A detailed analysis is presented of all pseudodifferential operators of orders up to 2 encountered in classical potential theory in two dimensions. In a sequel to this paper, the obtained apparatus will be used to construct stable discretizations of arbitrarily high order for a variety of boundary value problems for elliptic partial differential equations.

Quadruple and Octuple Layer Potentials in Two Dimensions I: Analytical Apparatus

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1 Introduction

Integral equations of classical potential theory are a tool for the solution of the Laplace equation; they have straightforward analogues for many other elliptic partial differential equations (PDEs). From the point of view of a modern mathematician, they are relatively simple objects. Indeed, a second kind integral equation (SKIE) is a sum of the unity operator and a compact operator; for most practical purposes, such an object behaves like a finite-dimensional system of linear algebraic equations, with the Fredholm alternative replacing the theory of determinants. Integral equations of the first kind (FKIEs) are a considerably more complicated object than those of the second kind. Since a first kind integral operator is compact, solving a first kind integral equation involves the application of the inverse of a compact operator to the right-hand side; depending on the right-hand side, the result might or might not be a function. Since the classical boundary value problems (Dirichlet, Neumann, and Robin) are easily reduced to SKIEs, the original creators of the potential theory simply ignored the FKIEs. Later, FKIEs of classical potential theory have also been investigated, and are now a fairly well-understood object.

In a nutshell, when the solution of a Dirichlet problem is represented by the potential of a single layer, the result is an FKIE; when the solution of a Dirichlet problem is represented by the potential of a double layer, the result is an SKIE. When the solution of a Neumann problem is represented by a single layer potential, the result is an SKIE; and when the solution of a Neumann problem is represented by a double layer potential, the result is not a classical integral equation, but rather an integro-pseudodifferential one (in computational electromagnetics, this particular object is known as a hypersingular equation). Once the integral equation is constructed, the question arises whether it has a solution, whether that solution is unique, etc. Generally, questions of this type are easily answered for the Laplace and Yukawa equations, and less so in other cases.

As a computational tool, SKIEs were popular before the advent of computers; between 1950 and 1970, they were almost completely replaced with Finite Differences and Finite Elements. The only areas where integral equations survived as a numerical tool were those where discretizing the whole area of definition of a PDE is impractical or very difficult, such as the radar scattering and certain areas of aerodynamics. The reasons for this lack of favor have to do with the fact that discretization of most integral equations of potential theory leads to dense systems of linear algebraic equations, while the Finite Elements and Finite Differences result in sparse matrices (hence the name "Finite Elements"). During the last 15 years or so, it has been discovered that many integral operators of potential theory can be applied to arbitrary vectors in a "fast" manner (for a cost proportional to n for the Laplace and Yukawa equations, and for a cost proportional to $n \cdot \log(n)$ for the Helmholtz equation, with n the number of nodes in the discretization of the integral operator). Detailed discussion of such numerical issues is outside the scope of this paper, and we refer the reader to [5, 6]. Here, we remark that the interest in integral formulations of problems of mathematical physics has been increasing, and that classical tools of potential theory turned out to be insufficient for dealing with many problems encountered in practice.

Specifically, many applications lead to integral formulations involving not only integral equations, but also integro-pseudodifferential ones. More frequently, while it is possible to formulate a problem as an FKIE or an SKIE, the numerical behavior (stability) of the resulting schemes leaves much to be desired. In such cases, it is sometimes possible to reformulate the problem as an integro-pseudodifferential equation with drastically improved stability properties (perhaps, after an appropriate preconditioning). A simple example of such a situation is the exterior Neumann problem for the Helmholtz equation, where the classical SKIE has so-called spurious resonances, coinciding with those for the *interior* Dirichlet problem on the same surface, and having nothing to do with the behavior of the exterior Neumann problem being solved. The so-called "combined field equation" solves the problem of spurious resonances at the expense of replacing an integral equation with an integro-pseudodifferential one (see, for example, [1, 12, 14, 17, 20]). Other examples of such situations include problems in scattering theory, in computational elasticity, in fluid dynamics, and in other fields.

In this paper, we investigate in detail the analytical structure of the integro-pseudodifferential equations obtained when Neumann problems are solved via double layer potentials, when Dirichlet problems are solved via quadruple layer potentials, when Neumann problems are solved via quadruple layer potentials, and in several other cases (see (11) – (29) in Section 2 for a detailed list). It turns out that the analytical structure of the obtained equations is quite simple, and involves several standard pseudodifferential operators (derivative, Hilbert transform, derivative of Hilbert transform, inverse of the derivative of the Hilbert transform, and the second derivative), composed (from the left or the right) with simple diagonal operators. We also show that the product of the standard hypersingular integral operator with the standard first kind integral operator of classical potential theory is a second kind integral operator; in other words, these two operators are perfect preconditioners for each other, asymptotically speaking.

Thus, the purpose of this paper is detailed analytical investigation of integro-pseudodifferential operators converting the densities of charge, dipole, quadrupole, and octapole distributions on a smooth curve in $\mathbb{R}^2$ into the potential, normal derivative of the potential, second normal derivative of the potential, and third normal derivative of the potential on that curve. It turns out that each of these operators is a sum of a standard operator (obtained by replacing the curve with a circle), an integral operator with a smooth kernel, and a diagonal operator. Once such expressions are obtained, it is quite easy to construct discretizations of the underlying integro-pseudodifferential operators that are adaptive, stable and of arbitrarily high order. Such discretizations (and resulting PDE solvers) have been constructed and will be reported in a sequel [10] to this paper.

Remark 1.1 *While the results reported here are easily generalized to three dimensions, it should be pointed out that there exist important classes of problems in three dimensions leading to integro-differential equations that are outside the scope of this paper. Specifically, when frequency-domain equations of electromagnetic scattering are reduced to integral equations on the boundary of the scatterer (yielding the so-called Stratton-Chew equations), the resulting integro-pseudodifferential operators are of a type not investigated here (in addition to normal derivatives on the boundary, they involve tangential derivatives); similarly, integral equations*

of elastic (as opposed to acoustic) scattering lead to integral expressions whose analysis is not a straightforward extension of that presented in this paper. Needless to say, such operators are frequently encountered in applications; they are currently under investigation.

The structure of this paper is as follows. In Section 2, we list the identities that are the purpose of this paper; the remainder of the paper is devoted to proving these identities. In Section 3 the necessary mathematical preliminaries are introduced. In Section 4 we present proofs of some of the results formulated in Section 2; when the proofs of several results are almost identical, we only prove one of them. Finally, in Section 5 we briefly discuss extensions of results of this paper to three dimensions, and to boundary conditions other than Dirichlet, Neumann, and Robin.

Remark 1.2 *The principal purpose of this paper is to present the explicit formulae (50) - (68), (89) - (93), (94) - (99), (100) - (107), to be used in the design of numerical tools for the solution of partial differential equations. The proofs of these formulae in Section 4 below are a fairly standard exercise in classical analysis, provided here for the sake of completeness. The authors expect that many readers will find it unnecessary to read this paper beyond Section 2.*

2 Statement of Results

2.1 Notation

We will be considering Dirichlet and Neumann problems for Laplace's equation in the interior or the exterior of an open region Ω bounded by a Jordan curve $\gamma(t) = (x_1(t), x_2(t))$ in $\mathbb{R}^2$ where $t \in [0, L]$. We will assume that γ is sufficiently smooth, and parametrized by its arclength. The image of γ will be denoted by Γ, so that $\partial\overline{\Omega} = \Gamma$. For a vector $y = (y_1, y_2) \in \mathbb{R}^2$ we will denote its Euclidean norm by $\|y\|$. Further, $c(t)$ will denote the curvature, and $N_\gamma(t)$ or simply $N(t)$, the exterior unit normal to Γ at $\gamma(t)$. Clearly,

$$N(t) = (x_2'(t), -x_1'(t))\,; \tag{1}$$

the situation is illustrated in Fig. 1.

A charge of unit intensity located at the point $x_0 \in \mathbb{R}^2$ generates a potential, $\Phi_{x_0} : \mathbb{R} \setminus \{x_0\} \to \mathbb{R}$, given by the expression

$$\Phi_{x_0}(x) = -\log(\|x - x_0\|)\,, \tag{2}$$

for all $x \neq x_0$. Further, the potential of a unit strength dipole located at $x_0 \in \mathbb{R}^2$, and oriented in the direction $h \in \mathbb{R}^2$, $\|h\| = 1$, is described by the formula

$$\Phi_{x_0,h}(x) = \frac{\langle h, x - x_0\rangle}{\|x - x_0\|^2}\,. \tag{3}$$

As is well known, the potential due to a point charge at $x_0 \in \mathbb{R}^2$, defined by formula (2), is harmonic in any region excluding the source point x_0.

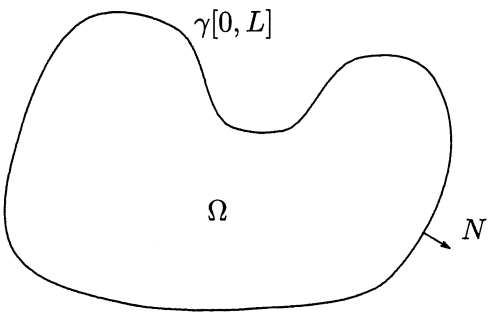

Figure 1: Boundary value problem in $\mathbb{R}^2$.

Definition 2.1 *Suppose that $\sigma : [0, L] \to \mathbb{R}$ is an integrable function. Then we will refer to the functions $p^0_{\gamma,\sigma} : \mathbb{R}^2 \to \mathbb{R}$ and $p^1_{\gamma,\sigma}, p^2_{\gamma,\sigma}, p^3_{\gamma,\sigma} : \mathbb{R}^2 \setminus \Gamma \to \mathbb{R}$, given by the formulae*

$$p^0_{\gamma,\sigma}(x) = \int_0^L \Phi_{\gamma(t)}(x)\,\sigma(t)\,dt, \tag{4}$$

$$p^1_{\gamma,\sigma}(x) = \int_0^L \frac{\partial \Phi_{\gamma(t)}(x)}{\partial N(t)}\,\sigma(t)\,dt, \tag{5}$$

$$p^2_{\gamma,\sigma}(x) = \int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(x)}{\partial N(t)^2}\,\sigma(t)\,dt, \tag{6}$$

$$p^3_{\gamma,\sigma}(x) = \int_0^L \frac{\partial^3 \Phi_{\gamma(t)}(x)}{\partial N(t)^3}\,\sigma(t)\,dt, \tag{7}$$

as the single, double, quadruple and octuple layer potentials, respectively.

Remark 2.1 *The functions $\frac{\partial \Phi_{\gamma(t)}}{\partial N(t)}, \frac{\partial^2 \Phi_{\gamma(t)}}{\partial N(t)^2}, \frac{\partial^3 \Phi_{\gamma(t)}}{\partial N(t)^3} : \mathbb{R}^2 \setminus \{\gamma(t)\} \to \mathbb{R}$ are often referred to as the dipole-, quadrupole- and octapole potentials, respectively. Obviously,*

$$\frac{\partial \Phi_{\gamma(t)}(x)}{\partial N(t)} = \frac{\langle N(t), x - \gamma(t)\rangle}{\|x - \gamma(t)\|^2}, \tag{8}$$

$$\frac{\partial^2 \Phi_{\gamma(t)}(x)}{\partial N(t)^2} = \frac{2\langle N(t), x - \gamma(t)\rangle^2}{\|x - \gamma(t)\|^4} - \frac{1}{\|x - \gamma(t)\|^2}, \tag{9}$$

$$\frac{\partial^3 \Phi_{\gamma(t)}(x)}{\partial N(t)^3} = \frac{8\langle N(t), x - \gamma(t)\rangle^3}{\|x - \gamma(t)\|^6} - \frac{6\,\langle N(t), x - \gamma(t)\rangle}{\|x - \gamma(t)\|^4}. \tag{10}$$

Clearly, the potentials $p^1_{\gamma,\sigma}$, $p^2_{\gamma,\sigma}$, $p^3_{\gamma,\sigma}$ are analytic in the interior of Ω for any integrable σ. However, for sufficiently smooth σ and γ, they can be extended to $\overline{\Omega}$ as smooth functions. Similarly, the potentials $p^1_{\gamma,\sigma}$, $p^2_{\gamma,\sigma}$, $p^3_{\gamma,\sigma}$ are analytic functions in the exterior $\mathbb{R}^2 \setminus \overline{\Omega}$ of Ω, and can be extended as smooth functions to $\mathbb{R}^2 \setminus \Omega$. Furthermore, the normal derivatives of these potentials also can be extended up to the boundary as smooth functions. Needless

to say, the interior and exterior extensions do not necessarily agree on the boundary Γ (with the obvious exception of $p^0_{\gamma,\sigma}(x)$), and we introduce the functions $p^{0,0}_{\gamma,\sigma}$, $p^{1,0}_{\gamma,\sigma,\mathrm{i}}$, $p^{1,0}_{\gamma,\sigma,\mathrm{e}}$, $p^{2,0}_{\gamma,\sigma,\mathrm{i}}$, $p^{2,0}_{\gamma,\sigma,\mathrm{e}}$, $p^{3,0}_{\gamma,\sigma,\mathrm{i}}$, $p^{3,0}_{\gamma,\sigma,\mathrm{e}}$, $p^{0,1}_{\gamma,\sigma,\mathrm{i}}$, $p^{0,1}_{\gamma,\sigma,\mathrm{e}}$, $p^{1,1}_{\gamma,\sigma,\mathrm{i}}$, $p^{1,1}_{\gamma,\sigma,\mathrm{e}}$, $p^{2,1}_{\gamma,\sigma,\mathrm{i}}$, $p^{2,1}_{\gamma,\sigma,\mathrm{e}}$, $p^{0,2}_{\gamma,\sigma,\mathrm{i}}$, $p^{0,2}_{\gamma,\sigma,\mathrm{e}}$, $p^{1,2}_{\gamma,\sigma,\mathrm{i}}$, $p^{1,2}_{\gamma,\sigma,\mathrm{e}}$, $p^{0,3}_{\gamma,\sigma,\mathrm{i}}$, $p^{0,3}_{\gamma,\sigma,\mathrm{e}} : [0,L] \to \mathbb{R}$ via the formulae

$$p^{0,0}_{\gamma,\sigma}(s) = \int_0^L \Phi_{\gamma(t)}(\gamma(s))\,\sigma(t)\,dt\,, \tag{11}$$

$$p^{1,0}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(t)}\,\sigma(t)\,dt\,, \tag{12}$$

$$p^{1,0}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(t)}\,\sigma(t)\,dt\,, \tag{13}$$

$$p^{2,0}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^2\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(t)^2}\,\sigma(t)\,dt\,, \tag{14}$$

$$p^{2,0}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^2\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(t)^2}\,\sigma(t)\,dt\,, \tag{15}$$

$$p^{3,0}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(t)^3}\,\sigma(t)\,dt\,, \tag{16}$$

$$p^{3,0}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(t)^3}\,\sigma(t)\,dt\,, \tag{17}$$

$$p^{0,1}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)}\,\sigma(t)\,dt\,, \tag{18}$$

$$p^{0,1}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)}\,\sigma(t)\,dt\,, \tag{19}$$

$$p^{1,1}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^2\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)\,\partial N(t)}\,\sigma(t)\,dt\,, \tag{20}$$

$$p^{1,1}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^2\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)\,\partial N(t)}\,\sigma(t)\,dt\,, \tag{21}$$

$$p^{2,1}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)\,\partial N(t)^2}\,\sigma(t)\,dt\,, \tag{22}$$

$$p^{2,1}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)\,\partial N(t)^2}\,\sigma(t)\,dt\,, \tag{23}$$

$$p^{0,2}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^2\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)^2}\,\sigma(t)\,dt\,, \tag{24}$$

$$p^{0,2}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^2\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)^2}\,\sigma(t)\,dt\,, \tag{25}$$

$$p^{1,2}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)^2\,\partial N(t)}\,\sigma(t)\,dt\,, \tag{26}$$

$$p^{1,2}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^2\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)^2\,\partial N(t)}\,\sigma(t)\,dt\,, \tag{27}$$

$$p^{0,3}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)^3}\,\sigma(t)\,dt\,, \tag{28}$$

$$p^{0,3}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)^3}\,\sigma(t)\,dt\,. \tag{29}$$

Remark 2.2 *Throughout the paper, the subscripts "i" and "e" will denote the limits from the interior and the exterior towards the boundary, respectively. Furthermore, the superscripts "i,j" (as, for example, in $p^{i,j}_{\gamma,\sigma,\mathrm{e}}(s)$) refers to i times and j times differentiation with respect to $N(t)$ and $N(s)$, respectively.*

Definition 2.2 *Suppose that the function $\sigma : [0,L] \to \mathbb{R}$ is twice continuously differentiable, and that γ is sufficiently smooth. Then we define the operators K^0_γ, $K^{1,0}_{\gamma,\mathrm{i}}$, $K^{1,0}_{\gamma,\mathrm{e}}$, $K^{2,0}_{\gamma,\mathrm{i}}$, $K^{2,0}_{\gamma,\mathrm{e}}$, $K^{3,0}_{\gamma,\mathrm{i}}$, $K^{3,0}_{\gamma,\mathrm{e}}$, $K^{0,1}_{\gamma,\mathrm{i}}$, $K^{0,1}_{\gamma,\mathrm{e}}$, $K^{1,1}_{\gamma,\mathrm{i}}$, $K^{1,1}_{\gamma,\mathrm{e}}$, $K^{2,1}_{\gamma,\mathrm{i}}$, $K^{2,1}_{\gamma,\mathrm{e}}$, $K^{0,2}_{\gamma,\mathrm{i}}$, $K^{0,2}_{\gamma,\mathrm{e}}$, $K^{1,2}_{\gamma,\mathrm{i}}$, $K^{1,2}_{\gamma,\mathrm{e}}$, $K^{0,3}_{\gamma,\mathrm{i}}$, $K^{0,3}_{\gamma,\mathrm{e}} : c^2[0,L] \to c[0,L]$ via the formulae*

$$K^0_\gamma(\sigma)(s) = p^{0,0}_{\gamma,\sigma}(s) = \int_0^L \Phi_{\gamma(t)}(\gamma(s))\,\sigma(t)\,dt\,, \tag{30}$$

$$K^{1,0}_{\gamma,\mathrm{i}}(\sigma)(s) = p^{1,0}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(t)}\,\sigma(t)\,dt\,, \tag{31}$$

$$K^{1,0}_{\gamma,\mathrm{e}}(\sigma)(s) = p^{1,0}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(t)}\,\sigma(t)\,dt\,, \tag{32}$$

$$K^{2,0}_{\gamma,\mathrm{i}}(\sigma)(s) = p^{2,0}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^2\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(t)^2}\,\sigma(t)\,dt\,, \tag{33}$$

$$K^{2,0}_{\gamma,\mathrm{e}}(\sigma)(s) = p^{2,0}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^2\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(t)^2}\,\sigma(t)\,dt\,, \tag{34}$$

$$K^{3,0}_{\gamma,\mathrm{i}}(\sigma)(s) = p^{3,0}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(t)^3}\,\sigma(t)\,dt\,, \tag{35}$$

$$K^{3,0}_{\gamma,\mathrm{e}}(\sigma)(s) = p^{3,0}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(t)^3}\,\sigma(t)\,dt\,, \tag{36}$$

$$K^{0,1}_{\gamma,\mathrm{i}}(\sigma)(s) = p^{0,1}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)}\,\sigma(t)\,dt\,, \tag{37}$$

$$K^{0,1}_{\gamma,\mathrm{e}}(\sigma)(s) = p^{0,1}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)}\,\sigma(t)\,dt\,, \tag{38}$$

$$K^{1,1}_{\gamma,\mathrm{i}}(\sigma)(s) = p^{1,1}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)\,\partial N(t)}\,\sigma(t)\,dt\,, \tag{39}$$

$$K^{1,1}_{\gamma,\mathrm{e}}(\sigma)(s) = p^{1,1}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)\,\partial N(t)}\,\sigma(t)\,dt\,, \tag{40}$$

$$K^{2,1}_{\gamma,\mathrm{i}}(\sigma)(s) = p^{2,1}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)\,\partial N(t)^2}\,\sigma(t)\,dt\,, \tag{41}$$

$$K^{2,1}_{\gamma,\mathrm{e}}(\sigma)(s) = p^{2,1}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)\,\partial N(t)^2}\,\sigma(t)\,dt\,, \tag{42}$$

$$K^{0,2}_{\gamma,\mathrm{i}}(\sigma)(s) = p^{0,2}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^2\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)^2}\,\sigma(t)\,dt\,, \tag{43}$$

$$K^{0,2}_{\gamma,\mathrm{e}}(\sigma)(s) = p^{0,2}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^2\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)^2}\,\sigma(t)\,dt\,, \tag{44}$$

$$K^{1,2}_{\gamma,\mathrm{i}}(\sigma)(s) = p^{1,2}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)^2\,\partial N(t)}\,\sigma(t)\,dt\,, \tag{45}$$

$$K^{1,2}_{\gamma,\mathrm{e}}(\sigma)(s) = p^{1,2}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)^2\,\partial N(t)}\,\sigma(t)\,dt\,, \tag{46}$$

$$K^{0,3}_{\gamma,\mathrm{i}}(\sigma)(s) = p^{0,3}_{\gamma,\sigma,\mathrm{i}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)^3}\,\sigma(t)\,dt\,, \tag{47}$$

$$K^{0,3}_{\gamma,\mathrm{e}}(\sigma)(s) = p^{0,3}_{\gamma,\sigma,\mathrm{e}}(s) = \lim_{h\to 0}\int_0^L \frac{\partial^3\Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)^3}\,\sigma(t)\,dt\,. \tag{48}$$

Remark 2.3 *Obviously, the operators* $K^{0,1}_{\gamma,\mathrm{i}}$, $K^{0,1}_{\gamma,\mathrm{e}}$, $K^{0,2}_{\gamma,\mathrm{i}}$, $K^{0,2}_{\gamma,\mathrm{e}}$, $K^{0,3}_{\gamma,\mathrm{i}}$, $K^{0,3}_{\gamma,\mathrm{e}}$, $K^{1,2}_{\gamma,\mathrm{i}}$, $K^{1,2}_{\gamma,\mathrm{e}}$ *given by the formulae (37), (38), (43) - (48) are the adjoints of the operators* $K^{1,0}_{\gamma,\mathrm{i}}$, $K^{1,0}_{\gamma,\mathrm{e}}$, $K^{2,0}_{\gamma,\mathrm{i}}$, $K^{2,0}_{\gamma,\mathrm{e}}$, $K^{3,0}_{\gamma,\mathrm{i}}$, $K^{3,0}_{\gamma,\mathrm{e}}$, $K^{2,1}_{\gamma,\mathrm{i}}$, $K^{2,1}_{\gamma,\mathrm{e}}$ *defined by (31) - (36), (41), (42), respectively. Furthermore,* K^0_γ, $K^{1,1}_{\gamma,\mathrm{i}}$, $K^{1,1}_{\gamma,\mathrm{e}}$ *defined by (30), (39), (40) are self-adjoint.*

2.2 Physical Interpretation

Formulae (30) – (48) have simple physical interpretations. Specifically, K^0_γ is the linear operator converting a charge distribution on the curve Γ into the potential of that charge distribution on Γ. The operator $K^{1,0}_{\gamma,\mathrm{i}}$ converts a dipole distribution on Γ into the potential created by that distribution on the *inside* of Γ; the operator $K^{1,0}_{\gamma,\mathrm{e}}$ converts a dipole distribution on Γ into the potential created by that distribution on the *outside* of Γ. The operator $K^{0,1}_{\gamma,\mathrm{e}}$ converts a charge distribution on Γ into the normal derivative of the potential created by that distribution on the *outside* of Γ, etc.

Generally, the first superscript denotes the number of differentiations at the source (charges, dipoles, quadrupoles, or octapoles); the second superscript denotes the number of differentiations at the point where the potential is evaluated (potential, normal derivative of the potential, second normal derivative of the potential, third normal derivative of the potential). In agreement with standard practice in the theory of pseudodifferential operators, we will define the

order k of either of the operators $K^{i,j}_{\gamma,\mathrm{i}}$ and $K^{i,j}_{\gamma,\mathrm{e}}$ by the formula

$$k = i + j - 1, \tag{49}$$

and observe that in this paper, we describe in detail all operators of potential theory whose order does not exceed 2. For example, we *do* investigate the operator $K^{1,2}_{\gamma,\mathrm{i}}$, converting a dipole distribution on Γ into its second normal derivative, but we *do not* investigate the operator $K^{2,2}_{\gamma,\mathrm{i}}$ converting a quadrupole distribution on Γ into its second normal derivative.

An examination of formulae (50) – (68) shows that the complexity of the expressions describing the operators (30) – (48) on the circle hardly increases as the order of the operator grows. On the other hand, the differences between the operators (30) – (48) on the circle and those on an arbitrary curve become more complicated with the growth of the order of the operator. For example, the operators K^{0}_{γ}, $K^{1,0}_{\gamma,\mathrm{i}}$, $K^{0,1}_{\gamma,\mathrm{i}}$, $K^{1,0}_{\gamma,\mathrm{e}}$, $K^{0,1}_{\gamma,\mathrm{e}}$ on an arbitrary smooth curve always differ from these operators on the circle by a compact operator (see formulae (89) – (93)). Similar differences for the operators $K^{2,0}_{\gamma,\mathrm{i}}$, $K^{2,0}_{\gamma,\mathrm{e}}$, $K^{1,1}_{\gamma,\mathrm{i}}$, $K^{1,1}_{\gamma,\mathrm{e}}$, $K^{0,2}_{\gamma,\mathrm{i}}$, $K^{0,2}_{\gamma,\mathrm{e}}$ involve the curvature of γ (see (94) – (99)). For the operators $K^{3,0}_{\gamma,\mathrm{i}}$, $K^{3,0}_{\gamma,\mathrm{e}}$, $K^{2,1}_{\gamma,\mathrm{i}}$, $K^{2,1}_{\gamma,\mathrm{e}}$, $K^{1,2}_{\gamma,\mathrm{i}}$, $K^{1,2}_{\gamma,\mathrm{e}}$, $K^{0,3}_{\gamma,\mathrm{i}}$, $K^{0,3}_{\gamma,\mathrm{e}}$, the corresponding formulae (100) – (107) already involve the square and the derivative of the curvature, as well as the Hilbert transform of the function.

Remark 2.4 *While it is certainly possible to derive explicit expressions for boundary integral operators of orders higher than 2, the complexity of the resulting formulae grows, while their numerical utility decreases. The authors have chosen to draw the line at the order 2, mostly because in the applications they anticipate, order 1 is sufficient.*

Remark 2.5 *While many of the facts presented in this paper can be obtained "automatically" from the standard theory of pseudodifferential operators, the purpose of this paper is to provide the explicit expressions (50) – (68) to be used in numerical calculations. Thus, we are ignoring the connections between the formulae (50) – (68), (89) – (93), (94) – (99), (100) – (107), and the more general theory of pseudodifferential operators.*

2.3 Results

The limits (12), (13), (18), (19) have been studied in detail in the literature (see, for example, [13, 11]). In Section 4, we conduct a similar investigation of (14) – (17), (20) – (29); first for a circle, and then for a sufficiently smooth Jordan curve. The following theorem provides explicit expressions for the operators (30) – (48) on the circle.

Theorem 2.6 *Suppose that γ is a circle of radius r parametrized by its arclength with the exterior unit normal denoted by N, k is an arbitrary integer, and $s \in [-\pi r, \pi r]$. Then,*

$$\begin{aligned} (a) \qquad K^{0}_{\gamma}(e^{ikt/r})(s) &= p^{0,0}_{\gamma,\, e^{ikt/r}}(s) = \int_{-\pi r}^{\pi r} \Phi_{\gamma(t)}(\gamma(s))\, e^{ikt/r}\, dt \\ &= \begin{cases} \pi\, |k|^{-1}\, r\, e^{iks/r}\,, & \text{for } k \neq 0\,, \\ -2\,\pi\, r\, \log(r)\,, & \text{for } k = 0\,, \end{cases} \end{aligned} \tag{50}$$

(b)

$$K^{1,0}_{\gamma,\mathrm{i}}(e^{ikt/r})(s) = p^{1,0}_{\gamma,e^{ikt/r},\mathrm{i}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial \Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(t)}\, e^{ikt/r}\, dt$$

$$= \begin{cases} -\pi\, e^{iks/r}\,, & \text{for } k\neq 0\,, \\ -2\,\pi\,, & \text{for } k=0\,, \end{cases} \tag{51}$$

$$K^{1,0}_{\gamma,\mathrm{e}}(e^{ikt/r})(s) = p^{1,0}_{\gamma,e^{ikt/r},\mathrm{e}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial \Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(t)}\, e^{ikt/r}\, dt$$

$$= \begin{cases} \pi\, e^{iks/r}\,, & \text{for } k\neq 0\,, \\ 0\,, & \text{for } k=0\,, \end{cases} \tag{52}$$

(c)

$$K^{2,0}_{\gamma,\mathrm{i}}(e^{ikt/r})(s) = p^{2,0}_{\gamma,e^{ikt/r},\mathrm{i}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(t)^2}\, e^{ikt/r}\, dt$$

$$= \begin{cases} \pi\,(|k|+1)\, r^{-1}\, e^{iks/r}\,, & \text{for } k\neq 0\,, \\ 2\,\pi\, r^{-1}\,, & \text{for } k=0\,, \end{cases} \tag{53}$$

$$K^{2,0}_{\gamma,\mathrm{e}}(e^{ikt/r})(s) = p^{2,0}_{\gamma,e^{ikt/r},\mathrm{e}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(t)^2}\, e^{ikt/r}\, dt$$

$$= \begin{cases} \pi\,(|k|-1)\, r^{-1}\, e^{iks/r}\,, & \text{for } k\neq 0\,, \\ 0\,, & \text{for } k=0\,, \end{cases} \tag{54}$$

(d)

$$K^{3,0}_{\gamma,\mathrm{i}}(e^{ikt/r})(s) = p^{3,0}_{\gamma,e^{ikt/r},\mathrm{i}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(t)^3}\, e^{ikt/r}\, dt$$

$$= \begin{cases} -\pi\,(|k|+1)\,(|k|+2)\, r^{-2}\, e^{iks/r}\,, & \text{for } k\neq 0\,, \\ -4\,\pi\, r^{-2}\,, & \text{for } k=0\,, \end{cases} \tag{55}$$

$$K^{3,0}_{\gamma,\mathrm{e}}(e^{ikt/r})(s) = p^{3,0}_{\gamma,e^{ikt/r},\mathrm{e}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(t)^3}\, e^{ikt/r}\, dt$$

$$= \begin{cases} \pi\,(|k|-1)\,(|k|-2)\, r^{-2}\, e^{iks/r}\,, & \text{for} k\neq 0\,, \\ 0\,, & \text{for } k=0\,, \end{cases} \tag{56}$$

(e)

$$K^{0,1}_{\gamma,\mathrm{i}}(e^{ikt/r})(s) = p^{0,1}_{\gamma,e^{ikt/r},\mathrm{i}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial \Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)}\, e^{ikt/r}\, dt$$

$$= \begin{cases} \pi\, e^{iks/r}\,, & \text{for } k\neq 0\,, \\ 0\,, & \text{for } k=0\,, \end{cases} \tag{57}$$

$$K^{0,1}_{\gamma,\mathrm{e}}(e^{ikt/r})(s) = p^{0,1}_{\gamma,e^{ikt/r},\mathrm{e}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial \Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)}\, e^{ikt/r}\, dt$$

$$= \begin{cases} -\pi\, e^{iks/r}\,, & \text{for } k\neq 0\,, \\ -2\,\pi\,, & \text{for } k=0\,, \end{cases} \tag{58}$$

(f)

$$K^{1,1}_{\gamma,\mathrm{i}}(e^{ikt/r})(s) = p^{1,1}_{\gamma,\,e^{ikt/r},\mathrm{i}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)\,\partial N(t)}\, e^{ikt/r}\,dt$$

$$= \begin{cases} -\pi\,|k|\,r^{-1}\,e^{iks/r}\,, & \text{for } k\neq 0\,, \\ 0\,, & \text{for } k=0\,, \end{cases} \tag{59}$$

$$K^{1,1}_{\gamma,\mathrm{e}}(e^{ikt/r})(s) = p^{1,1}_{\gamma,\,e^{ikt/r},\mathrm{e}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)\,\partial N(t)}\, e^{ikt/r}\,ds$$

$$= \begin{cases} -\pi\,|k|\,r^{-1}\,e^{iks/r}\,, & \text{for } k\neq 0\,, \\ 0\,, & \text{for } k=0\,, \end{cases} \tag{60}$$

(g)

$$K^{2,1}_{\gamma,\mathrm{i}}(e^{ikt/r})(s) = p^{2,1}_{\gamma,\,e^{ikt/r},\mathrm{i}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)\,\partial N(t)^2}\, e^{ikt/r}\,dt$$

$$= \begin{cases} \pi\,|k|\,(|k|+1)\,r^{-2}\,e^{iks/r}\,, & \text{for } k\neq 0\,, \\ 0\,, & \text{for } k=0\,, \end{cases} \tag{61}$$

$$K^{2,1}_{\gamma,\mathrm{e}}(e^{ikt/r})(s) = p^{2,1}_{\gamma,\,e^{ikt/r},\mathrm{e}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)\,\partial N(t)^2}\, e^{ikt/r}\,dt$$

$$= \begin{cases} -\pi\,|k|\,(|k|-1)\,r^{-2}\,e^{iks/r}\,, & \text{for } k\neq 0\,, \\ 0\,, & \text{for } k=0\,, \end{cases} \tag{62}$$

(h)

$$K^{0,2}_{\gamma,\mathrm{i}}(e^{ikt/r})(s) = p^{0,2}_{\gamma,\,e^{ikt/r},\mathrm{i}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)^2}\, e^{ikt/r}\,dt$$

$$= \begin{cases} \pi\,(|k|-1)\,r^{-1}\,e^{iks/r}\,, & \text{for } k\neq 0\,, \\ 0\,, & \text{for } k=0\,, \end{cases} \tag{63}$$

$$K^{0,2}_{\gamma,\mathrm{e}}(e^{ikt/r})(s) = p^{0,2}_{\gamma,\,e^{ikt/r},\mathrm{e}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)^2}\, e^{ikt/r}\,dt$$

$$= \begin{cases} \pi\,(|k|+1)\,r^{-1}\,e^{iks/r}\,, & \text{for } k\neq 0\,, \\ 2\,\pi\,r^{-1}\,, & \text{for } k=0\,, \end{cases} \tag{64}$$

(i)

$$K^{1,2}_{\gamma,\mathrm{i}}(e^{ikt/r})(s) = p^{1,2}_{\gamma,\,e^{ikt/r},\mathrm{i}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)^2\,\partial N(t)}\, e^{ikt/r}\,dt$$

$$= \begin{cases} -\pi\,|k|\,(|k|-1)\,r^{-2}\,e^{iks/r}\,, & \text{for } k\neq 0\,, \\ 0\,, & \text{for } k=0\,, \end{cases} \tag{65}$$

$$K^{1,2}_{\gamma,\mathrm{e}}(e^{ikt/r})(s) = p^{1,2}_{\gamma,\,e^{ikt/r},\mathrm{e}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)^2\,\partial N(t)}\, e^{ikt/r}\,dt$$

$$= \begin{cases} \pi\,|k|\,(|k|+1)\,r^{-2}\,e^{iks/r}\,, & \text{for } k\neq 0\,, \\ 0\,, & \text{for } k=0\,, \end{cases} \tag{66}$$

$$(j) \qquad K^{0,3}_{\gamma,\mathrm{i}}(e^{ikt/r})(s) = p^{0,3}_{\gamma,\,e^{ikt/r},\mathrm{i}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(s)^3}\, e^{ikt/r}\, dt$$

$$= \begin{cases} \pi\,(|k|-1)\,(|k|-2)\,r^{-2}\,e^{iks/r}\,, & \text{for } k\neq 0\,, \\ 0\,, & \text{for } k=0\,, \end{cases} \tag{67}$$

$$K^{0,3}_{\gamma,\mathrm{e}}(e^{ikt/r})(s) = p^{0,3}_{\gamma,\,e^{ikt/r},\mathrm{e}}(s) = \lim_{h\to 0}\int_{-\pi r}^{\pi r} \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(s)^3}\, e^{ikt/r}\, dt$$

$$= \begin{cases} -\pi\,(|k|+1)\,(|k|+2)\,r^{-2}\,e^{iks/r}\,, & \text{for } k\neq 0\,, \\ -4\,\pi\,r^{-2}\,, & \text{for } k=0\,. \end{cases} \tag{68}$$

Formulae (50) – (68) describe the action of the operators (30) – (48) on the circle for functions of the form $e^{ikt/r}$, with $k = 0, \pm 1, \pm 2, \ldots$ Now, it immediately follows from (50) – (68) that for any periodic function $\sigma : [0, L] \to \mathbb{C}$ given by its Fourier series

$$\sigma(t) = \sum_{k=-\infty}^{\infty} \widehat{\sigma}_k\, e^{2\pi ikt/L}\,, \tag{69}$$

the operators (30) – (48) (γ is the circle of radius $r = \frac{L}{2\pi}$) assume the form

$$\text{(a)} \qquad K^0_\gamma(\sigma)(s) = -L\,\log(\frac{L}{2\,\pi})\,\widehat{\sigma}_0 + \frac{L}{2}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{|k|}\widehat{\sigma}_k e^{2\pi iks/L}\,, \tag{70}$$

$$\text{(b)} \qquad K^{1,0}_{\gamma,\mathrm{i}}(\sigma)(s) = -2\,\pi\,\widehat{\sigma}_0 - \pi \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \widehat{\sigma}_k\, e^{2\pi iks/L}$$

$$= -\pi\,\sigma(s) - \pi\,\widehat{\sigma}_0\,, \tag{71}$$

$$K^{1,0}_{\gamma,\mathrm{e}}(\sigma)(s) = \pi \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \widehat{\sigma}_k\, e^{2\pi iks/L}$$

$$= \pi\,\sigma(s) - \pi\,\widehat{\sigma}_0\,, \tag{72}$$

$$\text{(c)} \qquad K^{2,0}_{\gamma,\mathrm{i}}(\sigma)(s) = \frac{4\,\pi^2}{L}\,\widehat{\sigma}_0 + \frac{2\,\pi^2}{L}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} (|k|+1)\,\widehat{\sigma}_k\, e^{2\pi iks/L}$$

$$= \frac{2\,\pi^2}{L}\,\sigma(s) + \pi\,H(\sigma')(s) + \frac{2\,\pi^2}{L}\,\widehat{\sigma}_0\,, \tag{73}$$

$$K^{2,0}_{\gamma,\mathrm{e}}(\sigma)(s) = \frac{2\,\pi^2}{L}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} (|k|-1)\,\widehat{\sigma}_k\, e^{2\pi iks/L}$$

$$= -\frac{2\,\pi^2}{L}\,\sigma(s) + \pi\,H(\sigma')(s) + \frac{2\,\pi^2}{L}\,\widehat{\sigma}_0\,, \tag{74}$$

(d)
$$K^{3,0}_{\gamma,\mathrm{i}}(\sigma)(s) = -\frac{16\,\pi^3}{L^2}\,\widehat{\sigma}_0 - \frac{4\,\pi^3}{L^2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}(|k|+1)\,(|k|+2)\,\widehat{\sigma}_k\,e^{2\pi iks/L}$$
$$= -\frac{8\,\pi^3}{L^2}\,\sigma(s) + \pi\,\sigma''(s) - \frac{6\,\pi^2}{L}\,H(\sigma')(s) - \frac{8\,\pi^3}{L^2}\,\widehat{\sigma}_0\,, \tag{75}$$
$$K^{3,0}_{\gamma,\mathrm{e}}(\sigma)(s) = \frac{4\,\pi^3}{L^2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}(|k|-1)\,(|k|-2)\,\widehat{\sigma}_k\,e^{2\pi iks/L}$$
$$= \frac{8\,\pi^3}{L^2}\,\sigma(s) - \pi\,\sigma''(s) - \frac{6\,\pi^2}{L}\,H(\sigma')(s) - \frac{8\,\pi^3}{L^2}\,\widehat{\sigma}_0\,, \tag{76}$$

(e)
$$K^{0,1}_{\gamma,\mathrm{i}}(\sigma)(s) = \pi\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\widehat{\sigma}_k\,e^{2\pi iks/L}$$
$$= \pi\,\sigma(s) - \pi\,\widehat{\sigma}_0\,, \tag{77}$$
$$K^{0,1}_{\gamma,\mathrm{e}}(\sigma)(s) = -2\,\pi\,\widehat{\sigma}_0 - \pi\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\widehat{\sigma}_k\,e^{2\pi iks/L}$$
$$= -\pi\,\sigma(s) - \pi\,\widehat{\sigma}_0\,, \tag{78}$$

(f)
$$K^{1,1}_{\gamma,\mathrm{i}}(\sigma)(s) = -\frac{2\,\pi^2}{L}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}|k|\,\widehat{\sigma}_k\,e^{iks/L}$$
$$= -\pi\,H(\sigma')(s)\,, \tag{79}$$
$$K^{1,1}_{\gamma,\mathrm{e}}(\sigma)(s) = -\frac{2\,\pi^2}{L}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}|k|\,\widehat{\sigma}_k\,e^{iks/L}$$
$$= -\pi\,H(\sigma')(s)\,, \tag{80}$$

(g)
$$K^{2,1}_{\gamma,\mathrm{i}}(\sigma)(s) = \frac{4\,\pi^3}{L^2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}|k|\,(|k|+1)\,\widehat{\sigma}_k\,e^{2\pi iks/L}$$
$$= -\pi\,\sigma''(s) + \frac{2\,\pi^2}{L}\,H(\sigma')(s)\,, \tag{81}$$
$$K^{2,1}_{\gamma,\mathrm{e}}(\sigma)(s) = -\frac{4\,\pi^3}{L^2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}|k|\,(|k|-1)\,\widehat{\sigma}_k\,e^{2\pi iks/L}$$
$$= \pi\,\sigma''(s) + \frac{2\,\pi^2}{L}\,H(\sigma')(s)\,, \tag{82}$$

(h)
$$K^{0,2}_{\gamma,\mathrm{i}}(\sigma)(s) = \frac{2\,\pi^2}{L}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}(|k|-1)\,\widehat{\sigma}_k\,e^{2\pi iks/L}$$

$$= -\frac{2\pi^2}{L}\sigma(s) + \pi H(\sigma')(s) + \frac{2\pi^2}{L}\widehat{\sigma}_0\,, \tag{83}$$

$$K^{0,2}_{\gamma,\mathrm{e}}(\sigma)(s) = \frac{4\pi^2}{L}\widehat{\sigma}_0 + \frac{2\pi^2}{L}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}(|k|+1)\,\widehat{\sigma}_k\, e^{2\pi iks/L}$$

$$= \frac{2\pi^2}{L}\sigma(s) + \pi H(\sigma')(s) + \frac{2\pi^2}{L}\widehat{\sigma}_0\,, \tag{84}$$

(i) $$K^{1,2}_{\gamma,\mathrm{i}}(\sigma)(s) = -\frac{4\pi^3}{L^2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}|k|\,(|k|-1)\,\widehat{\sigma}_k\, e^{2\pi iks/L}$$

$$= \pi\,\sigma''(s) + \frac{2\pi^2}{L}H(\sigma')(s)\,, \tag{85}$$

$$K^{1,2}_{\gamma,\mathrm{e}}(\sigma)(s) = \frac{4\pi^3}{L^2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}|k|\,(|k|+1)\,\widehat{\sigma}_k\, e^{2\pi iks/L}$$

$$= -\pi\,\sigma''(s) + \frac{2\pi^2}{L}H(\sigma')(s)\,, \tag{86}$$

(j) $$K^{0,3}_{\gamma,\mathrm{i}}(\sigma)(s) = \frac{4\pi^3}{L^2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}(|k|-1)\,(|k|-2)\,\widehat{\sigma}_k\, e^{2\pi iks/L}$$

$$= \frac{8\pi^3}{L^2}\sigma(s) - \pi\,\sigma''(s) - \frac{6\pi^2}{L}H(\sigma')(s) - \frac{8\pi^3}{L^2}\widehat{\sigma}_0\,, \tag{87}$$

$$K^{0,3}_{\gamma,\mathrm{e}}(\sigma)(s) = -\frac{16\pi^3}{L^2}\widehat{\sigma}_0 - \frac{4\pi^3}{L^2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}(|k|+1)\,(|k|+2)\,\widehat{\sigma}_k\, e^{2\pi iks/L}$$

$$= -\frac{8\pi^3}{L^2}\sigma(s) + \pi\,\sigma''(s) - \frac{6\pi^2}{L}H(\sigma')(s) - \frac{8\pi^3}{L^2}\widehat{\sigma}_0\,, \tag{88}$$

with $\widehat{\sigma}_k$ denoting the k-th Fourier coefficient of the function σ, and H the Hilbert transform (see (130) in Section 3.3).

Theorem 2.6 above is proved by direct evaluation of the relevant integrals (in Section 4 below, we compute these integrals via the theory of residues). Formulae (70) – (88) are an immediate consequence of Theorem 2.6; they provide explicit expressions for the operators (30) – (48) when γ is a circle.

The following theorem follows easily from well-known results (see, for example, [19, 13]), here stated in a slightly different form.

Theorem 2.7 *Suppose that $\gamma : [0, L] \to \mathbb{R}^2$ is a k times continuously differentiable Jordan curve parametrized by its arclength, and that $\eta : [0, L] \to \mathbb{R}^2$ denotes the circle of radius r. Then, for any sufficiently smooth function $\sigma : [0, L] \to \mathbb{R}$,*

(a) $$K^0_\gamma(\sigma)(s) = K^0_\eta(\sigma)(s) + M_0(\sigma)(s)\,, \tag{89}$$

$$
\begin{aligned}
(b) \qquad K^{1,0}_{\gamma,\mathrm{i}}(\sigma)(s) &= K^{1,0}_{\eta,\mathrm{i}}(\sigma)(s) + M_1(\sigma)(s) \\
&= -\pi\,\sigma(s) + N_1(\sigma)(s)\,, \qquad (90) \\
K^{1,0}_{\gamma,\mathrm{e}}(\sigma)(s) &= K^{1,0}_{\eta,\mathrm{e}}(\sigma)(s) + M_1(\sigma)(s) \\
&= \pi\,\sigma(s) + N_1(\sigma)(s)\,, \qquad (91) \\
(c) \qquad K^{0,1}_{\gamma,\mathrm{i}}(\sigma)(s) &= K^{0,1}_{\eta,\mathrm{i}}(\sigma)(s) + M_1^\star(\sigma)(s) \\
&= \pi\,\sigma(s) + N_1^\star(\sigma)(s)\,, \qquad (92) \\
K^{0,1}_{\gamma,\mathrm{e}}(\sigma)(s) &= K^{0,1}_{\eta,\mathrm{e}}(\sigma)(s) + M_1^\star(\sigma)(s) \\
&= -\pi\,\sigma(s) + N_1^\star(\sigma)(s)\,, \qquad (93)
\end{aligned}
$$

where $M_0, M_1, N_1 : c[0,L] \to c[0,L]$ are integral operators with kernels $m_0(s,t) \in c^{k-1}([0,L]\times[0,L])$, $m_1(s,t)$, $n_1(s,t) \in c^{k-2}([0,L]\times[0,L])$, respectively. Furthermore, $M_1^\star$, $N_1^\star$ are the adjoints of M_1, N_1, respectively, and the operator M_0 is self-adjoint.

Theorem 2.7 approximates the operators K^0_γ, $K^{1,0}_{\gamma,\mathrm{i}}$, $K^{1,0}_{\gamma,\mathrm{e}}$, $K^{0,1}_{\gamma,\mathrm{i}}$, $K^{0,1}_{\gamma,\mathrm{e}}$ for an arbitrary smooth Jordan curve by the same operators on the circle; Theorem 2.8 below extends these results to the operators (33), (34), (39), (40), (43), (44). While Theorem 2.7 is well-known, the authors failed to find Theorem 2.8 in the literature.

Theorem 2.8 *Suppose that $\gamma : [0,L] \to \mathbb{R}^2$ is a k times continuously differentiable Jordan curve parametrized by its arclength, and that $\eta : [0,L] \to \mathbb{R}^2$ denotes the circle of radius $\frac{L}{2\pi}$, also parametrized by its arclength. Then, for any sufficiently smooth function $\sigma : [0,L] \to \mathbb{R}$,*

$$
\begin{aligned}
(a) \qquad K^{2,0}_{\gamma,\mathrm{i}}(\sigma)(s) &= \Big(\pi\,c(s) - \frac{2\,\pi^2}{L}\Big)\sigma(s) + K^{2,0}_{\eta,\mathrm{i}}(\sigma)(s) + M_2(\sigma)(s) \\
&= \pi\,c(s)\,\sigma(s) + \pi\,H(\sigma')(s) + N_2(\sigma)(s)\,, \qquad (94) \\
K^{2,0}_{\gamma,\mathrm{e}}(\sigma)(s) &= -\Big(\pi\,c(s) - \frac{2\,\pi^2}{L}\Big)\sigma(s) + K^{2,0}_{\eta,\mathrm{e}}(\sigma)(s) + M_2(\sigma)(s) \\
&= -\pi\,c(s)\,\sigma(s) + \pi\,H(\sigma')(s) + N_2(\sigma)(s)\,, \qquad (95) \\
(b) \qquad K^{1,1}_{\gamma,\mathrm{i}}(\sigma)(s) &= K^{1,1}_{\eta,\mathrm{i}}(\sigma)(s) + G_2(\sigma)(s) \\
&= -\pi\,H(\sigma')(s) + G_2(\sigma)(s)\,, \qquad (96) \\
K^{1,1}_{\gamma,\mathrm{e}}(\sigma)(s) &= K^{1,1}_{\eta,\mathrm{e}}(\sigma)(s) + G_2(\sigma)(s) \\
&= -\pi\,H(\sigma')(s) + G_2(\sigma)(s)\,, \qquad (97) \\
(c) \qquad K^{0,2}_{\gamma,\mathrm{i}}(\sigma)(s) &= -\Big(\pi\,c(s) - \frac{2\,\pi^2}{L}\Big)\sigma(s) + K^{0,2}_{\eta,\mathrm{i}}(\sigma)(s) + M_2^\star(\sigma)(s) \\
&= -\pi\,c(s)\,\sigma(s) + \pi\,H(\sigma')(s) + N_2^\star(\sigma)(s)\,, \qquad (98) \\
K^{0,2}_{\gamma,\mathrm{e}}(\sigma)(s) &= \Big(\pi\,c(s) - \frac{2\,\pi^2}{L}\Big)\sigma(s) + K^{0,2}_{\eta,\mathrm{e}}(\sigma)(s) + M_2^\star(\sigma)(s) \\
&= \pi\,c(s)\,\sigma(s) + \pi\,H(\sigma')(s) + N_2^\star(\sigma)(s)\,, \qquad (99)
\end{aligned}
$$

where $c(s)$ denotes the curvature of γ at $\gamma(s)$, and M_2, N_2, $G_2 : c[0,L] \to c[0,L]$ are integral operators with kernels $m_2(s,t)$, $n_2(s,t)$, $g_2(s,t) \in c^{k-2}([0,L]\times[0,L])$, respectively. Furthermore, $M_2^\star$, $N_2^\star$ are the adjoints of M_2, N_2, the operator G_2 is self-adjoint, and H denotes the Hilbert transform (see (130) in Section 3.3).

Remark 2.9 *The formulae (90) - (93) above are somewhat misleading, in that they state very simple facts in a relatively complicated manner. Specifically, each of the operators $K^{1,0}_{\gamma,\mathrm{i}}$, $K^{1,0}_{\gamma,\mathrm{e}}$, $K^{0,1}_{\gamma,\mathrm{i}}$, $K^{0,1}_{\gamma,\mathrm{e}}$ is a second kind integral operator with smooth (c^{k-2}) kernel (see, for example, [13]). In the case of the circle, the kernels of the operators $K^{1,0}_{\eta,\mathrm{i}}$, $K^{1,0}_{\eta,\mathrm{e}}$, $K^{0,1}_{\eta,\mathrm{i}}$, $K^{0,1}_{\eta,\mathrm{e}}$ are identically equal to $-\frac{1}{2r}$. Thus, (90) - (93) state the trivial fact that the difference of two smooth kernels is smooth. We list (90) - (93) for compatibility with the formulae (89), (94) - (99).*

Observation 2.10 *Formulae (89) - (99) have a straightforward interpretation. Specifically, each of the operators K^0_γ, $K^{1,0}_{\gamma,\mathrm{i}}$, $K^{1,0}_{\gamma,\mathrm{e}}$, $K^{0,1}_{\gamma,\mathrm{i}}$, $K^{0,1}_{\gamma,\mathrm{e}}$, $K^{2,0}_{\gamma,\mathrm{i}}$, $K^{2,0}_{\gamma,\mathrm{e}}$, $K^{1,1}_{\gamma,\mathrm{i}}$, $K^{1,1}_{\gamma,\mathrm{e}}$, $K^{0,2}_{\gamma,\mathrm{i}}$, $K^{0,2}_{\gamma,\mathrm{e}}$, is a sum of a standard operator (the corresponding operator on the circle) and an integral operator with a smooth kernel.*

In Section 4 below, a proof of formulae (94) and (95) is given; the proofs of the formulae (94) – (99) in Theorem 2.8 are similar and are omitted. Theorem 2.11 below extends the results of Theorem 2.8 above to the operators $K^{3,0}_{\gamma,\mathrm{i}}$, $K^{3,0}_{\gamma,\mathrm{e}}$, $K^{2,1}_{\gamma,\mathrm{i}}$, $K^{2,1}_{\gamma,\mathrm{e}}$, $K^{1,2}_{\gamma,\mathrm{i}}$, $K^{1,2}_{\gamma,\mathrm{e}}$, $K^{0,3}_{\gamma,\mathrm{i}}$, $K^{0,3}_{\gamma,\mathrm{e}}$. Its proof is virtually identical to that of Theorem 2.8, and is omitted.

Theorem 2.11 *Suppose that $\gamma : [0,L] \to \mathbb{R}^2$ is a k times continuously differentiable Jordan curve parametrized by its arclength, and that $\eta : [0,L] \to \mathbb{R}^2$ denotes the circle of radius $\frac{L}{2\pi}$, also parametrized by its arclength. Then, for any sufficiently smooth function $\sigma : [0,L] \to \mathbb{R}$,*

$$
\begin{aligned}
(a) \qquad K^{3,0}_{\gamma,\mathrm{i}}(\sigma)(s) &= -\Big(2\pi\Big(c(s)\Big)^2 - \frac{4\pi^2}{L}\,c(s)\Big)\sigma(s) + \Big(\pi - \frac{L}{2}\,c(s)\Big)\,\sigma''(s) \\
&\quad -2\pi\,c'(s)\,H(\sigma)(s) + \frac{L}{2\pi}\,c(s)\,K^{3,0}_{\eta,\mathrm{i}}(\sigma)(s) + M_3(\sigma)(s) \\
&= -2\pi\Big(c(s)\Big)^2\sigma(s) + \pi\,\sigma''(s) - 2\pi\,c'(s)\,H(\sigma)(s) - 3\pi\,c(s)\,H(\sigma')(s) \\
&\quad +N_3(\sigma)(s)\,, \qquad (100) \\
K^{3,0}_{\gamma,\mathrm{e}}(\sigma)(s) &= \Big(2\pi\Big(c(s)\Big)^2 - \frac{4\pi^2}{L}\,c(s)\Big)\sigma(s) - \Big(\pi - \frac{L}{2}\,c(s)\Big)\,\sigma''(s) \\
&\quad -2\pi\,c'(s)\,H(\sigma)(s) + \frac{L}{2\pi}\,c(s)\,K^{3,0}_{\eta,\mathrm{e}}(\sigma)(s) + M_3(\sigma)(s) \\
&= 2\pi\Big(c(s)\Big)^2\sigma(s) - \pi\,\sigma''(s) - 2\pi\,c'(s)\,H(\sigma)(s) - 3\pi\,c(s)\,H(\sigma')(s) \\
&\quad +N_3(\sigma)(s)\,, \qquad (101) \\
(b) \qquad K^{2,1}_{\gamma,\mathrm{i}}(\sigma)(s) &= -\Big(\pi - \frac{L}{2}\,c(s)\Big)\,\sigma''(s) + \pi\,c'(s)\,H(\sigma)(s) + \frac{L}{2\pi}\,c(s)\,K^{2,1}_{\eta,\mathrm{i}}(\sigma)(s)
\end{aligned}
$$

$$
\begin{aligned}
& +G_3(\sigma)(s) \\
&= -\pi\,\sigma''(s) + \pi\,c'(s)\,H(\sigma)(s) + \pi\,c(s)\,H(\sigma')(s) + G_3(\sigma)(s)\,, \qquad (102)
\end{aligned}
$$

$$
\begin{aligned}
K^{2,1}_{\gamma,\mathrm{e}}(\sigma)(s) &= \Big(\pi - \frac{L}{2}\,c(s)\Big)\,\sigma''(s) + \pi\,c'(s)\,H(\sigma)(s) + \frac{L}{2\,\pi}\,c(s)\,K^{2,1}_{\eta,\mathrm{e}}(\sigma)(s) \\
&\quad +G_3(\sigma)(s) \\
&= \pi\,\sigma''(s) + \pi\,c'(s)\,H(\sigma)(s) + \pi\,c(s)\,H(\sigma')(s) + G_3(\sigma)(s)\,, \qquad (103)
\end{aligned}
$$

(c)

$$
\begin{aligned}
K^{1,2}_{\gamma,\mathrm{i}}(\sigma)(s) &= \Big(\pi - \frac{L}{2}\,c(s)\Big)\,\sigma''(s) + \frac{L}{2\,\pi}\,c(s)\,K^{1,2}_{\eta,\mathrm{i}}(\sigma)(s) + G^{\star}_3(\sigma)(s) \\
&= \pi\,\sigma''(s) + \pi\,c(s)\,H(\sigma')(s) + G_3(\sigma)(s)\,, \qquad (104)
\end{aligned}
$$

$$
\begin{aligned}
K^{1,2}_{\gamma,\mathrm{e}}(\sigma)(s) &= -\Big(\pi - \frac{L}{2}\,c(s)\Big)\,\sigma''(s) + \frac{L}{2\,\pi}\,c(s)\,K^{1,2}_{\eta,\mathrm{e}}(\sigma)(s) + G^{\star}_3(\sigma)(s) \\
&= -\pi\,\sigma''(s) + \pi\,c(s)\,H(\sigma')(s) + G_3(\sigma)(s)\,, \qquad (105)
\end{aligned}
$$

(d)

$$
\begin{aligned}
K^{0,3}_{\gamma,\mathrm{i}}(\sigma)(s) &= \Big(2\,\pi\,\big(c(s)\big)^2 - \frac{4\,\pi^2}{L}\,c(s)\Big)\sigma(s) - \Big(\pi - \frac{L}{2}\,c(s)\Big)\,\sigma''(s) \\
&\quad -\pi\,c'(s)\,H(\sigma)(s) + \frac{L}{2\,\pi}\,c(s)\,K^{0,3}_{\eta,\mathrm{i}}(\sigma)(s) + M^{\star}_3(\sigma)(s) \\
&= 2\,\pi\,\big(c(s)\big)^2\sigma(s) - \pi\,\sigma''(s) - \pi\,c'(s)\,H(\sigma)(s) - 3\,\pi\,c(s)\,H(\sigma')(s) \\
&\quad +N^{\star}_3(\sigma)(s)\,, \qquad (106)
\end{aligned}
$$

$$
\begin{aligned}
K^{0,3}_{\gamma,\mathrm{e}}(\sigma)(s) &= -\Big(2\,\pi\,\big(c(s)\big)^2 - \frac{4\,\pi^2}{L}\,c(s)\Big)\sigma(s) + \Big(\pi - \frac{L}{2}\,c(s)\Big)\,\sigma''(s) \\
&\quad -\pi\,c'(s)\,H(\sigma)(s) + \frac{L}{2\,\pi}\,c(s)\,K^{0,3}_{\eta,\mathrm{e}}(\sigma)(s) + M^{\star}_3(\sigma)(s) \\
&= -2\,\pi\,\big(c(s)\big)^2\sigma(s) + \pi\,\sigma''(s) - \pi\,c'(s)\,H(\sigma)(s) - 3\,\pi\,c(s)\,H(\sigma')(s) \\
&\quad +N^{\star}_3(\sigma)(s)\,, \qquad (107)
\end{aligned}
$$

where $c(s)$ denotes the curvature of γ at $\gamma(s)$, and M_3, N_3, $G_3 : c[0,L] \to c[0,L]$ are integral operators with kernels $m_3(s,t)$, $n_3(s,t)$, $g_3(s,t) \in c^{k-4}([0,L]\times[0,L])$, respectively. Furthermore, $M^{\star}_3$, N_3, $G^{\star}_3$ are the adjoints of M_3, N_3, G_3, and H denotes the Hilbert transform (see (130) in Section 3.3).

2.4 Computational Observations

In the numerical solution of elliptic PDEs, one is often confronted with the task of evaluating some (or all) of the operators (30) – (48) numerically. While this class of issues will be discussed in detail in a sequel to this paper, here we observe that an inspection of the formulae (50) – (68), (89) – (93), (94) – (99), (100) – (107) immediately shows that each of the operators (30) – (48) is a combination of the following: integral operators with smooth kernels, integral operators with the logarithmic singularity on the diagonal, the Hilbert transform, the derivative of the Hilbert

transform, and the second derivative. The techniques for the accurate integration of smooth functions have been available for hundreds of years, and the numerical evaluation of the second derivative presents no serious problems. Effective techniques for the numerical evaluation of the Hilbert transform are less well-known, but have also been available for many years (see, for example, [16]). Efficient integration of logarithmically singular functions is also not very difficult (see [15, 8, 2]). The only possible source of problems is the derivative of the Hilbert transform; quadrature rules for the evaluation of the latter have been constructed, and will be published in [10]. Thus, there exist rapidly convergent schemes for the numerical evaluation of all of the operators (30) – (48), and, therefore, for the discretization of any problem of mathematical physics that has been reduced to a set of integro-pseudodifferential equations involving any (or all) of the operators (30) – (48).

Of course, when a problem of mathematical physics is discretized, one of principal issues is the condition number of the obtained system of equations. An examination of the formulae (51), (57), (52), (58) immediately shows that the operators $K^{1,0}_{\gamma,\mathrm{i}}$, $K^{0,1}_{\gamma,\mathrm{i}}$, $K^{1,0}_{\gamma,\mathrm{e}}$, $K^{0,1}_{\gamma,\mathrm{e}}$ are asymptotically well-conditioned (being a sum of the identity and a compact operator). The spectrum of the operator K^0_γ decays as $1/k$ with k the sequence number of the eigenvalue (see (50)), and its n-point discretization will (asymptotically) have condition number $\sim n$. Each of the operators $K^{2,0}_{\gamma,\mathrm{i}}$, $K^{1,1}_{\gamma,\mathrm{i}}$, $K^{0,2}_{\gamma,\mathrm{i}}$, $K^{2,0}_{\gamma,\mathrm{e}}$, $K^{1,1}_{\gamma,\mathrm{e}}$, $K^{0,2}_{\gamma,\mathrm{e}}$ has a spectrum that grows linearly, and the n-point discretization of each of them will have condition number n. Finally, each of the operators $K^{3,0}_{\gamma,\mathrm{i}}$, $K^{2,1}_{\gamma,\mathrm{i}}$, $K^{1,2}_{\gamma,\mathrm{i}}$, $K^{0,3}_{\gamma,\mathrm{i}}$, $K^{3,0}_{\gamma,\mathrm{e}}$, $K^{2,1}_{\gamma,\mathrm{e}}$, $K^{1,2}_{\gamma,\mathrm{e}}$, $K^{0,3}_{\gamma,\mathrm{e}}$ has a spectrum that grows as k^2; an n-point discretization of any of them will have condition number $\sim n^2$. Thus, whenever the problem to be solved results in the discretization of any one of the operators K^0_γ, $K^{2,0}_{\gamma,\mathrm{i}}$, $K^{1,1}_{\gamma,\mathrm{i}}$, $K^{0,2}_{\gamma,\mathrm{i}}$, $K^{2,0}_{\gamma,\mathrm{e}}$, $K^{1,1}_{\gamma,\mathrm{e}}$, $K^{0,2}_{\gamma,\mathrm{e}}$, $K^{3,0}_{\gamma,\mathrm{i}}$, $K^{2,1}_{\gamma,\mathrm{i}}$, $K^{1,2}_{\gamma,\mathrm{i}}$, $K^{0,3}_{\gamma,\mathrm{i}}$, $K^{3,0}_{\gamma,\mathrm{e}}$, $K^{2,1}_{\gamma,\mathrm{e}}$, $K^{1,2}_{\gamma,\mathrm{e}}$, $K^{0,3}_{\gamma,\mathrm{e}}$ there is a potential for condition number problems, similar to those encountered with direct discretization of differential equations.

Fortunately, formulae (50) – (68) suggest a solution. Specifically, an examination of the formulae (50), (53), (89), (94) immediately indicates that each of the operators $K^0_\gamma \circ K^{2,0}_{\gamma,\mathrm{i}}$, $K^{2,0}_{\gamma,\mathrm{i}} \circ K^0_\gamma$ is a sum of multiplication by a constant with a compact operator, i.e.

$$K^0_\gamma \circ K^{2,0}_{\gamma,\mathrm{i}} = \pi^2 \cdot I + M^{00,20}_{\mathrm{i}}, \tag{108}$$

$$K^{2,0}_{\gamma,\mathrm{i}} \circ K^0_\gamma = \pi^2 \cdot I + M^{20,00}_{\mathrm{i}}, \tag{109}$$

with $M^{20,00}_{\mathrm{i}}$, $M^{00,20}_{\mathrm{i}}$ compact operators $L^2[0,L] \to L^2[0,L]$. Similarly,

$$K^0_\gamma \circ K^{2,0}_{\gamma,\mathrm{e}} = \pi^2 \cdot I + M^{00,20}_{\mathrm{e}}, \tag{110}$$

$$K^{2,0}_{\gamma,\mathrm{e}} \circ K^0_\gamma = \pi^2 \cdot I + M^{20,00}_{\mathrm{e}}, \tag{111}$$

and

$$K^0_\gamma \circ K^{1,1}_{\gamma,\mathrm{i}} = -\pi^2 \cdot I + M^{00,11}_{\mathrm{i}}, \tag{112}$$

$$K^{1,1}_{\gamma,\mathrm{i}} \circ K^0_\gamma = -\pi^2 \cdot I + M^{11,00}_{\mathrm{i}}, \tag{113}$$

$$K^0_\gamma \circ K^{1,1}_{\gamma,\mathrm{e}} = -\pi^2 \cdot I + M^{00,11}_{\mathrm{e}}, \tag{114}$$

$$K^{1,1}_{\gamma,\mathrm{e}} \circ K^0_\gamma = -\pi^2 \cdot I + M^{11,00}_{\mathrm{e}}, \tag{115}$$

and

$$K^0_\gamma \circ K^{0,2}_{\gamma,\mathrm{i}} = \pi^2 \cdot I + M^{00,02}_{\mathrm{i}}, \tag{116}$$

$$K^{0,2}_{\gamma,\mathrm{i}} \circ K^0_\gamma = \pi^2 \cdot I + M^{02,00}_{\mathrm{i}}, \tag{117}$$

$$K^0_\gamma \circ K^{0,2}_{\gamma,\mathrm{e}} = \pi^2 \cdot I + M^{00,02}_{\mathrm{e}}, \tag{118}$$

$$K^{0,2}_{\gamma,\mathrm{e}} \circ K^0_\gamma = \pi^2 \cdot I + M^{02,00}_{\mathrm{e}}; \tag{119}$$

all of the operators $M^{11,00}_{\mathrm{i}}$, $M^{11,00}_{\mathrm{e}}$, $M^{00,11}_{\mathrm{i}}$, $M^{00,11}_{\mathrm{e}}$, $M^{02,00}_{\mathrm{i}}$, $M^{02,00}_{\mathrm{e}}$, $M^{00,02}_{\mathrm{i}}$, $M^{00,02}_{\mathrm{e}}$ are compact. In other words, the operator K^0_γ is a perfect preconditioner (asymptotically speaking) for each of the second order pseudodifferential operators of potential theory in two dimensions; in turn, K^0_γ is preconditioned by each of the operators (94) – (99).

Expressions (100) – (107) contain the second derivative, and are, clearly, preconditioned by the operator of repeated integration $I_2 : L^2[0, L] \to L^2[0, L]$, defined by its action on the functions $e^{i \cdot m \cdot x / L}$ via the formula

$$I_2(e^{i \cdot m \cdot x / L}) = \frac{1}{m^2} \cdot e^{i \cdot m \cdot x / L}. \tag{120}$$

In other words, for each of the operators (30) – (48), there is available a straightforward preconditioner. Numerical implications of these (and related) observations will be discussed in [10].

3 Analytical Preliminaries

3.1 Principal Value Integrals

Integrals of the form

$$\int_a^b \frac{\varphi(t)}{t - s}\, dt, \tag{121}$$

where $s \in (a, b)$, do not exist in the classical sense, and are often referred to as *singular integrals.*

Definition 3.1 *Suppose that φ is a function $[a, b] \to \mathbb{R}$, $s \in (a, b)$, and the limit*

$$\lim_{\epsilon \to 0} \left(\int_a^{s-\epsilon} \frac{\varphi(t)}{t - s}\, dt + \int_{s+\epsilon}^b \frac{\varphi(t)}{t - s}\, dt \right) \tag{122}$$

exists and is finite. Then we will denote the limit (122) by

$$\text{p.v.} \int_a^b \frac{\varphi(t)}{t - s}\, dt, \tag{123}$$

and refer to it as a principal value integral.

Theorem 3.1 *Suppose that the function $\varphi : [a, b] \to \mathbb{R}$ is continuously differentiable in a neighborhood of $s \in (a, b)$. Then the principal value integral (123) exists.*

3.2 Finite Part Integrals

In this paper, we will be dealing with integrals of the form

$$\int_a^b \frac{\varphi(t)}{(t-s)^2}\,dt\,, \tag{124}$$

where $s \in (a,b)$, which are divergent in the classical sense. This type of integrals are often referred to as *hypersingular* or *strongly singular.*

Definition 3.2 *Suppose that φ is a function $[a,b] \to \mathbb{R}$, $s \in (a,b)$, and the limit*

$$\lim_{\epsilon \to 0} \left(\int_a^{s-\epsilon} \frac{\varphi(t)}{(t-s)^2}\,dt + \int_{s+\epsilon}^b \frac{\varphi(t)}{(t-s)^2}\,dt - \frac{2\varphi(s)}{\epsilon} \right) \tag{125}$$

exists and is finite. Then we will denote the limit (125) by

$$\text{f.p.} \int_a^b \frac{\varphi(t)}{(t-s)^2}\,dt\,, \tag{126}$$

and refer to it as a finite part integral (see, for example, [7]).

The following obvious theorem provides sufficient conditions for the existence of the finite part integral (125), and establishes a connection between finite part and principal value integrals.

Theorem 3.2 *Suppose that the function $\varphi : [a,b] \to \mathbb{R}$ is twice continuously differentiable in a neighborhood of $s \in (a,b)$. Then the finite part integral (126) exists, and*

$$\text{f.p.} \int_a^b \frac{\varphi(t)}{(t-s)^2}\,dt = \frac{d}{ds}\,\text{p.v.} \int_a^b \frac{\varphi(t)}{t-s}\,dt\,. \tag{127}$$

3.3 The Hilbert Transform

For an arbitrary periodic function $\varphi \in L^2[-\pi,\pi]$ and any integer k, we will denote by $\widehat{\varphi}_k$ the k-th Fourier coefficient of φ, defined by the formula,

$$\widehat{\varphi}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(s)\, e^{-iks}\,ds\,, \tag{128}$$

so that

$$\varphi(t) = \sum_{k=-\infty}^{\infty} \widehat{\varphi}_k\, e^{ikt}\,, \tag{129}$$

for all $t \in [-\pi,\pi]$.

Definition 3.3 *The Hilbert transform is the mapping $H : L^2[-\pi,\pi] \to L^2[-\pi,\pi]$, given by the formula*

$$H(\varphi)(s) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} -i\ \mathrm{sgn}(k)\,\widehat{\varphi}_k\, e^{iks}\,, \tag{130}$$

with $\varphi \in L^2[-\pi,\pi]$ an arbitrary function. The function $H(\varphi) : [-\pi,\pi] \to \mathbb{C}$ is often referred to as the conjugate function of φ.

The following theorem summarizes several well-known properties of the Hilbert transform (see, for example, [9]).

Theorem 3.3 *(a) The mapping $H : L^2[-\pi,\pi] \to L^2[-\pi,\pi]$ is bounded.*
(b) For any integrable φ, the identity

$$H(\varphi)(s) = \mathrm{p.v.}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\varphi(t)}{\tan\left(\frac{s-t}{2}\right)}\,dt\,, \tag{131}$$

holds almost everywhere.
(c) For any function $\varphi \in c^1[-\pi,\pi]$,

$$H(\varphi')(s) = \Big((H(\varphi))'\Big)(s) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |k|\,\widehat{\varphi}_k\, e^{iks}\,. \tag{132}$$

In other words,

$$HD = DH\,, \tag{133}$$

where $D = \frac{d}{ds}$ is the differentiation operator.

3.4 Boundary Integral Operators

In this subsection, we define boundary the integral operators $K_\gamma^{1,0}$, $K_\gamma^{2,0}$, $K_\gamma^{3,0}$, $K_\gamma^{0,1}$, $K_\gamma^{1,1}$, $K_\gamma^{2,1}$, $K_\gamma^{0,2}$, $K_\gamma^{1,2}$, $K_\gamma^{0,3}$, that are closely related to the operators (31) – (48) defined in Section 2.

Definition 3.4 *Suppose that the function $\sigma : [0,L] \to \mathbb{R}$ is sufficiently smooth. Then we denote by $K_\gamma^{1,0}, K_\gamma^{0,1} : c[0,L] \to c[0,L]$ and $K_\gamma^{2,0}, K_\gamma^{3,0}, K_\gamma^{1,1}, K_\gamma^{2,1}, K_\gamma^{0,2}, K_\gamma^{1,2}, K_\gamma^{0,3} : c^2[0,L] \to c[0,L]$ the operators defined by the formulae*

$$K_\gamma^{1,0}(\sigma)(s) = \int_0^L \frac{\partial \Phi_{\gamma(t)}(\gamma(s))}{\partial N(t)}\,\sigma(t)\,dt\,, \tag{134}$$

$$K_\gamma^{2,0}(\sigma)(s) = \mathrm{f.p.}\int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(t)^2}\,\sigma(t)\,dt\,, \tag{135}$$

$$K_\gamma^{3,0}(\sigma)(s) = \mathrm{f.p.}\int_0^L \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(t)^3}\,\sigma(t)\,dt\,, \tag{136}$$

$$K_\gamma^{0,1}(\sigma)(s) = \int_0^L \frac{\partial \Phi_{\gamma(t)}(\gamma(s))}{\partial N(s)} \sigma(t)\, dt\,, \tag{137}$$

$$K_\gamma^{1,1}(\sigma)(s) = \text{f.p.} \int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(s)\, \partial N(t)} \sigma(t)\, dt\,, \tag{138}$$

$$K_\gamma^{2,1}(\sigma)(s) = \text{f.p.} \int_0^L \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(s)\, \partial N(t)^2} \sigma(t)\, dt\,, \tag{139}$$

$$K_\gamma^{0,2}(\sigma)(s) = \text{f.p.} \int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(s)^2} \sigma(t)\, dt\,, \tag{140}$$

$$K_\gamma^{1,2}(\sigma)(s) = \text{f.p.} \int_0^L \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(s)^2\, \partial N(t)} \sigma(t)\, dt\,, \tag{141}$$

$$K_\gamma^{0,3}(\sigma)(s) = \text{f.p.} \int_0^L \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(s)^3} \sigma(t)\, dt\,, \tag{142}$$

respectively.

Remark 3.4 *Obviously, the operators $K_\gamma^{0,1}$, $K_\gamma^{0,2}$, $K_\gamma^{0,3}$, $K^{1,2}$ given by the formulae (137), (140) - (142) are the adjoints of the operators $K_\gamma^{1,0}$, $K_\gamma^{2,0}$, $K_\gamma^{3,0}$, $K_\gamma^{2,1}$ defined by (134) - (136), (139). Furthermore, $K_\gamma^{1,1}$, defined by (138), is self-adjoint.*

4 Proof of Results

In this section we prove the results from Section 2. The outline of this section is as follows: First, we consider the case where γ is a circle. We provide the proof for Theorem 2.6. In Lemma 4.2 we give explicit formulas for the boundary integral operators (134) – (140) for the case where γ is a circle. Then, by combining Theorem 2.6 and Lemma 4.2, we immediately get the so-called jump conditions for the operators (12) – (25) on a circle. These are stated in Theorem 4.3.

Next, we consider the case where γ is an arbitrary and sufficiently smooth Jordan curve. Since the proof of the identities (94) – (99) in Theorem 2.8 are similar, we only provide the proof for (94) and (95). In fact, (94) and (95) in Theorem 2.8 follow immediately from Theorem 4.7 and Lemma 4.6. The proof of Theorem 4.7 is based on Theorem 4.3 and the approximation (178) given in Lemma 4.5.

Proof of Theorem 2.6 Since the proofs for the identities (50) – (64) are nearly identical, we only provide the proof for the interior limit of the quadruple layer potential (53). Further, it is enough to prove (53) for the case $r = 1$; the general case follows by a simple transformation of variables. We choose the parametrization

$$\gamma(t) = (\cos(t), \sin(t))\,, \tag{143}$$

where $t \in [-\pi, \pi]$. It immediately follows from (143) that

$$\int_{-\pi}^{\pi} \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) - h \cdot N(s))}{\partial N(t)^2} e^{ikt}\, dt =$$
$$= \int_{-\pi}^{\pi} \frac{1 - 2 \cdot (1-h) \cdot \cos(t-s) + (1-h)^2 \cdot \cos(2\,(t-s))}{\left(1 + (1-h)^2 - 2 \cdot (1-h) \cdot \cos(t-s)\right)^2} e^{ikt}\, dt$$
$$= e^{iks} \cdot \int_{-\pi}^{\pi} \frac{1 - 2 \cdot (1-h) \cdot \cos(t) + (1-h)^2 \cdot \cos(2\,t)}{\left(1 + (1-h)^2 - 2 \cdot (1-h) \cdot \cos(t)\right)^2} e^{ikt}\, dt\,, \tag{144}$$

for any $s \in [-\pi, \pi]$. We will use calculus of residues to evaluate the integral (144). To this effect, the substitution

$$z = e^{it}\,, \tag{145}$$

converts (144) into

$$e^{iks} \cdot \int_{-\pi}^{\pi} \frac{1 - 2 \cdot (1-h) \cdot \cos(t) + (1-h)^2 \cdot \cos(2\,t)}{\left(1 + (1-h)^2 - 2 \cdot (1-h) \cdot \cos(t)\right)^2} e^{ikt}\, dt =$$
$$= e^{iks} \cdot \int_{|z|=1} \frac{-i}{z} \left(\frac{1 - (1-h)\,(z + z^{-1}) + \frac{1}{2}\,(1-h)^2\,(z^2 + z^{-2})}{\left(1 + (1-h)^2 - (1-h)\,(z + z^{-1})\right)^2} \right) \cdot z^k\, dz\,, \tag{146}$$

and after simple algebraic manipulation, we get

$$\frac{-i}{z} \left(\frac{1 - (1-h)\,(z + z^{-1}) + \frac{1}{2}\,(1-h)^2\,(z^2 + z^{-2})}{\left(1 + (1-h)^2 - (1-h)\,(z + z^{-1})\right)^2} \right) \cdot z^k =$$
$$= \frac{1}{2} \cdot \left(-\frac{iz^{k+1}}{\left((1-h) - z\right)^2} - \frac{iz^{k-1}}{\left(z\,(1-h) - 1\right)^2} \right). \tag{147}$$

Substituting (147) into (146), we obtain

$$\int_{-\pi}^{\pi} \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) - h \cdot N(s))}{\partial N(t)^2} e^{ikt}\, dt =$$
$$= e^{iks} \cdot \int_{|z|=1} \frac{1}{2} \cdot \left(-\frac{iz^{k+1}}{\left((1-h) - z\right)^2} - \frac{iz^{k-1}}{\left(z\,(1-h) - 1\right)^2} \right) dz\,. \tag{148}$$

Now, formula (53) for $r = 1$ follows by applying a standard residue calculation to (148). □

Remark 4.1 *Formulae (50) - (52), (57) - (58) follow from well-known results (see for example [11, 3]). While the derivation of (53) - (56), (59) - (64) is quite similar, the authors failed to find them in the literature.*

The operators $K_\gamma^{1,0}$, $K_\gamma^{2,0}$, $K_\gamma^{3,0}$, $K_\gamma^{1,1}$, $K_\gamma^{2,1}$, $K_\gamma^{0,1}$, $K_\gamma^{0,2}$, $K_\gamma^{0,3}$, $K_\gamma^{1,2}$ defined by (134) - (141), assume a particularly simple form on the circle. The following lemma follows immediately from an elementary computation.

Lemma 4.2 *Suppose that γ is a circle of radius r parametrized by its arclength with exterior unit normal denoted by N. Then, for any sufficiently smooth function $\sigma : [-\pi r, \pi r] \to \mathbb{C}$:*

$$\text{(a)} \qquad K_\gamma^{1,0}(\sigma)(s) = \int_{-\pi r}^{\pi r} -\frac{\sigma(t)}{2r}\,dt = -\pi\,\widehat{\sigma}_0\,, \tag{149}$$

$$\begin{aligned}\text{(b)} \qquad K_\gamma^{2,0}(\sigma)(s) &= \text{f.p.}\int_{-\pi r}^{\pi r}\left(\frac{1}{2r^2}+\frac{1}{2r^2\,\cos(\frac{t-s}{r})-2r^2}\right)\sigma(t)\,dt \\ &= \pi\,r^{-1}\,\widehat{\sigma}_0 + \pi\,H(\sigma')(s)\,,\end{aligned} \tag{150}$$

$$\begin{aligned}\text{(c)} \qquad K_\gamma^{3,0}(\sigma)(s) &= \text{f.p.}\int_{-\pi r}^{\pi r}\left(-\frac{1}{r^3}-\frac{3}{2r^3\,\cos(\frac{t-s}{r})-2r^3}\right)\sigma(t)\,dt \\ &= -2\,\pi\,r^{-2}\,\widehat{\sigma}_0 - 3\,\pi\,r^{-1}\,H(\sigma')(s)\,,\end{aligned} \tag{151}$$

$$\text{(d)} \qquad K_\gamma^{0,1}(\sigma)(s) = \int_{-\pi r}^{\pi r} -\frac{\sigma(t)}{2r}\,dt = -\pi\,\widehat{\sigma}_0\,, \tag{152}$$

$$\text{(e)} \qquad K_\gamma^{1,1}(\sigma)(s) = \text{f.p.}\int_{-\pi r}^{\pi r}\frac{\sigma(t)}{2r^2-2r^2\,\cos(\frac{t-s}{r})}\,dt = -\pi\,H(\sigma')(s)\,, \tag{153}$$

$$\text{(f)} \qquad K_\gamma^{2,1}(\sigma)(s) = \text{f.p.}\int_{-\pi r}^{\pi r}\frac{\sigma(t)}{2r^3\,\cos(\frac{t-s}{r})-2r^3}\,dt = \pi\,r^{-1}\,H(\sigma')(s)\,, \tag{154}$$

$$\begin{aligned}\text{(g)} \qquad K_\gamma^{0,2}(\sigma)(s) &= \text{f.p.}\int_{-\pi r}^{\pi r}\left(\frac{1}{2r^2}+\frac{1}{2r^2\,\cos(\frac{t-s}{r})-2r^2}\right)\sigma(t)\,dt \\ &= \pi\,r^{-1}\,\widehat{\sigma}_0 + \pi\,H(\sigma')(s)\,,\end{aligned} \tag{155}$$

$$\text{(h)} \qquad K_\gamma^{1,2}(\sigma)(s) = \text{f.p.}\int_{-\pi r}^{\pi r}\frac{\sigma(t)}{2r^3\,\cos(\frac{t-s}{r})-2r^3}\,dt = \pi\,r^{-1}\,H(\sigma')(s)\,, \tag{156}$$

$$\begin{aligned}\text{(i)} \qquad K_\gamma^{0,3}(\sigma)(s) &= \text{f.p.}\int_{-\pi r}^{\pi r}\left(-\frac{1}{r^3}-\frac{3}{2r^3\,\cos(\frac{t-s}{r})-2r^3}\right)\sigma(t)\,dt \\ &= -2\,\pi\,r^{-2}\,\widehat{\sigma}_0 - 3\,\pi\,r^{-1}\,H(\sigma')(s)\,,\end{aligned} \tag{157}$$

where H denotes the Hilbert transform (see (130) in Section 3.3).

The following theorem is an immediate consequence of Theorem 2.6 and Lemma 4.2. It summarizes the so-called jump conditions for the integrals (12) – (29) on the boundary Γ, where Γ is a circle.

Theorem 4.3 *Suppose that γ is a circle of radius r parametrized by its arclength with exterior unit normal denoted by N. Further, suppose that H denotes the Hilbert transform (130). Then, for any sufficiently smooth function $\sigma : [-\pi r, \pi r] \to \mathbb{C}$,*

$$\text{(a)} \qquad K_{\gamma,\mathrm{i}}^{1,0}(\sigma)(s) = -\pi\,\sigma(s) + K_\gamma^{1,0}(\sigma)(s)\,, \tag{158}$$

$$K^{1,0}_{\gamma,\mathrm{e}}(\sigma)(s) = \pi\,\sigma(s) + K^{1,0}_{\gamma}(\sigma)(s)\,, \tag{159}$$

(b)

$$K^{2,0}_{\gamma,\mathrm{i}}(\sigma)(s) = \pi\, r^{-1}\,\sigma(s) + K^{2,0}_{\gamma}(\sigma)(s)\,, \tag{160}$$

$$K^{2,0}_{\gamma,\mathrm{e}}(\sigma)(s) = -\pi\, r^{-1}\,\sigma(s) + K^{2,0}_{\gamma}(\sigma)(s)\,, \tag{161}$$

(c)

$$K^{3,0}_{\gamma,\mathrm{i}}(\sigma)(s) = -2\,\pi\, r^{-2}\,\sigma(s) + \pi\,\sigma''(s) + K^{3,0}_{\gamma}(\sigma)(s)\,, \tag{162}$$

$$K^{3,0}_{\gamma,\mathrm{e}}(\sigma)(s) = 2\,\pi\, r^{-2}\,\sigma(s) - \pi\,\sigma''(s) + K^{3,0}_{\gamma}(\sigma)(s)\,, \tag{163}$$

(d)

$$K^{0,1}_{\gamma,\mathrm{i}}(\sigma)(s) = \pi\,\sigma(s) + (K^{1,0}_{\gamma})^{\star}(\sigma)(s)\,, \tag{164}$$

$$K^{0,1}_{\gamma,\mathrm{e}}(\sigma)(s) = -\pi\,\sigma(s) + (K^{1,0}_{\gamma})^{\star}(\sigma)(s)\,, \tag{165}$$

(e)

$$K^{1,1}_{\gamma,\mathrm{i}}(\sigma)(s) = K^{1,1}_{\gamma,\mathrm{e}}(\sigma)(s) = K^{1,1}_{\gamma}(\sigma)(s) = -\pi\, H(\sigma')(s)\,, \tag{166}$$

(f)

$$K^{2,1}_{\gamma,\mathrm{i}}(\sigma)(s) = -\pi\,\sigma''(s) + K^{2,1}_{\gamma}(\sigma)(s)\,, \tag{167}$$

$$K^{2,1}_{\gamma,\mathrm{e}}(\sigma)(s) = \pi\,\sigma''(s) + K^{2,1}_{\gamma}(\sigma)(s)\,, \tag{168}$$

(g)

$$K^{0,2}_{\gamma,\mathrm{i}}(\sigma)(s) = -\pi\, r^{-1}\,\sigma(s) + K^{0,2}_{\gamma}(\sigma)(s)\,, \tag{169}$$

$$K^{0,2}_{\gamma,\mathrm{e}}(\sigma)(s) = \pi\, r^{-1}\,\sigma(s) + K^{0,2}_{\gamma}(\sigma)(s)\,, \tag{170}$$

(h)

$$K^{1,2}_{\gamma,\mathrm{i}}(\sigma)(s) = \pi\,\sigma''(s) + (K^{2,1}_{\gamma})^{\star}(\sigma)(s)\,, \tag{171}$$

$$K^{1,2}_{\gamma,\mathrm{e}}(\sigma)(s) = -\pi\,\sigma''(s) + (K^{2,1}_{\gamma})^{\star}(\sigma)(s)\,, \tag{172}$$

(i)

$$K^{0,3}_{\gamma,\mathrm{i}}(\sigma)(s) = 2\,\pi\, r^{-2}\,\sigma(s) - \pi\,\sigma''(s) + (K^{3,0}_{\gamma})^{\star}(\sigma)(s)\,, \tag{173}$$

$$K^{0,3}_{\gamma,\mathrm{e}}(\sigma)(s) = -2\,\pi\, r^{-2}\,\sigma(s) + \pi\,\sigma''(s) + (K^{3,0}_{\gamma})^{\star}(\sigma)(s)\,. \tag{174}$$

We now proceed to the case where γ is an arbitrary sufficiently smooth Jordan curve. The following obvious lemma can be found in most elementary textbooks on differential geometry (see, for example, [4]).

Lemma 4.4 *Suppose that $\gamma : [0, L] \to \mathbb{R}^2$ is a sufficiently smooth Jordan curve parametrized by its arclength with the exterior unit normal and the unit tangent vectors at $\gamma(s)$ denoted by $N(s)$ and $T(s)$, respectively. Then, there exist a positive real number a (dependent on γ), and two continuously differentiable functions $f, g : (-a, a) \to \mathbb{R}$ (dependent on γ), such that for any $s \in [0, L]$,*

$$\gamma(s+t) - \gamma(s) = \Big(t + t^3 \cdot f(t)\Big) \cdot T(s) - \left(\frac{c\,t^2}{2} + t^3 \cdot g(t)\right) \cdot N(s)\,, \tag{175}$$

for all $t \in (-a, a)$, where the coefficient c in (175) is the curvature of γ at the point $\gamma(s)$. Furthermore, for all $t \in (-a, a)$,

$$|f(t)| \leq \|\,\gamma'''(s)\|\,, \tag{176}$$

$$|g(t)| \leq \|\,\gamma'''(s)\|\,. \tag{177}$$

In the local parametrization (175), the potential of a quadrupole located at $\gamma(s)$ and oriented in the direction $N(s)$ assumes a particularly simple form, given by the following lemma.

Lemma 4.5 *Suppose that $\gamma : [0, L] \to \mathbb{R}^2$ is a sufficiently smooth Jordan curve parametrized by its arclength. Then, there exist real positive numbers A, a and h_0 such that for any $s \in [0, L]$*

$$\left| \frac{\partial^2 \Phi_{\gamma(s+t)}(\gamma(s) - h \cdot N(s))}{\partial N(s+t)^2} - \frac{h^2 - t^2}{(h^2+t^2)^2} - \frac{c\,h\,t^2\,(5\,h^2+t^2)}{(h^2+t^2)^3} \right| \leq A\,, \tag{178}$$

for all $t \in (-a, a)$, $0 \leq h < h_0$, where the coefficient c in (178) is the curvature of γ at the point $\gamma(s)$.

Proof. Without loss of generality, it is sufficient to prove the lemma for the case where $s = 0$, $\gamma(0) = 0$, and $\gamma'(0) = (1, 0)$. Substituting (175) into (9) and evaluating the result at $x = (0, h)$, we obtain

$$\frac{\partial^2 \Phi_{\gamma(t)}(x)}{\partial N(t)^2} \;=\; \frac{p_0(h,t)}{\left(h^2+t^2+r(h,t)\right)^2}\,, \tag{179}$$

where $p_0, r : \mathbb{R}^2 \to \mathbb{R}$ are functions given by the formulae

$$\begin{aligned}
p_0(h,t) \;=\; & \Big[h - t + c\,h\,t + \frac{c\,t^2}{2} - \frac{c^2\,t^3}{2} + 3\,h\,t^2\Big(f(t)+g(t)\Big) - 2\,t^3\Big(2\,f(t)-g(t)\Big) \\
& - \frac{c\,t^4}{2}\Big(f(t)+5\,g(t)\Big) + h\,t^3\Big(f'(t)+g'(t)\Big) - t^4\Big(f'(t)-g'(t)\Big) - 3\,t^5\Big(f(t)^2+g(t)^2\Big) \\
& - \frac{c\,t^5}{2}\Big(f'(t)+g'(t)\Big) - t^6\,f(t)\Big(f'(t)-g'(t)\Big) - t^6\,g(t)\Big(f'(t)+g'(t)\Big)\Big] \\
& \cdot \Big[h + t - c\,h\,t + \frac{c\,t^2}{2} + \frac{c^2\,t^3}{2} + 3\,h\,t^2\Big(f(t)-g(t)\Big) + 2\,t^3\Big(2\,f(t)+g(t)\Big) \\
& - \frac{c\,t^4}{2}\Big(f(t)-5\,g(t)\Big) + h\,t^3\Big(f'(t)-g'(t)\Big) + t^4\Big(f'(t)+g'(t)\Big) + 3\,t^5\Big(f(t)^2+g(t)^2\Big) \\
& - \frac{c\,t^5}{2}\Big(f'(t)-g'(t)\Big) + t^6\,f(t)\Big(f'(t)+g'(t)\Big) - t^6\,g(t)\Big(f'(t)-g'(t)\Big)\Big]\,,
\end{aligned} \tag{180}$$

$$r(h,t) \;=\; -c\,h\,t^2 - 2\,h\,t^3\,g(t) + \frac{c^2\,t^4}{4} + 2\,t^4\,f(t) + c\,t^5\,g(t) + t^6\Big(f(t)^2+g(t)^2\Big)\,. \tag{181}$$

We also introduce the notation

$$p_1(h,t) = \Big(h^2+t^2+r(h,t)\Big)^2 - \Big(h^2+t^2\Big)^2 = 2\,(h^2+t^2)\cdot r(h,t) + r(h,t)^2\,. \tag{182}$$

Obviously, (180) – (182) are algebraic combinations of f, g, f', g', t, and h, and an examination of formulae (180) – (182) immediately shows that there exist positive real numbers a, h_0, and

C (dependent on γ) such that

$$\begin{aligned}
\left|p_0(h,t) - h^2 + t^2 - 3\,c\,h\,t^2\right| &\leq C(h^2+t^2)^2\,, && (183)\\
\left|p_0(h,t)\cdot p_1(h,t) - 2\,c\,h\,t^2\,(h^2+t^2)\,(h^2-t^2)\right| &\leq C\,(h^2+t^2)^4\,, && (184)\\
\left|p_0(h,t)\cdot p_1(h,t)^2\right| &\leq C\,(h^2+t^2)^6\,, && (185)\\
\left|\frac{p_1(h,t)}{(h^2+t^2)^2}\right| &< 1\,, && (186)
\end{aligned}$$

for all $h < h_0$, $t \in (-a,a)$. Substituting (182) into (179), we have

$$\begin{aligned}
\frac{\partial^2 \Phi_{\gamma(t)}(x)}{\partial N(t)^2} &= \frac{p_0(h,t)}{(h^2+t^2)^2\left(1+\frac{p_1(h,t)}{(h^2+t^2)^2}\right)}\\
&= \frac{p_0(h,t)}{(h^2+t^2)^2}\sum_{k=0}^{\infty}(-1)^k\,\frac{p_1(h,t)^k}{(h^2+t^2)^{2k}}\,, && (187)
\end{aligned}$$

where the convergence of the series follows from (186). Combining (183) – (185), we obtain

$$\begin{aligned}
&\left|\frac{\partial^2 \Phi_{\gamma(t)}(x)}{\partial N(t)^2} - \frac{h^2-t^2}{(h^2+t^2)^2} - \frac{c\,h\,t^2\,(5\,h^2+t^2)}{(h^2+t^2)^3}\right| \leq \left|\frac{p_0(h,t) - h^2 + t^2 - 3\,c\,h\,t^2}{(h^2+t^2)^2}\right|\\
&\quad + \left|\frac{p_0(h,t)\cdot p_1(h,t) - 2\,c\,h\,t^2\,(h^2+t^2)\,(h^2-t^2)}{(h^2+t^2)^4}\right| + \sum_{k=2}^{\infty}\left|\frac{p_0(h,t)\cdot p_1(h,t)^k}{(h^2+t^2)^{2k+2}}\right|\\
&\leq\ 2\,C + C\cdot\frac{\alpha^2}{1-\alpha}\,, && (188)
\end{aligned}$$

with α defined by the formula

$$\alpha = \sup_{h<h_0\,,\,t\in(-a,a)}\left|\frac{p_1(h,t)}{(h^2+t^2)^2}\right|\,. \qquad (189)$$

Now, introducing the notation

$$A = 2\,C + C\cdot\frac{\alpha^2}{1-\alpha}\,, \qquad (190)$$

we obtain (178). □

Lemma 4.2 provides an explicit formula for the operator $K_\gamma^{2,0}$, defined in (135), in the case when γ is a circle. The following lemma shows that the operator $K_\gamma^{2,0}$ on an arbitrary sufficiently smooth Jordan curve of length L, is a compact perturbation of $K_\gamma^{2,0}$ on the circle of radius $\frac{L}{2\pi}$. Its proof is an immediate consequence of estimate (178) in Lemma 4.5.

Lemma 4.6 *Suppose that $\gamma : [0,L] \to \mathbb{R}^2$ is a sufficiently smooth Jordan curve parametrized by its arclength, and that $\eta : [0,L] \to \mathbb{R}^2$ denotes the circle of radius $\frac{L}{2\pi}$, also parametrized*

by its arclength. In addition, suppose that $\sigma : [0, L] \to \mathbb{R}$ *is a twice continuously differentiable function. Then,*

$$\text{f.p.} \int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(t)^2} \sigma(t)\, dt = \text{f.p.} \int_0^L \frac{\partial^2 \Phi_{\eta(t)}(\eta(s))}{\partial N(t)^2} \sigma(t)\, dt + M_2(\sigma)(s)\,, \tag{191}$$

where $M_2 : c[0, L] \to c[0, L]$ *is a compact operator defined by the formula*

$$M_2(\sigma)(s) = \int_0^L \left(\frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(t)^2} - \frac{\partial^2 \Phi_{\eta(t)}(\eta(s))}{\partial N(t)^2} \right) \sigma(t)\, dt\,. \tag{192}$$

Furthermore, for any $t \neq s$,

$$\begin{aligned} m_2(s,t) \;=\; & \frac{2\,\langle N(t), \gamma(s) - \gamma(t)\rangle^2}{\|\gamma(s) - \gamma(t)\|^4} - \frac{1}{2}\left(\frac{2\,\pi}{L}\right)^2 \\ & + \frac{\|\gamma(s) - \gamma(t)\|^2 - 2\left(\frac{L}{2\,\pi}\right)^2 \left(1 - \cos\left(\frac{2\,\pi}{L}(s-t)\right)\right)}{\|\gamma(s) - \gamma(t)\|^2\, 2\left(\frac{L}{2\,\pi}\right)^2 \left(1 - \cos\left(\frac{2\,\pi}{L}(s-t)\right)\right)}\,, \end{aligned} \tag{193}$$

and for $t = s$,

$$m_2(s,s) = \frac{5}{12}\left(c(s)\right)^2 - \frac{5}{12}\left(\frac{2\,\pi}{L}\right)^2\,, \tag{194}$$

where $c(s)$ *is the curvature of* γ *at the point* $\gamma(s)$, *and* $m_2 : [0, L] \times [0, L] \to \mathbb{R}$ *is the kernel of the operator* M_2.

The following theorem provides the so-called jump conditions for the operators (14) and (15) on the boundary Γ, when Γ is sufficiently smooth.

Theorem 4.7 *Suppose that* $\gamma : [0, L] \to \mathbb{R}^2$ *is a sufficiently smooth Jordan curve parametrized by its arclength. Then, for any sufficiently smooth function* $\sigma : [0, L] \to \mathbb{R}$,

$$\begin{aligned} & K^{2,0}_{\gamma,\mathrm{e}}(\sigma)(s) - K^{2,0}_{\gamma,\mathrm{i}}(\sigma)(s) = \\ & \quad = \lim_{h \to 0} \int_0^L \left(\frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) + h \cdot N(s))}{\partial N(t)^2} - \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) - h \cdot N(s))}{\partial N(t)^2} \right) \sigma(t)\, dt \\ & \quad = -2\,\pi\, c(s)\, \sigma(s)\,, \end{aligned} \tag{195}$$

and

$$\begin{aligned} & K^{2,0}_{\gamma,\mathrm{e}}(\sigma)(s) + K^{2,0}_{\gamma,\mathrm{i}}(\sigma)(s) = \\ & \quad = \lim_{h \to 0} \int_0^L \left(\frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) + h \cdot N(s))}{\partial N(t)^2} + \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) - h \cdot N(s))}{\partial N(t)^2} \right) \sigma(t)\, dt \\ & \quad = 2 \cdot \text{f.p.} \int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(t)^2} \sigma(t)\, dt\,, \end{aligned} \tag{196}$$

where $c(s)$ denotes the curvature of γ at $\gamma(s)$. In other words, the quadruple layer potential with density σ (see (6)), can be continuously extended from Ω to $\overline{\Omega}$ and from $\mathbb{R}^2 \setminus \overline{\Omega}$ to $\mathbb{R}^2 \setminus \Omega$, with the limiting values given by the formulae

$$p^{2,0}_{\gamma,\sigma,\mathrm{i}}(s) = K^{2,0}_{\gamma,\mathrm{i}}(\sigma)(s) = \pi\, c(s)\, \sigma(s) + \text{f.p.} \int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(t)^2}\, \sigma(t)\, dt\,, \tag{197}$$

$$p^{2,0}_{\gamma,\sigma,\mathrm{e}}(s) = K^{2,0}_{\gamma,\mathrm{e}}(\sigma)(s) = -\pi\, c(s)\, \sigma(s) + \text{f.p.} \int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(t)^2}\, \sigma(t)\, dt\,. \tag{198}$$

Proof. Without loss of generality, we can assume that $s \neq 0$ and $s \neq L$. We begin by proving (196). Suppose that $\eta : [0, L] \to \mathbb{R}^2$ is the circle of radius $\frac{L}{2\pi}$ parametrized by its arclength. We define the functions $\Sigma^h_\gamma, \Sigma^h_\eta : [0, L] \times [0, L] \to \mathbb{R}$ via the formulae

$$\Sigma^h_\gamma(s,t) = \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) + h \cdot N(s))}{\partial N(t)^2} + \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) - h \cdot N(s))}{\partial N(t)^2}\,, \tag{199}$$

$$\Sigma^h_\eta(s,t) = \frac{\partial^2 \Phi_{\eta(t)}(\eta(s) + h \cdot N(s))}{\partial N(t)^2} + \frac{\partial^2 \Phi_{\eta(t)}(\eta(s) - h \cdot N(s))}{\partial N(t)^2}\,, \tag{200}$$

and, substituting (199), (200) into (196), obtain the identity

$$K^{2,0}_{\gamma,\mathrm{e}}(\sigma)(s) + K^{2,0}_{\gamma,\mathrm{i}}(\sigma)(s) = \lim_{h\to 0} \int_0^L \Sigma^h_\eta(s,t)\, \sigma(t)\, dt + \lim_{h\to 0} \int_0^L \left(\Sigma^h_\gamma(s,t) - \Sigma^h_\eta(s,t)\right) \sigma(t)\, dt\,. \tag{201}$$

Substituting (160), (161) in Theorem 4.3 into (201), we have

$$K^{2,0}_{\gamma,\mathrm{e}}(\sigma)(s) + K^{2,0}_{\gamma,\mathrm{i}}(\sigma)(s) =$$
$$= 2 \cdot \text{f.p.} \int_0^L \frac{\partial^2 \Phi_{\eta(t)}(\eta(s))}{\partial N(t)^2}\, \sigma(t)\, dt + \lim_{h\to 0} \int_0^L \left(\Sigma^h_\gamma(s,t) - \Sigma^h_\eta(s,t)\right) \sigma(t)\, dt\,. \tag{202}$$

Due to Lemma 4.5, there exist positive real constants C_0, a, and h_0 such that for any $s \in [0, L]$

$$\left|\Sigma^h_\gamma(s,t) - \Sigma^h_\eta(s,t)\right| \leq C_0\,, \tag{203}$$

for all $|t - s| < a$, $0 \leq h < h_0$. For any $t \neq s$ and sufficiently small h, both $\Sigma^h_\gamma(s,t)$ and $\Sigma^h_\eta(s,t)$ are c^∞-functions. Therefore, there also exist positive real constants h_1, C_1 such that for any $s \in [0, L]$

$$\left|\Sigma^h_\gamma(s,t) - \Sigma^h_\eta(s,t)\right| \leq C_1\,, \tag{204}$$

for all $|t - s| > a$, $0 \leq h < h_1$. Now, applying Lebesgue's dominated convergence theorem (see, for example, [18]) to the second integral of the right hand side of (202), we obtain

$$\lim_{h\to 0} \int_0^L \left(\Sigma^h_\gamma(s,t) - \Sigma^h_\eta(s,t)\right) \sigma(t)\, dt =$$
$$= \int_0^L \lim_{h\to 0} \left(\Sigma^h_\gamma(s,t) - \Sigma^h_\eta(s,t)\right) \sigma(t)\, dt$$
$$= 2 \cdot \int_0^L \left(\frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(t)^2} - \frac{\partial^2 \Phi_{\eta(t)}(\eta(s))}{\partial N(t)^2}\right) \sigma(t)\, dt\,. \tag{205}$$

Finally, formula (196) immediately follows from the combination of (202), (205) with (191), (192) in Lemma 4.6.

We now proceed by proving formula (195). We define the functions $\Delta_\gamma^h, \Delta_\eta^h : [0,L]\times[0,L] \to \mathbb{R}$ via the formulae

$$\Delta_\gamma^h(s,t) = \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s)+h\cdot N(s))}{\partial N(t)^2} - \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s)-h\cdot N(s))}{\partial N(t)^2}, \tag{206}$$

$$\Delta_\eta^h(s,t) = \frac{\partial^2 \Phi_{\eta(t)}(\eta(s)+h\cdot N(s))}{\partial N(t)^2} - \frac{\partial^2 \Phi_{\eta(t)}(\eta(s)-h\cdot N(s))}{\partial N(t)^2}, \tag{207}$$

and, by substituting (206), (207) into (195), obtain the identity

$$\begin{aligned} &K^{2,0}_{\gamma,\mathrm{e}}(\sigma)(s) - K^{2,0}_{\gamma,\mathrm{i}}(\sigma)(s) = \\ &\quad = \frac{c(s)\,L}{2\,\pi}\cdot \lim_{h\to 0}\int_0^L \Delta_\eta^h(s,t)\,\sigma(t)\,dt + \lim_{h\to 0}\int_0^L \Big(\Delta_\gamma^h(s,t) - \frac{c(s)\,L}{2\,\pi}\cdot\Delta_\eta^h(s,t)\Big)\,\sigma(t)\,dt\,. \end{aligned} \tag{208}$$

Substituting (160), (161) in Theorem 4.3 into (208), we get

$$\begin{aligned} &K^{2,0}_{\gamma,\mathrm{e}}(\sigma)(s) - K^{2,0}_{\gamma,\mathrm{i}}(\sigma)(s) = \\ &\qquad = -2\,\pi\,c(s)\sigma(s) + \lim_{h\to 0}\int_0^L \Big(\Delta_\gamma^h(s,t) - \frac{c(s)\,L}{2\,\pi}\cdot\Delta_\eta^h(s,t)\Big)\,\sigma(t)\,dt\,. \end{aligned} \tag{209}$$

Due to Lemma 4.5, there exist positive real constants C_0, a, and h_0 such that for any $s\in[0,L]$

$$\left|\Delta_\gamma^h(s,t) - \frac{c(s)\,L}{2\,\pi}\cdot\Delta_\eta^h(s,t)\right| \le C_0\,, \tag{210}$$

for all $|t-s|<a$, $0\le h<h_0$. For any $t\neq s$ and sufficiently small h, both $\Delta_\gamma^h(s,t)$ and $\Delta_\eta^h(s,t)$ are c^∞-functions. Therefore, there also exist positive real constants h_1, C_1 such that for any $s\in[0,L]$

$$\left|\Delta_\gamma^h(s,t) - \frac{c(s)\,L}{2\,\pi}\cdot\Delta_\eta^h(s,t)\right| \le C_1\,, \tag{211}$$

for all $|t-s|>a$, $0\le h<h_1$. Applying Lebesgue's dominated convergence theorem (see, for example, [18]) to the second integral of the right hand side of (209), we have

$$\lim_{h\to 0}\int_0^L \Big(\Delta_\gamma^h(s,t) - \frac{c(s)\,L}{2\,\pi}\cdot\Delta_\eta^h(s,t)\Big)\,\sigma(t)\,dt = \int_0^L \lim_{h\to 0}\Big(\Delta_\gamma^h(s,t) - \frac{c(s)\,L}{2\,\pi}\cdot\Delta_\eta^h(s,t)\Big)\,\sigma(t)\,dt\,. \tag{212}$$

Examining (206), (207), we obviously have

$$\lim_{h\to 0}\Big(\Delta_\gamma^h(s,t) - \frac{c(s)\,L}{2\,\pi}\cdot\Delta_\eta^h(s,t)\Big) = 0\,. \tag{213}$$

Therefore, the integral on the right hand side of (212) is zero, from which (195) follows immediately. □

5 Generalizations

We have presented explicit (modulo an integral operator with a smooth kernel) formulae for integro-pseudodifferential operators of potential theory in two dimensions (up to order 2). The work presented here admits several obvious extensions.

a. Formulae (89) – (107) have their counterparts for elliptic PDEs other than the Laplace equation. Indeed, for any elliptic PDE in two dimensions, the Green's formula has the form

$$G(x,y) = \phi(x,y) \cdot \log(\|x-y\|) + \psi(x,y)\,, \tag{214}$$

with ϕ, ψ a pair of smooth functions; derivations of Section 4 are almost unchanged when $\log(\|x-y\|)$ is replaced with (214). In particular, the counterparts of the formulae (89) – (99) for the Helmholtz equation (with either real or complex Helmholtz coefficient) are identical to (89) – (99); the counterparts of the formulae (100) – (107) for the Helmholtz equation do not coincide with (100) – (107) exactly; instead, they assume the form

$$\begin{aligned}
\text{(a)} \qquad K^{3,0}_{\gamma,\mathrm{i}}(\sigma)(s) &= -2\pi\left(c(s)\right)^2\sigma(s) + 4\pi k^2\sigma(s) + \pi\sigma''(s) - 2\pi c'(s)\,H(\sigma)(s) \\
&\quad -3\pi c(s)\,H(\sigma')(s) + N_3(\sigma)(s)\,, && (215)\\
K^{3,0}_{\gamma,\mathrm{e}}(\sigma)(s) &= 2\pi\left(c(s)\right)^2\sigma(s) - 4\pi k^2\sigma(s) - \pi\sigma''(s) - 2\pi c'(s)\,H(\sigma)(s) \\
&\quad -3\pi c(s)\,H(\sigma')(s) + N_3(\sigma)(s)\,, && (216)\\
\text{(b)} \qquad K^{2,1}_{\gamma,\mathrm{i}}(\sigma)(s) &= -4\pi k^2\sigma(s) - \pi\sigma''(s) + \pi c'(s)\,H(\sigma)(s) + \pi c(s)\,H(\sigma')(s) \\
&\quad +G_3(\sigma)(s)\,, && (217)\\
K^{2,1}_{\gamma,\mathrm{e}}(\sigma)(s) &= 4\pi k^2\sigma(s) + \pi\sigma''(s) + \pi c'(s)\,H(\sigma)(s) + \pi c(s)\,H(\sigma')(s) \\
&\quad +G_3(\sigma)(s)\,, && (218)\\
\text{(c)} \qquad K^{1,2}_{\gamma,\mathrm{i}}(\sigma)(s) &= 4\pi k^2\sigma(s) + \pi\sigma''(s) + \pi c(s)\,H(\sigma')(s) + \widetilde{G}_3(\sigma)(s)\,, && (219)\\
K^{1,2}_{\gamma,\mathrm{e}}(\sigma)(s) &= -4\pi k^2\sigma(s) - \pi\sigma''(s) + \pi c(s)\,H(\sigma')(s) + \widetilde{G}_3(\sigma)(s)\,, && (220)\\
\text{(d)} \qquad K^{0,3}_{\gamma,\mathrm{i}}(\sigma)(s) &= 2\pi\left(c(s)\right)^2\sigma(s) - 4\pi k^2\sigma(s) - \pi\sigma''(s) - \pi c'(s)\,H(\sigma)(s) \\
&\quad -3\pi c(s)\,H(\sigma')(s) + \widetilde{N}_3(\sigma)(s)\,, && (221)\\
K^{0,3}_{\gamma,\mathrm{e}}(\sigma)(s) &= -2\pi\left(c(s)\right)^2\sigma(s) + 4\pi k^2\sigma(s) + \pi\sigma''(s) - \pi c'(s)\,H(\sigma)(s) \\
&\quad -3\pi c(s)\,H(\sigma')(s) + \widetilde{N}_3(\sigma)(s)\,, && (222)
\end{aligned}$$

where $k \in \mathbb{C}$ is the Helmholtz coefficient, and the operators N_3, G_3, $\widetilde{N}_3$, $\widetilde{G}_3 : L^2[0,L] \to L^2[0,L]$ are compact.

b. The derivation of the three-dimensional counterparts of formulae (89) – (107) is completely straightforward; such expressions have been obtained, and the paper reporting them is in preparation.

c. In certain areas of mathematical physics, one encounters integro-pseudodifferental equations whose analysis is outside the scope of this paper. An important example is the Stratton-Chew equations, to which Maxwell's equations are frequently reduced in computational electromagnetics. Another source of such problems is the scattering of elastic waves in solids. Problems of this type are currently under investigation.

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