Many algorithms that compute acoustic or electromagnetic fields scattered by surfaces of revolution require fast evaluation of the azimuthal Fourier components G_m of the Green's function for the Helmholtz equation in three dimensions. In this paper we derive a recurrence relation for the functions G_m and obtain explicit formulae for their partial derivatives. These observations significantly reduce the complexity of the computation of the scattered fields generated by axisymmetric scatterers.

On the Azimuthal Fourier Components of the Green's Function for the Helmholtz Equation in Three Dimensions

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1 Introduction

Numerical evaluation of scattered fields generated by surfaces of revolution requires fast and accurate computation of individual azimuthal Fourier components G_m of the free–space Green's function for the Helmholtz equation, and their partial derivatives (see, for example, Harrington and Mautz [9], Medgyesi–Mitschang and Putnam [11], Govind, Wilton, and Glisson [7], Berton and Bills [3], Gedney and Mitra [5], Bonnemason and Stupfel [4], Grannel, Shirron, and Couchman [8]). The functions G_m are defined in (10) below.

In this paper we derive a recurrence relation for the functions G_m which (unlike three–term recursion formulae for most classical special functions) turns out to involve five terms. In addition we obtain explicit formulae expressing partial derivatives of these functions through linear combinations of G_m–s. Such properties of the sequence $\{G_m\}$ are analogous to that of classical special functions and orthogonal polynomials (see, for example, Abramovitz and Stegun [1]), and, similarly to these classical objects, provide for fast and accurate computation of an individual G_m and any of its partial derivatives.

The plan of the paper is as follows. In Section 2 we introduce basic notation and definitions. In Section 3 we obtain the generating functions for the sequence $\{G_m\}$ and an auxiliary sequence $\{Q_m\}$ (see its definition in (13) below). In Section 4 we derive the recurrence relation for the functions G_m. In Section 5 we express partial derivatives of G_m as linear combinations of these functions. In Appendix we express certain infinite series involving functions G_m in a closed form, and obtain expansions of these functions in terms of spherical Bessel functions.

2 Notation and Definitions

The free-space Green's function for the three dimensional Helmholtz equation ψ satisfies the equation

$$\triangle\psi + k^2\psi = -\frac{4\pi}{k}\delta(|\mathbf{r}-\mathbf{r}'|), \tag{1}$$

where δ is Dirac's delta function, $\mathbf{r}$ and $\mathbf{r}'$ are vectors in $\mathbb{R}^3$,

$$k \stackrel{\text{def}}{=} 2\pi/\lambda, \tag{2}$$

and λ is the wavelength.

The solution of (1) satisfying the outgoing radiation condition is given by the formula

$$\psi(k|\mathbf{r}-\mathbf{r}'|) = \frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{(k|\mathbf{r}-\mathbf{r}'|)}. \tag{3}$$

Formulae (1)–(3) can be found (in a slightly different form) in Chap. 7 of Tikhonov and Samarskii [13].

In polar coordinates

$$\mathbf{r} = (\rho, \phi, z), \quad \mathbf{r}' = (\rho', \phi', z'), \tag{4}$$

we have

$$k^2|\mathbf{r}-\mathbf{r}'|^2 = a - b\cos(\phi - \phi'), \tag{5}$$

where

$$a \stackrel{\text{def}}{=} k^2[\rho^2 + (\rho')^2 + (z-z')^2], \tag{6}$$

and

$$b \stackrel{\text{def}}{=} 2 \cdot k^2 \cdot \rho \cdot \rho'. \tag{7}$$

Note that for any real ρ, ρ', z, and z',

$$a \geq b. \tag{8}$$

Combining (3) and (5) we have

$$\psi(k|\mathbf{r}-\mathbf{r}'|) = \frac{\exp\left(i(a - b\cos(\phi-\phi'))^{1/2}\right)}{(a - b\cos(\phi-\phi'))^{1/2}}. \tag{9}$$

The functions G_m are defines as Fourier coefficients of (9) with respect to the azimuthal angle ϕ and have the form

$$G_m(a,b) \stackrel{\text{def}}{=} \int_0^{2\pi} \psi(k|\mathbf{r}-\mathbf{r}'|) \cdot \exp(im\phi)d\phi = \int_0^{2\pi} \frac{\exp\left(i(a - b\cos\phi)^{1/2}\right)}{(a-b\cos\phi)^{1/2}} \cdot \exp(im\phi)d\phi. \tag{10}$$

Obviously, for any $a > b$ and integer m,

$$G_m(a,b) = G_{-m}(a,b). \tag{11}$$

It follows from (10) that

$$\frac{\exp\left(i(a - b\cos\phi)^{1/2}\right)}{(a - b\cos\phi)^{1/2}} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} G_m(a,b)\exp(im\phi). \tag{12}$$

We will also consider the sequence of functions $\{Q_m\}$, that for any integer m and $a > b$ are defined by the formula

$$Q_m(a,b) \stackrel{\text{def}}{=} \int_0^{2\pi} \exp\left(i(a - b\cos\phi)^{1/2}\right) \cdot \exp(im\phi)d\phi, \tag{13}$$

which implies that

$$\exp\left(i(a - b\cos\phi)^{1/2}\right) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} Q_m(a,b)\exp(im\phi). \tag{14}$$

3 The Generating Functions

The main result of this section is Theorem 3.1 below where we obtain the generating function G for the sequence $\{G_m\}$ and the generating function Q for the sequence $\{Q_m\}$. It is well known that generating functions are normally used for the derivation of recurrences and certain other properties of classical special functions and orthogonal polynomials (see, for example, Chap. 7 of Lebedev [10], Chap. 11 of Seaborn [12]). The corresponding computations for the sequence $\{G_m\}$ require analysis of two generating functions G and Q, which is a departure from the standard scheme for classical special functions where usually only one generating function is involved.

We define the generating function P for a certain sequence $\{P_m\}$ in a standard way, i. e. as a function whose Laurent expansion

$$P(x,t) = \sum_{m=-\infty}^{\infty} P_m(x) \cdot t^m \tag{15}$$

converges uniformly in a certain ring of the complex t–plane.

Theorem 3.1. *Let for any $a > b$ the functions G and Q be defined by the formulae*

$$G(a,b,t) \stackrel{\text{def}}{=} 2\pi \cdot \frac{\exp\left(i(a - b(t+t^{-1})/2)^{1/2}\right)}{(a - b(t+t^{-1})/2)^{1/2}}, \tag{16}$$

and

$$Q(a,b,t) \stackrel{\text{def}}{=} 2\pi \cdot \exp\left(i(a - b(t+t^{-1})/2)^{1/2}\right). \tag{17}$$

Then

$$G(a,b,t) = \sum_{m=-\infty}^{\infty} G_m(a,b) \cdot t^m, \tag{18}$$

and

$$Q(a,b,t) = \sum_{m=-\infty}^{\infty} Q_m(a,b) \cdot t^m. \tag{19}$$

The series (18) and (19) converge uniformly in any domain lying entirely within the ring

$$t_- < |t| < t_+, \tag{20}$$

where

$$t_- \stackrel{\text{def}}{=} c^{-1} - \left(c^{-2} - 1\right)^{1/2}, \quad t_+ \stackrel{\text{def}}{=} c^{-1} + \left(c^{-2} - 1\right)^{1/2}, \tag{21}$$

and

$$c \stackrel{\text{def}}{=} \frac{b}{a}. \tag{22}$$

Proof. Since the quadratic equation

$$a - b(t + t^{-1})/2 = 0 \tag{23}$$

has the roots t_- and t_+, it is easy to see that the functions G and Q are analytic in the ring (20). By Laurent's theorem each of them has a Laurent expansion that converges uniformly in any domain that lies wholly in this ring. Now we obtain the formulae (18) and (19) by substituting

$$t = \exp(i\phi) \tag{24}$$

into (12) and (14), respectively. •

Remark 3.1. Obviously, the functions G and Q are connected via the relation

$$G(a,b,t) = \frac{Q(a,b,t)}{(a - b(t+t^{-1})/2)^{1/2}}, \tag{25}$$

which follows from (16) and (17). •

4 The Recurrence Relation for the Functions G_m

The main result of this section is the recurrence relation for the functions G_m. Namely, in Theorem 4.1 below we show that for all $a > b$ and $|m| > 1$,

$$aG_m(a,b) - \frac{b}{2}(G_{m+1}(a,b) + G_{m-1}(a,b)) - \mu_m(G_{m+1}(a,b) - G_{m-1}(a,b)) +$$

$$\mu_m^2 \left(\frac{m}{m+1} G_{m+2}(a,b) - \frac{2m^2}{m^2-1} G_m(a,b) + \frac{m}{m-1} G_{m-2}(a,b) \right) = 0, \tag{26}$$

where

$$\mu_m \stackrel{\text{def}}{=} \frac{b}{4m}. \tag{27}$$

We begin with two preliminary results summarized in Lemma 4.1 and Corollary 4.1 below.

Lemma 4.1. *For any integer m and $a > b$,*

$$Q_m(a,b) = i\mu_m(G_{m+1}(a,b) - G_{m-1}(a,b)). \tag{28}$$

Proof. The combination of (16) and (17) yields

$$\frac{\partial Q(a,b,t)}{\partial t} = -\frac{ib}{4} \cdot (1 - t^{-2}) \cdot G(a,b,t). \tag{29}$$

Substituting (18) and (19) into (29) we have

$$\sum_{m=-\infty}^{\infty} Q_m(a,b) \cdot m \cdot t^{m-1} = -\frac{ib}{4} \sum_{m=-\infty}^{\infty} (G_m(a,b) - G_{m+2}(a,b)) \cdot t^m. \tag{30}$$

Equating the coefficients of equal powers of t in (30) we immediately obtain (28). •

The following corollary is an immediate consequence of the formula (28).

Corollary 4.1. *For any integer $|m| > 1$,*

$$Q_{m+1}(a,b) - Q_{m-1}(a,b) =$$

$$i\mu_m \left(\frac{m}{m+1} G_{m+2}(a,b) - \frac{2m^2}{m^2-1} G_m(a,b) + \frac{m}{m-1} G_{m-2}(a,b) \right). \bullet \tag{31}$$

Theorem 4.1. *For any $a > b$ and $|m| > 1$ the relation (26) holds.*

Proof. We start with the identity

$$\left(a-\frac{b}{2}(t+t^{-1})\right)\cdot G(a,b,t)=\left(a-\frac{b}{2}(t+t^{-1})\right)^{1/2}\cdot Q(a,b,t), \tag{32}$$

which is an immediate consequence of (25). Differentiation of (32) with respect to t in combination with (25) yields

$$-\frac{b}{2}\cdot(1-t^{-2})\cdot G(a,b,t)+\left(a-\frac{b}{2}(t+t^{-1})\right)\cdot\frac{\partial G(a,b,t)}{\partial t}=$$
$$-\frac{b}{4}\cdot(1-t^{-2})\cdot G(a,b,t)-\frac{ib}{4}\cdot(1-t^{-2})\cdot Q(a,b,t). \tag{33}$$

Substituting (18) and (19) into (33) we have

$$-\frac{b}{4}\cdot\sum_{m=-\infty}^{\infty}(G_m(a,b)-G_{m+2}(a,b))\cdot t^m+$$
$$\sum_{m=-\infty}^{\infty}\left(a\cdot m\cdot G_m(a,b)-\frac{m\cdot b}{2}\cdot(G_{m+1}(a,b)+G_{m-1}(a,b))\right)\cdot t^{m-1}=$$
$$-\frac{ib}{4}\cdot\sum_{m=-\infty}^{\infty}(Q_m(a,b)-Q_{m+2}(a,b))\cdot t^m. \tag{34}$$

Equating the coefficients of equal powers of t in (34) and using (31) we immediately obtain (26). •

Remark 4.1. An important for applications property of any recurrence relation is its numerical stability, i. e. the sensitivity of the $(m+n)$–th term of the sequence ($m=0,\pm1,\pm2,\ldots$, $n=0,\pm1,\pm2,\ldots$), computed via n steps of recursion, to small perturbations of its m-th term (see, for example, Chap. 9 of Abramovitz and Stegun [1]). A somewhat involved analysis shows that (26) is stable for both upward and downward recurrences if

$$1<m<r_-, \tag{35}$$

is stable for downward and unstable for upward recurrences if

$$r_-<m<r_+, \tag{36}$$

and is unstable for both upward and downward recurrences if

$$m>r_+. \tag{37}$$

In (35)–(37),

$$r_- \stackrel{\text{def}}{=} \left(\frac{bt_-}{2}\right)^{1/2}, \quad r_+ \stackrel{\text{def}}{=} \left(\frac{bt_+}{2}\right)^{1/2}, \tag{38}$$

where t_- and t_+ are defined in (21). The proofs of these results will be reported elsewhere. •

Remark 4.2. Simple analysis indicates that in the region (37) the functions G_m are almost always (numerically) small. In fact, the definition (10) and Riemann's lemma yield

$$\lim_{m\to\infty} G_m(a,b) = 0. \tag{39}$$

Combining formulae (9.3.1) and (10.1.1) of [1] with (67) and (68) of Appendix it is easy to show that the decay of functions G_m for m satisfying (37) can by approximately described by the formula

$$G_m(a,b) \sim \left(\frac{r_-}{r_+}\right)^m. \tag{40}$$

Therefore the functions G_m in the region (37) are small unless $r_+ \approx r_-$, i. e. when $a \approx b$. •

5 Partial Derivatives of the Functions G_m

In this section we show that partial derivatives of functions G_m can be expressed via linear combinations of these functions.

Lemma 5.1. *For any $a > b$,*

$$\frac{\partial G_m(a,b)}{\partial \rho} = 2\cdot k^2\cdot(\rho+\rho')\cdot\frac{\partial G_m(a,b)}{\partial a} + 2\cdot k^2\cdot\rho'\cdot\frac{\partial G_m(a,b)}{\partial b}, \tag{41}$$

and

$$\frac{\partial G_m(a,b)}{\partial z} = 2\cdot k^2\cdot(z-z')\cdot\frac{\partial G_m(a,b)}{\partial a}. \tag{42}$$

Furthermore, for any integer m,

$$\frac{\partial G_m(a,b)}{\partial b} = -\frac{1}{2}\left(\frac{\partial G_{m+1}(a,b)}{\partial a} + \frac{\partial G_{m-1}(a,b)}{\partial a}\right). \tag{43}$$

Proof. The formulae immediately follow from (6) and (7). The formula (43) can be easily obtained by substituting the expansion (18) into the obvious relation

$$-\frac{1}{2}\cdot(t+t^{-1})\cdot\frac{\partial G(a,b,t)}{\partial a}=\frac{\partial G(a,b,t)}{\partial b}, \tag{44}$$

and equating the coefficients of equal powers of t. •

Throughout the proofs of Lemma 5.2 and Theorem 5.1 below we will write for brevity

$$X_m \overset{\text{def}}{=} \frac{\partial G_m(a,b)}{\partial a}. \tag{45}$$

Lemma 5.2. *For any $a>b$ and $n>0$,*

$$aX_n - bX_{n-1} = p_n, \tag{46}$$

where

$$p_n \overset{\text{def}}{=} (n-1/2)\cdot G_n(a,b)-\frac{1}{2}\cdot\mu_n\cdot(G_{n+1}(a,b)-G_{n-1}(a,b)). \tag{47}$$

Proof. Substituting the expansion (18) into the obvious relation

$$-\frac{b}{2}\cdot(1-t^{-2})\cdot\frac{\partial G(a,b,t)}{\partial a}=\frac{\partial G(a,b,t)}{\partial t}, \tag{48}$$

and equating the coefficients of equal powers of t we have

$$\frac{1}{2}G_n(a,b)=\mu_n\cdot(X_{n+1}-X_{n-1}). \tag{49}$$

Next, differentiation of (32) with respect to t yields

$$G(a,b,t)+\left(a-\frac{b}{2}(t+t^{-1})\right)\cdot\frac{\partial G(a,b,t)}{\partial t}=\frac{1}{2}G(a,b,t)+\frac{i}{2}Q(a,b,t). \tag{50}$$

Substituting the expansions (18) and (19) into (50) and equating the coefficients of equal powers of t we have

$$aX_n-\frac{b}{2}(X_{n+1}+X_{n-1})+\frac{1}{2}\cdot G_n(a,b)=\frac{i}{2}\cdot Q_n(a,b). \tag{51}$$

Combining (28) and (51) we obtain

$$-\frac{1}{2}G_n(a,b)-\frac{1}{2}\mu_n(G_{n+1}(a,b)-G_{n-1}(a,b))=aX_n-\frac{b}{2}(X_{n+1}+X_{n-1}). \tag{52}$$

Now (46) is an immediate consequence of (49) and (52). •

Theorem 5.1. *For any $a > b$ and $m \geq 2$,*

$$\frac{\partial G_{m-2}(a,b)}{\partial a} = \frac{A_m}{m(a^2-b^2)^2 + a^4 - b^4}, \tag{53}$$

where

$$\begin{aligned} A_m &\stackrel{\text{def}}{=} (2ma^2 - (m-1)b^2)\cdot(bp_{m-1} + ap_m) - \\ &\quad a^2\cdot(m-1)\cdot(bp_{m+1} + ap_{m+2}) + 4\cdot(m^2-1)\cdot a^4\cdot s_m, \end{aligned} \tag{54}$$

$$s_m \stackrel{\text{def}}{=} \frac{1}{2b}(G_{m+1}(a,b) - G_{m-1}(a,b)), \tag{55}$$

and μ_m and p_m are defined in (27) and (47), respectively.

Proof. Differentiating (26) with respect to a we obtain

$$\begin{aligned} &G_n(a,b) + aX_n - \frac{b}{2}(X_{n+1} + X_{n-1}) - \mu_n(X_{n+1} - X_{n-1}) + \\ &\mu_n^2\left(\frac{n}{n+1}X_{n+2} - \frac{2n^2}{n^2-1}X_n + \frac{n}{n-1}X_{n-2}\right) = 0. \end{aligned} \tag{56}$$

Substituting (49) and (52) into (56) we have

$$\frac{1}{n-1}X_{n-2} - \frac{2n}{n^2-1}X_n + \frac{1}{n+1}X_{n+2} = s_n, \tag{57}$$

where s_n is defined in (55).

The combination of (46) for $n = m-1, m, m+1$ and $m+2$ with (57) for $n = m$ yields the following system of linear equations:

$$\left.\begin{aligned} &aX_{m-1} - bX_{m-2} = p_{m-1}, \\ &aX_m - bX_{m-1} = p_m, \\ &aX_{m+1} - bX_m = p_{m+1}, \\ &aX_{m+2} - bX_{m+1} = p_{m+2}, \\ &\frac{1}{m-1}X_{m-2} - \frac{2m}{m^2-1}X_m + \frac{1}{m+1}X_{m+2} = s_m. \end{aligned}\right\} \tag{58}$$

Evaluating X_{m-2} from (58) and using the definition (45) we obtain (53)–(55). •

Remark 5.1. Clearly, for all integer m and $a > b$ any partial derivative $\partial^{k+l} G_m / \partial \rho^k \partial z^l$ (k, $l = 0, 1, 2...$) can be expressed as a linear combination of a finite number of functions G_m, which is a consequence of (11), (41)–(43), and (53)–(55). •

6 Conclusions

We have demonstrated that the functions G_m satisfy a recurrence relation and showed that their partial derivatives can be expressed via linear combinations of these functions themselves. Such properties of the sequence of functions $\{G_m\}$ are similar to that of all the families of classical special functions and orthogonal polynomials.

The use of the recurrence (26) significantly reduces the complexity of the numerical computation of the scattered fields generated by surfaces of revolution. As was observed by many authors such computations can be reduced to the solution of M integral equations on the generating curve of the scatterer (see, for example, [7]). Here $M \sim k$ is the number of nonvanishing (with the given accuracy) azimuthal Fourier components of the incident field computed on the surface of the scatterer, and k is defined in (2). After appropriate discretization these integral equations are reduced to a sequence of linear systems with dense matrices $Z^{(m)}$ ($m = 0, 1, ..., M$). Any element of these matrices $Z_{ij}^{(m)}$ is a linear combination of a finite number of the functions G_m and their partial derivatives, with the arguments of these functions ρ, z, ρ', and z' belonging to the mesh on the generating curve of the scatterer (see, for example, [3], [5]).

It is well known that evaluating the matrix elements $Z_{ij}^{(m)}$ consumes a major portion of the CPU time (see, for example, [5]). Indeed, simple arguments based on the Nyquist theorem show that the number of elements in $Z^{(m)}$ is at least $O(N^2)$, where N is the size of the scatterer in wavelengths. Note that $N \sim k$, which is a consequence of (2). The computation of one $Z_{ij}^{(m)}$ requires $O(N)$ operations which immediately follows from (10). Therefore the CPU time T_1 for computing all the matrix elements has the estimate

$$T_1 = O\left(M \cdot N^3\right). \tag{59}$$

Next, solving the linear system for every m by means of some iterative technique normally requires application of the system's matrix to a certain sequence of vectors (see, for example, Chap. 10 of Golub and Van Loan [6]), i. e. can be done for $O(N^2)$ operations. Thus the CPU time T_{sol} for solving the M linear systems has the estimate

$$T_{sol} = O\left(M \cdot N^2\right). \tag{60}$$

Comparing (59) and (60) we see that the complexity of computing the scattered field is dominated by the cost of generating matrices $Z^{(m)}$ $(m = 0, 1, ..., M)$ and thus has the CPU time estimate T_1.

Now we turn to estimating the CPU time for the generation of the matrix elements by means of (26); this method consists of two stages. On the first stage we must evaluate the starting values of the recursion, i. e. $O(N^2)$ elements of the four initial matrices, which requires $O(N^3)$ operations. On the second stage we compute all other matrices using the recursion (26) which requires $O(M \cdot N^2)$ operations. Therefore the CPU time T_2 of computing the matrices by means of (26) has the estimate

$$T_2 = O\left(N^3\right) + O\left(M \cdot N^2\right). \tag{61}$$

Obviously, now the CPU time estimate of the computation of the scattered field has the form (61) which immediately follows from (60) and (61).

For large–scale problems we have $N \gg 1$ and $M \gg 1$, which in combination with (59) and (61) yields

$$T_2 \ll T_1. \tag{62}$$

Currently a fast algorithm for generating matrices $Z^{(m)}$ using (26) is being implemented and the corresponding results will be reported at a later date.

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8 Appendix

In Appendix we discuss certain consequences of the formulae (10) and (12).

Theorem A.1. *For any $a > b$,*

$$\sum_{m=-\infty}^{\infty} G_m(a,b) = G_0(a,b) + 2\sum_{m=1}^{\infty} G_m(a,b) = 2\pi \frac{\exp\left(i(a-b)^{1/2}\right)}{(a-b)^{1/2}}, \tag{63}$$

$$\sum_{m=-\infty}^{\infty} (-1)^m G_m(a,b) = G_0(a,b) + 2\sum_{m=1}^{\infty} (-1)^m G_m(a,b) = 2\pi \frac{\exp\left(i(a+b)^{1/2}\right)}{(a+b)^{1/2}}, \tag{64}$$

$$\sum_{m=-\infty}^{\infty} |G_m(a,b)|^2 = |G_0(a,b)|^2 + 2\sum_{m=1}^{\infty} |G_m(a,b)|^2 = \frac{4\pi^2}{(a^2-b^2)^{1/2}}. \tag{65}$$

Proof. Formulae (63) and (64) immediately follow from (12) after substituting $\phi = 0$ and $\phi = \pi$, respectively. The formula (65) is easily obtained by applying Parseval's theorem to (12) and using the relation

$$\int_0^{2\pi} \frac{d\phi}{a - b\cos\phi} = \frac{2\pi}{(a^2-b^2)^{1/2}}. \bullet \tag{66}$$

Theorem A.2. *For any $a > b$ and integer $l \geq 0$,*

$$G_{2l}(a,b) = 2i \sum_{n=l}^{\infty} (4n+1) \cdot \Lambda(n-l) \cdot \Lambda(n+l) \cdot j_{2n}(r_-) \cdot h_{2n}^{(1)}(r_+), \tag{67}$$

$$G_{2l+1}(a,b) = 2i \sum_{n=l}^{\infty} (4n+3) \cdot \Lambda(n-l) \cdot \Lambda(n+l+1) \cdot j_{2n+1}(r_-) \cdot h_{2n+1}^{(1)}(r_+), \tag{68}$$

where and j and $h^{(1)}$ are spherical Bessel functions of the first and third kind, respectively, r_+ and r_- are given in (38), and Λ is defined by the formula

$$\Lambda(x) \stackrel{\text{def}}{=} \frac{\Gamma(x+1/2)}{\Gamma(x+1)}. \tag{69}$$

Proof. Combining formulae (10.1.1), (10.1.45), and (10.1.46) of [1] we have

$$\frac{\exp\left(i(a-b\cos\phi)^{1/2}\right)}{(a-b\cos\phi)^{1/2}} = i \sum_{n=0}^{\infty} (2n+1) \cdot j_n(r_-) \cdot h_n^{(1)}(r_+) \cdot P_n(\cos\phi), \tag{70}$$

where P_n are Legendre polynomials. Following Alpert and Rokhlin [2] we can write for any integer $p \geq 0$

$$P_{2p}(\cos\phi) = \frac{1}{\pi} \sum_{q=0}^{p} (2 - \delta_{0q}) \cdot \Lambda(p-q) \cdot \Lambda(p+q) \cdot \cos(2q\phi), \tag{71}$$

and

$$P_{2p+1}(\cos\phi) = \frac{2}{\pi}\sum_{q=0}^{p} \Lambda(p-q)\cdot\Lambda(p+q+1)\cdot\cos((2q+1)\phi), \tag{72}$$

where δ_{nm} is Kronecker's delta. Now substituting (70), (71), and (72) into (10) we immediately obtain (67) and (68). •

The formula (67) for $l = 0$ is given in [3].

Remark A.1. Obviously, the function Λ is closely related to the beta function B (see, for example, Chap. 6 of [1]). In fact, comparing the formula (6.2.2) of [1] with (69) we have

$$\Lambda(x) = \frac{B(x, 1/2)}{\Gamma(1/2)}. \tag{73}$$

Note, that the function Λ satisfies the recurrence relation

$$\Lambda(0) = \Gamma(1/2) = \pi^{1/2}, \quad \Lambda(n+1) = \Lambda(n)\cdot\frac{n+1/2}{n+1} \quad \text{for all } n = 0, 1, \ldots, \tag{74}$$

which is an immediate consequence of (69). •

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