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An amusing sequence of functions

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AN AMUSING SEQUENCE OF FUNCTIONS

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Abstract. We consider the amusing sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ given by

$$f_n(x) = \sum_{k=1}^{n} \frac{|\sin(k\pi x)|}{k}.$$

Every rational point is eventually the location of a strict local minimum of f_n: more precisely, f_n has a strict local minimum in all rational points $x = p/q \in \mathbb{Q}$ with $|q| \leq \sqrt{n}$.

1. Introduction

The purpose of this short note is to introduce

$$f_n(x) = \sum_{k=1}^{n} \frac{|\sin(k\pi x)|}{k}.$$

Theorem. *The function $f_n(x)$ has a strict local minimum in $x = p/q$ for all $n \geq q^2$.*

The asymptotically sharp scaling is given by $n \geq (1 + o(1))q^2/\pi$. We believe that this curious result is a good indicator that this sequence might have all sorts of other nice properties and could be of some interest. A natural question would be whether anything can be said about the location of local maxima: it is tempting to conjecture that they cannot be too well approximated by rationals with small denominators (because that's where the minima are).

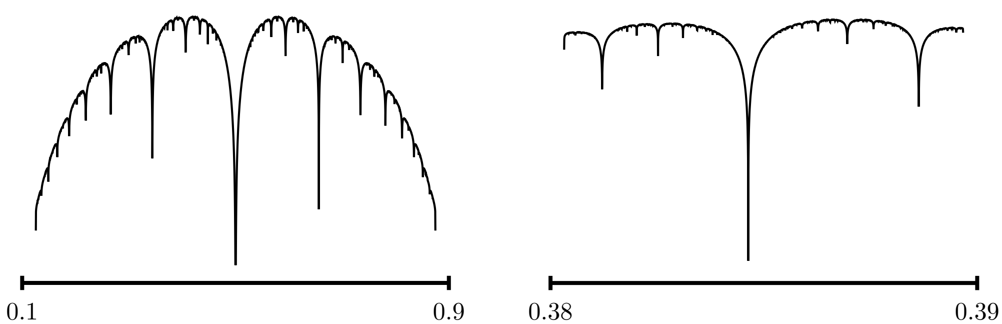

Figure 1. The function $f_{50.000}$ on $[0.1, 0.9]$ and zoomed in (right). The big cusp in the right picture is located at $x = 5/13$, the two smaller cusps are at $x = 8/21$ and $x = 7/18$.

Both definition of the function as well as its graph are reminiscent of the Takagi function τ, which is a continuous but nowhere differentiable function that was first considered by Takagi [8] in 1901 (with independent re-discoveries by van der Waerden [9] in 1930 and de Rham [4] in 1957): if $d(x)$ denotes the distance from x to the nearest integer, then τ is given by

$$\tau(x) = \sum_{k=0}^{\infty} \frac{d(2^k x)}{2^k}.$$

$\tau(x)$ has since appeared in connection to inequalities for digit sums [1], the Riemann hypothesis [3] and extremal combinatorics [5] (many more results can be found in the surveys [2, 7]). We emphasize a 1959 result of Kahane [6] who proved that the set of local minima are exactly the dyadic rational numbers.

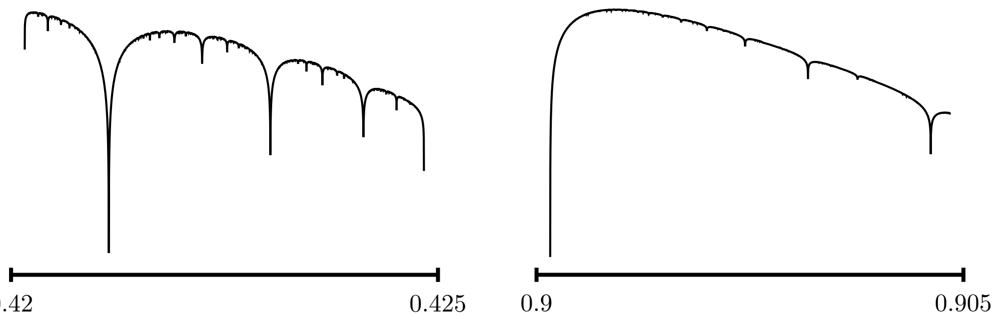

FIGURE 2. The function $f_{50.000}$ in two other locations.

2. Proof

Proof. We observe that for $x \in \mathbb{R}$ and $\varepsilon \to 0$

$$|\sin(x+\varepsilon)| - |\sin(x)| = \begin{cases} |\varepsilon| + \mathcal{O}(\varepsilon^2) & \text{if } x/\pi \in \mathbb{Z} \\ \varepsilon \operatorname{sgn}(\sin(x))\cos(x) + \mathcal{O}(\varepsilon^2) & \text{otherwise.} \end{cases}$$

Here, sgn denotes the signum function

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Let us now consider the function in $x = p/q$ with $\gcd(p,q) = 1$. We have

$$\sum_{k=1}^{n} \frac{|\sin(k\pi(p/q+\varepsilon))|}{k} - \frac{|\sin(k\pi(p/q))|}{k} = \pi\varepsilon \sum_{k=1}^{n} \operatorname{sgn}\left(\sin\left(\frac{k\pi p}{q}\right)\right)\cos\left(\frac{k\pi p}{q}\right)$$
$$+ \pi|\varepsilon| \# \{1 \le k \le n : k(p/q) \in \mathbb{Z}\} + \mathcal{O}(\varepsilon^2).$$

We analyze these two coefficients and show that the first one is bounded (the second term is clearly unbounded). The function $\operatorname{sgn}(\sin(x))\cos(x)$ has period π and the map $k \to k \cdot p$ is a permutation on $\mathbb{Z}_q$. It is then easy to see that the symmetries of sine and cosine imply

$$\sum_{k=1}^{q} \operatorname{sgn}\left(\sin\left(\frac{k\pi p}{q}\right)\right)\cos\left(\frac{k\pi p}{q}\right) = 0 \quad \text{and thus} \quad \sum_{k=m+1}^{m+q} \operatorname{sgn}\left(\sin\left(\frac{k\pi p}{q}\right)\right)\cos\left(\frac{k\pi p}{q}\right) = 0$$

for all $m \in \mathbb{N}$. The periodicity and $k \to k \cdot p$ being a permutation give

$$\inf_{n \in \mathbb{N}} \sum_{k=1}^{n} \operatorname{sgn}\left(\sin\left(\frac{k\pi p}{q}\right)\right)\cos\left(\frac{k\pi p}{q}\right) = \min_{1 \le n \le q} \sum_{k=1}^{n} \operatorname{sgn}\left(\sin\left(\frac{k\pi p}{q}\right)\right)\cos\left(\frac{k\pi p}{q}\right)$$
$$\ge -\max_{1 \le n \le q} \sum_{k=1}^{n} \cos\left(\frac{k\pi}{q}\right) \ge -\frac{q}{2},$$

while, at the same time, we obviously have

$$\# \{1 \le k \le n : k(p/q) \in \mathbb{Z}\} \ge \left\lfloor \frac{n}{q} \right\rfloor$$

from which positivity follows for $n \ge q^2/2$.

□

Remarks. A more careful analysis shows that

$$-\max_{1\leq n\leq q}\sum_{k=1}^{n}\cos\left(\frac{k\pi}{q}\right)=-(1+o(1))\frac{q}{2}\frac{2}{\pi}\int_{0}^{\pi/2}\cos x dx=-(1+o(1))\frac{q}{\pi}$$

from which we get that, asymptotically, $n\geq(1+o(1))q^2/\pi$ suffices. We observe that this is optimal: if $p=q-1$ and $1\leq k\leq q/2$, then

$$\operatorname{sgn}\left(\sin\left(\frac{k\pi p}{q}\right)\right)\cos\left(\frac{k\pi p}{q}\right)=-\cos\left(\frac{k\pi}{q}\right)\quad\text{and}\quad\sum_{k=1}^{\lfloor q/2\rfloor}-\cos\left(\frac{k\pi}{q}\right)\sim-\frac{q}{\pi}.$$

The main argument only appealed to certain fairly elementary symmetry properties of the trigonometric functions and easily extends to various other functions. A particularly nice example comes from replacing the sine by the cosine

$$g_n(x)=\sum_{k=1}^{n}\frac{|\cos(k\pi x)|}{k}.$$

Whether $x=p/q$ is the location of a local minimum or maximum now depends on the parity of q. What other phenomena can be found?

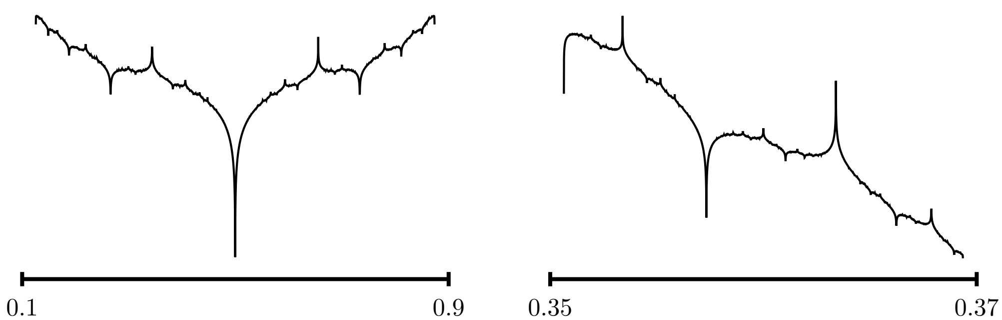

FIGURE 3. The function $g_{50.000}$ (left) and zoomed in (right).

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