(\theta) = \log |f(\theta)|$.
- (2) Compute the analytic signal from g , i.e.

$$h(\theta) = (g + iHg)(\theta).$$

- (3) Then we have the Blaschke factorization $F = B \cdot G$, where

$$G(\theta) = e^{h(\theta)} \quad \text{and} \quad B(\theta) = F(\theta)/G(\theta)$$

on the unit circle.

Clearly, the algorithm won't work whenever there is a root on the boundary of the unit disk because $\log |f(\theta)|$ will be unbounded; also, whenever $|f(\theta)|$ becomes very small, the algorithm naturally becomes unstable. Various ways for additional stabilization have been proposed: the stabilizing effect of adding a small constant has been investigated by Nahon [13] whereas Letelier & Saito [11] propose adding a small pure sinusoid.

3.3. Removal of multiplicative noise. We return to the analogy

$$B \sim \text{frequency and } G \sim \text{amplitude.}$$

B is constructed from F by its roots inside the unit disk; conversely, given $F(e^{i\cdot})$ as boundary data, we can uniquely reconstruct the values of F inside \mathbb{D} by convolving with the boundary data with the Poisson kernel. This compact integral operator enjoys a variety of smoothing properties; as a consequence it is stable against all sorts of perturbations (assuming they roughly preserve the local averages).

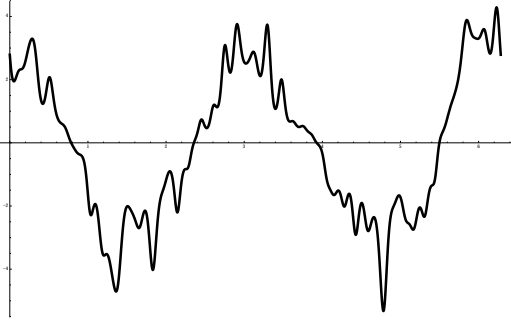


FIGURE 6. A multiplicatively perturbed pure frequency.

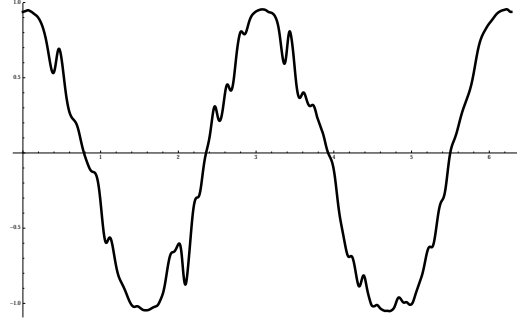


FIGURE 7. The Blaschke product after two iterations.

A particular example given above consists of a function

$$f(\theta) = \cos(2\theta) \left(\sum_{n=1}^{50} \alpha_n \frac{\cos n\theta}{\sqrt{n}} + \beta_n \frac{\sin n\theta}{\sqrt{n}} \right),$$

where α, β are instances of i.i.d. $\mathcal{N}(0, 1)$ random variables. Applying the Blaschke decomposition twice, we see that the arising frequency part recovers the underlying pure frequency.

3.4. As a versatile tool. We believe that the stability properties of the unwinding series (coming from the relationship of the roots of F inside \mathbb{D} with the boundary values of F on $\partial\mathbb{D}$ via the Poisson kernel as mentioned above) makes it a valuable tool. We give one particular example from a presentation of Healy [9, 10] who used the method to investigate the Doppler effect.

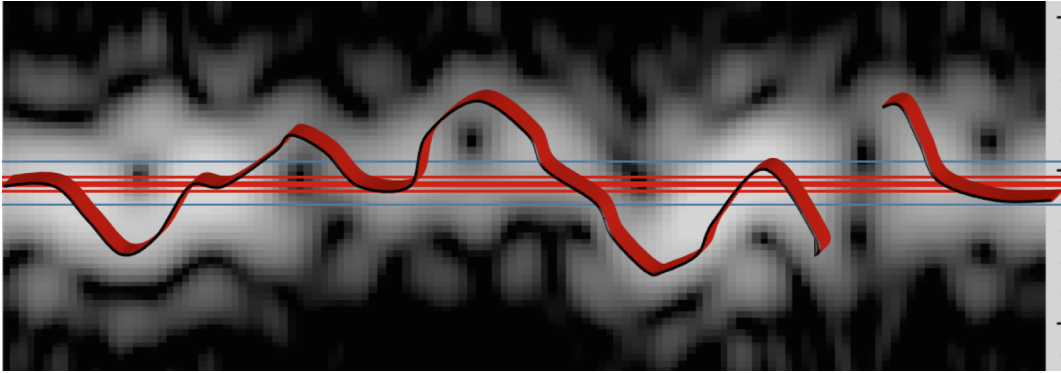


FIGURE 8. Standard approximation (red) of the Doppler effect with short-time Fourier transform (ground truth between the thin red lines).

The first picture shows the standard approach via the short-time Fourier transform; while the ground truth is within the thin red lines (and almost flat), the approximation deviates quite a bit. Note, in contrast, the increased stability from the series.

3.5. The instantaneous phase problem. Given a complex signal, we may write it in polar coordinates as

$$F(\theta) = a(\theta)e^{i\phi(\theta)}.$$

It is of interest in practice to understand how fast the frequency changes; naturally

$$F' = |F'|e^{i\phi} + i\phi'|F|e^{i\phi}$$

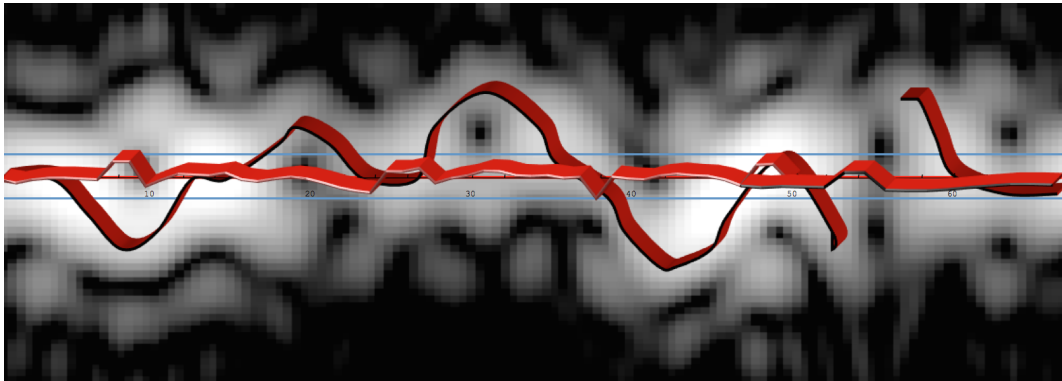


FIGURE 9. Approximation via nonlinear phase unwinding superimposed.

and thus

$$\frac{F'}{F} = \frac{|F'|}{|F|} + i\phi'.$$

However, the direct computation via this identity can be challenging and numerically unstable; various methods have been proposed (including one using Blaschke products due to Picinbono [14]) Blaschke products are well known to have the following particularly nice property: if

$$e^{im\theta} \prod_{k \in I} \frac{\bar{a}_k}{|a_k|} \frac{e^{i\theta} - a_k}{1 - \bar{a}_k e^{i\theta}} = a(\theta) e^{i\phi(\theta)},$$

then one has

$$\phi'(\theta) = m + \sum_{k \in I} \frac{1 - |a_k|^2}{|e^{i\theta} - a_k|^2} > m \geq 0.$$

The unwinding series is therefore an approximation using strictly increasing frequencies, which greatly stabilizes numerical computation (see [13] for details).

4. PROOFS

4.1. Proof of Theorem 1.

Proof. We study the action of moving a single root from inside the unit disk \mathbb{D} to the outside (by inversion along the unit circle). Let $|\alpha| < 1$ be the root; we compare

$$f(z) = (z - \alpha)F(z) \quad \text{and} \quad g(z) = (1 - \bar{\alpha}z)F(z)$$

on the boundary $\partial\mathbb{D}$. Expanding $F|_{\partial\mathbb{D}}$ into a Fourier series

$$F(z)|_{\partial\mathbb{D}} = \sum_{n=0}^{\infty} a_n z^n,$$

we immediately get

$$\begin{aligned} f(z)|_{\partial\mathbb{D}} &= -\alpha a_0 + \sum_{n=1}^{\infty} (a_{n-1} - \alpha a_n) z^n \\ g(z)|_{\partial\mathbb{D}} &= a_0 + \sum_{n=1}^{\infty} (a_n - \bar{\alpha} a_{n-1}) z^n. \end{aligned}$$

From the definition of $\|\cdot\|_X$

$$\begin{aligned} \|f(z)|_{\partial\mathbb{D}}\|_X^2 - \|g(z)|_{\partial\mathbb{D}}\|_X^2 &= -\gamma_0(1-|\alpha|^2)|a_0|^2 + \sum_{n=1}^{\infty} \gamma_n(|a_{n-1} - \alpha a_n|^2 - |a_n - \bar{\alpha}a_{n+1}|^2) \\ &= -\gamma_0(1-|\alpha|^2)|a_0|^2 + (1-|\alpha|^2) \sum_{n=1}^{\infty} \gamma_n(|a_{n-1}|^2 - |a_n|^2) \\ &= (1-|\alpha|^2) \sum_{n=0}^{\infty} (\gamma_{n+1} - \gamma_n) |a_n|^2 \\ &= (1-|\alpha|^2) \|F|_{\partial\mathbb{D}}\|_Y^2. \end{aligned}$$

Suppose now that we apply the result to $F = B \cdot G$ and assume furthermore that $F(\alpha) = 0$. We may then first remove all other roots one after the other which can only decrease the norm. Finally, we apply this result in the last step and obtain the desired statement. \square

The last part of the argument highlights a fundamental difficulty: while there is an effective gain every time we move a root to the outside, it is not clear to us how the sum of these gains could be properly controlled (which is why we only take the last one). This we only managed to do in the case of the Dirichlet space, where an additional (algebraic) simplification takes place.

4.2. Proof of Theorem 2. We study again the action of moving a single root to the outside by inversion along the unit circle. The computation resembles the computation in the more general case except that we are able to invoke Green's formula at the end of the argument.

Lemma 1. *Let $F \in \mathcal{H}^1(\mathbb{D})$ and $a \in \mathbb{C}$ with $|a| < 1$. If*

$$f = (z - a)F \quad \text{and} \quad g = (1 - \bar{a}z)F$$

then

$$\int_0^{2\pi} |g'(e^{i\theta})|^2 dt \leq \int_0^{2\pi} |f'(e^{i\theta})|^2 dt - (1 - |a|^2) \int_0^{2\pi} |F(e^{it})|^2 dt$$

whenever all terms are defined and finite.

Proof. Obviously

$$f' = F + (z - a)F'$$

and thus

$$|f'|^2 = |F|^2 + F\overline{(z - a)F'} + \overline{F}(z - a)F' + |z - a|^2|F'|^2.$$

At the same time

$$\begin{aligned} g' &= -\bar{a}F + (1 - \bar{a}z)F' \\ |g'|^2 &= |a|^2|F|^2 - a\overline{F}(1 - \bar{a}z)F' - \bar{a}F\overline{(1 - \bar{a}z)F'} + |1 - \bar{a}z|^2|F'|^2. \end{aligned}$$

If $|z| = 1$, then $|z - a|^2 = |1 - \bar{a}z|^2$ and since we only integrate over $\partial\mathbb{D}$, we get

$$\int_{\partial\mathbb{D}} |f'|^2 - |g'|^2 = (1 - |a|^2) \int_{\partial\mathbb{D}} |F|^2 + \int_{\partial\mathbb{D}} F\overline{(z - a)F'} + \overline{F}(z - a)F' + a\overline{F}(1 - \bar{a}z)F' + \bar{a}F\overline{(1 - \bar{a}z)F'}.$$

This is already almost what we want, it remains to show that

$$\int_{\partial\mathbb{D}} F\overline{(z - a)F'} + \overline{F}(z - a)F' + a\overline{F}(1 - \bar{a}z)F' + \bar{a}F\overline{(1 - \bar{a}z)F'} \geq 0.$$

The expression can be rewritten as

$$\int_{\partial\mathbb{D}} F\overline{F'}(\overline{(z - a)} + \overline{a(1 - \bar{a}z)}) + \overline{F}F'((z - a) + a(1 - \bar{a}z)),$$

which is

$$(1 - |a|^2) \int_{\partial\mathbb{D}} F\overline{F'}\bar{z} + \overline{F}F'z,$$

Now we go back from the classical derivative $f'(z)$ to the angular derivative along the boundary of the disk $\dot{f}(z)$. As before

$$\frac{d}{d\theta}f(e^{i\theta}) = f'(e^{i\theta})e^{i\theta}i$$

which can be rewritten as

$$f'(z) = -i\bar{z}\dot{f}(z) \quad \text{whenever } |z| = 1.$$

Using this, we can rewrite the terms as

$$F\bar{z}\bar{F}' = F\bar{z}(-i)\bar{z}\dot{F} = iF\dot{F} \quad \text{whenever } |z| = 1$$

$$F'\bar{z}\bar{F} = -i\bar{z}\bar{z}\dot{F}\bar{F} = -i\dot{F}\bar{F} \quad \text{whenever } |z| = 1.$$

We need to show that

$$i \int_{\partial\mathbb{D}} F\dot{F}\bar{F} - \dot{F}\bar{F} \geq 0.$$

If we write

$$F(e^{it}) = x(t) + iy(t),$$

then

$$i(F\dot{F}\bar{F} - \dot{F}\bar{F}) = 2(x(t)\dot{y}(t) - \dot{x}(t)y(t)).$$

The problem consists now of evaluating

$$\int_{\partial\mathbb{D}} 2(x(t)\dot{y}(t) - \dot{x}(t)y(t))dt.$$

This corresponds to integrating the vectorfield

$$2(-y, x)$$

over the curve

$$\gamma(t) = (x(t), y(t)).$$

Green's theorem states that this implies

$$\int_{\partial\mathbb{D}} 2(x(t)\dot{y}(t) - \dot{x}(t)y(t))dt = 4A \geq 0,$$

where A is the area of the domain enclosed by the curve γ (weighted at each point with the winding number with respect to γ). Note that Green's theorem is only valid as long as the integration along the curve is counterclockwise, which is the case because $x(t) + iy(t)$ comes from a holomorphic function. \square

If $F = B \cdot G$ has more than one root in \mathbb{D} , the Lemma can be applied iteratively.

Proof of Theorem 2. The previous language establishes a relationship between f, g and F . However, only the *modulus* of F ever appears in the argument: exploiting that

$$|z - \alpha_i| = |1 - \bar{\alpha}_i z| \quad \text{whenever } |z| = 1$$

allows for an iterative application of the previous Lemma inverting several roots along the unit circle while keeping a closed form expression (the function F changes but its modulus is pointwise constant throughout the process). Note that by construction we can always set $\alpha_1 = 0$ and therefore the gain is always at least

$$\int_0^{2\pi} |G(e^{it})|^2 \sum_i \frac{1 - |a_i|^2}{|e^{it} - \alpha_i|^2} dt \geq \int_0^{2\pi} |G(e^{it})|^2 dt.$$

\square

4.3. Proof of Corollary 1. The proof of Corollary 1 is quite interesting in itself. While we have that

$$\int_0^{2\pi} |G'_{n+1}(e^{i\theta})|^2 d\theta \leq \int_0^{2\pi} |G'_n(e^{i\theta})|^2 d\theta - \int_{\partial\mathbb{D}} |G_n(e^{i\theta}) - G_n(0)|^2 d\theta,$$

there is no way of turning this into a quantitative decay estimate because the gain

$$\int_{\partial\mathbb{D}} |G_n(e^{i\theta}) - G_n(0)|^2 d\theta \quad \text{need not be proportional to the size of} \quad \int_0^{2\pi} |G'_n(e^{i\theta})|^2 d\theta.$$

Put geometrically, $G_n(e^{i\theta})$ may wind around $G_n(0)$ very quickly while $|G_n(e^{i\theta}) - G_n(0)|$ could be quite small all the time. The crucial insight is as follows: if that were actually case and $\|G_n(e^{i\theta}) - G_n(0)\|_{L^2([0,2\pi])}$ is small, then one would certainly hope that $\|G_n(e^{i\theta}) - G_n(0)\|_{L^\infty([0,2\pi])}$ is also small. Now we reverse the order of that argument: suppose that $\|G_n(e^{i\theta}) - G_n(0)\|_{L^\infty([0,2\pi])}$ is *not* small. This means that $|G_n(e^{i\theta}) - G_n(0)|$ is big for some θ , which does not at all mean that the function is large in L^2 , it could just be big in one place and very small everywhere else: this, however, would imply that the L^2 -norm of the gradient is large and we know it cannot exceed that of the initial data.

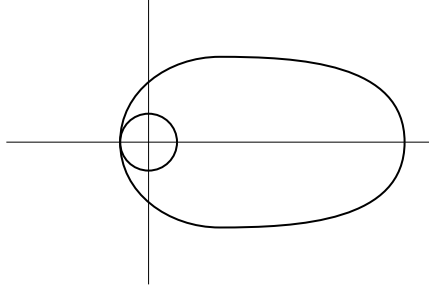


FIGURE 10. A curve γ with $\|\gamma\|_{L^\infty}$ large but $\|\gamma\|_{L^2}$ small has $\|\gamma'\|_{L^2}$ large.

4.3.1. Sobolev embedding. The second ingredient of the argument may be formulated as follows: let $h : \mathbb{T} \rightarrow \mathbb{R}$ be differentiable. If $\|h_x\|_{L^2}$ is not very big and $\|h\|_{L^\infty}$ has a certain size, then $\|h\|_{L^2}$ cannot be arbitrarily small (depending on the first two quantities): the only way to be big in L^∞ but small in L^2 is quick decay around the point where the supremum is assumed. The inequality is merely the classical embedding of the sobolev space $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$.

Lemma 2. *Let $h : \mathbb{T} \rightarrow \mathbb{R}$ be a differentiable function which changes sign. Then*

$$\|h\|_{L^2}^2 \geq \frac{1}{16} \frac{\|h\|_{L^\infty}^4}{\|h_x\|_{L^2}^2}.$$

Proof. Assume w.l.o.g. that $h(0) = 0$. Assume x to be such that $|h(x)| = \|h\|_{L^\infty}$. Then

$$\|h\|_{L^\infty}^4 = |h(x)|^4 = \left| \int_0^x 4h(z)^3 h'(z) dz \right| \leq 4\|h\|_{L^\infty}^2 \int_0^x |h(z)h'(z)| dz.$$

Using Cauchy-Schwarz, we get

$$\|h\|_{L^\infty}^2 = |h(x)|^2 \leq 4 \left(\int_0^x |h(z)|^2 dz \right)^{\frac{1}{2}} \left(\int_0^x |h'(z)|^2 dz \right)^{\frac{1}{2}}.$$

Squaring both sides, we get the statement. \square

4.3.2. *Random projections.* We apply the statement to a curve $\gamma_F : \partial\mathbb{D} \rightarrow \mathbb{C}$, which is different object than a periodic function $h : \mathbb{T} \rightarrow \mathbb{R}$. The natural approach to reduce one to the other would be to fix a vector $n \in \mathbb{R}^2$ with unit length $|n| = 1$ and consider the projection

$$h(t) = \langle \gamma(t), n \rangle.$$

The next Lemma states that there exists a vector n such this reduction does not change the L^2 -norm and L^∞ -norm by more than an absolute constant

$$\int_0^{2\pi} |\gamma(t)|^2 dt \leq 6 \int_0^{2\pi} |\langle n, \gamma(t) \rangle|^2 dt. \quad \text{and} \quad |h(0)| = |\langle \gamma(0), n \rangle| \geq \frac{1}{\sqrt{2}} |\gamma(0)|.$$

It is easy to see by Cauchy-Schwarz that h varies slower than γ

$$\left| \frac{d}{dt} h(t) \right| = |\langle \gamma'(t), n \rangle| \leq \|\gamma'(t)\|.$$

Therefore, after having established the existence of such a vector n and reparametrizing the curve in such a way that $|h(0)| \geq 2^{-1/2} \max_t \|\gamma(t)\|$, we could deduce that

$$\int_0^{2\pi} |\gamma(t)|^2 dt \geq \int_0^{2\pi} |h(t)|^2 dt \geq \frac{1}{16} \frac{\|h\|_{L^\infty}^4}{\|h_x\|_{L^2}^2} \geq \frac{1}{64} \frac{\max_t |\gamma(t)|^4}{\int_0^{2\pi} |\gamma'(z)|^2 dz}$$

which is a quantitative version of our intuition described above: in order for the function to be big at some point, it cannot be too small on average. Let us now prove the statement. The argument says that it is sufficient to take that vector at random to have the desired property to be true on average (in particular, there exists at least one vector for which it is true).

Lemma 3. *Let $\gamma : \mathbb{T} \rightarrow \mathbb{R}^2$ be a periodic curve in the plane and assume $\gamma(0) \neq (0, 0)$. Then there exists a unit vector $|n| = 1$ with*

$$|\langle \gamma(0), n \rangle| \geq \frac{1}{\sqrt{2}} |\gamma(0)|$$

as well as

$$\int_0^{2\pi} |\gamma(t)|^2 dt \leq 6 \int_0^{2\pi} |\langle n, \gamma(t) \rangle|^2 dt.$$

Proof. The line from the origin to $\gamma(0)$ defines a unique angle α . Let us now chose n randomly from $\alpha - \pi/4$ and $\alpha + \pi/4$. Any such vector satisfies the first condition and the expectation of the L^2 -norm is

$$\begin{aligned} \frac{2}{\pi} \int_{\alpha - \frac{\pi}{4}}^{\alpha + \frac{\pi}{4}} \int_0^{2\pi} |\langle (\cos s, \sin s), \gamma(t) \rangle|^2 dt ds &\geq \frac{2}{\pi} \int_0^{2\pi} |\gamma(t)|^2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |\langle (\cos s, \sin s), (0, 1) \rangle|^2 ds dt \\ &= \int_0^{2\pi} |\gamma(t)|^2 \frac{2}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2(s) ds dt \\ &= \frac{\pi - 2}{2\pi} \int_0^{2\pi} |\gamma(t)|^2 dt \geq \frac{1}{6} \int_0^{2\pi} |\gamma(t)|^2 dt \end{aligned}$$

Since a random vector has that expectation, there exists at least one vector with that value. \square

4.3.3. Proof of Corollary 1.

Proof. Our monotonicity formula implies that

$$\int_0^{2\pi} |(G_n(e^{it}) - G_n(0))'|^2 dt = \int_0^{2\pi} |G_n'(e^{it})|^2 dt \quad \text{is monotonically decreasing in } n.$$

Suppose that for some n and some $z \in \partial\mathbb{D}$

$$|G_n(z) - G_n(0)| \geq \varepsilon.$$

We can identify $G_n(z) - G_n(0) : \partial\mathbb{D} \rightarrow \mathbb{C}$ with a curve $\gamma : \mathbb{T} \rightarrow \mathbb{R}^2$ and reparametrize it using our Lemma so that

$$|\langle \gamma(0), n \rangle| \geq \frac{\varepsilon}{\sqrt{2}}$$

and

$$\int_0^{2\pi} |\gamma(t)|^2 dt \leq 6 \int_0^{2\pi} |\langle n, \gamma(t) \rangle|^2 dt.$$

We can now apply our second Lemma to the function

$$h(t) = \langle \gamma(t), n \rangle.$$

Since $G_n(z) - G_n(0)$ has winding number at least 1, so has γ and therefore h vanishes at least in two points. h and $G_n(z) - G_n(0)$ have comparable L^∞ -norm (up to a factor of $\sqrt{2}$) and comparable L^2 -norm (up to a factor of 6) and elementary geometric considerations (projections make vectors only smaller) show that the derivative of h satisfies

$$|h'(t)| \leq |\dot{\gamma}(t)|.$$

We can now apply our second Lemma and conclude that

$$\int_0^{2\pi} |(G_n(e^{it}) - G_n(0))|^2 dt \sim \|h\|_{L^2}^2 \gtrsim \frac{\|h\|_{L^\infty}^4}{\|h_x\|_{L^2}^2} \geq \frac{\varepsilon^4}{\int_0^{2\pi} |F'(e^{i\theta})|^2 d\theta},$$

where the second inequality follows from the assumption that $|G_n(z) - G_n(0)| \geq \varepsilon$ and the fact that the L^2 -norm of the derivative is decreasing. Now, let's look at the next step in the algorithm, where we decompose

$$G_n(z) - G_n(0) = BG_{n+1}.$$

Our inequality tells us that the squared L^2 -norm of the derivative decreases at least by a factor of (using $|B| = 1$ for Blaschke products)

$$\int_0^{2\pi} |G_{n+1}(e^{it})|^2 dt = \int_0^{2\pi} |G_n(e^{it}) - G_n(0)|^2 dt \gtrsim \frac{\varepsilon^4}{\int_0^{2\pi} |F'(e^{i\theta})|^2 d\theta}.$$

This yields

$$\begin{aligned} \int_0^{2\pi} |G'_{n+1}(e^{i\theta})|^2 dt &\leq \int_0^{2\pi} |G'_n(e^{i\theta})|^2 dt - \int_0^{2\pi} |G_n(e^{it}) - G_n(0)|^2 dt \\ &\leq \int_0^{2\pi} |G'_n(e^{i\theta})|^2 dt - c \frac{\varepsilon^4}{\int_0^{2\pi} |F'(e^{i\theta})|^2 d\theta} \end{aligned}$$

for some universal constant $c > 0$. However, since all the involved quantities are nonnegative, this immediately implies that number of $n \in \mathbb{N}$ for which

$$|G_n(z) - G_n(0)| \geq \varepsilon \quad \text{for some } z \in \partial\mathbb{D}$$

is bounded from above by

$$\left(\int_0^{2\pi} |F'(e^{i\theta})|^2 d\theta \right) / \left(c \frac{\varepsilon^4}{\int_0^{2\pi} |F'(e^{i\theta})|^2 d\theta} \right) = \frac{1}{c} \frac{\left(\int_0^{2\pi} |F'(e^{i\theta})|^2 d\theta \right)^2}{\varepsilon^4}.$$

□

4.4. Proof of Corollary 2.

Proof. We recall the action of removing one root, i.e. comparing

$$f(z) = (z - \alpha)F(z) \quad \text{and} \quad g(z) = (1 - \bar{\alpha}z)F(z).$$

As was shown in the proof of Theorem 1, we have the identity

$$\|f(z)|_{\partial\mathbb{D}}\|_X^2 - \|g(z)|_{\partial\mathbb{D}}\|_X^2 = (1 - |\alpha|^2) \|F|_{\partial\mathbb{D}}\|_Y^2.$$

In the Dirichlet space $X = \mathcal{D}$, we have $\gamma_n = n$ and thus $Y = L^2$. We will now illustrate the effect of applying the identity twice (to remove two roots α, β from \mathbb{D}). The arising functions are

$$f_1(z) = (z - \alpha)(z - \beta)F(z), \quad f_2(z) = (1 - \bar{\alpha}z)(z - \beta)F(z) \quad \text{and} \quad f_3(z) = (1 - \bar{\alpha}z)(1 - \bar{\beta}z)F(z).$$

Applying the identity twice yields

$$\begin{aligned}\|f_1(z)|_{\partial\mathbb{D}}\|_{\mathcal{D}}^2 - \|f_2(z)|_{\partial\mathbb{D}}\|_{\mathcal{D}}^2 &= \|(z - \beta)F|_{\partial\mathbb{D}}\|_{L^2}^2 \\ \|f_2(z)|_{\partial\mathbb{D}}\|_{\mathcal{D}}^2 - \|f_3(z)|_{\partial\mathbb{D}}\|_{\mathcal{D}}^2 &= \|(1 - \bar{\alpha}z)F|_{\partial\mathbb{D}}\|_{L^2}^2\end{aligned}$$

The crucial ingredient is now an algebraic identity on $\partial\mathbb{D}$ for all $|\alpha|, |\beta| < 1$ and $z \in \partial\mathbb{D}$

$$|z - \alpha||z - \beta| = |1 - \bar{\alpha}z||z - \beta| = |1 - \bar{\alpha}z||1 - \bar{\beta}z|.$$

This algebraic identity is merely another way of writing $|B(e^{i\theta})| = 1$ for a general Blaschke product with two factors; generalizations to three and more factors hold for the same reason. This allows us to simplify

$$\|(z - \beta)F|_{\partial\mathbb{D}}\|_{L^2}^2 + \|(1 - \bar{\alpha}z)F|_{\partial\mathbb{D}}\|_{L^2}^2 = \int_{\partial\mathbb{D}} |f_3(z)|^2 \left(\frac{1}{|z - \alpha|^2} + \frac{1}{|z - \beta|^2} \right),$$

where the f_3 could be just as well replaced by f_1 or f_2 . The general case follows analogously. \square

The argument could be easily summarized as saying that the algebraic structure of $X = \mathcal{D}$ implies that $Y = L^2$; the additional algebraic ingredient is $|B(e^{i\theta})| = 1$ which implies that the various $\|F|_{\partial\mathbb{D}}\|_Y^2$ one gets from successive removal of roots can actually be summed up in closed form.

4.5. Proof of Theorem 3.

Proof. The statement is pointwise and invariant under multiplication with polynomials having all roots outside of \mathbb{D} : it thus suffices to prove it for polynomials having all their roots inside of \mathbb{D} . We write

$$F_1 = \prod_{i=1}^n (z - \alpha_i) \quad \text{and} \quad F_2 = \prod_{i=1}^n (z - \beta_i).$$

Obviously

$$B_1 = \prod_{i=1}^n \frac{z - \alpha_i}{1 - \bar{\alpha}_i z} \quad \text{as well as} \quad B_2 = \prod_{i=1}^n \frac{z - \beta_i}{1 - \bar{\beta}_i z}$$

and thus

$$G_1 = \prod_{i=1}^n (1 - \bar{\alpha}_i z) \quad \text{as well as} \quad G_2 = \prod_{i=1}^n (1 - \bar{\beta}_i z).$$

An explicit computation yields that

$$G_1(z) - G_2(z) = \sum_{k=0}^n z^k \left(\sum_{\substack{A \subset \{1, \dots, n\} \\ |A|=k}} \prod_{j \in A} (-\bar{\alpha}_j) - \sum_{\substack{B \subset \{1, \dots, n\} \\ |B|=k}} \prod_{j \in B} (-\bar{\beta}_j) \right)$$

while

$$F_1(z) - F_2(z) = \sum_{k=0}^n z^k \left(\sum_{\substack{A \subset \{1, \dots, n\} \\ |A|=n-k}} \prod_{j \in A} (-\alpha_j) - \sum_{\substack{B \subset \{1, \dots, n\} \\ |B|=n-k}} \prod_{j \in B} (-\beta_j) \right).$$

Altogether, this implies that if

$$G_1(z) - G_2(z) = \sum_{k=1}^n c_k z^k \quad \text{then} \quad F_1(z) - F_2(z) = \sum_{k=1}^n \bar{c}_{n-k} z^k.$$

It remains to show that both quantities have the same norm if $|z| = 1$

$$\begin{aligned}|F_1(z) - F_2(z)| &= \left| \sum_{k=1}^n \bar{c}_{n-k} z^k \right| = \left| \frac{1}{z^n} \sum_{k=1}^n \bar{c}_{n-k} z^k \right| = \left| \sum_{k=1}^n \bar{c}_{n-k} \left(\frac{1}{z} \right)^{n-k} \right| \\ &= \left| \sum_{k=1}^n \bar{c}_{n-k} \bar{z}^{n-k} \right| = \left| \sum_{k=1}^n c_{n-k} z^{n-k} \right| = |G_1(z) - G_2(z)|.\end{aligned}$$

\square

4.6. Proof of Theorem 4.

Proof. We imitate the argument in the case of Fourier series and again study the effect of removing one root by comparing

$$f(z) = (z - \alpha)F(z) \quad \text{and} \quad g(z) = (z - \bar{\alpha})F(z).$$

Note that

$$\hat{f}(\xi) = i \frac{d}{d\xi} \hat{F}(\xi) - \alpha \hat{F}(\xi) \quad \text{and} \quad \hat{g}(\xi) = i \frac{d}{d\xi} \hat{F}(\xi) - \bar{\alpha} \hat{F}(\xi).$$

and therefore

$$\|f\|_X^2 - \|g\|_X^2 = \int_0^\infty \left(\left| i \frac{d}{d\xi} \hat{F}(\xi) - \alpha \hat{F}(\xi) \right|^2 - \left| i \frac{d}{d\xi} \hat{F}(\xi) - \bar{\alpha} \hat{F}(\xi) \right|^2 \right) \psi(\xi) d\xi$$

After simple computation we arrive at

$$\left| i \frac{d}{d\xi} \hat{F}(\xi) - \alpha \hat{F}(\xi) \right|^2 - \left| i \frac{d}{d\xi} \hat{F}(\xi) - \bar{\alpha} \hat{F}(\xi) \right|^2 = i(\alpha - \bar{\alpha}) [F' \bar{F} + F \bar{F}'] .$$

Writing $F(\xi) = a(\xi) + ib(\xi)$ as real and imaginary part, we see that

$$F' \bar{F} + F \bar{F}' = 2(a' a + b' b).$$

This implies that we can write

$$\left| i \frac{d}{d\xi} \hat{F}(\xi) - \alpha \hat{F}(\xi) \right|^2 - \left| i \frac{d}{d\xi} \hat{F}(\xi) - \bar{\alpha} \hat{F}(\xi) \right|^2 = i(\alpha - \bar{\alpha}) \frac{d}{d\xi} |\hat{F}(\xi)|^2$$

and therefore with integration by parts

$$\|f\|_X^2 - \|g\|_X^2 = \int_0^\infty \left(i(\alpha - \bar{\alpha}) \frac{d}{d\xi} |\hat{F}(\xi)|^2 \right) \psi(\xi) d\xi = 2\Im \alpha \int_0^\infty |\hat{F}(\xi)|^2 \psi'(\xi) d\xi,$$

which is clearly nonnegative because $\alpha \in \mathbb{C}_+$. It remains to study the special case of the Dirichlet space: if $\psi(\xi) = \xi$ we have with Plancherel that

$$\int_0^\infty |\hat{F}(\xi)|^2 \psi'(\xi) d\xi = \int_0^\infty |\hat{F}(\xi)|^2 d\xi = \int_{\mathbb{R}} |F(\xi)|^2 d\xi.$$

The key ingredient is again of an algebraic nature: the difference can be quantified in terms of a quantity whose behavior can be controlled while removing several roots one after the other. Let us illustrate this again with

$$f_1(z) = (z - \alpha)(z - \beta)F(z), \quad f_2(z) = (z - \bar{\alpha})(z - \beta)F(z) \quad \text{and} \quad f_3(z) = (z - \bar{\alpha})(z - \bar{\beta})F(z).$$

Applying the identity twice yields

$$\begin{aligned} \|f_1(z)\|_{\mathcal{D}}^2 - \|f_2(z)\|_{\mathcal{D}}^2 &= \|(z - \beta)F\|_{L^2}^2 \\ \|f_2(z)\|_{\mathcal{D}}^2 - \|f_3(z)\|_{\mathcal{D}}^2 &= \|(z - \bar{\alpha})F\|_{L^2}^2 \end{aligned}$$

and we see once more that the sum of the gain can be controlled. This yields the statement. \square

REFERENCES

- [1] A. Beurling and J. Deny, Espaces de Dirichlet. I. Le cas elementaire. Acta Math. 99 1958 203-224.
- [2] A. Beurling and J. Deny, Dirichlet spaces. Proc. Nat. Acad. Sci. U.S.A. 45 1959 208-215.
- [3] L. Carleson, A representation formula for the Dirichlet integral. Math. Z. 73 1960 190-196.
- [4] R. Coifman and G. Weiss, A kernel associated with certain multiply connected domains and its applications to factorization theorems. Studia Math. 28 1966/1967 31-68.
- [5] T. Eisner and M. Pap, Discrete orthogonality of the Malmquist Takenaka system of the upper half plane and rational interpolation. J. Fourier Anal. Appl. 20 (2014), no. 1, 1-16.
- [6] O. El-Fallah, K. Kellay, J. Mashreghi and T. Ransford, A primer on the Dirichlet space. Cambridge Tracts in Mathematics, 203. Cambridge University Press, Cambridge, 2014. xiv+211
- [7] H. Feichtinger and M. Pap, Hyperbolic wavelets and multiresolution in the Hardy space of the upper half plane. Blaschke products and their applications, 193-208, Fields Inst. Commun., 65, Springer, New York, 2013.
- [8] D. Gabor, Theory of Communication, J. Inst. Electrical Engineers. Part III: Radio and Communication Engineering, 1946, vol. 93, no. 26, pp. 429-457.

- [9] D. Healy Jr., Multi-Resolution Phase, Modulation, Doppler Ultrasound Velocimetry, and other Trendy Stuff, talk, slides via personal communication
- [10] D. Healy, Phase analysis, Talk given at the University of Maryland, slides as private communication
- [11] J. Letelier and N. Saito, Amplitude and Phase Factorization of Signals via Blaschke Product and Its Applications, talk given at JSIAM09, [HTTPS://WWW.MATH.UCDAVIS.EDU/ SAITO/TALKS/JSIAM09.PDF](https://www.math.ucdavis.edu/~saito/talks/jsiam09.pdf)
- [12] S. Mallat, Group invariant scattering. *Comm. Pure Appl. Math.* 65 (2012), no. 10, 1331-1398.
- [13] M. Nahon, Phase Evaluation and Segmentation, Ph.D. Thesis, Yale University, 2000.
- [14] B. Picinbino, On Instantaneous Amplitude and Phase of Signals, *IEEE Transactions on Signal Processing*, vol 45., 1997, 552–560.
- [15] M. Pap, Hyperbolic wavelets and multiresolution in $H^2(\mathbb{T})$. *J. Fourier Anal. Appl.* 17 (2011), no. 5, 755-776.
- [16] W. Mi, T. Qian and F. Wan, A Fast Adaptive Model Reduction Method Based on Takenaka-Malmquist Systems, *Systems & Control Letters*. Volume 61, Issue 1, January 2012, Pages 223–230.
- [17] T. Qian, Adaptive Fourier Decomposition, Rational Approximation, Part 1:Theory, invited to be included in a special issue of *International Journal of Wavelets, Multiresolution and Information Processing*.
- [18] T. Qian, I. T. Ho, I. T. Leong and Y. B. Wang, Adaptive decomposition of functions into pieces of non-negative instantaneous frequencies, *International Journal of Wavelets, Multiresolution and Information Processing*, 8 (2010), no. 5, 813-833.
- [19] T. Qian, L.H. Tan and Y.B. Wang, Adaptive Decomposition by Weighted Inner Functions: A Generalization of Fourier Serie, *Journal of Fourier Analysis and Applications*, 2011, 17(2): 175-190.
- [20] T. Qian and L. Zhang, Mathematical theory of signal analysis vs. complex analysis method of harmonic analysis, *Appl. Math. J. Chinese Univ*, 2013, 28(4): 505-530.
- [21] T. Qian, L. Zhang and Z. Li, Algorithm of Adaptive Fourier Decomposition, *IEEE Transactions on Signal Processing*, Issue Date: Dec. 2011 Volume: 59 Issue:12 On page(s): 5899 - 5906.
- [22] T. Rado. A lemma on the topological index. *Fund. Math.*, 27:212-225, 1936.
- [23] D.E.Vakman, On the Definition of Concepts of Amplitude, Phase and Instantaneous Frequency of a Signal, *Radiotekhnika i Elektronika*, 1972, vol. 17, no. 5, pp. 972-978
- [24] G. Weiss and M. Weiss, A derivation of the main results of the theory of H^p -spaces. *Rev. Un. Mat. Argentina* 20 1962 63-71.

(Ronald R. Coifman) DEPARTMENT OF MATHEMATICS, PROGRAM IN APPLIED MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CT 06510, USA
E-mail address: `coifman@math.yale.edu`

(Stefan Steinerberger) DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CT 06510, USA
E-mail address: `stefan.steinerberger@yale.edu`