We observe that solutions of a large class of highly oscillatory second order linear ordinary differential equations can be approximated using nonoscillatory phase functions. In addition, we describe numerical experiments which illustrate computational implications of this fact. For example, many special functions of interest — such as the Bessel functions J_ν and Y_ν — can be evaluated accurately using a number of operations which is $O(1)$ in the order ν. The present paper is devoted to the development of an analytical apparatus. Numerical aspects of this work will be reported at a later date.

On the existence of non-oscillatory phase functions for second order ordinary differential equations in the high-frequency regime

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1. Introduction

Given a differential equation

$$y''(t) + \lambda^2 q(t)y(t) = 0 \quad \text{for all } 0 \leqslant t \leqslant 1, \tag{1}$$

where λ is a real number and $q : [0,1] \to \mathbb{R}$ is smooth and strictly positive, a sufficiently smooth $\alpha : [0,1] \to \mathbb{R}$ is a phase function for (1) if the pair of functions u, v defined by the formulas

$$u(t) = \frac{\cos(\alpha(t))}{|\alpha'(t)|^{1/2}} \tag{2}$$

and

$$v(t) = \frac{\sin(\alpha(t))}{|\alpha'(t)|^{1/2}} \tag{3}$$

form a basis in the space of solutions of (1). Phase functions have been extensively studied: they were first introduced in [9], play a key role in the theory of global transformations of ordinary differential equations [3, 10], and are an important element in the theory of special functions [16, 6, 11, 1].

Despite this long history, an important property of phase functions appears to have been overlooked. Specifically, that when the function q is nonoscillatory, solutions of the equation (1) can be accurately represented using a nonoscillatory phase function.

This is somewhat surprising since α is a phase function for (1) if and only if it satisfies the third order nonlinear ordinary differential equation

$$(\alpha'(t))^2 = \lambda^2 q(t) - \frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2 \quad \text{for all } 0 \leqslant t \leqslant 1. \tag{4}$$

The equation (4) was introduced in [9], and and we will refer to it as Kummer's equation. The form of (4) and the appearance of λ in it suggests that its solutions will be oscillatory — and most of them are. However, Bessel's equation

$$y''(t) + \left(1 - \frac{\lambda^2 - 1/4}{t^2}\right)y(t) = 0 \quad \text{for all } 0 < t < \infty \tag{5}$$

furnishes a nontrivial example of an equation which admits a nonoscillatory phase function regardless of the value of λ. If we define u, v by the formulas

$$u(t) = \sqrt{\frac{\pi t}{2}}J_\lambda(t) \tag{6}$$

and

$$v(t) = \sqrt{\frac{\pi t}{2}}Y_\lambda(t), \tag{7}$$

where J_λ and Y_λ denote the Bessel functions of the first and second kinds of order λ, and let α be defined by the relations (2),(3), then

$$\alpha'(t) = \frac{2}{\pi t}\frac{1}{J_\lambda^2(t) + Y_\lambda^2(t)}. \tag{8}$$

It can be easily verified that (8) is a strictly negative, monotonically decreasing function of t

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on $(0, \infty)$. That this nonoscillatory phase function for Bessel's equation exists is the basis of several methods for the evaluation of Bessel functions of large orders and for the computation of their zeros [6, 8, 15].

The general situation is not quite so favorable: there need not exist a nonoscillatory function α such that (2) and (3) are exact solutions of (1). However, assuming that q is nonoscillatory and λ is sufficiently large, there exists a nonoscillatory function α such that (2), (3) approximate solutions of (1) with spectral accuracy (i.e., the approximation errors decay exponentially with λ).

To see that this claim is plausible, we apply Newton's method for the solution of nonlinear equations to Kummer's equation (4). In doing so, it will be convenient to move the setting of our analysis from the interval $[0, 1]$ to the real line so that we can use the Fourier transform to quantity the notion of "nonoscillatory." Suppose that the extension of q to the real line is smooth and strictly positive, and that the Fourier transform of $\log(q)$ is smooth and decays rapidly. Letting

$$(\alpha'(t))^2 = \lambda^2 \exp(r(t)) \tag{9}$$

in (4) yields the logarithm form of Kummer's equation:

$$r''(t) - \frac{1}{4}(r'(t))^2 + 4\lambda^2 (\exp(r(t)) - q(t)) = 0 \quad \text{for all } t \in \mathbb{R}. \tag{10}$$

We use $\{r_n\}$ to denote the sequence of Newton iterates for the equation (10) obtained from the initial guess

$$r_0(t) = \log(q(t)). \tag{11}$$

The function r_0 corresponds to the first order WKB approximations for (1). That is to say that if we insert the associated phase function

$$\alpha_0(t) = \lambda \int_0^t \exp\left(\frac{1}{2}r_0(u)\right) \, du = \lambda \int_0^t \sqrt{q(u)} du \tag{12}$$

into (2),(3), then

$$u(t) = q^{-1/4}(t) \cos\left(\lambda \int_0^t \sqrt{q(u)} \, du\right) \tag{13}$$

and

$$v(t) = q^{-1/4}(t) \sin\left(\lambda \int_0^t \sqrt{q(u)} \, du\right). \tag{14}$$

For each $n \geqslant 0$, r_{n+1} is obtained from r_n by solving the linearized equation

$$h''(t) - \frac{1}{2}r_n'(t)h'(t) + 4\lambda^2 \exp(r_n(t)) h(t) = f_n(t) \quad \text{for all } t \in \mathbb{R}, \tag{15}$$

where

$$f_n(t) = -r_n''(t) + \frac{1}{4}(r_n'(t))^2 - 4\lambda^2 (\exp(r_n(t)) - q(t)), \tag{16}$$

and letting

$$r_{n+1}(t) = r_n(t) + h(t). \tag{17}$$

By introducing the change of variables

$$x(t) = \int_0^t \exp\left(\frac{r_n(u)}{2}\right) \, du \tag{18}$$

into (15), we transform it into the inhomogeneous Helmholtz equation

$$h''(x) + 4\lambda^2 h(x) = g_n(x) \quad \text{for all } x \in \mathbb{R}, \tag{19}$$

where

$$g_n(x) = \exp\left(-r_n(x)\right) f_n(x). \tag{20}$$

Suppose that $\widehat{g_n}$ decays rapidly (when $n = 0$, this is a consequence of our assumption that $\log(q)$ has a rapidly decaying Fourier transform) and let h^* be the solution of (19) whose Fourier transform is

$$\widehat{h^*}(\xi) = \frac{\widehat{g_n}(\xi)}{4\lambda^2 - \xi^2}. \tag{21}$$

Since $\widehat{h^*}(\xi)$ is singular when $\xi = \pm 2\lambda$, h^* will necessarily have a component which oscillates at frequency 2λ. However, according to (21), the $L^\infty(\mathbb{R})$ norm of that component is

$$\frac{\widehat{g_n}(2\lambda)}{4\lambda}. \tag{22}$$

In fact, by rearranging (21) as

$$\widehat{h^*}(\xi) = \frac{1}{4\lambda} \left(\frac{\widehat{g_n}(\xi)}{2\lambda - \xi} + \frac{\widehat{g_n}(\xi)}{2\lambda + \xi} \right) \tag{23}$$

and decomposing each of the terms on the right-hand side of (23) as

$$\frac{\widehat{g_n}(\xi)}{2\lambda \pm \xi} = \frac{1}{4\lambda} \left(\frac{\widehat{g_n}(\xi) - \widehat{g_n}(\mp 2\lambda) \exp\left(-(2\lambda \pm \xi)^2\right)}{2\lambda \pm \xi} + \widehat{g_n}(\mp 2\lambda) \frac{\exp\left(-(2\lambda \pm \xi)^2\right)}{2\lambda \pm \xi} \right), \tag{24}$$

we obtain

$$h^*(x) = h_0(x) + h_1(x), \tag{25}$$

where h_0 is defined by the formula

$$\widehat{h_0}(\xi) = \frac{1}{4\lambda} \left(\frac{\widehat{g_n}(\xi) - \widehat{g_n}(-2\lambda) \exp\left(-(2\lambda + \xi)^2\right)}{2\lambda + \xi} + \frac{\widehat{g_n}(\xi) - \widehat{g_n}(2\lambda) \exp\left(-(2\lambda - \xi)^2\right)}{2\lambda - \xi} \right), \tag{26}$$

and h_1 is defined by the formula

$$\widehat{h_1}(\xi) = \frac{1}{4\lambda} \left(\widehat{g_n}(-2\lambda) \frac{\exp\left(-(2\lambda + \xi)^2\right)}{2\lambda + \xi} + \widehat{g_n}(2\lambda) \frac{\exp\left(-(2\lambda - \xi)^2\right)}{2\lambda - \xi} \right). \tag{27}$$

Since the factor in the denominator in (26) has been canceled and both $\widehat{g_n}$ and the Gaussian function are smooth and rapidly decaying, $\widehat{h_0}$ is also smooth and rapidly decaying. Meanwhile, a straightforward calculation shows that the Fourier transform of

$$\frac{1}{2i} \operatorname{erf}\left(\frac{x}{2}\right) \exp(2\lambda i x) \tag{28}$$

is

$$\frac{\exp\left(-(2\lambda - \xi)^2\right)}{2\lambda - \xi}, \tag{29}$$

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so that (27) implies that

$$h_1(x) = \frac{1}{4\lambda} \left(\widehat{g}_n(-2\lambda) \frac{1}{2i} \operatorname{erf} \left(\frac{x}{2} \right) \exp(2\lambda i x) - \widehat{g}_n(2\lambda) \frac{1}{2i} \operatorname{erf} \left(\frac{x}{2} \right) \exp(-2\lambda i x) \right). \tag{30}$$

Since g_n is real-valued, $\widehat{g}_n(2\lambda) = \widehat{g}_n(-2\lambda)$. Inserting this into (30) yields

$$h_1(x) = \frac{\widehat{g}(2\lambda)}{4\lambda} \operatorname{erf} \left(\frac{x}{2} \right) \sin(2\lambda x), \tag{31}$$

which makes it clear that the $L^\infty(\mathbb{R})$ norm of h_1 is $(4\lambda)^{-1} \widehat{g}_n(2\lambda)$.

In (25), the solution of (19) is decomposed as the sum of a nonoscillatory function h_0 and a highly oscillatory function h_1 of small magnitude. However, the solution of (15) is actually given by the function

$$h^*(x(t)) = h_0(x(t)) + h_1(x(t)) \tag{32}$$

obtained by reversing the change of variables (18). But since $x(t)$ is nonoscillatory and the Fourier transform of $h_0(x)$ decays rapidly, we expect that the composition $h_0(x(t))$ will also have a rapidly decaying Fourier transform. The $L^\infty(\mathbb{R})$ norm of $h_1(x(t))$ is, of course, the same as that of $h_1(x)$. So the solution of the linearized equation (15) can be written as the sum of a nonoscillatory function $h_0(x(t))$ and a highly oscillatory function $h_1(x(t))$ of negligible magnitude.

If, in each iteration of the Newton procedure, we approximate the solution of (15) by constructing $h^*(x(t))$ and discarding the oscillatory term $h_1(t(x))$ of small magnitude, then it is plausible that we will arrive at an approximate solution $r(t)$ of the logarithm form of Kummer's equation which is nonoscillatory, assuming the Fourier transform of $r_0(t) = \log(q(t))$ decays rapidly enough and λ is sufficiently large.

Most of the remainder of this paper is devoted to developing a rigorous argument to replace the preceding heuristic discussion. In Section 2, we summarize a number of well-known mathematical facts to be used throughout this article. In Section 3, we reformulate Kummer's equation as a nonlinear integral equation. Once that is accomplished, we are in a position to state the principal result of the paper and discuss its implications; this is done in Section 4. The proof of this principal result is contained in Sections 5, 6, 7 and 8. Section 9 contains an elementary proof of a relevant error estimate for second order differential equations of the form (1).

In Section 10, we present the results of numerical experiments concerning the evaluation of special functions. The details of our numerical algorithm will be reported at a later date.

We conclude with a few brief remarks in Section 11.

2. Preliminaries

2.1. Modified Bessel functions

The modified Bessel function $K_\nu(t)$ of the first kind of order ν is defined for $t \in \mathbb{R}$ and $\nu \in \mathbb{C}$ by the formula

$$K_\nu(t) = \int_0^\infty \exp\left(-t \cosh(t) \right) \cosh(\nu t) \, dt. \tag{33}$$

The following bound on the ratio of $K_{\nu+1}$ to K_ν can be found in [14].

Theorem 1. *Suppose that $t > 0$ and $\nu > 0$ are real numbers. Then*

$$\frac{K_{\nu+1}(t)}{K_\nu(t)} < \frac{\nu + \sqrt{\nu^2 + t^2}}{t} \leqslant \frac{2\nu}{t} + 1. \tag{34}$$

2.2. The binomial theorem

A proof of the following can be found in [13], as well as many other sources.

Theorem 2. *Suppose that r is a real number, and that y is a real number such that $|y| < 1$. Then*

$$(1 + y)^r = \sum_{k=0}^\infty \frac{\Gamma(r + 1)}{\Gamma(k + 1)\Gamma(r - k + 1)} y^k. \tag{35}$$

2.3. The Lambert W function

The Lambert W function or product logarithm is the multiple-valued inverse of the function

$$f(z) = z \exp(z). \tag{36}$$

We follow [5] in using W_0 to denote the branch of W which is real-valued and greater than or equal to -1 on the interval $[-1/e, -\infty)$ and W_{-1} to refer to the branch which is real-valued and less than or equal to -1 on $[-1/e, 0)$.

We will make use of the following elementary facts concerning W_0 and W_{-1} (all of which can be found in [5] and its references).

Theorem 3. *Suppose that $y \geqslant -1/e$ is a real number. Then*

$$x \exp(x) \leqslant y \quad \textit{if and only if} \quad x \leqslant W_0(y). \tag{37}$$

Theorem 4. *Suppose that $0 < y \leqslant 1/e$ is a real number. Then*

$$x \exp(-x) \leqslant y \quad \textit{if and only if} \quad x \geqslant -W_{-1}(-y). \tag{38}$$

Theorem 5. *Suppose that $0 \leqslant x \leqslant 1$ is a real number. Then*

$$\frac{x}{2} \leqslant W_0(x) \leqslant x. \tag{39}$$

Theorem 6. *Suppose that $x > \exp(1)$ is a real number. Then*

$$\log(x) \leqslant -W_{-1}\left(-\frac{1}{x}\right) \leqslant 2\log(x). \tag{40}$$

2.4. Fréchet derivatives and the contraction mapping principle

Given Banach spaces X, Y and a mapping $f : X \to Y$ between them, we say that f is Fréchet differentiable at $x \in X$ if there exists a bounded linear operator $X \to Y$, denoted by f'_x, such that

$$\lim_{h \to 0} \frac{\|f(x + h) - f(x) - f'_x[h]\|}{\|h\|} = 0. \tag{41}$$

Theorem 7. *Suppose that X and Y are a Banach spaces and that $f : X \to Y$ is Fréchet differentiable at every point of X. Suppose also that D is a convex subset of X, and that there exists a real number $M > 0$ such that*

$$\|f'_x\| \leqslant M \tag{42}$$

for all $x \in D$. Then

$$\|f(x) - f(y)\| \leqslant M\|x - y\| \tag{43}$$

for all x and y in D.

Suppose that $f : X \to X$ is a mapping of the Banach space X into itself. We say that f is contractive on a subset D of X if there exists a real number $0 < \alpha < 1$ such that

$$\|f(x) - f(y)\| \leqslant \alpha\|x - y\| \tag{44}$$

for all $x, y \in D$. Moreover, we say that $\{x_n\}_{n=0}^{\infty}$ is a sequence of fixed point iterates for f if $x_{n+1} = f(x_n)$ for all $n \geqslant 0$.

Theorem 7 is often used to show that a mapping is contractive so that the following result can be applied.

Theorem 8. *(The Contraction Mapping Principle) Suppose that D is a closed subset of a Banach space X. Suppose also that $f : X \to X$ is contractive on D and $f(D) \subset D$. Then the equation*

$$x = f(x) \tag{45}$$

has a unique solution $\sigma^* \in D$. Moreover, any sequence of fixed point iterates for the function f which contains an element in D converges to σ^*.

A discussion of Fréchet derivatives and proofs of Theorems 7 and 8 can be found, for instance, in [18].

2.5. Gronwall's inequality

The following well-known inequality can be found in, for instance, [2].

Theorem 9. *Suppose that f and g are continuous functions on the interval $[a, b]$ such that*

$$f(t) \geqslant 0 \ \text{ and } \ g(t) \geqslant 0 \ \text{ for all } \ a \leqslant t \leqslant b. \tag{46}$$

Suppose further that there exists a real number $C > 0$ such that

$$f(t) \leqslant C + \int_s^t f(s)g(s)\,ds \ \text{ for all } \ a \leqslant t \leqslant b. \tag{47}$$

Then

$$f(t) \leqslant C \exp \left(\int_a^t g(s) \, ds \right) \quad \text{for all } a \leqslant t \leqslant b. \tag{48}$$

2.6. *Schwarzian derivatives*

The Schwarzian derivative of a smooth function $f : \mathbb{R} \to \mathbb{R}$ is

$$\{f, t\} = \frac{f'''(t)}{f'(t)} - \frac{3}{2} \left(\frac{f''(t)}{f'(t)} \right)^2. \tag{49}$$

If the function $x(t)$ is a diffeomorphism of the real line (that is, a smooth, invertible mapping $\mathbb{R} \to \mathbb{R}$), then the Schwarzian derivative of $x(t)$ can be related to the Schwarzian derivative of its inverse $t(x)$; in particular,

$$\{x, t\} = - \left(\frac{dx}{dt} \right)^2 \{t, x\}. \tag{50}$$

The identity (50) can be found, for instance, in Section 1.13 of [11].

3. Integral equation formulation

In this section, we reformulate Kummer's equation

$$(\alpha'(t))^2 = \lambda^2 q(t) - \frac{1}{2} \frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4} \left(\frac{\alpha''(t)}{\alpha'(t)} \right)^2 \tag{51}$$

as a nonlinear integral equation. As in the introduction, we assume that the function q has been extended to the real line and we seek a function α which satisfies (51) on the real line.

By letting

$$(\alpha'(t))^2 = \lambda^2 \exp(r(t)) \tag{52}$$

in (51), we obtain the equation

$$r''(t) - \frac{1}{4} (r'(t))^2 + 4\lambda^2 (\exp(r(t)) - q(t)) = 0 \quad \text{for all } t \in \mathbb{R}. \tag{53}$$

We next take r to be of the form

$$r(t) = \log(q(t)) + \delta(t), \tag{54}$$

which results in

$$\delta''(t) - \frac{1}{2} \frac{q'(t)}{q(t)} \delta'(t) - \frac{1}{4} (\delta'(t))^2 + 4\lambda^2 q(t) (\exp(\delta(t)) - 1) = q(t)p(t), \quad \text{for all } t \in \mathbb{R}, \tag{55}$$

where p is defined by the formula

$$p(t) = \frac{1}{q(t)} \left(\frac{5}{4} \left(\frac{q'(t)}{q(t)} \right)^2 - \frac{q''(t)}{q(t)} \right). \tag{56}$$

Note that the function p appears in the standard error analysis of WKB approximations (see, for instance, [12]). Expanding the exponential in a power series and rearranging terms yields

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the equation

$$\delta''(t) - \frac{1}{2}\frac{q'(t)}{q(t)}\delta'(t) + 4\lambda^2 q(t)\delta(t) - \frac{1}{4}(\delta'(t))^2 + 4\lambda^2 q(t)\left(\frac{(\delta(t))^2}{2} + \frac{(\delta(t))^3}{3!} + \cdots\right) = q(t)p(t). \quad (57)$$

Applying the change of variables

$$x(t) = \int_0^t \sqrt{q(u)}\, du \quad (58)$$

transforms (57) into

$$\delta''(x) + 4\lambda^2\delta(x) - \frac{1}{4}(\delta'(x))^2 + 4\lambda^2\left(\frac{(\delta(x))^2}{2} + \frac{(\delta(x))^3}{3!} + \cdots\right) = p(x) \quad \text{for all} \quad x \in \mathbb{R}. \quad (59)$$

At first glance, the relationship between the function $p(x)$ appearing in (59) and the coefficient $q(t)$ in the ordinary differential equation (1) is complex. However, the function $p(t)$ defined via (56) is related to the Schwarzian derivative (see Section 2.6) of the function $x(t)$ defined in (58) via the formula

$$p(t) = -\frac{2}{q(t)}\{x, t\} = -2\left(\frac{dt}{dx}\right)^2\{x, t\}. \quad (60)$$

It follows from (60) and Formula (50) in Section 2.6 that

$$p(x) = 2\{t, x\}. \quad (61)$$

That is to say: p, when viewed as a function of x, is simply twice the Schwarzian derivative of t with respect to x.

It is also notable that the part of (59) which is linear in δ is a constant coefficient Helmholtz equation. This suggests that we form an integral equation for (59) using a Green's function for the Helmholtz equation. To that end, we define the integral operator T for functions $f \in L^1(\mathbb{R})$ via the formula

$$T[f](x) = \frac{1}{4\lambda}\int_{-\infty}^{\infty}\sin(2\lambda|x-y|)f(y)\, dy \quad (62)$$

The following theorem summarizes the relevant properties of the operator T.

Theorem 10. *Suppose that $\lambda > 0$ is a real number, and that the operator T is defined as in (62). Suppose also that $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$. Then:*

1. *$T[f](x)$ is an element of $C^2(\mathbb{R})$;*

2. *$T[f](x)$ is a solution of the ordinary differential equation*
$$y''(x) + 4\lambda^2 y(x) = f(x) \quad \text{for all } x \in \mathbb{R}; \text{ and}$$

3. *the Fourier transform of $T[f](x)$ is the principal value of*
$$\frac{\widehat{f}(\xi)}{4\lambda^2 - \xi^2} = \frac{1}{4\lambda}\left(\frac{\widehat{f}(\xi)}{2\lambda - \xi} + \frac{\widehat{f}(\xi)}{2\lambda + \xi}\right).$$

Proof. We observe that

$$T\left[f\right](x) = \frac{1}{4\lambda} \int_{-\infty}^{x} \sin\left(2\lambda\left(x-y\right)\right) f(y)\, dy + \frac{1}{4\lambda} \int_{x}^{\infty} \sin\left(2\lambda(y-x)\right) f(y)\, dy$$

$$= \frac{1}{4\lambda} \sin(2\lambda x) \int_{-\infty}^{x} \cos(2\lambda y) f(y)\, dy - \frac{1}{4\lambda} \cos(2\lambda x) \int_{-\infty}^{x} \sin(2\lambda y) f(y)\, dy \qquad (63)$$

$$+ \frac{1}{4\lambda} \cos(2\lambda x) \int_{x}^{\infty} \sin(2\lambda y) f(y)\, dy - \frac{1}{4\lambda} \sin(2\lambda x) \int_{x}^{\infty} \cos(2\lambda y) f(y)\, dy$$

for all $x \in \mathbb{R}$. We differentiate both sides of (63) with respect to x, apply the Lebesgue dominated convergence theorem to each integral (this is permissible since the sine and cosine functions are bounded and $f \in L^{1}\left(\mathbb{R}\right)$) and combine terms in order to conclude that $T\left[f\right]$ is differentiable everywhere and

$$\frac{d}{dx} T\left[f\right](x) = = \frac{1}{2} \int_{-\infty}^{\infty} \cos\left(2\lambda\left|x-y\right|\right) \operatorname{sign}(x-y) f(y)\, dy \qquad (64)$$

for all $x \in \mathbb{R}$. In the same fashion, we conclude that

$$\left(\frac{d}{dx}\right)^{2} T\left[f\right](x) = f(x) - \lambda \int_{-\infty}^{\infty} \sin\left(2\lambda\left|x-y\right|\right) f(y)\, dy \qquad (65)$$

for all $x \in \mathbb{R}$. Since f is continuous by assumption and the second term appearing on the right-hand side in (65) is a continuous function of x by the Lebesgue dominated convergence theorem, we see from (65) that $T\left[f\right]$ is twice continuously differentiable. By combining (65) and (62), we conclude that $T\left[f\right]$ is a solution of the ordinary differential equation

$$y''(x) + 4\lambda^{2} y(x) = f(x) \quad \text{for all } x \in \mathbb{R}. \qquad (66)$$

We now define the function g through the formula

$$\widehat{g}(\xi) = \frac{1}{4\lambda} \left(\frac{1}{2\lambda - \xi} + \frac{1}{2\lambda + \xi}\right). \qquad (67)$$

It is well known that the Fourier transform of the principal value of $1/x$ is the function

$$-i\pi \operatorname{sign}(x); \qquad (68)$$

see, for instance, [17] or [7]. It follows readily that the inverse Fourier transform of the principal value of

$$\frac{1}{2\lambda \pm \xi} \qquad (69)$$

is

$$\pm \frac{1}{2i} \exp\left(\mp 2\lambda i x\right) \operatorname{sign}(x). \qquad (70)$$

From this and (67), we conclude that

$$g(x) = \frac{1}{4\lambda} \left(\frac{1}{2i} \exp\left(-2\lambda i x\right) \operatorname{sign}(x) - \frac{1}{2i} \exp\left(2\lambda i x\right) \operatorname{sign}(x)\right)$$

$$= \frac{1}{4\lambda} \sin\left(2\lambda|x|\right). \qquad (71)$$

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In particular, $T[f]$ is the convolution of f with g. As a consequence,

$$\widehat{T[f]}(\xi) = \widehat{g}(\xi)\widehat{f}(\xi) = \frac{1}{4\lambda}\left(\frac{\widehat{f}(\xi)}{2\lambda - \xi} + \frac{\widehat{f}(\xi)}{2\lambda + \xi}\right), \tag{72}$$

which is the third and final conclusion of the theorem. $\qquad\square$

In light of Theorem 10, it is clear that introducing the representation

$$\delta(x) = T[\sigma](x) \tag{73}$$

into (59) yields the nonlinear integral equation

$$\sigma(x) = S[T[\sigma]](x) + p(x) \quad \text{for all } x \in \mathbb{R}, \tag{74}$$

where S is the operator defined for functions $f \in C^1(\mathbb{R})$ by the formula

$$S[f](x) = \frac{(f'(x))^2}{4} - 4\lambda^2\left(\frac{(f(x))^2}{2!} + \frac{(f(x))^3}{3!} + \frac{(f(x))^4}{4!} + \cdots\right). \tag{75}$$

The following theorem is immediately apparent from the procedure used to transform Kummer's equation (51) into the nonlinear integral equation (74).

Theorem 11. *Suppose that $\lambda > 0$ is a real number, that $q : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable, strictly positive function, that $x(t)$ is defined by (58), and that $p(x)$ is defined via (61). Suppose also that $\sigma \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ is a solution of the integral equation (74), that δ is defined via the formula*

$$\delta(x) = T[\sigma](x) = \frac{1}{4\lambda}\int_{-\infty}^{\infty} \sin(2\lambda|x - y|)\,\sigma(y)\,dy, \tag{76}$$

and that the function α is defined by the formula

$$\alpha(t) = \lambda\int_0^t \sqrt{q(u)}\exp\left(\frac{\delta(x(u))}{2}\right)\,du. \tag{77}$$

Then:

1. *$\delta(x)$ is a twice continuously differentiable solution of (59);*

2. *$\delta(x(t))$ is a twice continuously differentiable solution of of (57);*

3. *$\alpha(t)$ is three times continuously differentiable solution of (51); and*

4. *$\alpha(t)$ is a phase function for the ordinary differential equation*

$$y''(t) + \lambda^2 q(t)y(t) = 0 \quad \text{for all } 0 \leqslant t \leqslant 1. \tag{78}$$

4. Overview and statement of the principal result

The composition operator $S \circ T$ appearing in (74) does not map any Lebesgue space $L^p(\mathbb{R})$ or Hölder space $C^{k,\alpha}(\mathbb{R})$ to itself, which complicates the analysis of (74). Moreover, the integral defining $T[\sigma]$ only exists if either $\sigma \in L^1(\mathbb{R})$ or $\widehat{\sigma}(\pm 2\lambda) = 0$. Even if both of these conditions are satisfied, it is not necessarily the case that $S[T[\sigma]] + p$ will satisfy either condition. This casts doubts on whether (74) is solvable for arbitrary p.

We avoid a detailed discussion of which spaces and in what sense (74) admits solutions and instead show that, under mild conditions on p and λ, there exists a function p_b which approximates p and such that the equation

$$\sigma(x) = S\left[T\left[\sigma\right]\right](x) + p_b(x) \quad \text{for all } x \in \mathbb{R} \tag{79}$$

admits a solution. Moreover, we prove that if p is nonoscillatory then the solution of (79) is also nonoscillatory and $\|p - p_b\|_\infty$ decays exponentially in λ. The next theorem, which is the principal result of this article, makes these statements precise. Its proof is given in Sections 5, 6, 7 and 8.

Theorem 12. *Suppose that $q \in C^\infty(\mathbb{R})$ is a strictly positive, and that $x(t)$ is defined by the formula*

$$x(t) = \int_0^t \sqrt{q(u)}\, du. \tag{80}$$

Suppose also that $p(x)$ is defined via the formula

$$p(x) = 2\{t, x\}; \tag{81}$$

that is, $p(x)$ is twice the Schwarzian derivative of the variable t with respect to the variable x defined via (80). Suppose furthermore that there exist real numbers $\lambda > 0$, $\Gamma > 0$ and $a > 0$ such that

$$\lambda \geqslant 6 \max\left\{\frac{1}{a}, \Gamma\right\} \tag{82}$$

and

$$|\widehat{p}(\xi)| \leqslant \Gamma \exp\left(-a|\xi|\right) \quad \text{for all } \xi \in \mathbb{R}. \tag{83}$$

Then there exist functions $p_b \in C^\infty(\mathbb{R})$ and $\sigma_b \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$ such that σ_b is a solution of (79),

$$|\widehat{\sigma_b}(\xi)| \leqslant 2\Gamma \exp\left(-\frac{5}{6}a|\xi|\right) \quad \text{for all } \xi \in \mathbb{R}, \tag{84}$$

and

$$\|p - p_b\|_\infty < \frac{24\Gamma}{5a} \exp\left(-\frac{5}{6}a\lambda\right). \tag{85}$$

According to Theorem 11, if σ is a solution of the integral equation (74), then the function α defined by (77) is a phase function for the differential equation (1). We define α_b in analogy with (77) using the solution σ_b of the modified equation (79). That is, we let δ_b be the function defined by the formula

$$\delta_b(x) = T\left[\sigma_b\right](x) = \frac{1}{4\lambda} \int_{-\infty}^\infty \sin(2\lambda|x - y|)\sigma_b(y)\, dy, \tag{86}$$

and then define α_b via

$$\alpha_b(t) = \lambda \int_0^t \sqrt{q(u)} \exp\left(\frac{\delta_b(x(u))}{2}\right) du. \tag{87}$$

Obviously, α_b is not a phase function for the equation (1). However, if we define $q_b : \mathbb{R} \to \mathbb{R}$ by the formula

$$\frac{1}{q_b(t)}\left(\frac{5}{4}\left(\frac{q_b'(t)}{q_b(t)}\right)^2 - \frac{q_b''(t)}{q_b(t)}\right) = p_b(t), \tag{88}$$

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then Theorem 11 implies that α_b is a phase function for the differential equation

$$y''(t) + \lambda^2 q_b(t)y(t) = 0 \quad \text{for all } 0 \leqslant t \leqslant 1. \tag{89}$$

Since $\|q - q_b\|_\infty$ is bounded in terms of $\|p - p_b\|_\infty$ and the solutions of (89) closely approximate those of (1) when $\|q - q_b\|_\infty$ is small, the function α_b can be used to approximation solutions of (1) when $\|p - p_b\|_\infty$ is small.

Theorem 13, which appears below, gives a relevant error estimate. Given $\epsilon > 0$, it specifies a bound on $\|p - p_b\|_\infty$ which ensures that solutions of (89) approximate those of (1) to relative precision ϵ. Its proof appears in Section 9.

Defintion 1. *We say that α is an ϵ-approximate phase function for the ordinary differential equation (1) if there exists a basis of solutions $\{\widetilde{u}, \widetilde{v}\}$ of (1) such that*

$$|u(t) - \widetilde{u}(t)| \leqslant \epsilon \sup_{0 \leqslant t \leqslant 1} |\widetilde{u}(t)| \quad \text{for all } 0 \leqslant t \leqslant 1 \tag{90}$$

and

$$|v(t) - \widetilde{v}(t)| \leqslant \epsilon \sup_{0 \leqslant t \leqslant 1} |\widetilde{v}(t)| \quad \text{for all } 0 \leqslant t \leqslant 1, \tag{91}$$

where u, v are defined by

$$u(t) = \frac{\sin(\alpha(t))}{|\alpha'(t)|^{1/2}} \tag{92}$$

and

$$v(t) = \frac{\cos(\alpha(t))}{|\alpha'(t)|^{1/2}}. \tag{93}$$

Theorem 13. *Suppose that $q \in C^\infty(\mathbb{R})$ is strictly positive, that the function p is defined by the formula (56), and that there exist real numbers $0 < \eta_1 < \eta_2$ such that*

$$\eta_1 \leqslant q(t) \leqslant \eta_2 \quad \text{for all } 0 \leqslant t \leqslant 1, \tag{94}$$

$$\left| \frac{q'(t)}{q(t)} \right| \leqslant \eta_2 \quad \text{for all } 0 \leqslant t \leqslant 1, \tag{95}$$

and

$$|p(t)| \leqslant \eta_2 \quad \text{for all } 0 \leqslant t \leqslant 1. \tag{96}$$

Suppose also that $\lambda > 0$, $\epsilon > 0$ are real numbers such that

$$0 < \epsilon < \lambda \exp\left(\frac{\eta_2^{3/4}}{4} \right), \tag{97}$$

and that k is the real number defined by

$$k = 20 \left(\frac{\eta_2}{\eta_1} \right)^2 + 8\eta_2^2 + 10\frac{\eta_2}{\eta_1} + 1. \tag{98}$$

Suppose furthermore that $p_b : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable function such that

$$\|p - p_b\|_\infty \leqslant \frac{1}{2}\frac{\eta_1}{\lambda} \exp(-k) \exp\left(-\frac{\eta_2^{3/4}}{4} \right) \epsilon. \tag{99}$$

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Then there exists a function $q_b : [0, 1] \rightarrow \mathbb{R}$ such that (88) holds and any phase function for (89) is an ϵ-approximate phase function for (1).

By combining Theorems 11, 12 and 13 we obtain the following.

Corollary 1. *Suppose that q, $x(t)$, p, η_1, η_2, λ, ϵ, k, Γ, a and η are as in the hypotheses of Theorem 12 and 13. Suppose that, in addition,*

$$\lambda \geqslant \frac{6}{5a} \left(k + \frac{\eta_2^{3/4}}{4} + \log \left(\frac{12\Gamma}{\eta_1 a^2} \frac{1}{\epsilon} \right) \right). \tag{100}$$

Then there exists a function $\sigma_b \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$ such that

$$|\widehat{\sigma}_b(\xi)| \leqslant 2\Gamma \exp\left(-\frac{5}{6} a|\xi| \right) \quad \text{for all } \xi \in \mathbb{R} \tag{101}$$

and the function $\alpha_b(t)$ defined by

$$\alpha_b(t) = \lambda \int_0^t \sqrt{q(u)} \exp\left(\frac{\delta_b(x(u))}{2} \right) \, du, \tag{102}$$

where δ_b is the function defined through the formula

$$\delta_b(x) = \frac{1}{4\lambda} \int \sin(2\lambda|x - y|)\sigma_b(y) \, dy, \tag{103}$$

is an ϵ-approximate phase function for the ordinary differential equation (1).

Remark 1. *The hypothesis (100) can be replaced with the weaker condition*

$$\lambda > \frac{6}{5a} W_{-1} \left(-\frac{25}{288} \frac{\eta_1 a^2}{\Gamma} \exp(-k) \exp\left(-\frac{\eta_2^{3/4}}{4} \right) \epsilon \right), \tag{104}$$

where W_{-1} denotes the branch of the Lambert W function which is real-valued and less than -1 on the interval $[-1/e, 0)$ (see Section 2.3).

The proof of Theorem 12 is divided amongst Sections 5, 6, 7 and 8. The principal difficulty lies in constructing a function p_b which approximates p and for which (79) admits a solution. We accomplish this by introducing a modified integral equation

$$\sigma(x) = S\left[T_b\left[\sigma \right] \right](x) + p(x), \tag{105}$$

where T_b is a "band-limited" version of T. That is, $T_b\left[f \right]$ is defined via the formula

$$\widehat{T_b\left[f \right]}(\xi) = \widehat{T\left[f \right]}(\xi)b(\xi), \tag{106}$$

where $b(\xi)$ is a $C_c^\infty(\mathbb{R})$ bump function. This modified integral equation is introduced in Section 5.

In Section 6, we show that under mild conditions on p and λ, (105) admits a solution σ. The argument proceeds by applying the Fourier transform to (105) and using the contraction mapping principle to show that the resulting equation admits a solution.

In Section 7, we estimate the Fourier transform of the solution σ of (105) under the assumption that $\widehat{p}$ is exponentially decaying. We show that $\widehat{\sigma}$ is also exponentially decaying, albeit at a slightly slower rate.

In Section 8, we define a band-limited version σ_b of the solution σ of (105) via the formula

$$\widehat{\sigma_b}(\xi) = \widehat{\sigma}(\xi)b(\xi). \tag{107}$$

By combining the observation that $T_b\left[\sigma\right] = T\left[\sigma_b\right]$ with (105), we obtain

$$\sigma(x) = S\left[T\left[\sigma_b\right]\right](x) + p(x). \tag{108}$$

We then define p_b by the formula

$$p_b(x) = p(x) + \sigma_b(x) - \sigma(x) \tag{109}$$

and rearrange (108) as

$$\sigma_b(x) = S\left[T\left[\sigma_b\right]\right](x) + p_b(x). \tag{110}$$

The decay estimate on $\widehat{\sigma}$ is then used to bound $\|p - p_b\|_\infty = \|\sigma - \sigma_b\|_\infty \leqslant \|\widehat{\sigma} - \widehat{\sigma_b}\|_1$.

5. Band-limited integral equation

In this section, we introduce a "band-limited" version of the operator T and use it to form an alternative to the integral equation (74).

Let $b(\xi)$ be any infinitely differentiable function such that

1. $b(\xi) = 1$ for all $|\xi| \leqslant \lambda$,

2. $0 \leqslant b(\xi) \leqslant 1$ for all $\lambda \leqslant |\xi| \leqslant \sqrt{2}\lambda$, and

3. $b(\xi) = 0$ for all $|\xi| \geqslant \sqrt{2}\lambda$.

We define $T_b\left[f\right]$ for functions $f \in L^1\left(\mathbb{R}\right)$ via the formula

$$\widehat{T_b\left[f\right]}(\xi) = \widehat{f}(\xi)\frac{b(\xi)}{4\lambda^2 - \xi^2} \tag{111}$$

We will refer to T_b as the band-limited version of the operator T and and we call the nonlinear integral equation

$$\sigma(x) = S\left[T_b\left[\sigma\right]\right](x) + p(x) \quad \text{for all} \ \ x \in \mathbb{R} \tag{112}$$

obtained by replacing T with T_b in (74) the "band-limited" version of (74).

Since T_b is a Fourier multiplier, it is convenient to analyze (112) in the Fourier domain rather than the space domain. We now introduce notation which will allow us to write down the equation obtained by applying the Fourier transform to both sides of (112).

We let W_b and $\widetilde{W}_b$ be the linear operators defined for $f \in L^1\left(\mathbb{R}\right)$ via the formulas

$$W_b\left[f\right](\xi) = f(\xi)\frac{b(\xi)}{4\lambda^2 - \xi^2} \tag{113}$$

and

$$\widetilde{W}_b\left[f\right](\xi) = f(\xi)\frac{b(\xi)i\xi}{4\lambda^2 - \xi^2}, \tag{114}$$

where $b(\xi)$ is the function used to define the operator T_b.

For functions $f \in L^1(\mathbb{R})$, it is standard to denote the Fourier transform of the function $\exp(f(x))$ by $\exp^*[f]$; that is,

$$\exp^*[f](\xi) = \delta(\xi) + f(\xi) + \frac{f * f(\xi)}{2!} + \frac{f * f * f(\xi)}{3!} + \cdots . \tag{115}$$

In (115), δ defers to the delta distribution and $f * f * \cdots * f$ denotes repeated convolution of the function f with itself. The Fourier transform of $\exp(f(x))$ never appears in this paper; however, we will encounter the Fourier transforms of the functions

$$\exp(f(x)) - 1 \tag{116}$$

and

$$\exp(f(x)) - f(x) - 1. \tag{117}$$

So, in analogy with the definition (115), we define $\exp_1^*[f]$ for $f \in L^1(\mathbb{R})$ by the formula

$$\exp_1^*[f](\xi) = f(\xi) + \frac{f * f(\xi)}{2!} + \frac{f * f * f(\xi)}{3!} + \cdots , \tag{118}$$

and we define $\exp_2^*[f]$ for $f \in L^1(\mathbb{R})$ via the formula

$$\exp_2^*[f](\xi) = \frac{f * f(\xi)}{2!} + \frac{f * f * f(\xi)}{3!} + \cdots . \tag{119}$$

That is, $\exp_1^*[f]$ is obtained by truncating the leading term of $\exp^*[f]$ and $\exp_2^*[f]$ is obtained by truncated the first two leading terms of $\exp^*[f]$.

Finally, we define functions $\psi(\xi)$ and $v(\xi)$ using the formulas

$$\psi(\xi) = \widehat{\sigma}(\xi) \tag{120}$$

and

$$v(\xi) = \widehat{p}(\xi). \tag{121}$$

Applying the Fourier transform to both sides of (112) results in the nonlinear equation

$$\psi(\xi) = R[\psi](\xi), \tag{122}$$

where $R[f]$ is defined for $f \in L^1(\mathbb{R})$ by the formula

$$R[f](\xi) = \frac{1}{4}\widetilde{W}_b[f] * \widetilde{W}_b[f](\xi) - 4\lambda^2 \exp_2^*[W_b[f]](\xi) + v(\xi). \tag{123}$$

6. Existence of solutions of the band-limited equation.

In this section, we give conditions under which the sequence $\{\psi_n\}_{n=0}^\infty$ of fixed point iterates for (122) obtained by using the function v defined by (121) as an initial approximation converges. More explicitly, ψ_0 is defined by the formula

$$\psi_0(\xi) = v(\xi), \tag{124}$$

and for each integer $n \geqslant 0$, ψ_{n+1} is obtained from ψ_n via

$$\psi_{n+1}(\xi) = R[\psi_n](\xi). \tag{125}$$

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Theorem 14. *Suppose that $\lambda > 0$ is a real number, and that $v \in L^1(\mathbb{R})$ such that*

$$\|v\|_1 \le \frac{\lambda^2}{18}. \tag{126}$$

Theen the sequence $\{\psi_n\}_{n=0}^{\infty}$ defined by (124) and (125) converges in $L^1(\mathbb{R})$ norm to a function ψ which satisfies the equation (122) for almost all $\xi \in \mathbb{R}$.

Proof. We observe that the Fréchet derivative (see Section 2.4) of R at f is the linear operator $R'_f : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ given by the formula

$$R'_f[h](\xi) = \frac{\widetilde{W}_b[f] * \widetilde{W}_b[h](\xi)}{2} - 4\lambda^2 \exp_1^*[W_b[f]] * W_b[h](\xi). \tag{127}$$

From formulas (113) and (114) and the definition of $b(\xi)$ we see that

$$\|W_b[f]\|_1 \le \frac{\|f\|_1}{2\lambda^2} \tag{128}$$

and

$$\left\|\widetilde{W}_b[f]\right\|_1 \le \frac{\|f\|_1}{\sqrt{2}\lambda} \tag{129}$$

for all $f \in L^1(\mathbb{R})$. From (127), (128) and (129) we conclude that

$$
\begin{aligned}
\left\|R'_f[h]\right\|_1 &\le \frac{1}{2}\left\|\widetilde{W}_b[f]\right\|_1 \left\|\widetilde{W}_b[h]\right\|_1 + 4\lambda^2 \|W_b[f]\|_1 \exp\left(\|W_b[f]\|_1\right)\|W_b[h]\|_1 \\
&\le \frac{\|f\|_1\|h\|_1}{4\lambda^2} + 4\lambda^2 \frac{\|f\|_1}{2\lambda^2}\frac{\|h\|_1}{2\lambda^2}\exp\left(\frac{\|f\|_1}{2\lambda^2}\right) \\
&\le \left(\frac{\|f\|_1}{4\lambda^2} + \frac{\|f\|_1}{\lambda^2}\exp\left(\frac{\|f\|_1}{2\lambda^2}\right)\right)\|h\|_1
\end{aligned}
\tag{130}
$$

for all f and h in $L^1(\mathbb{R})$. Similarly, by combining (123), (128) and (129) we conclude that

$$
\begin{aligned}
\|R[f]\|_1 &\le \frac{1}{4}\left\|\widetilde{W}_b[f]\right\|_1^2 + 4\lambda^2 \|W_b[f]\|_1^2 \exp\left(\|W_b[f]\|_1\right) + \|v\|_1 \\
&\le \frac{\|f\|_1^2}{8\lambda^2} + \frac{\|f\|_1^2}{\lambda^2}\exp\left(\frac{\|f\|_1}{2\lambda^2}\right) + \|v\|_1
\end{aligned}
\tag{131}
$$

whenever $f \in L^1(\mathbb{R})$.

Now we set $r = \lambda^2/9$ and let B denote the closed ball of radius r centered at 0 in $L^1(\mathbb{R})$. Since

$$\frac{1}{9} < W_0\left(\frac{1}{8}\right), \tag{132}$$

where W_0 denotes the branch of the Lambert W function which is real-valued and greater than or equal to -1 on the interval $[-1/e, \infty)$ (see Section 2.3), we conclude that

$$\frac{r}{\lambda^2}\exp\left(\frac{r}{2\lambda^2}\right) < \frac{1}{4}. \tag{133}$$

According to Theorem 5 of Section 2.3 and (132), it also the case that

$$\frac{r}{\lambda^2} < \frac{1}{8}. \tag{134}$$

We combine (133), (134) with (130) to conclude that

$$\left\| R'_f[h] \right\|_1 \leqslant \left(\frac{r}{4\lambda^2} + \frac{r}{\lambda^2} \exp\left(\frac{r}{2\lambda^2} \right) \right) \|h\|_1 < \left(\frac{1}{32} + \frac{1}{4} \right) \|h\|_1 < \frac{1}{2} \|h\|_1 \tag{135}$$

for all $h \in L^1(\mathbb{R})$ and $f \in B$. In other words, the $L^1(\mathbb{R}) \to L^1(\mathbb{R})$ operator norm of the linear operator R'_f is bounded by $1/2$ whenever f is in the ball B.

Similarly, we insert (133), (134) and (126) into (131) to conclude that

$$\begin{aligned}
\|R[f]\|_1 &\leqslant \frac{r^2}{8\lambda^2} + \frac{r^2}{\lambda^2} \exp\left(\frac{r}{2\lambda^2} \right) + \frac{r}{2} \\
&\leqslant r \left(\frac{r}{8\lambda^2} + \frac{r}{\lambda^2} \exp\left(\frac{r}{2\lambda^2} \right) + \frac{1}{2} \right) \\
&\leqslant r \left(\frac{1}{64} + \frac{1}{4} + \frac{1}{2} \right) \\
&\leqslant r
\end{aligned} \tag{136}$$

for all $f \in B$.

Together with Theorem 7 of Section 2.4, formula (135) implies that the operator R is a contraction on the ball B while (136) says that it maps the ball B into itself. We now apply the contraction mapping theorem (Theorem 8 in Section 2.4) to conclude that any sequence of fixed point iterates for (122) which originates in B will converge to a solution of (122). Since $\{\psi_n\}$ is such a sequence, we are done. □

If ψ is a solution of (122) then the function σ defined by the formula

$$\sigma(x) = \int_{-\infty}^{\infty} \exp(ix\xi)\psi(\xi)\, d\xi \tag{137}$$

is clearly a solution of the band-limited integral equation (112). Note that because $\psi \in L^1(\mathbb{R})$, the integral in (137) is well-defined and σ is an element of the space $C_0(\mathbb{R})$ of continuous functions which vanish at infinity. We record this observation as follows:

Theorem 15. *Suppose that $\lambda > 0$ is a real number. Suppose also that $p \in L^1(\mathbb{R})$ such that $\widehat{p} \in L^1(\mathbb{R})$ and*

$$\|\widehat{p}\|_1 < \frac{\lambda^2}{18}. \tag{138}$$

Then there exists a function $\sigma \in C_0(\mathbb{R})$ which is a solution of the integral equation (112).

Remark 2. *Since σ is not necessarily in $L^1(\mathbb{R})$, the integral*

$$\int_{-\infty}^{\infty} \exp(-ix\xi)\sigma(x)\, dx \tag{139}$$

need not exist. Nor is the existence of the improper integral

$$\lim_{R \to \infty} \int_{-R}^{R} \exp(-ix\xi)\sigma(x)\, dx \tag{140}$$

guaranteed. However, when viewed as a tempered distribution, the Fourier transform of σ exists and

is ψ; that is to say,

$$\int_{-\infty}^{\infty} \psi(x) f(x) \, dx = \int_{-\infty}^{\infty} \sigma(x) \widehat{f}(x) \, dx \tag{141}$$

for all functions $f \in S(\mathbb{R})$.

In the next section we will prove that under additional assumptions on v, ψ lies in $L^2(\mathbb{R})$. This implies that $\sigma \in L^2(\mathbb{R})$ and ensures the convergence of the improper Riemann integrals (140).

7. Fourier estimate

In this section, we derive a pointwise estimate on the solution ψ of Equation (122) under additional assumptions on the function v.

Lemma 1. *Suppose that a and C are real numbers such that*

$$0 \leqslant C < a. \tag{142}$$

Suppose also that $f \in L^1(\mathbb{R})$, and that

$$|f(\xi)| \leqslant C \exp(-a|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \tag{143}$$

Then

$$|\exp_2^*[f](\xi)| \leqslant \frac{C^2}{2\pi} \exp(-a|\xi|) \frac{1 + a|\xi|}{a} \exp\left(\frac{C}{2\pi a}\right) \exp\left(\frac{C}{2\pi}|\xi|\right) \quad \text{for all } \xi \in \mathbb{R}, \tag{144}$$

where $\exp_2^*$ is the operator defined in (119).

Proof. Let

$$g(\xi) = C \exp(-a|\xi|) \tag{145}$$

and for each integer $m > 0$, denote by g_m the m-fold convolution product of the function g. That is to say that g_1 is defined via the formula

$$g_1(\xi) = g(\xi) \tag{146}$$

and for each integer $m > 0$, g_{m+1} is defined in terms of g_m by the formula

$$g_{m+1}(\xi) = g_m * g(\xi). \tag{147}$$

We observe that for each integer $m > 0$ and all $\xi \in \mathbb{R}$,

$$g_m(\xi) = 2\sqrt{aC} \left(\frac{C|\xi|}{2\pi}\right)^{m-1/2} \frac{K_{m-1/2}(a|\xi|)}{\Gamma(m)}, \tag{148}$$

where K_ν denotes the modified Bessel function of the second kind of order ν (see Section 2.1). By repeatedly applying Theorem 1 of Section 2.1, we conclude that for all integers $m > 0$ and all real t,

$$
\begin{aligned}
K_{m-1/2}(t) &\leqslant K_{1/2}(t) \prod_{j=1}^{m-1} \left(\frac{2\left(j - \frac{1}{2}\right)}{t} + 1\right) \\
&= K_{1/2}(t) \left(\frac{2}{t}\right)^{m-1} \frac{\Gamma\left(\frac{1+t}{2} + m - 1\right)}{\Gamma\left(\frac{1+t}{2}\right)}.
\end{aligned}
\tag{149}
$$

We insert the identity

$$K_{1/2}(t) = \sqrt{\frac{\pi}{2t}} \exp(-t) \tag{150}$$

into (148) in order to conclude that for all integers $m > 0$ and all real numbers $t > 0$,

$$K_{m-1/2}(t) \leqslant \frac{\sqrt{\pi}}{2} \left(\frac{t}{2}\right)^{1/2-m} \exp(-t) \frac{\Gamma\left(\frac{1+t}{2} + m - 1\right)}{\Gamma\left(\frac{1+t}{2}\right)}. \tag{151}$$

By combining (151) and (148) we conclude that

$$g_m(\xi) \leqslant C \exp(-a|\xi|) \left(\frac{C}{\pi a}\right)^{m-1} \frac{\Gamma\left(\frac{1+a|\xi|}{2} + m - 1\right)}{\Gamma(m)\Gamma\left(\frac{1+a|\xi|}{2}\right)} \tag{152}$$

for all integers $m > 0$ and all $\xi \neq 0$. Moreover, the limit as $\xi \to 0$ of each side of (152) is finite and the two limits are equal, so (152) in fact holds for all $\xi \in \mathbb{R}$. We sum (152) over $m = 2, 3, \ldots$ in order to conclude that

$$\exp_2^*[g](\xi) \leqslant C \exp(-a|\xi|) \sum_{m=2}^{\infty} \left(\frac{C}{\pi a}\right)^{m-1} \frac{\Gamma\left(\frac{1+a|\xi|}{2} + m - 1\right)}{\Gamma(m+1)\Gamma(m)\Gamma\left(\frac{1+a|\xi|}{2}\right)}$$

$$= C \exp(-a|\xi|) \sum_{m=1}^{\infty} \left(\frac{C}{\pi a}\right)^{m} \frac{\Gamma\left(\frac{1+a|\xi|}{2} + m\right)}{\Gamma(m+2)\Gamma(m+1)\Gamma\left(\frac{1+a|\xi|}{2}\right)} \tag{153}$$

for all $\xi \in \mathbb{R}$. Now we observe that

$$\frac{1}{\Gamma(m+2)} \leqslant \left(\frac{1}{2}\right)^m \quad \text{for all } m = 0, 1, 2, \ldots. \tag{154}$$

Inserting (154) into (153) yields

$$\exp_2^*[g](\xi) \leqslant C \exp(-a|\xi|) \sum_{m=1}^{\infty} \left(\frac{C}{2\pi a}\right)^{m} \frac{\Gamma\left(\frac{1+a|\xi|}{2} + m\right)}{\Gamma(m+1)\Gamma\left(\frac{1+a|\xi|}{2}\right)} \tag{155}$$

for all $\xi \in \mathbb{R}$. Now we apply the binomial theorem (Theorem 2 of Section 2.2), which is justified since $C < a$, to conclude that

$$\exp_2^*[g](\xi) \leqslant C \exp(-a|\xi|) \left(\left(1 - \frac{C}{2\pi a}\right)^{-\frac{1+a|\xi|}{2}} - 1\right)$$

$$= C \exp(-a|\xi|) \left(\exp\left(\frac{1 + a|\xi|}{2} \log\left(\frac{1}{1 - \frac{C}{2\pi a}}\right)\right) - 1\right) \tag{156}$$

for all $\xi \in \mathbb{R}$. We observe that

$$\exp(x) - 1 \leqslant x \exp(x) \quad \text{for all } x \geqslant 0, \tag{157}$$

and

$$1 \leqslant \log\left(\frac{1}{1-x}\right) \leqslant 2x \quad \text{for all } 0 \leqslant x \leqslant \frac{1}{2\pi}. \tag{158}$$

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By combining (157) and (158) with (156) we conclude that

$$\exp_2^*[g](\xi) \leqslant C \exp(-a|\xi|) \frac{1 + a|\xi|}{2} \log\left(\frac{1}{1 - \frac{C}{2\pi a}}\right) \exp\left(\frac{1 + a|\xi|}{2} \log\left(\frac{1}{1 - \frac{C}{2\pi a}}\right)\right)$$

$$\leqslant \frac{C^2}{2\pi} \exp(-a|\xi|) \frac{1 + a|\xi|}{a} \exp\left(\frac{C}{2\pi a}\right) \exp\left(\frac{C}{2\pi}|\xi|\right) \tag{159}$$

for all $\xi \in \mathbb{R}$. Owing to (143),

$$|\exp_2^*[f](\xi)| \leqslant \exp_2^*[g](\xi) \quad \text{for all } \xi \in \mathbb{R}. \tag{160}$$

By combining this observation with (159), we obtain (144), which completes the proof. $\qquad\square$

Remark 3. *Kummer's confluent hypergeometric function $M(a, b, z)$ is defined by the series*

$$M(a, b, z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \frac{(a)_3 z^3}{(b)_3 3!} + \cdots, \tag{161}$$

where $(a)_n$ is the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} = a(a + 1)(a + 2)\dots(a + n - 1). \tag{162}$$

By comparing the definition of $M(a, b, z)$ with (153), we conclude that

$$|\exp_2^*[f](\xi)| \leqslant C \exp(-a|\xi|) \left(M\left(\frac{1 + a|\xi|}{2}, 2, \frac{C}{\pi a}\right) - 1\right) \quad \text{for all } \xi \in \mathbb{R} \tag{163}$$

provided

$$|f(\xi)| \leqslant C \exp(-a|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \tag{164}$$

The weaker bound (144) is sufficient for our immediate purposes, but formula (163) might serve as a basis for improved estimates on solutions of Kummer's equation.

The following lemma is a special case of Formula (148).

Lemma 2. *Suppose that $C \geqslant 0$ and $a > 0$ are real numbers, and that $f \in L^1(\mathbb{R})$ such that*

$$|f(\xi)| \leqslant C \exp(-a|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \tag{165}$$

Then

$$|f * f(\xi)| \leqslant C^2 \exp(-a|\xi|) \left(\frac{1 + a|\xi|}{a}\right) \quad \text{for all } \xi \in \mathbb{R}. \tag{166}$$

We will also make use of the following elementary observation.

Lemma 3. *Suppose that $a > 0$ is a real number. Then*

$$\exp(-a|\xi|)|\xi| \leqslant \frac{1}{a \exp(1)} \quad \text{for all } \xi \in \mathbb{R}. \tag{167}$$

We combine Lemmas 1 and 2 with (128) and (129) in order to obtain the following key estimate.

Theorem 16. *Suppose that $\Gamma > 0$, $\lambda > 0$, $a > 0$ and $C \geqslant 0$ are real numbers such that*

$$0 \leqslant C < 2a\lambda^2. \tag{168}$$

Suppose also that $f \in L^1(\mathbb{R})$ such that

$$|f(\xi)| \leqslant C \exp(-a|\xi|) \quad \text{for all } |\xi| \leqslant \sqrt{2}\lambda, \tag{169}$$

and that $v \in L^1(\mathbb{R})$ such that

$$|v(\xi)| \leqslant \Gamma \exp(-a|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \tag{170}$$

Suppose further that R is the operator defined via (123). Then

$$|R[f](\xi)| \leqslant \exp(-a|\xi|) \left(\frac{C^2}{\lambda^2} \left(\frac{1 + a|\xi|}{a} \right) \left(\frac{1}{8} + \frac{1}{2\pi} \exp\left(\frac{C}{4\pi\lambda^2 a} \right) \exp\left(\frac{C}{4\pi\lambda^2}|\xi| \right) \right) + \Gamma \right) \tag{171}$$

for all $\xi \in \mathbb{R}$.

Proof. We define the operator R_1 via the formula

$$R_1[f](\xi) = \frac{1}{4}\widetilde{W}_b[f] * \widetilde{W}_b[f](\xi) \tag{172}$$

and R_2 by the formula

$$R_2[f](\xi) = -4\lambda^2 \exp_2^*[W_b[f]](\xi), \tag{173}$$

where W_b and $\widetilde{W}_b$ are defined as in Section 5. Then

$$R[f](\xi) = R_1[f](\xi) + R_2[f](\xi) + v(\xi) \tag{174}$$

for all $\xi \in \mathbb{R}$. We observe that

$$\left| \widetilde{W}_b[f](\xi) \right| \leqslant \frac{C}{\sqrt{2}\lambda} \exp(-a|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \tag{175}$$

By combining Lemma 2 with (175) we obtain

$$|R_1[f](\xi)| \leqslant \frac{C^2}{8\lambda^2} \exp(-a|\xi|) \left(\frac{1 + a|\xi|}{a} \right) \quad \text{for all } \xi \in \mathbb{R}. \tag{176}$$

Now we observe that

$$|W_b[f](\xi)| \leqslant \frac{C}{2\lambda^2} \exp(-a|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \tag{177}$$

Combining Lemma 1 with (177) yields

$$|R_2[f](\xi)| \leqslant \frac{C^2}{2\pi\lambda^2} \exp(-a|\xi|) \left(\frac{1 + a|\xi|}{a} \right) \exp\left(\frac{C}{4\pi\lambda^2 a} \right) \exp\left(\frac{C}{4\pi\lambda^2}|\xi| \right) \tag{178}$$

for all $\xi \in \mathbb{R}$. Note that (168) ensures that the hypothesis (142) in Lemma 1 is satisfied. We combine (176) with (178) and (170) in order to obtain (171), and by so doing we complete the proof. $\qquad\square$

Remark 4. *Note that Theorem 16 only requires that $f(\xi)$ satisfy a bound on the interval $[-\sqrt{2}\lambda, \sqrt{2}\lambda]$ and not on the entire real line.*

In the next theorem, we use Theorem 16 to bound the solution of (122) under an assumption on the decay of v.

Theorem 17. *Suppose that $\lambda > 0$, $a > 0$ and $\Gamma \geqslant 0$ are real numbers such that*

$$\lambda \geqslant 6 \max \left\{ \Gamma, \frac{1}{a} \right\}. \tag{179}$$

Suppose also that $v \in L^1(\mathbb{R})$ such that

$$|v(\xi)| \leqslant \Gamma \exp(-a|\xi|) \quad \text{for all} \ \ \xi \in \mathbb{R}. \tag{180}$$

Then there exists a solution of $\psi(\xi)$ of equation (122) such that

$$|\psi(\xi)| \leqslant 2\Gamma \exp\left(-\left(a - \frac{1}{\lambda} \right) |\xi| \right) \quad \text{for all} \ \ \xi \in \mathbb{R}. \tag{181}$$

Proof. Due to (179) and (180),

$$\|v\|_1 \leqslant \frac{\lambda^2}{18}. \tag{182}$$

It follows from Theorem 14 and (182) that a solution $\psi(\xi)$ of (122) is obtained as the limit of the sequence of fixed point iterates $\{\psi_n(\xi)\}$ defined by the formula

$$\psi_0(\xi) = v(\xi) \tag{183}$$

and the recurrence

$$\psi_{n+1}(\xi) = R\left[\psi_n\right](\xi). \tag{184}$$

We now derive pointwise estimates on the iterates $\psi_n(\xi)$ in order to establish (181).

Let $\{\beta_k\}_{k=0}^{\infty}$ be the sequence of real numbers be generated by the recurrence relation

$$\beta_{k+1} = \frac{\beta_k^2}{2\lambda} + \Gamma \tag{185}$$

with the initial value

$$\beta_0 = \Gamma. \tag{186}$$

It can be established by induction that (179) implies that this sequence is bounded above by 2Γ and monotonically increasing, and hence β_k converges to a real number β such that $0 \leqslant \beta \leqslant 2\Gamma$.

Now suppose that $n \geqslant 0$ is an integer, and that

$$|\psi_n(\xi)| \leqslant \beta_n \exp(-a|\xi|) \quad \text{for all} \ \ |\xi| \leqslant \sqrt{2}\lambda. \tag{187}$$

When $n = 0$, this is simply the assumption (180). The function $\psi_{n+1}(\xi)$ is obtained from $\psi_n(\xi)$ via the formula

$$\psi_{n+1}(\xi) = R\left[\psi\right](\xi). \tag{188}$$

We combine Theorem 16 with (188) and (187) to conclude that

$$|\psi_{n+1}(\xi)| \leqslant \exp(-a|\xi|) \left(\frac{\beta_n^2}{\lambda^2} \left(\frac{1 + a|\xi|}{a} \right) \left(\frac{1}{8} + \frac{1}{2\pi} \exp\left(\frac{\beta_n}{4\pi\lambda^2 a} \right) \exp\left(\frac{\beta_n}{4\pi\lambda^2} |\xi| \right) \right) + \Gamma \right) \tag{189}$$

for all $\xi \in \mathbb{R}$. The application of Theorem 16 is justified: the hypothesis (168) is satisfied since

$$\beta_n \leqslant 2\Gamma \leqslant 2\lambda^2 a \tag{190}$$

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for all integers $n \geqslant 0$. We restrict ξ to the interval $[-\sqrt{2}\lambda, \sqrt{2}\lambda]$ in (189) and use the fact that

$$\frac{1}{a\lambda} < \frac{1}{6}, \tag{191}$$

which is a consequence of (179), in order to conclude that

$$|\psi_{n+1}(\xi)| \leqslant \exp(-a|\xi|) \left(\frac{\beta_n^2}{\lambda^2} \left(\frac{1 + a\sqrt{2}\lambda}{a} \right) \left(\frac{1}{8} + \frac{1}{2\pi} \exp\left(\frac{\beta_n}{4\pi\lambda^2 a} \right) \exp\left(\frac{\beta_n}{4\pi\lambda^2} \sqrt{2}\lambda \right) \right) + \Gamma \right)$$

$$\leqslant \exp(-a|\xi|) \left(\frac{\beta_n^2}{\lambda} \left(\frac{1}{6} + \sqrt{2} \right) \left(\frac{1}{8} + \frac{1}{2\pi} \exp\left(\frac{\beta_n}{24\pi\lambda} \right) \exp\left(\frac{\beta_n}{2\sqrt{2}\pi\lambda} \right) \right) + \Gamma \right) \tag{192}$$

for all $|\xi| \leqslant \sqrt{2}\lambda$. Now we combine (192) with the inequality

$$\frac{\beta_n}{\lambda} \leqslant \frac{2\Gamma}{\lambda} < \frac{1}{3} \tag{193}$$

and the observation that

$$\left(\frac{1}{6} + \sqrt{2} \right) \left(\frac{1}{8} + \frac{1}{2\pi} \exp\left(\frac{1}{72\pi} \right) \exp\left(\frac{1}{6\sqrt{2}\pi} \right) \right) \leqslant \frac{1}{2} \tag{194}$$

in order to conclude that

$$|\psi_{n+1}(\xi)| \leqslant \exp(-a|\xi|) \left(\frac{\beta_n^2}{\lambda} \left(\frac{1}{6} + \sqrt{2} \right) \left(\frac{1}{8} + \frac{1}{2\pi} \exp\left(\frac{1}{72\pi} \right) \exp\left(\frac{1}{6\sqrt{2}\pi} \right) \right) + \Gamma \right)$$

$$\leqslant \left(\frac{\beta_n^2}{2\lambda} + \Gamma \right) \exp(-a|\xi|) \tag{195}$$

$$= \beta_{n+1} \exp(-a|\xi|)$$

for all $|\xi| \leqslant \sqrt{2}\lambda$. We conclude by induction that (187) holds for all integers $n \geqslant 0$.

The sequence $\{\psi_n(\xi)\}$ converges to $\psi(\xi)$ in $L^1(\mathbb{R})$ norm (and hence a subsequence of $\psi_n(\xi)$ converges to $\psi(\xi)$ pointwise almost everywhere) and (187) holds for all integers $n \geqslant 0$. Moreover, for all integers $n \geqslant 0$, $\beta_{n+1} \leqslant 2\Gamma$. From these observations we conclude that

$$|\psi(\xi)| \leqslant 2\Gamma \exp(-a|\xi|) \tag{196}$$

for almost all $|\xi| \leqslant \sqrt{2}\lambda$. We also observe that $\psi(\xi)$ is a fixed point of the operator R, so that

$$\psi(\xi) = R\left[\psi\right](\xi) \tag{197}$$

for all $\xi \in \mathbb{R}$. Clearly,

$$R\left[f\right](\xi) = R\left[g\right](\xi) \tag{198}$$

for all $\xi \in \mathbb{R}$ if $f(\xi) = g(\xi)$ for almost all $|\xi| \leqslant \sqrt{2}\lambda$, so (196) in fact holds for all $|\xi| \leqslant \sqrt{2}\lambda$.

We now apply Theorem 16 to the function $\psi(\xi)$ (which is justified since $2\Gamma < 2\lambda^2 a$) to conclude that

$$|\psi(\xi)| \leqslant \exp(-a|\xi|) \left(\frac{4\Gamma^2}{\lambda^2} \left(\frac{1 + a|\xi|}{a} \right) \left(\frac{1}{8} + \frac{1}{2\pi} \exp\left(\frac{2\Gamma}{4\pi\lambda^2 a} \right) \exp\left(\frac{2\Gamma}{4\pi\lambda^2} |\xi| \right) \right) + \Gamma \right) \tag{199}$$

for all $\xi \in \mathbb{R}$. Note the distinction between (187) and (199) is that the former only holds for all ξ in the interval $[-\sqrt{2}\lambda, \sqrt{2}\lambda]$, while the later holds for all ξ on the real line. It follows from (179) that

$$\frac{1}{\lambda a} < \frac{1}{6} \quad \text{and} \quad \frac{\Gamma}{\lambda} < \frac{1}{6}. \tag{200}$$

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We insert these bounds into (199) in order to conclude that

$$|\psi(\xi)| \leqslant \Gamma \exp(-a|\xi|) \left(\frac{2}{3\lambda} \left(\frac{1 + a|\xi|}{a} \right) \left(\frac{1}{8} + \frac{1}{2\pi} \exp\left(\frac{1}{72\pi} \right) \exp\left(\frac{1}{12\pi\lambda} |\xi| \right) \right) + 1 \right) \qquad (201)$$

for all $\xi \in \mathbb{R}$. Now we observe that

$$\frac{1}{2\pi} \exp\left(\frac{1}{72\pi} \right) < \frac{1}{6}, \qquad (202)$$

which, when combined with (201), yields

$$|\psi(\xi)| \leqslant \Gamma \exp(-a|\xi|) \left(\left(\frac{1}{9} + \frac{2|\xi|}{3\lambda} \right) \left(\frac{1}{8} + \frac{1}{6} \exp\left(\frac{1}{12\pi\lambda} |\xi| \right) \right) + 1 \right) \quad \text{for all } \xi \in \mathbb{R}. \qquad (203)$$

By rearranging the right-hand side of (203) as

$$\Gamma \exp(-a|\xi|) \left(\frac{1}{72} + \frac{1}{56} \exp\left(\frac{1}{12\pi\lambda} |\xi| \right) + \frac{|\xi|}{12\lambda} + \frac{|\xi|}{9\lambda} \exp\left(\frac{1}{12\pi\lambda} |\xi| \right) + 1 \right)$$

$$= \Gamma \exp\left(-\left(a - \frac{1}{\lambda} \right) |\xi| \right) \exp\left(-\frac{1}{\lambda} |\xi| \right) \cdot$$

$$\left(\frac{1}{72} + \frac{1}{56} \exp\left(\frac{1}{12\pi\lambda} |\xi| \right) + \frac{|\xi|}{12\lambda} + \frac{|\xi|}{9\lambda} \exp\left(\frac{1}{12\pi\lambda} |\xi| \right) + 1 \right)$$

and applying Lemma 3, we arrive at the inequality

$$|\psi(\xi)| \leqslant \Gamma \exp\left(-\left(a - \frac{1}{\lambda} \right) |\xi| \right) \left(\frac{1}{72} + \frac{1}{56} + \frac{1}{12 \exp(1)} + \frac{4}{33 \exp(1)} + 1 \right) \quad \text{for all } \xi \in \mathbb{R}, \qquad (204)$$

from which (181) follows immediately. □

Suppose that $\psi \in L^1(\mathbb{R})$ is a solution of (122). Then the function σ defined by the formula

$$\sigma(x) = \int_{-\infty}^{\infty} \exp(ix\xi) \psi(\xi) \, d\xi \qquad (205)$$

is a solution of the integral equation (112). However, the Fourier transform of (205) might only be defined in the sense of tempered distributions and not as a Lebesgue or improper Riemann integral. If, however, we assume the function p appearing in (122) is an element of $L^1(\mathbb{R})$ and impose the hypotheses of Theorem 18 on the Fourier transform of p, then $\psi \in L^2(\mathbb{R})$, from which we conclude that σ is also an element of $L^2(\mathbb{R})$. In this event, there is no difficulty in defining the Fourier transform of σ. We record these observations in the following theorem.

Theorem 18. *Suppose that there exist real numbers $\lambda > 0$, $\Gamma > 0$ and $a > 0$ such that*

$$\lambda > 6 \max \left\{ \Gamma, \frac{1}{a} \right\}. \qquad (206)$$

Suppose also that $p \in L^1(\mathbb{R})$ such that and

$$|\hat{p}(\xi)| \leqslant \Gamma \exp(-a|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \qquad (207)$$

Then there exists a solution $\sigma(x)$ of the integral equation (112) such that

$$|\hat{\sigma}(\xi)| \leqslant 2\Gamma \exp\left(-\left(a - \frac{1}{\lambda} \right) |\xi| \right) \quad \text{for all } \xi \in \mathbb{R} \qquad (208)$$

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Remark 5. *The bound (208) implies that $\widehat{\sigma}$ decays faster than any polynomial, from which we conclude that σ is infinitely differentiable.*

8. Approximate solution of the original equation

We would like to insert the solution σ of (112) into the original equation (74). However, we have no guarantee that σ is in $L^1(\mathbb{R})$, nor do we expect that $\widehat{\sigma}(\pm 2\lambda) = 0$. As a consequence, the integrals defining $T[\sigma]$ might not exist.

To remedy this problem, we define a "band-limited" version σ_b of σ by the formula

$$\widehat{\sigma}_b(\xi) = \widehat{\sigma}(\xi)b(\xi), \tag{209}$$

where $b(\xi)$ is the function used to define the operator T_b. We observe that there is no difficulty in applying T to σ_b since

$$\widehat{\sigma}_b(\pm 2\lambda) = 0. \tag{210}$$

Moreover, $T_b[\sigma] = T[\sigma_b]$, so that

$$\sigma(x) = S[T[\sigma_b]](x) + p(x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{211}$$

Rearranging (211), we obtain

$$\sigma_b(x) = S[T[\sigma_b]](x) + p_b(x) \quad \text{for all} \quad x \in \mathbb{R}, \tag{212}$$

where $p_b(x)$ is defined the formula

$$p_b(x) = p(x) + \sigma_b(x) - \sigma(x). \tag{213}$$

Using (208) and (213), we conclude that under the hypotheses of Theorem 18,

$$
\begin{aligned}
\|p - p_b\|_\infty &\leqslant \|\widehat{\sigma}_b - \widehat{\sigma}\|_1 \\
&\leqslant 2\Gamma \int_{|\xi| \geqslant \lambda} \exp\left(-\left(a - \frac{1}{\lambda}\right)|\xi|\right) d\xi \\
&\leqslant \frac{4\Gamma}{a - \frac{1}{\lambda}} \exp\left(-\left(a - \frac{1}{\lambda}\right)\lambda\right) \\
&\leqslant \frac{24\Gamma}{5a} \exp\left(-\frac{5}{6}a\lambda\right).
\end{aligned}
\tag{214}
$$

Together Theorem 18 and (214) imply Theorem 12.

9. Backwards error estimate

In this section, we prove Theorem 13.

Although both p and p_b are defined on the real line, we are only concerned with solutions of (1) on the interval $[0, 1]$, and so we only require estimates there. Accordingly, throughout this section we use $\|\cdot\|_\infty$ to denote the $L^\infty([0,1])$ norm.

We omit the proof of the following lemma, which is somewhat long and technical but entirely elementary (it can be established with the techniques found in ordinary differential equation textbooks; see, for instance, Chapter 1 of [4]).

Lemma 4. *Suppose that $q : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and strictly positive, that $p(t)$ is defined by (56), and that there exist real numbers $\eta_1 > 0$ and $\eta_2 > 0$ such that*

$$\eta_1 \leq q(t) \leq \eta_2 \quad \text{for all } 0 \leq t \leq 1, \tag{215}$$

and

$$|p(t)|, |q'(t)| \leq \eta_2 \quad \text{for all } 0 \leq t \leq 1. \tag{216}$$

Let

$$k = 20 \left(\frac{\eta_2}{\eta_1}\right)^2 + 8\eta_2^2 + 10\frac{\eta_2}{\eta_1} + 1 \tag{217}$$

and suppose also that $\epsilon > 0$ is a real number such that

$$\epsilon < \frac{\eta_1}{2}. \tag{218}$$

Suppose furthermore that $p_b : [0, 1] \to \mathbb{R}$ is an infinitely differential function such that

$$\|p - p_b\|_\infty \leq \epsilon \exp(-k). \tag{219}$$

Then there exists an infinitely differentiable function $q_b : [0, 1] \to \mathbb{R}$ such that

$$\frac{1}{q_b(t)} \left(\frac{5}{4} \left(\frac{q_b'(t)}{q_b(t)} \right)^2 - \frac{q_b''(t)}{q_b(t)} \right) = p_b(t) \quad \text{for all } 0 \leq t \leq 1 \tag{220}$$

and

$$\|q - q_b\|_\infty \leq \epsilon. \tag{221}$$

We derive a bound on the change in solutions of the ordinary differential equation (1) when the coefficient $q(t)$ is perturbed.

Lemma 5. *Suppose that $\lambda > 0$, $\epsilon > 0$, $\eta_1 > 0$ and $\eta_2 > 0$ are real numbers. Suppose also that $q : [0, 1] \to \mathbb{R}$ is a continuously differentiable function such that*

$$\eta_1 \leq q(t) \leq \eta_2 \quad \text{for all } 0 \leq t \leq 1, \tag{222}$$

that $p(t)$ is defined by the formula (56), and that

$$|p(t)| \leq \eta_2 \quad \text{for all } 0 \leq t \leq 1. \tag{223}$$

Suppose furthermore that $q_b : [0, 1] \to \mathbb{R}$ is a continuously differentiable function such that

$$|q(t) - q_b(t)| \leq \frac{1}{2} \frac{\eta_1}{\lambda} \exp\left(-\frac{\eta_2^{3/4}}{4} \right) \epsilon \quad \text{for all } 0 \leq t \leq 1. \tag{224}$$

If $z(t)$ is a solution of the ordinary differential equation

$$z''(t) + \lambda^2 q(t) z(t) = 0 \quad \text{for all } 0 \leq t \leq 1 \tag{225}$$

and $z_0(t)$ is the unique solution of the ordinary differential equation

$$z_0''(t) + \lambda^2 q_b(t) z_0(t) = 0 \quad \text{for all } 0 \leq t \leq 1 \tag{226}$$

such that $z_0(0) = z(0)$ and $z_0'(0) = z'(0)$, then

$$\|z - z_0\|_\infty \leqslant \epsilon \|z\|_\infty. \tag{227}$$

Proof. We start by observing that the function $\psi(t) = z_0(t) - z(t)$ is the unique solution of the initial value problem

$$\begin{cases} \psi''(t) + \lambda^2 q(t)\psi(t) = \lambda^2 f(t) & \text{for all } 0 \leqslant t \leqslant 1 \\ \psi(0) = \psi'(0) = 0, \end{cases} \tag{228}$$

where

$$f(t) = (q(t) - q_0(t)) (\psi(t) + z(t)). \tag{229}$$

We now apply the well-known Liouville-Green transformation to (228) in two steps. First, we introduce the function

$$\phi(t) = (q(t))^{1/4} \psi(t), \tag{230}$$

which is the solution of the initial value problem

$$\begin{cases} \phi''(t) - \dfrac{q'(t)}{2q(t)}\phi'(t) + \lambda^2 q(t)\phi(t) = (q(t))^{1/4} \lambda^2 f(t) - \dfrac{1}{4}q(t)p(t)\phi(t) & \text{for all } 0 \leqslant t \leqslant 1 \\ \phi(0) = \phi'(0) = 0. \end{cases} \tag{231}$$

Next we introduce the change of variables

$$x(t) = \int_0^t \sqrt{q(u)}\, du, \tag{232}$$

which transforms (231) into

$$\begin{cases} \phi''(x) + \lambda^2 \phi(x) = (q(x))^{-3/4} \lambda^2 f(x) - \dfrac{1}{4}p(x)\phi(x) & \text{for all } 0 \leqslant x \leqslant x_1 \\ \phi(0) = \phi'(0) = 0, \end{cases} \tag{233}$$

where

$$x_1 = \int_0^1 \sqrt{q(u)}\, du. \tag{234}$$

We now use the Green's function for the initial value problem (233) obtained from the Liouville-Green transform to conclude that for all $0 \leqslant x \leqslant x_1$,

$$\phi(x) = \int_0^x \frac{\sin(\lambda(x - y))}{\lambda} \left((q(y))^{-3/4} \lambda^2 f(y) - \frac{1}{4}p(y)\phi(y) \right) dy. \tag{235}$$

By inserting (229) into (235) and we obtain the inequality

$$|\phi(x)| \leqslant \left(\|q - q_b\|_\infty \frac{\lambda}{\eta_1} + \frac{\eta_2}{4} \right) \int_0^x |\phi(y)|\, dy + \frac{\lambda}{\eta_1^{3/4}} \|q - q_b\|_\infty \|z\|_\infty \quad \text{for all } 0 \leqslant x \leqslant x_1. \tag{236}$$

We apply Gronwall's inequality (Theorem 9 in Section 2.5) to (236) in order to conclude that

$$|\phi(x)| \leqslant \frac{\lambda}{\eta_1^{3/4}} \|q - q_b\|_\infty \|z\|_\infty \exp \left(\left(\|q - q_b\|_\infty \frac{\lambda}{\eta_1} + \frac{\eta_2}{4} \right) x \right) \quad \text{for all } 0 \leqslant x \leqslant x_1. \tag{237}$$

Combing (230), (237) and the observation that

$$x_1 = \int_0^1 \sqrt{q(u)}\, du \leqslant \eta_2^{1/2} \tag{238}$$

yields the inequality

$$|\phi(t)| \leqslant \frac{\lambda}{\eta_1}\|q - q_b\|_\infty \exp\left(\frac{\lambda}{\eta_1}\|q - q_b\|_\infty \eta_2^{1/2}\right) \exp\left(\frac{\eta_2^{3/4}}{4}\right)\|z\|_\infty \tag{239}$$

for all $0 \leqslant t \leqslant 1$. Now

$$\frac{\lambda}{\eta_1}\|q - q_b\|_\infty \exp\left(\frac{\lambda}{\eta_1}\|q - q_b\|_\infty \eta_2^{1/2}\right) \exp\left(\frac{\eta_2^{3/4}}{4}\right) < \epsilon \tag{240}$$

if and only if

$$\frac{\lambda}{\eta_1}\eta_2^{1/2}\|q - q_b\|_\infty < W_0\left(\epsilon \eta_2^{1/2}\exp\left(-\frac{\eta_2^{3/4}}{4}\right)\right), \tag{241}$$

where W_0 is the branch of the Lambert W function which is real-valued and greater than -1 on the interval $[-1/e, \infty)$ (see Section 2.3). According to Theorem 5,

$$\frac{\lambda}{\eta_1}\eta_2^{1/2}\|q - q_b\|_\infty < \frac{1}{2}\left(\epsilon \eta_2^{1/2}\exp\left(-\frac{\eta_2^{3/4}}{4}\right)\right), \tag{242}$$

implies (241). We algebraically simplify (242) in order to conclude that

$$\|q - q_b\|_\infty < \frac{1}{2}\frac{\eta_1}{\lambda}\exp\left(-\frac{\eta_2^{3/4}}{4}\right)\epsilon \tag{243}$$

implies (239). □

By combining Lemmas 4 and 5 we obtain Theorem 13.

10. Numerical experiments

In this section, we describe numerical experiments which, *inter alia*, illustrate one of the important consequences of the existence of nonoscillatory phase functions. Namely, that a large class of special functions can be evaluated to high accuracy using a number of operations which does not grow with order.

Although the proof of Theorem 12 suggests a numerical procedure for the construction of nonoscillatory phase functions, we utilize a different procedure here. It has the advantage that the coefficient q in the ordinary differential equation (1) need not be extended outside of the interval on which the nonoscillatory phase function is constructed. A paper describing this work is in preparation.

The code we used for these calculations was written in Fortran and compiled with the Intel Fortran Compiler version 12.1.3. All calculations were carried out in double precision arithmetic on a desktop computer equipped with an Intel Xeon X5690 CPU running at 3.47 GHz.

10.1. A nonoscillatory solution of the logarithm form of Kummer's equation.

In this experiment, we illustrate Theorem 12 in Section 4. We first construct a nonoscillatory solution r of the logarithm form of Kummer's equation

$$r''(t) - \frac{1}{4}\left(r'(t)\right)^2 + 4\lambda^2\left(\exp(r(t)) - q(t)\right) = 0 \tag{244}$$

on the interval $[-1, 1]$, where $\lambda = 1{,}000$ and q is the function $[-1, 1] \to \mathbb{R}$ defined by the formula

$$q(t) = \left(3 + \frac{1}{1 + 10t^2} + t^3 \cos(5t)\right). \tag{245}$$

Then we compute the 500 leading Chebyshev coefficients of q and r.

We display the results of this experiment in Figures 1 and 2. Figure 1 contains plots of the functions q and r, while Figure 2 contains a plot of the base-10 logarithms of the absolute values of the leading Chebyshev coefficients of q and r.

We observe that, consistent with Theorem 12, the Chebyshev coefficients of both r and q decay exponentially, although those of r decay at a slightly slower rate.

10.2. Evaluation of Legendre polynomials.

In this experiment, we compare the cost of evaluating Legendre polynomials of large order using the standard recurrence relation with the cost of doing so with a nonoscillatory phase function.

For any integer $n \geqslant 0$, the Legendre polynomial $P_n(x)$ of order n is a solution of the second order differential equation

$$(1 - t^2)y''(t) - 2ty'(t) + n(n + 1)y(t) = 0. \tag{246}$$

Equation (246) can be put into the standard form

$$\psi''(t) + \left(\frac{1 + n - nt^2 - n^2(t^2 - 1)}{(1 - t^2)^2}\right)\psi(t) = 0 \tag{247}$$

by introducing the transformation

$$\psi(t) = \sqrt{1 - t^2}\, y(t). \tag{248}$$

Legendre polynomials satisfy the well-known three term recurrence relation

$$(n + 1)P_{n+1}(t) = (2n + 1)tP_n(t) - nP_{n-1}(t). \tag{249}$$

See, for instance, [11] for a discussion of the these and other properties of Legendre polynomials.

For each of 9 values of n, we proceed as follows. We sample 1000 random points

$$t_1, t_2, \ldots, t_{1000} \tag{250}$$

from the uniform distribution on the interval $(-1, 1)$. Then we evaluate the Legendre polynomial of order n using the recurrence relation (249) at each of the points $t_1, t_2, \ldots, t_{1000}$. Next, we construct a nonoscillatory phase function for the ordinary differential equation (247) and

use it evaluate the Legendre polynomial of order n at each of the points $t_1, t_2, \ldots, t_{1000}$. Finally, for each integer $j = 1, \ldots, 1000$, we compute the error in the approximation of $P_n(t_j)$ obtained from the nonoscillatory phase function by comparing it to the value obtained using the recurrence relation (we regard the recurrence relation as giving the more accurate approximation).

The results of this experiment are shown in Table 1. There, each row correponds to value of n. That value is listed n, as is the time required to compute each phase function for that value of n, the average time required to evaluate the Legendre polynomial of order n using the recurrence relation, the average cost of evaluating the Legendre polynomial of order n with the nonoscillatory phase function, and the largest of the absolute errors in the approximations of the quantities

$$P_n(t_1), P_n(t_2), \ldots, P_n(t_{1000})$$

obtained via the phase function method.

This experiment reveals that, as expected, the cost of evaluating $P_n(t)$ using the recurrence relation (249) grows as $O(n)$ while the cost of doing so with nonoscillatory phase function is indepedet of order.

However, it also exposes a limitation of phase functions. The values of $P_n(t)$ are obtained in part by evaluating sine and cosine of a phase function whose magnitude is on the order of n. This imposes limitations on the accuracy of the method due to the well-known difficulties in evaluating periodic functions of large arguments.

Figure 3 contains a plot of the nonoscillatory phase function for the equation (247) when $n =$1,000,000.

10.3. Evaluation of Bessel functions.

In this experiment, we compare the cost of evaluating Bessel functions of integer order via the standard recurrence relation with that of doing so using a nonoscillatory phase function.

We will denote by J_ν the Bessel function of the first kind of order ν. It is a solution of the second order differential equation

$$t^2 y''(t) + t y'(t) + (t^2 - \nu^2) y(t) = 0, \tag{251}$$

which can be brought into the standard form

$$\psi''(t) + \left(1 - \frac{\lambda^2 - 1/4}{t^2}\right) \psi(t) = 0 \tag{252}$$

via the transformation

$$\psi(t) = \sqrt{t}\, y(t). \tag{253}$$

An inspection of (252) reveals that J_ν is nonoscillatory on the interval

$$\left(0, \frac{1}{2}\sqrt{4\nu^2 - 1}\right) \tag{254}$$

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and oscillatory on the interval

$$\left(\frac{1}{2}\sqrt{4\nu^2 - 1}, \infty\right). \tag{255}$$

The Bessel functions satisfy the three-term recurrence relation

$$J_{\nu+1}(t) = \frac{2\nu}{t} J_\nu(t) - J_{\nu-1}(t). \tag{256}$$

The recurrence (256) is numerically unstable in the forward direction; however, when evaluated in the direction of decreasing index, it yields a stable mechanism for evaluating Bessel functions of integer order (see, for instance, Chapter 3 of [11]).

For each of 9 values of n, we proceed as follows. First, we sample 1000 random points

$$t_1, t_2, \ldots, t_{1000} \tag{257}$$

from the uniform distribution on the interval $[2n, 3n]$. We then use the recurrence relation (256) to evaluate the Bessel function J_n of order n at the points $t_1, t_2, \ldots, t_{1000}$. Next, we construct a nonoscillatory phase function for the equation (253) on the interval $[2n, 3n]$ and use it to evaluate J_n at the points $t_1, t_2, \ldots, t_{1000}$. Finally, for each integer $j = 1, \ldots, 1000$, we compute the error in the approximation of $J_n(t_j)$ obtained from the nonoscillatory phase function by comparing it to the value obtained using the recurrence relation (once again we regard the recurrence relation as giving the more accurate approximation).

The results of this experiment are displayed in Table 2. There, each row corresponds to one value of n. In addition to that value of n, it lists the time required to compute the phase function at order n, the average cost of evaluating J_n using the recurrence relation, the average cost of evaluating it with the nonoscillatory phase function, and the largest of the absolute errors in the approximations of the quantities

$$J_n(t_1), J_n(t_2), \ldots, J_n(t_{1000})$$

obtained via the phase function method.

We observe that while the cost of evaluating J_n using the recurrence relation (256) grows as $O(n)$, the time taken by the nonoscillatory phase function approach scales as $O(1)$. We also note that, as in the case of Legendre polynomials, there is some loss of accuracy with the phase function method due to the difficulties of evaluating trigonometric functions of large arguments.

11. Conclusions

We have shown that the solutions of a large class of second order differential equations can be accurately represented using nonoscillatory phase functions.

We have also presented the results of numerical experiments which demonstrate one of the applications of nonoscillatory phase functions: the evaluation of special functions at a cost which is independent of order. An efficient algorithm for the evaluation of highly oscillatory special functions will be reported at a later date.

A number of open issues and questions related to this work remain. Most obviously, a further investigation of the integral equation (74) and the conditions under which it admits an

exact solution is warranted. Moreover, there are applications of nonoscillatory phase functions beyond the evaluation of special functions which should be explored. And, of course, the generalization of these results to higher dimensions is of great interest. The authors are vigorously pursuing these avenues of research.

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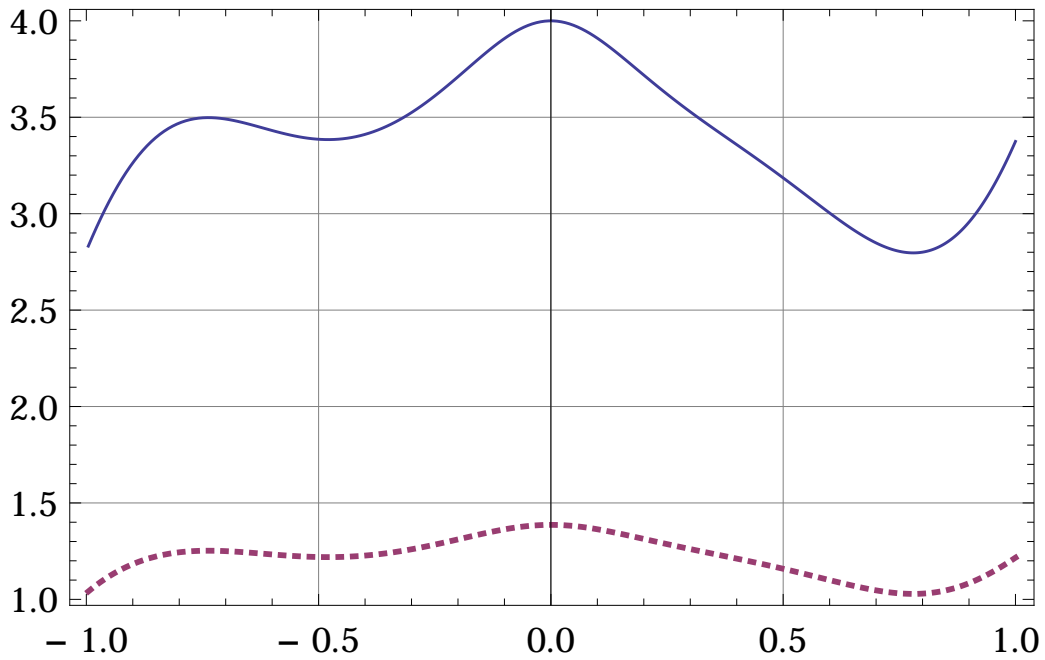
Figure 1: The function q defined by formula (245) in Section 10.1 (solid line) and the corresponding solution r of the logarithm form of Kummer's equation (53) when $\lambda = 1,000$ (dotted line).
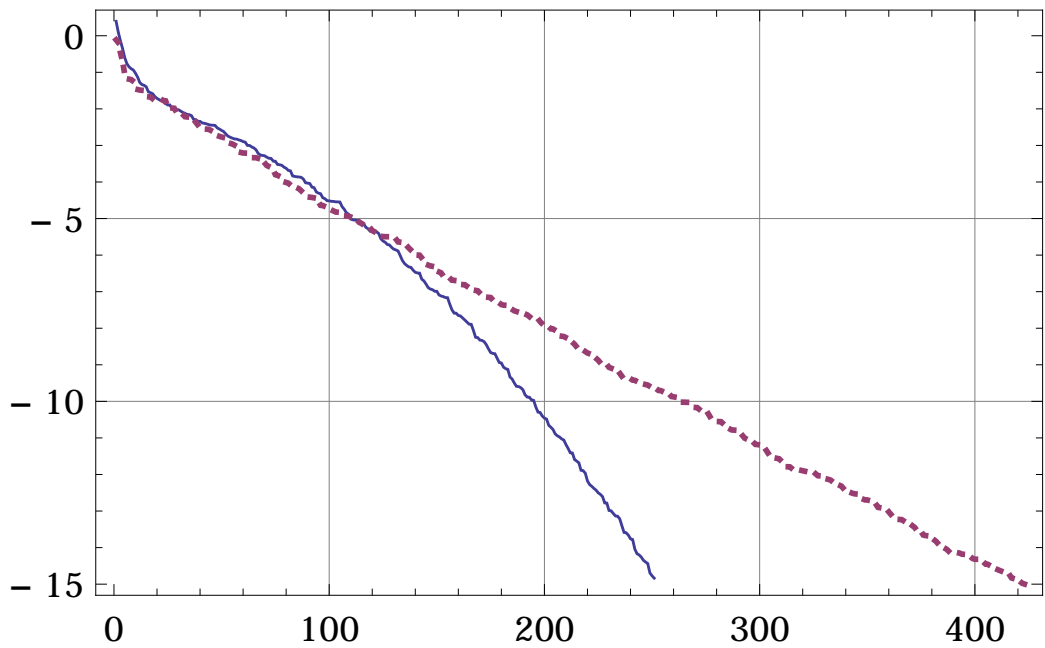

Figure 2: The base-10 logarithms of the leading Chebyshev coefficients of the function q defined by formula (245) in Section 10.1 (solid line) and of the associated nonoscillatory solution r of equation the logarithm form of Kummer's equation (53) when $\lambda = 1,000$ (dotted line).

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n	Phase function construction time	Avg. phase function evaluation time	Avg. recurrence evaluation time	Largest absolute error
10^1	1.55×10^{-1} secs	1.29×10^{-6} secs	5.82×10^{-8} secs	5.16×10^{-14}
10^2	1.76×10^{-1} secs	1.29×10^{-6} secs	9.73×10^{-7} secs	1.59×10^{-13}
10^3	1.57×10^{-1} secs	1.29×10^{-6} secs	1.03×10^{-5} secs	6.13×10^{-13}
10^4	1.55×10^{-1} secs	1.29×10^{-6} secs	1.04×10^{-4} secs	1.20×10^{-12}
10^5	1.56×10^{-1} secs	1.31×10^{-6} secs	1.04×10^{-3} secs	9.79×10^{-12}
10^6	1.58×10^{-1} secs	1.40×10^{-6} secs	9.81×10^{-3} secs	2.40×10^{-11}
10^7	1.65×10^{-1} secs	1.40×10^{-6} secs	9.69×10^{-2} secs	8.59×10^{-11}
10^8	1.87×10^{-1} secs	1.42×10^{-6} secs	9.68×10^{-1} secs	1.71×10^{-10}
10^9	2.05×10^{-1} secs	1.34×10^{-6} secs	9.68×10^{-0} secs	6.11×10^{-10}

Table 1: **The evaluation of Legendre polynomials**. A comparison of the time required to evaluate the Legendre polynomial of order n using the standard recurrence relation and the time necessary to evaluate it using a nonoscillatory phase function. The recurrence relation approach scales as $O(n)$ while the phase function approach scales as $O(1)$.
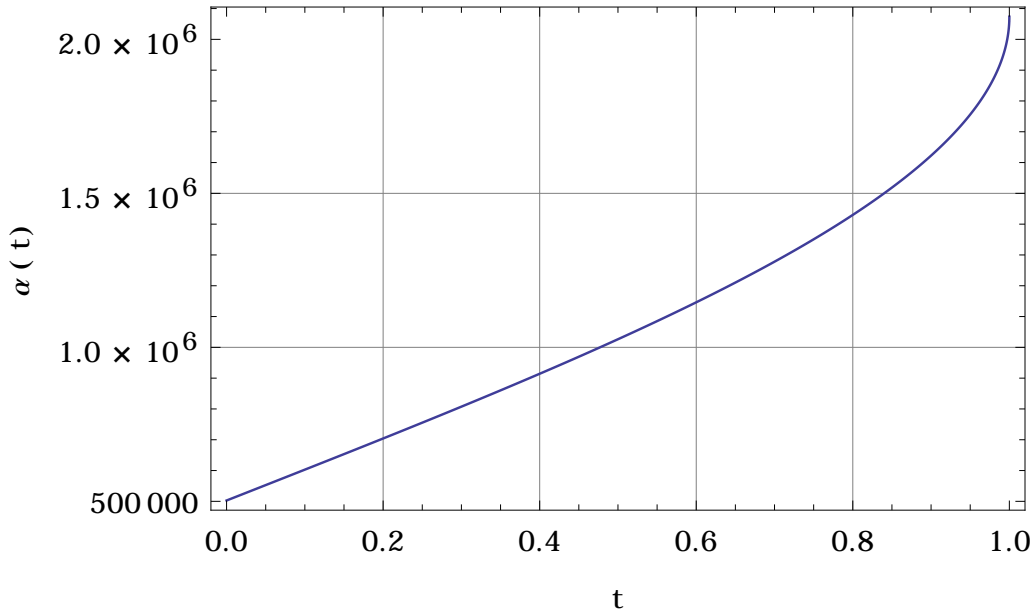

Figure 3: **A phase function for Legendre's differential equation**. A plot of the nonoscillatory phase function associated with Legendre's equation (246) at order $n = 1{,}000{,}000$. It is sufficient to construct the phase function on the interval $[0, 1)$ due to the symmetry properties of Legendre's differential equation.

n	Phase function construction time	Avg. phase function evaluation time	Avg. recurrence evaluation time	Largest absolute error
10^1	5.23×10^{-1} secs	1.30×10^{-6} secs	1.99×10^{-6} secs	2.81×10^{-14}
10^2	5.39×10^{-1} secs	1.31×10^{-6} secs	7.29×10^{-6} secs	7.85×10^{-14}
10^3	5.36×10^{-1} secs	1.37×10^{-6} secs	4.87×10^{-5} secs	2.40×10^{-13}
10^4	5.52×10^{-1} secs	1.33×10^{-6} secs	4.35×10^{-4} secs	1.01×10^{-12}
10^5	5.46×10^{-1} secs	1.49×10^{-6} secs	4.11×10^{-3} secs	3.18×10^{-12}
10^6	5.81×10^{-1} secs	1.44×10^{-6} secs	4.24×10^{-2} secs	8.57×10^{-12}
10^7	6.41×10^{-1} secs	1.45×10^{-6} secs	4.36×10^{-1} secs	5.98×10^{-11}
10^8	7.00×10^{-1} secs	1.35×10^{-6} secs	$4.39\times10^{+0}$ secs	1.14×10^{-10}
10^9	$1.26\times10^{+0}$ secs	1.41×10^{-6} secs	$4.42\times10^{+1}$ secs	2.43×10^{-10}

Table 2: **The evaluation of Bessel functions**. A comparison of the time required to evaluate the Bessel function J_n using the standard recurrence relation with that required to evaluate it using a nonoscillatory phase function. All of the points at which J_n was evaluated were in the interval $[2n, 3n]$. The recurrence relation approach scales as $O(n)$ in the order n while the time required by the phase function method is $O(1)$.