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Department of Computer Science

YALE UNIVERSITY

New Haven, Connecticut 06520

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A RITZ METHOD FOR AN OPTIMAL CONTROL PROBLEM

by

Martin H. Schultz

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In this paper we generalize the results and greatly simplify the proofs of the basic papers of Bosarge and Johnson cf. [2], [3], and [4], on a variational method for approximating the solution of the "state regulator problem" in optimal control. In particular, we consider the Lagrange formulation of the problem and show that the Lagrange multiplier can be characterized as the solution of the variational problem of minimizing a quadratic, positive definite functional, F, over an appropriate function space, Φ^n.

We obtain approximate solutions by using Ritz's idea of minimizing F over finite dimensional subspaces of Φ^n and derive general a priori error bounds for this procedure in terms of quantities in "approximation theory." Finally, we apply these results to obtain asymptotic error bounds for the special case of spline type subspaces of Φ^n.

Let $Q(t)$ and $R(t)$ be an $n \times r$ symmetric, positive definite matrix and an $r \times r$ symmetric, positive definite matrix both of which are continuous functions of $t \in [0,T]$. For each $k \geq 0$ let Φ^k denote the set of functions from $[0,T]$ to R^k which are piecewise differentiable with bounded derivative.

The state regulator problem in optimal control is to find $u^* \in \Phi^r$ and $x^* \in \Phi^n$ which minimize

$$(1) \quad J[u,x] \equiv 1/2 \int_0^T \{\langle x(t),Q(t)x(t)\rangle_n + \langle u(t),R(t)u(t)\rangle_r\}dt$$

over all $u \in \Phi^r$, where $x(t)$ is given by

$$(2) \quad \dot{x}(t) = A(t)x(t) + B(t)u(t) \quad , \ t > 0$$

and

$$(3) \quad x(0) = x_0 \quad ,$$

$$|y|_n^2 \equiv \langle y,y\rangle_n \equiv \sum_{i=1}^{n} y_i^2 \qquad \text{for all } y \in R^n ,$$

$$|z|_r^2 \equiv \langle z,z\rangle_r \equiv \sum_{i=1}^{r} z_i^2 , \text{ for all } z \in R^r ,$$

$A(t)$ is an $n \times n$ matrix and $B(t)$ is an $n \times r$ matrix both of which are piecewise continuous with respect to $t \in [0,T]$.

By standard arguments in the calculus of variations, cf. [1], one can show that the above problem is equivalent to the variational problem of finding $\lambda^* \in \Phi^n$ which minimizes

$$(4) \quad -L[u,x;\lambda,\gamma] \equiv J[u,x] + \int_0^T \langle \lambda(t), -\dot{x}(t) + A(t)x(t) + B(t)u(t) \rangle_n dt$$

$$+ \langle \gamma, x(0) - x_0 \rangle_n .$$

subject to the constraint

$$(5) \quad \lambda(T) = 0,$$

where γ, $u(t)$, and $x(t)$ are given by

$$(6) \quad \gamma = -\lambda(0),$$

$$(7) \quad u(t) = -R^{-1}(t)\, B^T(t)\lambda(t) , \quad \text{for all } t \in [0,T], \text{ and}$$

$$(8) \quad x(t) = -Q^{-1}(t) \quad (\dot{\lambda}(t) + A^T(t)\lambda(t)), \text{ for all } t \in [0,T].$$

Using the characterizations (6), (7), and (8), we can express $-L[u,x;\lambda,\gamma]$ in terms of only λ. In fact,

$$- L[u;x;\lambda,\gamma] = - J[u,x] + \langle \lambda(t), x(t) \rangle_n \Big|_0^T - \int_0^T \langle \dot{\lambda} + A^T\lambda, x \rangle\, dt$$

$$- \int_0^T \langle B^T\lambda, u \rangle_r \, dt + \langle \lambda(0), x(0) - x_0 \rangle_n = J[u,x] - \langle \lambda(0), x_0 \rangle_n .$$

But $1/2 \int_0^T \langle u, Ru \rangle_r dt = 1/2 \int_0^T \langle R^{-1}B^T\lambda, RR^{-1} B^T\lambda \rangle_r$

$= 1/2 \int_0^T \langle BR^{-1}B^T\lambda, \lambda \rangle_n \, dt$

and

$$1/2 \int_0^T \langle x, Qx \rangle_n dt = 1/2 \int_0^T \langle Q^{-1}A^T\lambda + Q^{-1}\mathring{\lambda}, A^T \lambda + \mathring{\lambda} \rangle_n dt$$

$$= 1/2 \int_0^T \langle Q^{-1}\mathring{\lambda}, \mathring{\lambda} \rangle \, dt + 1/2 \int_0^T \langle Q^{-1}A^T\lambda, A^T\lambda \rangle_n dt$$

$$+ 1/2 \int_0^T \langle Q^{-1}A^T\lambda, \mathring{\lambda} \rangle \, dt + 1/2 \int_0^T \langle Q^{-1}\mathring{\lambda}, A^T\lambda \rangle_n \, dt$$

$$= 1/2 \int_0^T \langle Q^{-1}\mathring{\lambda}, \mathring{\lambda} \rangle_n dt + 1/2 \int_0^T \langle AQ^{-1}A^T\lambda, \lambda \rangle_n \, dt$$

$$+ \int_0^T \langle AQ^{-1}\mathring{\lambda}, \lambda \rangle_n \, dt.$$ Thus,

$$F[\lambda] \equiv - L[u,x;\lambda,\delta] = 1/2 \int_0^T \langle Q^{-1}\mathring{\lambda}, \mathring{\lambda} \rangle \, dt + 1/2 \int_0^T \langle AQ^{-1}A^T\lambda, \lambda \rangle_n \, dt$$

$$+ 1/2 \int_0^T \langle BR^{-1}B^T\lambda, \lambda \rangle_n dt + \int_0^T \langle AQ^{-1}\mathring{\lambda}, \lambda \rangle_n dt \quad - \langle \lambda(0), x_0 \rangle_n.$$

If we define

$$(9)\quad [\lambda,\eta] \equiv \int_0^T \langle Q^{-1}\dot{\lambda},\dot{\eta}\rangle_n dt + \int_0^T \langle AQ^{-1}A^T\lambda,\lambda\rangle_n dt$$

$$+ \int_0^T \langle BR^{-1}BT\lambda,\lambda\rangle_n dt + \int_0^T \{\langle AQ^{-1}\dot{\lambda},\eta\rangle_n + \langle AQ^{-1}\dot{\eta},\lambda\rangle_n\} dt,$$

for all λ and η in Φ^n, then

$$(10)\qquad F[\lambda] = \frac{1}{2}[\lambda,\lambda] - \langle \lambda(0), x_0\rangle_n .$$

If we use the notation that for any $t \times t$ matrix M, $t > 0$,

$$|M|_t \equiv \max\{\, |Mx|_t \mid x \in R^t \text{ and } |x|_t = 1 \,\},$$

we may prove

Theorem 1. The optimal Lagrange multiplier exists and is the unique solution in $\Phi_0^n \equiv \{\phi \in \Phi^n \mid \phi(T) = \underline{0}\}$ of

$$(11)\qquad [\lambda,\eta] = \langle \eta(0), x_0\rangle_n, \quad \text{for all } \eta \in \Phi_0^n .$$

Moreover, $[\lambda,\eta]$ is symmetric,

$$(12)\quad \|\dot{\lambda}\|_2^2 \equiv \int_0^T \langle \dot{\lambda},\dot{\lambda}\rangle_n dt \le 2\lambda_Q^{-1}[\, \|Q\|_2 + \rho\|A^T\|_2\,]^2[\lambda,\lambda] ,$$

and

$$(13)\qquad \|\lambda\|_2^2 \le 2\lambda_Q^{-1}\rho^2[\lambda,\lambda] ,$$

where $\lambda_Q \equiv \min_{0 \le t \le T} \{ \lambda(t) | \lambda(t)$ is an eigenvalue of $Q(t)\}$,

$$\|Q\|_2^2 \equiv \int_0^T |Q(t)|_n^2 \, dt, \quad \|A^T\|_2^2 \equiv \int_0^T |A^T(t)|_n^2 \, dt, \quad \|A^T\|_\infty \equiv \sup_{0 \le t \le T} |A^T(t)|_n$$

and $\rho \equiv \|Q\|_2 \left(2\|A^T\|_\infty \right)^{-1/2} e^{\|A^T\|_\infty T}$.

Proof. The existence part of the Theorem is a standard result in optimal control theory , cf. [1]. If $\eta \in \Phi_0^n$ and $\alpha \in R$, $F[\lambda^* + \alpha\eta] \ge F[\lambda^*]$ with equality if and only if $\alpha = 0$. Hence, we must have $\frac{\partial F}{\partial \alpha} [\lambda^* + \alpha\eta] (0) = 0$ and this implies that (11) holds.

Clearly, $[\lambda, \eta]$ is symmetric in λ and η and

$$[\lambda,\lambda] = 1/2 \int_0^T \langle u, Ru \rangle_r \, dt + 1/2 \int_0^T \langle x, Qx \rangle_n \, dt$$

$$\ge 1/2 \int_0^T \langle x, Qx \rangle_n \, dt \ge 1/2 \, \lambda_Q \int_0^T \langle x, x \rangle_n \, dt ,$$

where u and x are given by (7) and (8). From (8) we have

$$|\lambda(t)| \equiv \langle \lambda(t), \lambda(t) \rangle_n^{1/2} \le \|Q\|_2 \, \|x\|_2 + \int_t^T |A^T(s)| \, |\lambda(s)| \, ds.$$

By Gronwall's megnality, cf. [5], this implies that

$$\|\lambda\|_2 \le \|Q\|_2 \left(2 \|A^T\|_\infty \right)^{-1/2} e^{\|A^T\|_\infty T} \|x\|_2 \equiv \rho \|x\|_2 .$$

Thus $[\lambda,\lambda] \ge 1/2 \lambda_Q \rho^{-2} \|\lambda\|_2^2$, which proves (13).

Moreover,

$$\|\mathring{\lambda}\|_2 \leq \|Q\|_2 \|x\|_2 + \|A^T\|_2 \|\lambda\|_2 \leq [\|Q\|_2 + \rho \|A^T\|_2] \|x\|_2$$

and hence $[\lambda, \lambda] \geq 1/2\lambda_Q \; [\|Q\|_2 + \rho \|A^T\|_2]^{-2}$, which proves (12).

Moreover, if λ and μ both satisfy (11), then

$0 = [\lambda-\mu, \lambda-\mu] \geq 1/2 \lambda_Q \rho^{-2} \|\lambda-\mu\|_2^2$ and $\lambda=\mu$ which proves the uniqueness result. Q.E.D.

To define the Ritz approximation method, let S be any finite dimensional subspace of Φ_0^n. We find an approximation, λ_S, to λ^* by minimizing F over S and determine an approximation, u_S, to u via equation (7). When we apply the computed control we obtain the state x_S determined by (2). It is important to note that x_S is not the state which can be computed via equation (8).

We now show that the Ritz procedure yields a unique approximation.

Theorem 2. There exists a unique $\lambda_S \in S$ which minimizes F over S.

Proof. Let $\{B_i(t)\}_{i=1}^M$ be a basis for S.

Considering $F\left[\sum_{i=1}^M \beta_i B_i\right] = 1/2 \left[\sum_{i=1}^M \beta_i B_i, \sum_{i=1}^M \beta_i B_i\right] - \langle \sum_{i=1}^M \beta_i B_i(0), x_0 \rangle_n$

as a function of $\underline{\beta} \in R^M$, it is clear that F is twice continuously differentiable and hence F has a minimum at $\underline{\beta}^*$ if and only if

(14) $$\frac{\partial F}{\partial \beta_i} [\underline{\beta}^*] = 0, \quad \text{for all } \underline{1 \le i \le} M,$$

and the Hessian matrix of F, $H \equiv \left[\frac{\partial^2 F}{\partial x_i \partial x_j} \right]$, is positive definite.

Calculating the equations (14) we obtain

(15) $$\frac{\partial F}{\partial \beta_i} [\underline{\beta}^*] = \sum_{j=1}^{M} \beta_j \, [B_i, B_j] - \langle B_i(0), x_0 \rangle_n , \quad \underline{1 \le i \le} M,$$

or

(16) $$A\underline{\beta}^* = k$$

where

(17) $$A \equiv \Big[[B_i, B_j] \Big]$$

and

(18) $$\underline{k} \equiv [\langle B_i(0), x_0 \rangle_n].$$

Clearly A is symmetric and positive definite. In fact, if $\underline{\beta} \neq 0$, then

$$\underline{\beta}^T A\underline{\beta} = \left[\sum_{i=1}^{M} \beta_i B_i, \sum_{i=1}^{M} \beta_i B_i \right] \geq 1/2 \, \lambda_Q \rho^{-2} \, \Big\| \sum_{i=1}^{M} \beta_i B_i \Big\|_2 > 0,$$

where we have used (13). Moreover, it follows from (15) that H = A and hence $\underline{\beta}^*$ is the unique minimum of F over R^M. Q.E.D.

We now obtain a general error bound.

Theorem 3. If λ_S denotes the minimizing element of F over S,

$$(19) \qquad |\lambda^* - \lambda_S| \equiv [\lambda^* - \lambda_S, \lambda^* - \lambda_S]^{1/2} = \inf_{w \in S} |\lambda^* - w|.$$

Proof. If $w \in S$, $F[w] = 1/2\,[w,w] - \langle w(0), x_0 \rangle_n$ and

$$F[w] - F[\lambda^*] = 1/2\,[w,w] - 1/2\,[\lambda^*,\lambda^*] + \langle x_0, \lambda^*(0) - w(0) \rangle.$$

But taking $\eta = \lambda^*$ in (11) gives that $[\lambda^*, \lambda^*] = \langle x_0\ \lambda^*(0) \rangle_n$ and hence

$$F[w] - F[\lambda^*] = 1/2\,[w,w] + 1/2\,[\lambda^*,\lambda^*] + \langle x_0, - w(0) \rangle_n.$$

Taking $\eta = w$ in (11) gives that $[\lambda^*, w] = \langle x_0, w(0) \rangle_n$ and hence

$$F[w] - F[\lambda^*] = 1/2[w,w] + 1/2\,[\lambda^*,\lambda^*] - [\lambda^*, w]$$

$$= 1/2\,[\lambda^* - w, \lambda^* - w] = 1/2|\lambda^* - w|^2 .$$

Thus, $|\lambda^* - \lambda_S|^2 = 2(F[\lambda_S] - F[\lambda^*]) \leq 2(F[w] - F[\lambda^*]) = |\lambda^* - w|^2,$

and we have $\inf_{w \in S} |\lambda^* - w| \leq |\lambda^* - \lambda_S| \leq \inf_{w \in S} |\lambda^* - w|$. Q.E.D.

Combining Theorems 1 and 3, we have the following

Corollary. If λ_S denotes the minimizing element of F over S,

$$(20) \qquad \|\lambda^* - \lambda_S\|_2 \leq (2\lambda_Q^{-1})^{1/2} \rho \quad \inf_{w \in S} |\lambda^* - w|$$

and

$$(21)\quad \|\dot{\lambda}^* - \dot{\lambda}_S\|_2 \leq (2\lambda_Q^{-1})^{1/2}(\|\dot{Q}\|_2 + \rho\|A^T\|_2) \inf_{w \in S} |\lambda^* - w| .$$

Using the results of this Corollary we may prove the following results.

<u>Theorem 4.</u> If $u_S(t) \equiv -R^{-1}(t)B^T(t)\lambda_S(t), \quad 0 \leq t \leq T,$ is the computed optimal control,

$$(22)\quad \|u^* - u_S\|_2 \leq \|R^{-1} B^T\|_\infty (2\lambda_Q^{-1})^{1/2}\rho \inf_{w \in S} |\lambda^* - w| ,$$

where $\|R^{-1} B^T\|_\infty \equiv \sup_{0\leq t\leq T} |R^{-1}(t)B^T(t)|_r$.

<u>Proof.</u> In fact, $\delta_S(t) \equiv u^*(t) - u_S(t)$ satisfies the equation

$$\delta_S(t) = -R^{-1}(t)B^T(t)(\lambda^*(t) - \lambda_S(t))$$

and (22) follows from

$$\|\delta_S\|_2 = \|R^{-1} B^T(\lambda^* - \lambda_S)\|_2 \leq \|R^{-1} B^T\|_\infty \|\lambda^* - \lambda_S\|_2$$

and (20). Q.E.D.

<u>Theorem 5.</u> If $\dot{x}_S(t) = A(t)x_S(t) + B(t)u_S(t), \quad 0\leq t\leq T$ and $x_S(0) = x_0$,

$$(23)\quad \|x^* - x_S\|_2 \leq \Gamma\|R^{-1} B^T\|_\infty(2\lambda_Q^{-1})^{1/2}\rho \inf_{w \in S} |\lambda^* - w|$$

and

$$(24)\quad \|\dot{x}^* - \dot{x}_S\|_2 \leq (\Gamma\|A\|_\infty + \|B\|_\infty)\|R^{-1} B^T\|_\infty(2\lambda_Q^{-1})^{1/2}\rho \inf_{w \in S} |\lambda^* - w| ,$$

where $\Gamma \equiv T\|B\|_2 \, e^{\int_0^T |A(z)|_n dz}$, $\|A\|_\infty \equiv \sup_{0\le t\le T} |A(t)|_n$,

and $\|B\|_\infty \equiv \sup_{0\le t\le T} \{ \, |B(t)w|_n \mid w \in R^r \text{ and } |w|_r = 1 \}$.

Proof. Let $\varepsilon_S(t) \equiv x^*(t) - x_S(t)$, $0 \le t \le T$. Then

$$\dot{\varepsilon}_S(t) = A(t)\varepsilon_S(t) + B(t)(u^*(t) - u_S(t)), \quad 0 \le t \le T,$$

and $\varepsilon_S(0) = 0$. This implies that

$$\varepsilon_S(t) = \int_0^t A(z)\varepsilon_S(z)dz + \int_0^t B(z)\delta_S(z)dz.$$

By the Gronwall inequality, cf. [5],

$$|\varepsilon_S(t)|_n \le T^{1/2}\|B\|_\infty \|\delta_S\|_2 \, e^{\int_0^T |A(z)|_n dz}$$

and

$$\|\varepsilon_S\|_2^2 \equiv \int_0^T |\varepsilon_S(t)|_n^2 \, dt \le T^2\|B\|_2^2 \, \|\delta_S\|_2^2 \, e^{2\int_0^T |A(z)|_n dz}$$

$$\equiv \Gamma^2\|u^* - u_S\|_2^2 ,$$

which when combined with (22), proves (23). Moreover,

$$|\dot{\varepsilon}_S(t)|_n \le \|A\|_\infty \, |\varepsilon_S(t)|_n + \|B\|_\infty \, |u^*(t) - u_S(t)|_r , \quad 0 \le t \le T,$$

and hence

$$\|\dot{\varepsilon}_S\|_2 \le \|A\|_\infty \|\varepsilon_S\|_2 + \|B\|_\infty \|u^* - u_S\|_2 \le (\Gamma\|A\|_\infty + \|B\|_\infty)\|u^* - u_S\|_2.$$

Inequality (24) follows by using (22) to bound $\|u^* - u_S\|_2$. Q.E.D.

We now prove a result which gives us an error bound for the cost criteria, i.e., if we actually use the computed control $u_S(t)$ and the system behaves according to $x_S(t)$ how does $J[u_S, x_S]$ compare with $J[u^*, x^*]$. The proof is essentially the same as the one for the analogous result in [4].

Theorem 6. Under the above hypotheses,

$$(25) \quad J[u^*,x^*] \leq J[u_S,x_S]$$

$$\leq J[u^*,x^*] + \|R^{-1}B^T\|_\infty^2 \, \lambda_Q^{-1} \rho^2 \inf_{w\in S} |\lambda^* - w|^2 \, (\|Q\|_\infty \Gamma^2 + \|R\|_\infty).$$

Proof. If $\delta_S(t) \equiv u^*(t) - u_S(t)$, $0 \leq t \leq T$, and

$$\varepsilon_S(t) \equiv x^*(t) - x_S(t), \quad 0 \leq t \leq T,$$

$$J[u_S,x_S] = 1/2 \int_0^T \langle x_S(t), Q(t)x_S(t)\rangle_n \, dt + 1/2 \int_0^T \langle u_S(t), R(t)u_S(t)\rangle_r \, dt$$

$$= 1/2 \int_0^T \langle x^* + \varepsilon_S, Q(x^* + \varepsilon_S)\rangle_n \, dt + 1/2 \int_0^T \langle u^* + \delta_S, R(u^* + \delta_S)\rangle_r dt$$

$$= J[u^*,x^*] + \int_0^T \langle \delta_S, Ru^*\rangle_r \, dt + \int_0^T \langle \varepsilon_S, Qx^*\rangle_n \, dt$$

$$+ 1/2 \int_0^T \langle \delta_S, R\delta_S\rangle_r \, dt + 1/2 \int_0^T \langle \varepsilon_S, Q\varepsilon_S\rangle_n \, dt.$$

But since (7) must hold for the optimal λ^* and u^*, we have

$$(26)\quad \int_0^T \langle \delta_S, Ru^* \rangle_r\, dt = -\int_0^T \langle \delta_S, B^T\lambda^* \rangle_n\, dt = -\int_0^T \langle B\delta_S, \lambda^* \rangle_n .$$

However, from the dynamical equation (2) we have that $\dot{\varepsilon}_S(t) = A(t)\varepsilon_S(t) + B(t)\,\delta_S(t)$ and combining this with (26) yields

$$(27)\quad \int_0^T \langle \delta_S, Ru^* \rangle_r\, dt = -\int_0^T \langle \dot{\varepsilon}_S - A\varepsilon_S, \lambda^* \rangle_n\, dt .$$

Integrating the right-hand side of (27) by parts, using the boundary conditions on ε_S and λ^*, and using (8) for λ^* and x^* yields

$$\int_0^T \langle \delta_S, Ru^* \rangle_r\, dt = \int_0^T \{\langle \varepsilon_S, \dot{\lambda}^* \rangle_n + \langle \varepsilon_S, A^T\lambda^* \rangle\} dt = -\int_0^T \langle \varepsilon_S, Qx^* \rangle_n\, dt.$$

Thus,

$$J[u_S, x_S] = J[u^*, x^*] + 1/2 \int_0^T \langle \delta_S, R\delta_S \rangle_r\, dt + 1/2 \int_0^T \langle \varepsilon_S, Q\varepsilon_S \rangle_n\, dt$$

$$\leq J[u^*, x^*] + 1/2\, \|R\|_\infty\, \|\delta_S\|_2^2 + 1/2\, \|Q\|_\infty\, \|\varepsilon_S\|_2^2$$

$$\leq J[u^*, x^*] + 1/2 \|Q\|_\infty \Gamma^2\, \|R^{-1}B^T\|_\infty^2\, (2\lambda_Q)^{-1} \rho^2 \inf_{w \in S} |\lambda^* - w|^2$$

$$+ 1/2\, \|R^{-1}B^T\|_\infty^2\, \lambda_Q^{-1}\, \rho^2 \inf_{w \in S} |\lambda^* - w|^2 \, (\|Q\|_\infty \Gamma^2 + \|R\|_\infty),$$

where we have used (22) and (23). Q.E.D.

We now consider how these error bounds can be used for the example of finite dimensional spaces of smooth polynomial spline functions. Let Δ: $0 = x_0 < x_1 < \cdots < x_{N+1} = T$ be a partition of $[0,T]$, $S_0(d,\Delta) \equiv \{$piecewise polynomials, $S(x)$, of degree d with respect to $\Delta \mid S(x) \in C^d [0,T]$ and $\dot{S}(T) = 0\}$, and $h \equiv \max_{0 \le i \le N} (x_{i+1} - x_i)$.

As is well known, cf. [6], these spline spaces have a convenient set of basis functions, $\{B_{d,i}(t)\}_{i=1}^{p}$, $P \equiv \dim S_0(d,\Delta)$, which have small support i.e., supp $B_i(t) \equiv \overline{\{t \in [0,T] \mid B_i(t) \neq 0\}}$ is "thin" in $[0,T]$. Thus, if we use a finite dimensional subspace of Φ_0^n of the form $S_d(\Delta) \equiv \{\sum_{i=1}^{p} \underline{\beta}_i B_{d,i}(t) \mid \underline{\beta}_i \in R^n, 1 \le i \le p\}$ we will obtain for the matrix A in (16) a sparse block-banded matrix, cf. [2],[3], and [4]. Moreover, if every component of λ^* is d+1 times piecewise, continuously differentiable with respect to t there exists a positive constant, K_d, independent of Δ, such that $\inf_{w \in S_d} |\lambda^* - \lambda| \le K_d h^d \|D^{d+1}\lambda^*\|_2$, where K_d can be explicitly determined, cf. [6]. Combining these results we obtain

Theorem 7. If each component of λ^* is d+1 times piecewise continuously differentiable with respect to t, there exists a positive constant, K_d, independent of Δ, such that

$$(27) \quad \|\lambda^* - \lambda_{S_d(\Delta)}\|_2 \leq (2\lambda_Q^{-1})^{1/2} \rho K_d h^d \|D^{d+1} \lambda^*\|_2,$$

$$(28) \quad \|u^* - u_{S_d(\Delta)}\|_2 \leq \|R^{-1}B^T\|_\infty (2\lambda_Q^{-1})^{1/2} \rho K_d h^d \|D^{d+1}\lambda^*\|_2,$$

and

$$(29) \quad J[u^*,x^*] \leq J[u_{S_d(\Delta)}, x_{S_d(\Delta)}] \leq J[u^*,x^*]$$

$$+ \|R^{-1}B^T\|_\infty^2 \lambda_Q^{-1} \rho^2 (\|Q\|_\infty \Gamma^2 + \|R\|_\infty) K_d^2 h^{2d} \|D^{d+1}\lambda^*\|_2^2.$$

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