A Renormalization Group Theory for the Mind

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Abstract: We address the barrier of infinite regress associated to the study of mind and consciousness by using set theoretic methods that include transfinite aspects. Information processing in neural networks (as modeled by McCulloch-Pitts equation dynamics, for example) is abstracted to construct various other dynamical systems on sets and classes. To these is applied a transfinite form of the Renormalization Group theory of physics. This novel form of the renormalization methodology is used to develop a framework and theory of limit points (in both the countable realm and the transfinite) of the constructed dynamical systems as well as to develop the related notions of fixed point, basin (of attraction), phase and phase diagram. Such features are set and class theoretic analogs of correspondents in physics. In previous work, we introduced axioms for what we called consciousness operators, which generalize the operator arising in the discussion of Russell's paradox. In the current work, we take steps toward the construction and classification of all consciousness operators. We reformulate the classification problem in the language of an abstract renormalization group flow. These results (whose semantic equivalents in the study of mind and consciousness were heretofore unknown) are used to inform and augment the authors' axiomatic theory of consciousness (experience and awareness) framed in the language of sets and operators.

Table of Contents

1. Introduction

Mental processing such as reasoning and perception supervene on interaction of our senses with the environment. An affirmation of this assertion can be found in Kant, 1781, "*All our knowledge begins with the senses, proceeds then to understanding and ends with reason… It is beyond a doubt that all our knowledge begins with experience*."[1]

However, the limitations of our senses bound the capability, both scope and style, of those mental processes. While we are able to perceive, model and analyze much of our environment, both physical and mental, there are aspects of nature that confound our abilities. Our mental processes support an intuitive notion of such fundamental aspects of our experience as time and space. Yet sensory limitations inhibit development of a deep understanding of them. Our perception of time and space could very well be a delusory basis upon which to frame an understanding of nature. The same limitations deprive us of meaningful understanding of mind and consciousness. The construct of the homunculus (Watson, Berry, 2003) characterizes a critical shortcoming (stemming from its self referential nature) that our senses impose on our understanding of consciousness that, in particular, takes form of an infinite regress.

We use Cantor's set theory, including its 20th century refinements (Aczel, 1988, Devlin, 1993, Hrbacek, Jech, 1999, Moschovakis, 2005 …) to penetrate the barrier of infinite regress in the study of mind and consciousness. We exhibit new features and constructs that transcend this barrier in both the countable realm and in the transfinite, the latter a highly structured universe. Motivated by the dynamics of mental activity in Miranker, Zuckerman, 2009b, we use renormalization (Goldenfeld, 1992, Miranker, 2008) to develop such transcendent features. This development exposes existence of dynamical systems in the transfinite realm. These systems support extension of the mathematical theory of consciousness developed in MZ1[2] and of the applications of that theory (that include modeling of neural network activity) elaborated in MZ2. (A prior reading of MZ1 and MZ2 is helpful but not essential in order to access the present text.) We shall show in analogy with the study of phases of matter that renormalization uncovers existence of novel set theoretic constructs (including a phase diagram for sets) in the transfinite. We develop semantic equivalents of such constructs (*that were heretofore unknown*) to advance the modeling and analysis of mind and consciousness.

Outline

Section 2. In this section of preliminaries we assemble properties of classes of sets and operators on sets that we shall require. Emphasis is placed on operators called selectors. Following that the constructs of metaclasses and metaoperators are described.

Section 3. The notion and properties of a perfect class are developed. The selector associated to a perfect class is (under an additional hypothesis) a consciousness operator. The collection of consciousness operators (a metaclass) forms a commutative semigroup. Then accretion dynamics, a Platonic[3] dynamics on sets associated to a choice of selector,

[1] See Remark 4.20 and footnote 13 for an analytic instantiation of these Kantian ideas.
[2] The citations Miranker, Zuckerman, 2009a/b will hereafter be referred to as MZ1/MZ2, resp.
[3] By Platonic is meant ideal or the virtual as opposed to physical or the palpable.

is introduced. The accretion dynamics forecast a notion of a transfinite limit point to be introduced in Sect. 4.2. Any consciousness operator is a selector (MZ1). Each step of accretion dynamics appends to (augments) the set being mapped, this being a critical feature of the development.

Section 4. We develop a pair of associated Platonic dynamical systems. These are metadynamics(a) on the metaclass called *Classes* and metadynamics(a') on the metaclass called *Selectors*. The steps of both of these dynamics are composed of accretive class operations. This feature enables introduction of a transfinite limit process that generates the constructs enumerated in the title of Sect. 4. To metadynamics(a') is associated a transfinite renormalization metaoperator $\mathscr{X}$. We study the associated limit points of $\mathscr{X}$ and their basins of attraction. A basin is a set theoretic analog of a phase of matter in a physical system. To metadynamics(a') we associate a transfinite sequence of powers of an operator, these used subsequently to generate transfinite limit points of the dynamics. This framework is a key feature for our renormalization theory. Connections between such limit points and consciousness operators are developed. We shall use the terminology, renormalization (semi-) group (RG) from physics. With this terminology, we shall mean the successive stages of a trajectory of an appropriate dynamical system, including especially its initial point and its transfinite limit point.

Section 5. We introduce a **new** dynamical system called metadynamics(b) defined on the class $\mathcal{S}$ of all sets. Its fixed points are called finitely perfect sets. With the latter is associated the limit point operator $\mathscr{L}$, which we employ to characterize the associated limit points and their basins of attraction. These considerations form a bridge between the metadynamics(a) of Sect. 4 and metadynamics(c) specified in Sect. 6. (The latter have relevance to neural networks.) We introduce $\mathscr{F}$ and $\mathscr{H}$, the classes of fixed points and their basins, generated by metadynamics(b). A decomposition of the class of sets, $\mathcal{S}$ in terms of $\mathscr{F}$ and $\mathscr{H}$ generates specification of a phase diagram for sets. Properties of these constructs are developed. Structure of the power set $\mathscr{P}(F)$ of a fixed point F is conceptualized diagrammatically and cardinal invariants of a set $F \in \mathscr{F}$ are discussed.

Section 6. We introduce the set of finitely pictured sets (denoted **FP**). *This set forms a bridge between our axiomatic developments and neural networks in the brain* (MZ1, MZ2), *the modeling of the latter being naturally framed in* **FP**. The **FP** notion motivates specialization of metadynamics(b) to a metadynamics(c) on the subsets of **FP** all of whose elements are normal. Then the notions of fixed point, basin of attraction, phase and a phase diagram associated to metadynamics(c) are introduced. In this way, aspects of the theory developed become available for application to neural networks.

Section 7. We introduce special consciousness operators and sets of those consciousness operators, and then we develop their associated properties. We show how these special consciousness operators have a fundamental correspondence to the limit points and phases of both metadynamics(a) and metadynamics(c).

Section 8. The renormalization of a selector, a result of a transfinite process, is defined. A commutative diagram illustrating this renormalization and the restriction of a consciousness operator to **FP** is given. That diagram illustrates relationships between the transfinite limits of metadynamics(a) and ordinary limits of metadynamics(c).

Section 9. A clearer picture of the constructs in this work is developed by application to the consciousness related notions of experience and awareness elaborated in MZ1 and MZ2. To do this, we introduce the construct of a quale of a set x, and we show that the collection of all qualia, $\mathcal{Q}(x)$, is itself a set. This permits definition of the qualia operator $\mathcal{Q}$. Properties of $\mathcal{Q}(x)$ and of $\mathcal{Q}$ are developed, including when restrictions to **FP** (made for neural net applicability) are imposed. Several diagrams are given to conceptualize the theory and semantics of these new constructs.

Section 10. To critique the dynamical systems constructed in this development, we assemble them into a hierarchy that includes the McCulloch-Pitts model of neural network activity (Haykin, 2009, Hertz, Krogh, Palmer, 1991). The expression of analogies between the mental (set theoretic) and the physical (material), concerning renormalization and phase diagrams, in particular, gives context to a relationship between renormalization and consciousness. A table of these analogies along with another called the Syntax-Semantic Dictionary are employed to supply semantic interpretations for the syntactic developments of our theory. A semantic discussion of infinite regress in the context of consciousness is given along with a table of supplementary semantic interpretations. Finally we comment on future work.

Appendix: We defer presentation of proofs and demonstrations, assembling them in the appendix. Prior appendix entries are a glossary, a glossary of operators and classes, a glossary of metaoperators and the list of the consciousness operator axioms of MZ1.

2. Preliminaries; operators and classes, metaoperators and metaclasses

We assemble required properties of classes of sets and operators on sets with emphasis on operators called selectors[4]. Then metaclasses and metaoperators are introduced.

2.1 Operators and classes

We begin by defining constructs of interest. (See the Glossary of Classes and Operators.)

Definition 2.1 (Order of a pair of operators, lattice): The collection of operators is partially ordered. In particular, if $\forall x$, $\mathcal{O}_1 x \subseteq \mathcal{O}_2 x$, we write $\mathcal{O}_1 \subseteq \mathcal{O}_2$. Further, with the notions $\cup$ and $\cap$, the collection becomes a lattice (Birkhoff, 1967).

Definition 2.2 (Subset of a class): If A is a set and $\mathcal{C}$ is a class, then $A \subseteq \mathcal{C}$ if and only if $\forall y \in A, y \in \mathcal{C}$.

[4] The Russell operator, denoted $\mathcal{R}$ is an example of a selector. $\mathcal{R}$ is also a consciousness operator (see Sect. 3), taking a fundamental role in the foundations of consciousness developed in MZ1.

Definition 2.3: (Selector (MZ1))**):** An operator $\mathcal{O}$ is a selector if for a fixed class $\mathcal{C}$, we have $\mathcal{O}x = \mathcal{C} \cap x$. Equivalently, $\mathcal{O}$ is a selector if $x \subseteq y \Rightarrow \mathcal{O}x = x \cap \mathcal{O}y$.

Remark 2.4: *The class $\mathcal{S}$ of all sets is referred to as the Aczel universe, which includes the class $\mathcal{S}_{nwf}$ of non-well-founded sets, in order to emphasize that we employ the anti-foundation axiom in lieu of Von Neumann's axiom of foundation* (MZ1 and Aczel, 1988).

Definition 2.5: (Class associated to a selector; selector associated to a class): If $\mathcal{O}$ is a selector, the class $\mathcal{C}(\mathcal{O})$ associated to it is defined by[5] $\mathcal{C}(\mathcal{O}) = \{y \in \mathcal{S} \mid \mathcal{O}\mathcal{B}y = \mathcal{B}y\}$. Given a class $\mathcal{C}$, the selector $\mathcal{O}(\mathcal{C})$ associated to it is defined by $\mathcal{O}(\mathcal{C})x = \mathcal{C} \cap x$.

Properties of selectors and classes are given in the next four statements. (Recall that all proofs are given in the appendix.)

Lemma 2.6: If $\mathcal{O}$ is a selector, then $\forall x$, $\mathcal{O}x = \mathcal{C}(\mathcal{O}) \cap x$. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are selectors, then $\mathcal{O}_1\mathcal{O}_2$ is a selector, and $\mathcal{O}_1\mathcal{O}_2 = \mathcal{O}_1 \cap \mathcal{O}_2 = \mathcal{O}_2\,\mathcal{O}_1$. (Compare MZ1, Prop. 3.11.)

Proposition 2.7: Under composition, the collection of selectors is a commutative semi-group, the identity operator $\mathcal{I}$ is the identity element, and the elimination operator $\mathcal{E}$ is the zero element.

Lemma 2.8: Let $\mathcal{Z}$ be a selector, then $\mathcal{O}(\mathcal{C}(\mathcal{Z})) = \mathcal{Z}$. Let $\mathcal{D}$ be an arbitrary class, then $\mathcal{C}(\mathcal{O}(\mathcal{D})) = \mathcal{D}$.

Remark 2.9: Lemma 2.8 establishes a 1-1 correspondence between classes and selectors.

2.2 Metaoperators and metaclasses

A metaclass is a collection whose elements are classes (MacLane, 1970). We shall be especially concerned with two metaclasses[6], *Classes,* the collection of classes and *Selectors,* the collection of selectors. Metaoperators, two of which ($\mathcal{O} \mapsto \mathcal{C}(\mathcal{O})$ and $\mathcal{C} \mapsto \mathcal{O}(\mathcal{C})$) introduced in Sect. 2.1, are mappings between a pair of metaclasses. (See Aczel, 1988, Chaps. 6 and 7.) Other examples will appear in the development to follow. Note that the semi-group identified in Prop. 2.6 is a metaclass.

3. Consciousness operators, perfect classes, accretion dynamics

The notion and properties of a perfect class are developed. An operator associated to a perfect class (under an additional hypothesis) is a consciousness operator. The collection of all consciousness operators forms a commutative semi-group. We introduce accretion

[5] $\mathcal{B}$ is the brace operator. Recall that a glossary of operators and classes is found in the appendix.
[6] For clarity, metaclasses are capitalized and italicized.

dynamics, a Platonic dynamics on sets associated to a choice of selector. Accretion dynamics forecast a notion of a transfinite limit point to be introduced in Sect. 4.2.

3.1 Perfect classes and consciousness operators

The following lemma concerning consciousness operators (recall that any consciousness operator[7] is a selector (MZ1)) establishes the framework for specifying a perfect class. We then define that construct as well as the one of a derived class.

Lemma 3.1: If $\mathscr{K}$ is a consciousness operator, and $A \subseteq \mathcal{C}(\mathscr{K})$, then $A \in \mathcal{C}(\mathscr{K})$.

Definition 3.2 (Perfect class): A class $\mathcal{C}$ is *perfect* if $A \subseteq \mathcal{C} \Rightarrow A \in \mathcal{C}$.

Definition 3.3 (Derived class): The derived class $\mathcal{C}$'of a class $\mathcal{C}$, is given by $\mathcal{C}' = \{y \in \mathcal{S} | \forall z \in y, z \in \mathcal{C}\}$.

Properties and examples of perfect classes are developed in the next several statements.

Lemma 3.4: a) If $A \in \mathcal{C}'$, then $A \subseteq \mathcal{C}$.
b) If $\mathcal{C}' \subseteq \mathcal{C}$, then $\mathcal{C}$ is perfect and conversely.
c) If $\mathcal{C}$ is perfect, then so is $\mathcal{C}'$.

Lemma 3.5: a) A perfect class is a proper class. b) The intersection of two perfect classes is a perfect class.

Example 3.6 (Perfect classes): (Recall that all demonstrations are in the appendix.)

1. The class $\mathcal{S}$ of all sets.
2. The class $\mathcal{N} = \{x | x \notin x\}$ of normal sets.
3. The class $\mathcal{S}_{wf}$ (specified in Def. 2.2 in MZ2.) of well-founded sets. $\mathcal{S}_{wf}$ is referred to as the Von Neumann universe.
4. The derived class $\mathcal{N}'$.

Each of these perfect classes corresponds to an operator (a selector). In particular,

$$\mathscr{I}x = \mathcal{S} \cap x, \quad \mathscr{R}x = \mathcal{N} \cap x, \quad \mathscr{W}x = \mathcal{S}_{wf} \cap x \text{ and } \mathscr{M}x \overset{def}{=} \mathcal{N}' \cap x,$$

where $\mathscr{I}$ is the identity operator, $\mathscr{R}$ is the Russell operator and $\mathscr{W}$ is the well-founded operator (see the Glossary of operators and classes). Note that $\mathscr{W} \subset \mathscr{M} \subset \mathscr{R} \subset \mathscr{I}$.

Lemma 3.7: The perfect class $\mathcal{S}_{wf}$ is a subclass of every other perfect class.

Remark 3.8: The class $\mathcal{A}$ of abnormal sets and the class of all finite sets are each proper but not perfect.

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[7] The consciousness operator axioms of MZ1 are listed in the appendix.

Next we develop properties of the metaclass of consciousness operators, including, in particular, a connection to perfect classes.

Lemma 3.9: If $\mathscr{K}$ is a consciousness operator, then $\mathscr{K} \subseteq \mathscr{R}$.

Definition 3.10 (Operator generated by a perfect class): If $\mathscr{C} \subseteq \mathscr{N}$ is a perfect class, let the operator $\mathscr{K}(\mathscr{C})$ be specified by $\mathscr{K}(\mathscr{C})x = \mathscr{C} \cap x, \forall x$. (Compare Def. 2.6.)

The following Theorem 3.11establishes a one-to-one correspondence between perfect subclasses of $\mathscr{N}$ and consciousness operators.

Theorem 3.11: Suppose $\mathscr{C} \subseteq \mathscr{N}$ is a perfect class (Def. 3.2). Then $\mathscr{K}(\mathscr{C})$ is a consciousness operator. Conversely, suppose $\mathscr{K}$ is a consciousness operator. Then the class $\mathscr{C}(\mathscr{K})$ is perfect, and $\mathscr{C}(\mathscr{K}) \subseteq \mathscr{N}$. (This theorem is illustrated in Fig. 8.1.)

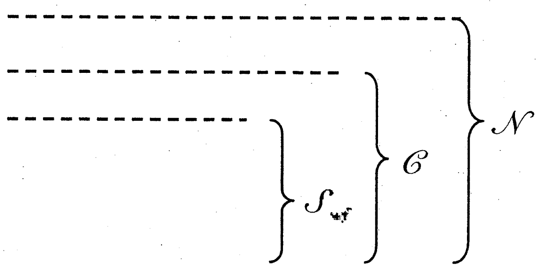

Figure 3.1: The perfect subclasses $\mathscr{S}_{wf} \subseteq \mathscr{C} \subseteq \mathscr{N}$,
illustrating Lemma 3.7 and Theorem 3.11

Theorem 3.11 leads to Cor. 3.12 and Prop. 3.13 in which striking results concerning the algebra of consciousness operators are given.

Corollary 3.12: If $\mathscr{K}_1$ and $\mathscr{K}_2$ are consciousness operators, then so is $\mathscr{K}_1\mathscr{K}_2$. Moreover $\mathscr{K}_1\mathscr{K}_2 = \mathscr{K}_1 \cap \mathscr{K}_2 = \mathscr{K}_2\,\mathscr{K}_1$.

Proposition 3.13: The metaclass of consciousness operators is a commutative semi-group with the Russell operator $\mathscr{R}$ as the identity element and with the well-founded operator $\mathscr{W}$ as the zero element. (Compare Prop. 2.7.)

Example 3.14: An example of a selector that is not a consciousness operator is $\mathscr{O}(\mathscr{D})$, where $\mathscr{D} = \mathscr{S}_{wf} \cup \{\{\varnothing,\Omega\}\}$ and Ω is the Quine atom. Moreover, $\mathscr{W} \subset \mathscr{O}(\mathscr{D}) \subset \mathscr{R}$.

A schematic illustrating the location of the metaclasses *Selectors* and *Consciousness Operators* within the theory of metaclasses is shown in Fig. 3.2.

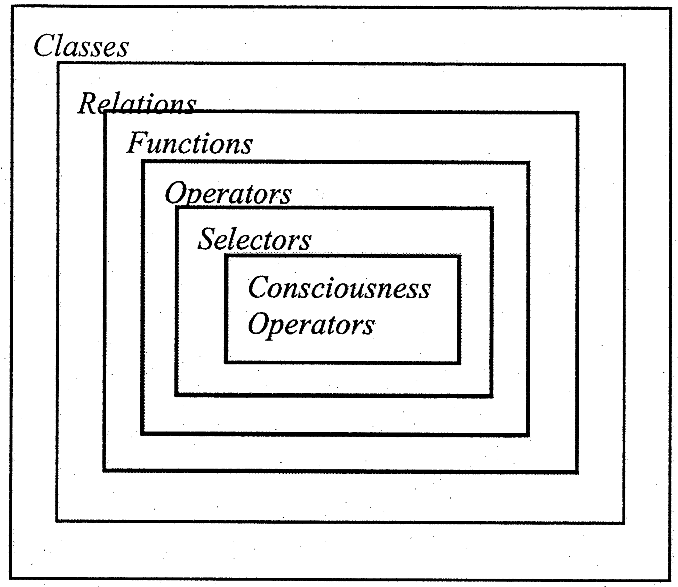

Figure 3.2: Nesting of metaclasses. See Example 3.14

3.2 Accretion dynamics

We now introduce accretion dynamics, a step of which is a generalization of Von Neumann's successor operator from the theory of ordinals. Accretion dynamics, the first of a sequence of Platonic dynamical systems defined on abstract constructs, is associated to a choice of selector, the latter possibly being a consciousness operator. Each step of accretion dynamics appends to (augments) the set being mapped. This device forecasts a notion of a transfinite limit point to be developed subsequently.

Definition 3.15 (Accretion dynamics): The dynamical system[8] $(\mathscr{S}, 1 \cup \mathscr{B}\mathscr{O})$, where[9] $\mathscr{O}$ is a selector, is called accretion dynamics. We write the associated evolution equation as

$$x(s+1) = (1 \cup \mathscr{B}\mathscr{O})x(s), s \in \mathbf{N}.$$

Since accretion dynamics flow in $\mathscr{S}$ and so do not model an apparently physical process, we regard s as a Platonic time variable that we refer to as accretion time.

Example 3.16: If $\mathscr{O}=1$, then $1 \cup \mathscr{B}\mathscr{O}$ is the Von Neumann successor operator (generator of the ordinals). The only fixed point of $1 \cup \mathscr{B}$ is the Quine atom Ω.

We conclude Sect. 3 with the following remark that specifies when fixed points of accretion dynamics are absent.

[8] Throughout a dynamical system will be denoted by an ordered pair, such as $(\mathscr{A}, \mathscr{T})$. The first member of the pair is the domain of the system and the second is the propagator.
[9] For clarity we shall hereafter use 1 in place of $\mathscr{I}$ to denote the identity operator. The relevant meaning of the symbol 1, so commonly overloaded, is clear from context.

Remark 3.17: If $\mathscr{C} \subsetneq \mathscr{N}$ is a perfect class, the accretion dynamics associated to the consciousness operator $\mathscr{K}(\mathscr{C})$, a selector, has no fixed points.

4. Metadynamics(a), metadynamics(a'), transfinite iterations, fixed point, limit point, renormalization group (RG), metaoperator, phase and basin of attraction

We develop a pair of associated Platonic dynamical systems, namely, metadynamics(a) on the metaclass called *Classes* (in Sect. 4.1) and metadynamics(a') on the metaclass called *Selectors* (in Sect. 4.2). The steps of both of these systems are composed of augmenting class operations (as with accretion dynamics). This feature enables introduction of a transfinite limit process that generates the constructs enumerated in the title of this section. Metadynamics(a') is associated with a transfinite renormalization metaoperator $\mathscr{X}$. We study the associated limit points of $\mathscr{X}$ and their basins of attraction. Such a basin is a set theoretic analog of a phase of matter in a physical system.

4.1 Metadynamics(a), transfinite limit and transfinite limit points

We specify metadynamics(a) and then introduce a transfinite sequence of powers of an operator, these used subsequently to generate transfinite limit points of the dynamics. Connections between such fixed points and consciousness operators are developed. We begin with two definitions used in the specification of metadynamics(a) that follows.

Definition 4.1 (Power class transformation): The power class transformation is given by[10] $\mathscr{P}\mathscr{A} = \mathscr{A}' = \{A \in \mathscr{S} | A \subseteq \mathscr{A}\} = \{A \in \mathscr{S} | \forall x \in A, x \in \mathscr{A}\}$. (See Def. 3.3.) Note that $\mathscr{P}$ is an isotonic metaoperator.

Definition 4.2 (Transformation $1 \cup \mathscr{P}$): The transformation $1 \cup \mathscr{P}: \mathscr{A} \mapsto (1 \cup \mathscr{P})\mathscr{A}$ is defined as follows.

$$\mathscr{A} \mapsto \mathscr{A} \cup \text{ class of subsets of } \mathscr{A}.$$

Definition 4.3 (Metadynamics(a), evolution equation, metatime): The transformation $1 \cup \mathscr{P}$ generates the dynamical system, (*Classes*, $1 \cup \mathscr{P}$), called metadynamics(a) whose associated equation of evolution is

$$\mathscr{A}(r+1) = (1 \cup \mathscr{P})\mathscr{A}(r).$$

We see that metadynamics(a) are accretive ($(1 \cup \mathscr{P})\mathscr{A} \supset \mathscr{A}$). Since these dynamics flow in *Classes*, r is a non-physical Platonic time that we refer to as metatime.

Remark 4.4: $\mathscr{C}$ is a perfect class if and only if $(1 \cup \mathscr{P})\mathscr{C} = \mathscr{C}$, that is, if and only if $\mathscr{C}$ is a fixed point of $1 \cup \mathscr{P}$.

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[10] The $\mathscr{P}$ used in Def. 4.1 is a metaoperator since it acts on a class. Context should eliminate confusion between the two uses of $\mathscr{P}$. Note that $1 \cup \mathscr{P}$ applied to a set/class delivers a set/class.

The following Theorem 4.5 shows a critical role played by the fixed points of metadynamics(a) in the theory of consciousness operators.

Theorem 4.5: A selector $\mathcal{O}$ is a consciousness operator if and only if $\mathcal{O} \subseteq \mathcal{R}$ and $(1 \cup \mathcal{P})\mathcal{C}(\mathcal{O}) = \mathcal{C}(\mathcal{O})$. (See Def. 2.5.)

We shall be concerned with the transfinite powers of operators, and so we review the following axioms of transfinite induction (for operators), wherein $\mathcal{S}_{ord}$ denotes the class of ordinals (finite and infinite) (Hrbacek, Jech,1999, Chap. 6).

1. $\forall \alpha \in \mathcal{S}_{ord}$, $\mathcal{O}^{\alpha+1} = {}^{\alpha+1} = \mathcal{O}(\mathcal{O}^{\alpha})$, equivalently $\mathcal{O}^{\alpha+1}x = \mathcal{O}(\mathcal{O}^{\alpha}x)$, $\forall x \in \mathcal{S}$.

2. If β is a limit ordinal (a limit ordinal is an ordinal that is not a successor of an ordinal), $\mathcal{O}^{\beta} = \bigcup_{\alpha<\beta} \mathcal{O}^{\alpha}$, equivalently $\mathcal{O}^{\beta}x = \bigcup_{\alpha<\beta} \mathcal{O}^{\alpha}x$, $\forall x \in \mathcal{S}$.

Using these axioms, we now develop properties of transfinite sequences of powers of operators, including, in particular, transfinite limit points (of metadynamics(a)). We also develop properties of these constructs and give examples. We begin with Theorem 4.6, concerning transfinite sequences of powers of an operator.

Theorem 4.6: Any operator $\mathcal{O}: \mathcal{S} \rightarrow \mathcal{S}$ leads to a transfinite sequence of its powers.

Next using the transfinite powers of the operator $1 \cup \mathcal{P}$, we shall show how to construct the powers of the **metaoperator** $1 \cup \mathcal{P}$.

Definition 4.7 (Transfinite sequence of metaoperators): $\forall \alpha \in \mathcal{S}_{ord}$ and $\forall$ class $\mathcal{A}$, let

$$\Phi_{\alpha} \mathcal{A} \overset{def}{=} \bigcup_{w \in \mathcal{P}\mathcal{A}} (1 \cup \mathcal{P})^{\alpha} w$$

(See Footnote 10.)

Lemma 4.8: $\forall \alpha \in \mathcal{S}_{ord}$, then

a) Φ_{α} is an isotonic metaoperator,

b) if $x \in \mathcal{S}$, $\Phi_{\alpha}x = (1 \cup \mathcal{P})^{\alpha} x$,

c) if β is a limit ordinal, $\Phi_{\beta} = \bigcup_{\alpha<\beta} \Phi_{\alpha}$,

d) $(1 \cup \mathcal{P})\Phi_{\alpha} = \Phi_{\alpha+1}$.

Definition 4.9 (Transfinite powers of the metaoperator $1 \cup \mathcal{P}$): $\forall \alpha \in \mathcal{S}_{ord}$, let

$$(1 \cup \mathcal{P})^{\alpha} \overset{def}{=} \Phi_{\alpha}$$

We shall call $\{(1\cup\mathscr{P})^{\alpha} | \alpha \in \mathcal{S}_{ord}\}$ the sequence of transfinite powers of the metaoperator $1\cup\mathscr{P}$.

Definition 4.10 (Transfinite limit of iterations of $1\cup\mathscr{P}$, *notation): If $\mathcal{A}$ is a class, let $\mathcal{A}^* = \bigcup_{\alpha\in\mathcal{S}_{ord}} (1\cup\mathscr{P})^{\alpha}\mathcal{A}$ denote the transfinite limit point of the iteration of $1\cup\mathscr{P}$ with initial point $\mathcal{A}$. For any operator $\mathcal{O}$, we define $\mathcal{C}^*(\mathcal{O}) = (\mathcal{C}(\mathcal{O}))^*$.

Proposition 4.11: $\mathcal{A}^*$ is a class.

Next we give several results, showing that a transfinite limit point has a fixed point property, and then that there is a critical connection of a transfinite limit point to a consciousness operator.

Lemma 4.12: Let $\mathcal{C}$ and $\mathcal{D}$ be classes. Then $\mathcal{C} \subseteq \mathcal{D} \Rightarrow \mathcal{C}^* \subseteq \mathcal{D}^*$. In particular, $\varnothing^* \subseteq \mathcal{C}^*$ for any class $\mathcal{C}$. Moreover, $\mathcal{C} \subseteq \mathcal{C}^*$.

Example 4.13: a) $\varnothing^* = \mathcal{S}_{wf}$. b) $\mathcal{S}^*_{ord} = \mathcal{S}_{wf}$. Recall that $\mathcal{S}_{wf}$ is a proper class.

Theorem 4.14: Corresponding to any class $\mathcal{A}$, the class $\mathcal{A}^*$ is a fixed point of $1\cup\mathscr{P}$. That is, a limit point is a fixed point. Conversely, a fixed point of $1\cup\mathscr{P}$ is a limit point thereof. Moreover, $\mathcal{A}$ is perfect if and only if $\mathcal{A}^* = \mathcal{A}$ (Remark 4.4).

Corollary 4.15: A selector $\mathcal{O}$ is a consciousness operator if and only if $\mathcal{O} \subseteq \mathcal{R}$ and $\mathcal{C}(\mathcal{O})$ is a limit point of metadynamics(a).

Corollary 4.16: If $\mathcal{O}$ is a selector, $\mathcal{K}(\mathcal{C}^*(\mathcal{O}) \cap \mathcal{N})$ (Def 3.10) is a consciousness operator. The metaoperator $\mathcal{O} \mapsto \mathcal{K}(\mathcal{C}^*(\mathcal{O}) \cap \mathcal{N})$ is a **projection** of *Selectors* onto *Consciousness operators.* (See Def. 8.1.)

Now we specialize the notion of transfinite limit to sets, and along with that we introduce the notions of the phase of a set and then of the transfinite trajectory of a set.

Definition 4.17 (Limit point of a set, phase of a set): The limit point of a set x is the perfect (and proper; see Lemma 3.5) class x^* (Def. 4.10). We shall call x^* the phase of x relative to metadynamics(a). For $C \in \mathcal{S}$, the metaclass associated to C is called the *Phase*(C), where $Phase(C) = \{x^* | x \in C\}$.

Definition 4.18 (Transfinite trajectory of a set): Let $x_{\alpha} = (1\cup\mathscr{P})^{\alpha} x$ where $x \in \mathcal{S}$ and $\alpha \in \mathcal{S}_{ord}$. The transfinite trajectory of metadynamics(a) that emanates from x is the transfinite sequence of sets $\{x_{\alpha} | \alpha \in \mathcal{S}_{oed}\}$.

Aspects of metadynamics(a) are conceptualized in Fig. 4.1.

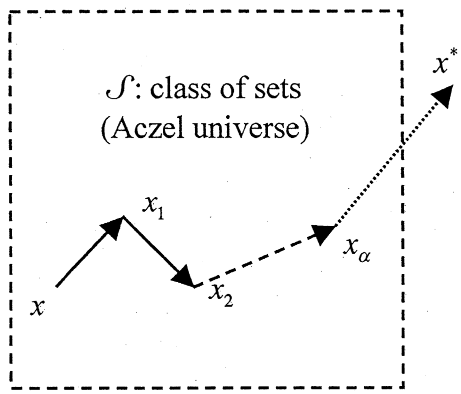

Figure 4.1: Illustration of aspects of metadynamics(a). The limit point x^* resulting from a set x is a class lying outside of the class $\mathcal{S}$. We refer to the class x^* as the *illusion*[11] generated from the set x (experience) by metadynamics(a).

As a reminder of the semantic goals of our study, we introduce a particular consciousness operator arising from experience.

Definition 4.19 (Particular consciousness operator): $\mathcal{K}\langle y\rangle = \mathcal{O}(y^* \cap \mathcal{N})$ is a consciousness operator depending on a set y (i.e., arising from an experience y (Def 9.1)).

Remark 4.20: $\mathcal{K}\langle y\rangle x = y^* \cap \mathcal{N} \cap x$. Note that "$y \mapsto \mathcal{K}\langle y\rangle$" is a metaoperator. Note that $(x,y) \mapsto \mathcal{K}\langle y\rangle x$ is an operator , say $\mathcal{M}$, from ordered pairs of sets to sets[12].

Now we define the notion of a basin of attraction, and along with that, a representation of *Classes* as a union of basins.

Definition 4.21 (Basin of attraction of a fixed point class): The basin of attraction of a fixed point class $\mathcal{G}$, namely *Basin*($\mathcal{G}$) of metadynamics(a) is a metaclass, i.e., the collection of all classes $\mathcal{C}$ such that $\mathcal{C}^* = \mathcal{G}$.

Corollary 4.22: *Classes* = $\bigcup_{\mathit{fixed\ points}}$ Basin of attraction of the fixed point. Moreover, the union is disjoint.

Next we specify the basin of attraction operator, $\mathcal{V}$, corresponding to metadynmics(a)).

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[11] The syntax and semantics of the notion of illusion will be elaborated in forthcoming work.
[12] An interpretation of the action of $\mathcal{M}$ may be derived from Kant, 1781 (see Sect. 1). In particular, that experience, namely (x,y) leads to knowledge, namely $\mathcal{K}\langle y\rangle x$.

Definition 4.23 (Basin of attraction metaoperator $\mathcal{V}$ for metadynmics(a)): The basin of attraction metaoperator $\mathcal{V}$ (constructed from metadynamics(a)) that takes *Classes* to *Classes* is given by

$$\mathcal{V}(\mathcal{C})=\{z\in\mathcal{S}\,|\,z^*=\mathcal{C}\}.$$

$\mathcal{V}(\mathcal{C})$ representing all sets attracted to $z^*=\mathcal{C}$ by metadynamics(a).

Question 4.24: We leave open the question: When is $\mathcal{V}(\mathcal{C})$ nonempty? Note that a necessary condition for $\mathcal{V}(\mathcal{C})$ to be nonempty is that $\mathcal{C}$ be a fixed point of $\mathbf{1}\cup\mathcal{P}$.

We now develop properties of $\mathcal{V}$ that connect basins, phases and equivalence classes of limit points.

Lemma 4.25: Let $W\in\mathcal{S}$. Then $\mathcal{V}(W^*)\subseteq W^*$ and $\mathcal{V}(W^*)$ is a proper class.

Definition 4.26 (The equivalence relation $\equiv_*$): $z_1\equiv_* z_2 \Leftrightarrow z_1^*=z_2^*$. That is, z_1 and z_2 belong to the same phase. (See Def. 4.1.)

Remark 4.27: 1. If $W_1^*\neq W_2^*$, then $\mathcal{V}(W_1^*)\cap\mathcal{V}(W_2^*)=\varnothing$. 2. $\mathcal{V}(\varnothing^*)=\varnothing^*$ 3. If $\varnothing^*\subset W^*$, then $\mathcal{V}(W^*)\subset W^*$.

Proposition 4.28: $\mathcal{V}(W^*)=Basin(W^*)\cap\mathcal{S}$ is the $\equiv_*$ equivalence class of W in $\mathcal{S}$.

4.2 Metadynamics(a'), RG metaoperator $\mathcal{X}$, basin, phase, phase diagram

Metadynamics(a') arises from metadynamics(a) by re-expressing the dynamics on *Classes* as a dynamics on *Selectors*[13]. We shall use the terminology renormalization (semi-)group (RG) from physics, meaning the steps of a trajectory of a dynamical system, including its initial and transfinite limit points. We introduce metadynamics(a') denoted by (*Selectors*, $\mathcal{X}$). The metaoperator $\mathcal{X}$ yields a characterization of consciousness operators and the notion of the phase of a selector. (In the terminology of the RG theory of physics, the term phase denotes the phase of physical matter.) Physical (semantic) interpretations of this section are assembled in Table 10.2.

Definition 4.29 (Metadynamics(a'), RG metaoperator $\mathcal{X}$): Referring to Remark 2.8, we see that corresponding to metadynamics(a) on *Classes,* there is a metadynamics(a') on *Selectors* denoted by (*Selectors*, $\mathcal{X}$), where $\mathcal{X}$ denotes the RG metaoperator (Def. 2.4.)

$$\mathcal{X}(\mathcal{Z})=\mathcal{O}((1\cup\mathcal{P})\mathcal{C}(\mathcal{Z})).$$

The associated evolution equation (described by the commutative diagram in Fig. 4.2) is

$$\mathcal{O}(r+1)=\mathcal{X}(\mathcal{O}(r)).$$

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[13] The developments of Sect. 4.2, which shall be discussed in greater detail in future work, are not referred to again until the Sect. 10 on semantics.

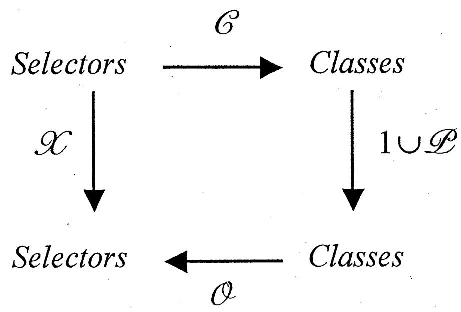

Figure 4.2: A commutative diagram illustrating metadynamics(a')

Lemma 4.30: Let $\mathcal{O}$ be a selector. Then a) $\mathcal{O} \subseteq \mathcal{X}(\mathcal{O})$ and b) $\mathcal{X}(\mathcal{O}) = \mathcal{O} \Leftrightarrow (1 \cup \mathcal{P})\, \mathcal{C}(\mathcal{O}) = \mathcal{C}(\mathcal{O})$.

Example 4.31: From Lemma 4.28 and referring to Example 3.6, we see that the distinct selectors $\mathcal{W}, \mathcal{M}, \mathcal{R}$ and $\mathcal{S}$ are fixed points of $\mathcal{X}$.

Employing $\mathcal{R}$ and $\mathcal{X}$, Cor. 4.32 establishes a key characterization of a consciousness operator, a first step toward the construction and classification of such operators.

Corollary 4.32: $\mathcal{K}$ is a consciousness operator if and only if a) $\mathcal{K}$ is a selector, b) $\mathcal{K} \subseteq \mathcal{R}$ and c) $\mathcal{X}(\mathcal{K}) = \mathcal{K}$.

We now introduce the constructs of a transfinite trajectory of a selector (relevant to metadynamics(a')) and the transfinite limit of a trajectory. Properties of such limits are developed and examples given. We begin with the following definition.

Definition 4.33 (Transfinite powers of $\mathcal{X}$): Let $\mathcal{X}$ be a selector. Then

$$\mathcal{X}^{\alpha}(\mathcal{Z}) \overset{def}{=} \mathcal{O}\left((1 \cup \mathcal{P})^{\alpha}\, \mathcal{C}(\mathcal{Z})\right).$$

Definition 4.34 (Transfinite trajectory, transfinite limit): Given an initial selector $\mathcal{O}$, the transfinite trajectory of $\mathcal{O}$ is the transfinite sequence $\{\mathcal{X}^{\alpha}(\mathcal{O}) | \alpha \in \mathcal{S}_{ord}\}$. We call the selector $\mathcal{O}^{**} = \bigcup_{\alpha \in \mathcal{S}_{ord}} \mathcal{X}^{\alpha}(\mathcal{O}) \overset{def}{=} \lim_{\alpha \in \mathcal{S}_{ord}} \mathcal{X}^{\alpha}(\mathcal{O})$ the transfinite limit of this trajectory.

Lemma 4.35: Let $\mathcal{O}$ and $\mathcal{Z}$ be selectors. If $\mathcal{O} \subseteq \mathcal{Z}$, then $\mathcal{O}^{**} \subseteq \mathcal{Z}^{**}$ and $\mathcal{O}^{**} = \mathcal{O}(\mathcal{C}^{*}(\mathcal{O}))$.

Example 4.36: If the selector $\mathcal{O} \subseteq \mathcal{W}$, then $\mathcal{O}^{**} = \mathcal{W}$. (See Example 3.6.)

The following definition of phase and basin of attraction of a selector are motivated by RG physics.

Definition 4.37 (Phase and basin of attraction of a selector): Let $\mathcal{O}$ be a selector. The phase of $\mathcal{O}$ is the limit selector $\mathcal{O}^{**}$. If $\mathcal{X}(\mathcal{O}) = \mathcal{O}$, then the basin of attraction of $\mathcal{O}$ is the metaclass of those selectors $\mathcal{Z}$ such that $\mathcal{Z}^{**} = \mathcal{O}$.

Properties of basins and limit points, including especially their connection to consciousness operators are now developed.

Proposition 4.38: Let $\mathcal{Z}$ be a selector. Then a) $\mathcal{X}(\mathcal{Z}^{**}) = \mathcal{Z}^{**}$, b) if $\mathcal{X}(\mathcal{Z}) = \mathcal{Z}$, then $\mathcal{Z}^{**} = \mathcal{Z}$, and c) if $\beta \in \mathcal{S}_{ord}$, $(\mathcal{X}^{\beta}(\mathcal{Z}))^{**} = \mathcal{Z}^{**}$ and the entire transfinite trajectory of $\mathcal{Z}$ lies in the basin of the limit point $\mathcal{Z}^{**}$.

Corollary 4.39: The transfinite trajectory of a selector does not pass through every operator in the metaclass *Selectors.*

We connect limits points of metadynamics(a') to consciousness operators by reformulating Cor. 4.16 as Cor. 4.40, which delivers a second step towards the transfinite construction of consciousness operators.

Corollary 4.40: If $\mathcal{O}$ is a selector, then $(\mathcal{O} \cap \mathcal{R})^{**}$ is a consciousness operator.

A decomposition (representation) of the metaclass *Selectors* given in Prop. 4.41 leads to the specification of a phase diagram.

Proposition 4.41: The metaclass, $Selectors = \bigcup_{\mathcal{O}:\, \mathcal{X}(\mathcal{O}) = \mathcal{O}} \text{Basin of attraction of } \mathcal{O}$.

Moreover the union is disjoint.

Definition 4.42 (Phase diagram): The decomposition displayed in Prop. 4.41 is called the phase diagram of Metadynamics(a').

5. Finitely perfect sets, metadynamics(b), trajectories, limit points, fixed points, phase diagrams

We introduce a **new** dynamical system called metadynamics(b) defined on the class $\mathcal{S}$ of all sets. Its fixed points, called finitely perfect sets, are located in the countable realm. Properties of the constructs associated to metadynamics(a) are developed. These include limit points, basins, phase and a phase diagram. Recall that a basin is a set theoretic analog of a phase of matter in a physical system. These considerations form a mathematical bridge between the metadynamics(a) of Sect. 4 and metadynamics(c) (the latter supplying a link to neural nets) to be specified in Sect. 6.

5.1. Metadynamics(b)

We begin with the introduction of finitely perfect sets and metadynamics(b), an augmenting dynamics on such sets. Then we study the associated limit points and their basins of attraction. Metadyanamics(a) differs from metadynamics(b), since the former generates transfinite limit points while the latter generates ordinary limit points (what we shall call limit points in the countable realm). A notion of a phase diagram for sets emerges from these considerations.

Definition 5.1 (Set of finite subsets): For any set $A \in \mathcal{S}$, $\mathscr{P}_f A$ will denote the set of finite subsets of A. Note that $\mathscr{P}_f$ is isotone.

Definition 5.2 (Finitely perfect set): Set D is finitely perfect if $\mathscr{P}_f D \subseteq D$. Equivalently if $\forall n \in \mathbf{N}$ and for every mapping $g : n \mapsto D$, $\{g(m|m<n)\} \in D$. (See Remark 5.8.)

Remark 5.3: a) No finite set can be finitely perfect. Hence a finitely perfect set is infinite. b) Moreover the operator $\mathscr{P}_f$ is one to one.

Definition 5.4 (Metadynamics(b) and trajectories): The pair $(\mathcal{S}, 1 \cup \mathscr{P}_f)$ specifies metadynamics(b). Given an initial point $x \in \mathcal{S}$, the trajectory of x is the *ordered* set $\{(1 \cup \mathscr{P}_f)^n x | n \in \mathbf{N}\}$. Taking r as metatime, the associated equation of evolution is

$$A(r+1) = (1 \cup \mathscr{P}_f) A(r).$$

(Compare Def. 4.3.) We note that metadynamics(b) are accretive.

Next we introduce the notion of a limit point of metdynamics(b) and an associated limit point operator $\mathscr{L}$. Properties of these constructs are developed and examples given.

Definition 5.5 (Limit point of metadynamics(b), limit point operator $\mathscr{L}$, ~ notation): Let A be a set. The set $\tilde{A}$ denotes a limit point of metadynamics(b): $(\mathcal{S}, 1 \cup \mathscr{P}_f)$. $\tilde{A}$ is given by $\tilde{A} = \bigcup_{n \in \mathbf{N}} (1 \cup \mathscr{P}_f)^n A$, equivalently, $\tilde{A} = \sup_{n \to \infty} (1 \cup \mathscr{P}_f)^n A = (1 \cup \mathscr{P}_f)^\omega A$. $\tilde{A}$ is a set. The limit point operator $\mathscr{L}$ is specified by $\mathscr{L}(A) = \tilde{A}$.

Remark 5.6: $A \supseteq B \Rightarrow \tilde{A} \supseteq \tilde{B}$, from which it follows that $\tilde{A} \supseteq \tilde{\varnothing}$, $\forall A \in \mathcal{S}$. Note that $\tilde{\varnothing} = \mathbf{H}$, where $\mathbf{H}$ is the set of hereditarily finite sets (Hrbacek, Jech, 1999, Chap.6). Moreover, $A \subseteq \tilde{A}$.

Lemma 5.7: $\mathscr{P}_f \tilde{A} \subseteq \tilde{A}$. That is, any finite subset of $\tilde{A}$ is an element of $\tilde{A}$. (Compare Def. 5.2.) Thus $\tilde{A}$ is a finitely perfect set.

Remark 5.8: $\tilde{\varnothing} = \mathbf{H} \subset \mathcal{S}_{wf}$ is an example of a finitely perfect set.

We now show that fixed points of metadynamics(b) are its limit points.

Definition 5.9 (Fixed point of $1\cup\mathscr{P}_f$): We say that $D \in \mathcal{S}$ is a fixed point of $1\cup\mathscr{P}_f$ if $(1\cup\mathscr{P}_f)D = D$.

Lemma 5.10: If $(1\cup\mathscr{P}_f)A = A$, then $\tilde{A} = A$.

Example 5.11: Examples of fixed points of $1\cup\mathscr{P}_f$ are a) **H**, b) **FP** (see Def. 6.1) and c) **FP** $\cap\,\mathcal{N}$.

Now we develop a collection of properties of trajectories of metadynamics(b). These include the interrelationship of various associated constructs, such as finitely perfect sets, fixed points, phase of a set, basins of attraction and the basin operator $\mathscr{J}$ (Def 5.16).

Proposition 5.12: a) The set $\tilde{A}$ is finitely perfect for any set (any initial condition) $A \in S$. b) Moreover $\tilde{A}$ is a fixed point of $1\cup\mathscr{P}_f$. c) Conversely, any fixed point of $1\cup\mathscr{P}_f$ is finitely perfect. d) Moreover, for any set *A*, $\tilde{\tilde{A}} = \tilde{A}$.

Interpretation 5.13: For any initial condition $A \in \mathcal{S}$, a fixed point $\tilde{A}$ of the dynamical system $(\mathcal{S}, 1\cup\mathscr{P}_f)$ is reached in a countable (possibly finite) number of steps.

Definition 5.14 (Basin of attraction of a fixed point; phase of a set): Let the set *F* be a fixed point of $1\cup\mathscr{P}_f$. The basin of attraction of *F* is the class $\{A|\tilde{A} = F\}$; if *B* is a set, $\tilde{B}$ is called the phase of *B* relative to metadynamics(b).

Proposition 5.15: Let the set *F* be a fixed point of $1\cup\mathscr{P}_f$. Then the class of sets $\{A|\tilde{A} = F\}$ is also a set. Moreover, there exists a set *B* such that $\tilde{B} = F$, but $B \neq F$. That is, the basin of *F* contains a set that is not a fixed point.

Definition 5.16 (Basin of attraction of a set and associated basin operator $\mathscr{J}$): For any set *Y*, let $\mathscr{J}(Y)$, the basin of attraction of *Y*, be the set given by $\mathscr{J}(Y) = \{A|\tilde{A} = Y\}$. (Contrast this with $\mathscr{V}$ in Def. 4.23.)

Remark 5.17: a) $\mathscr{J}(Y) \neq \varnothing \Leftrightarrow Y$ is a fixed point of $1\cup\mathscr{P}$. Moreover, b) $\mathscr{J}(Y) \subseteq \mathscr{P}Y$, and c) if *F* is a fixed point of $1\cup\mathscr{P}$, then $\mathscr{J}(F)$ contains a set *A* that is not a fixed point.

Summary 5.18 (Properties of metadynamics(b)):

1. Every initial point (set) has a unique limit point (set) that is reached in a countable (possibly finite) number of steps. (Def 5.5)
2. No fixed point is universally repelling. (Prop. 5.15)
3. Each fixed point is a limit point. (Lemma 5.10)
4. Every limit point is a fixed point. (Lemma 5.10)
5. No trajectory passes through every point in $\mathcal{S}$.
6. There is no initial point such that its trajectory fills the basin of attraction of the limit point of that trajectory. (The proof of this property will be presented in later work.)

5.2. Classes of fixed points, classes of basins, phase diagram

We introduce $\mathscr{F}$ and $\mathscr{H}$, the classes of fixed points and of their basins, respectively, generated by metadynamics(b). A decomposition of $\mathscr{S}$ in terms of $\mathscr{F}$ and $\mathscr{H}$ generates specification of a phase diagram for sets. Properties of these constructs are developed.

Definition 5.19 (Class of fixed points): Let $\mathscr{F} = \{F | (1 \cup \mathscr{P}_f) F = F\}$ denote the class of fixed points of $1 \cup \mathscr{P}_f$, that is, of metadynamics(b).

Definition 5.20 (Class of basins): Let $\mathscr{H} = \{ \mathscr{J} F | F \in \mathscr{F}\}$ denote the class of basins of fixed points $F \in \mathscr{F}$.

Regarding $\mathscr{J}$ as an operator with domain $\mathscr{F}$ and range $\mathscr{H}$ motivates Prop. 5.21.

Proposition 5.21: $\mathscr{J}$ is an invertible operator. Moreover $\mathscr{J}^{-1} = \mathscr{U}\mathscr{O}(\mathscr{F})$, where $\mathscr{U}$ is the monadic set union operator.

Corollary 5.22: $\mathscr{F} = \{\tilde{A} | A \in \mathscr{S}\}$.

The decompositions described in the following proposition are key for specifying the notion of a phase diagram for metadynamics(b). (Compare Prop. 4.41.)

Proposition 5.23: $\mathscr{S} = \bigcup_{F \in \mathscr{F}} \mathscr{J}(F)$. Moreover this union is disjoint. Equivalently, $\mathscr{S} = \bigcup_{B \in \mathscr{H}} B = \mathscr{U}\mathscr{H}$ (Def. 5.20), where $\mathscr{U}$ is the monadic union metaoperator.

Definition 5.24 (Phase diagram): The decomposition displayed in Prop. 5.23 is called the phase diagram of metadynamics(b). (Compare Def. 4.42.)

Corollary 5.25: The following three classes are identical and proper.

1. $\mathscr{F}$
2. $\{F | F$ is finitely perfect $\}$
3. $\{\tilde{A} | A \in \mathscr{S}\}$

5.3 Properties and relationships

We develop results that demonstrate properties and relationships among the constructs $\mathscr{F}, \mathscr{L}, \mathscr{J}, \mathscr{B}, \mathscr{S}, \mathscr{P}_f$, ~ and **H**. The structure of $\mathscr{P}(F)$ for a typical fixed point $F \in \mathscr{F}$ is conceptualized diagrammatically and cardinal invariants of $F \in \mathscr{F}$ are specified. We begin with the following two propositions. The first assembles properties of $\mathscr{J}$ and $\mathscr{L}$. The second gives an optimization property of a limit point of metadynamics(b).

Proposition 5.26:

i) $\mathscr{L} = \sup_{n\in\mathbf{N}} (1 \cup \mathscr{P}_f)^n$.

ii) $(1 \cup \mathscr{P}_f)\mathscr{L} = \mathscr{L}$.

iii) $\mathscr{L}(1 \cup \mathscr{P}_f) = \mathscr{L}$.

iv) $A \subseteq B \Rightarrow \mathscr{L}A \subseteq \mathscr{L}B$.

v) $1 \subset \mathscr{L}$ and $\mathscr{L}^2 = \mathscr{L}$.

vi) $\mathscr{J}(Y) = \{A | \mathscr{L}(A) = Y\}$.

vii) $\mathscr{L}(A) \in \mathscr{J}(\mathscr{L}(A))$.

viii) $\mathscr{B}\mathscr{L} \subseteq \mathscr{J}\mathscr{L}$.

ix) If $B \in \mathscr{J}(F)$, then $\mathscr{L}(B) = F$.

Proposition 5.27: For any set A, $\tilde{A} = \bigcap_{\{F \in \mathscr{F},\ A \subseteq F\}} F$. (Restating: $\tilde{A}$ is the least fixed point G such that $A \subseteq G$).

Next we describe the cardinal invariants of $F \in \mathscr{F}$.

Proposition 5.28: The following is a list of six cardinal invariants of $F \in \mathscr{F}$ along with associated properties. (Recall that $\mathscr{P}_f(F) \subseteq F$.)

a) $|\mathscr{P}(F)| = 2^{|F|}$,

b) $|\mathscr{P}_f(F)| \leq |F|$

c) $|F - \mathscr{P}_f(F)| \leq |F|$,

d) $|\mathscr{J}(F)| \leq 2^{|F|}$,

e) $|\mathscr{P}(F) - \mathscr{J}(F)| \leq 2^{|F|}$,

f) $|\mathscr{F} \cap \mathscr{P}(F)| \leq |\mathscr{P}(F)|$.

Fig. 5.1 shows the structure of a fixed point F of metadynamics(b), wherein a triangle whose vertex is labeled F is a representation of $\mathscr{P}(F)$.

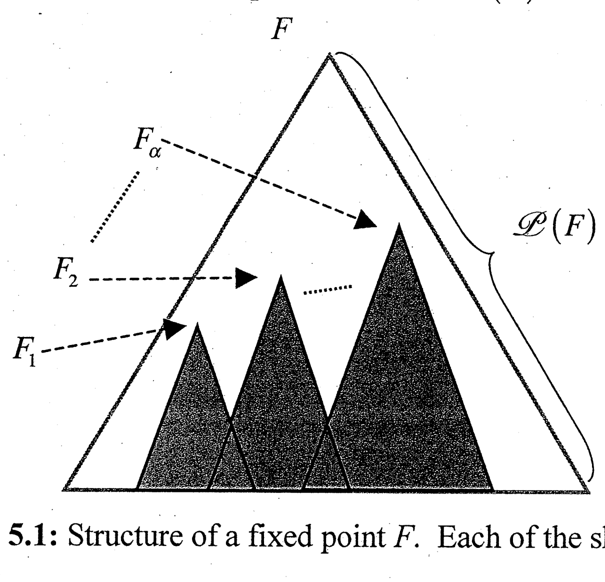

Figure 5.1: Structure of a fixed point F. Each of the shaded triangles is the power set of the associated fixed point $F_\alpha \in \mathscr{P}F - \{F\}$, $\alpha \in \mathscr{S}_{ord}$

We introduce a structure operator $\mathcal{Y}$ associated to metadynamics(b), using it with the basin operator $\mathcal{J}$ (Def. 5.16) to develop the structure of fixed points of those dynamics.

Definition 5.29 (Structure operator $\mathcal{Y}$ associated to metadynamics(b)): Let $\mathcal{Y}: \mathcal{S} \to \mathcal{P}\mathcal{F}$ be specified by $\mathcal{Y}A = \{F \in \mathcal{F} | F \subset A\}$. ($\mathcal{Y}A$ is a set by the axiom of comprehension.)

Remark 5.30: a) $\mathcal{Y}A \subset \mathcal{P}A$ and b) $\mathcal{Y} = \mathcal{O}(\mathcal{F})$ ($\mathcal{P} - \mathcal{B}$).

Lemma 5.31: If $F \in \mathcal{F}$, then $\bigcup_{\{G \in \mathcal{Y} F\}} \mathcal{J}(G)$ is a set.

Theorem 5.32: Let $F \in \mathcal{F}$. Then

a) $\mathcal{J}(F) = \mathcal{P}F - \bigcup_{\{G \in \mathcal{Y}F\}} \mathcal{J}(G)$. (This is a recursion relation[14] for $\mathcal{J}$.)

b) $\mathcal{J}(F) = \mathcal{P}F - \bigcup_{\{G \in \mathcal{Y}F\}} \mathcal{P}(G)$.

Remark 5.33: Theorem 5.32,b shows that $\mathcal{J}(\boldsymbol{F})$ is the unshaded portion of Fig. 5.1.

Remark 5.34: Since **H** is the least fixed point of $(\mathcal{S}, 1 \cup \mathcal{P}_f)$, the interior of its associated triangle is empty (see Fig. 5.2).

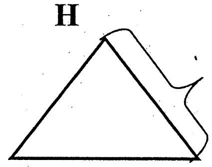

Figure 5.2: Illustration of **H** as the least fixed point of $(\mathcal{S}, 1 \cup \mathcal{P}_f)$

Lemma 5.35: If $F \in \mathcal{F}$ and $\mathcal{P}_f F \subseteq A \subseteq F$, where $A \in \mathcal{S}$. Then $A \in \mathcal{F}$.

We conclude Sect. 5. with Lemma 5.36 and Prop. 5.37, which show that dynamics on the class $\mathcal{S}$ may be analyzed in terms of dynamics on the sets $\mathcal{P}\,G$ where $G \in \mathcal{F}$. These results also provide a bridge between developments in Sect. 5 and those in Sect. 6 that deal with metadynamics(c), the latter providing a bridge to neural net circuitry.

Lemma 5.36: If $G \in \mathcal{F}$, then $(\mathcal{P}\,G, 1 \cup \mathcal{P}_f)$ is a sub dynamical system of $(\mathcal{S}, 1 \cup \mathcal{P}_f)$.

Proposition 5.37: If $G \in \mathcal{F}$, then $\mathcal{P}\,G = \bigcup_{H \in \mathcal{F} \cap \mathcal{P}G} \mathcal{J}(H)$. Moreover the union (the phase diagram of the sub dynamical system $(\mathcal{P}G, 1 \cup \mathcal{P}_f)$) is disjoint.

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[14] This recursion is reminiscent of a Volterra integral equation.

6. Finitely pictured sets (the bridge to neural networks), metadynamics(c)

We introduce the set of finitely pictured sets (denoted **FP**). This set forms a bridge between our axiomatic developments and the modeling of neural networks in the brain (MZ1, MZ2), modeling of the latter being naturally framed in **FP**. The **FP** notion motivates specialization of metadynamics(b) (developed in Sect. 5) to $\mathscr{P}(\mathbf{FP} \cap \mathscr{N})$ where it becomes metadynamics(c). The notions of fixed point, basin of attraction, phase and a phase diagram associated to metadynamics(c) are introduced. Thus aspects of the theory developed are available for application to neural network modeling.

6.1 Finitely pictured sets (FP)

We begin by recalling the theory of finitely pictured sets (Aczel, 1988).

Definition 6.1 (FP, finitely pictured sets): A set x is finitely pictured $(x \in \mathbf{FP})$ if x is the decoration of the point of an accessible pointed graph (see the notion of a DLAPG in MZ1) with finitely many nodes. (Compare the notion of hereditarily finite non-well-founded sets (Aczel, 1988, page 7)).

The remainder of Sect. 6.1 consists of the development of constructs and their properties (associated to **FP**). The main result is Theorem 6.14 where a fundamental property of a mapping onto the set of finitely perfect subsets of **FP** is established. Theorem 6.14 establishes a connection between Sect. 4 and Sect. 6. We begin with the following Prop. 6.2 that specifies the cardinality of **FP**.

Proposition 6.2: The collection **FP** of finitely pictured sets is a countably infinite set.

Remark 6.3: If $x \in \mathbf{FP}$, then $\mathscr{P}_f x \in \mathbf{FP}$ (Def. 5.1). Hence $\mathbf{FP} \in \mathscr{F}$. (See Def. 5.17.)

Proposition 6.4: (a) If $y \in \mathbf{FP}$, then y is a finite subset of **FP**, and (b) $\overline{y}$ (Def. 5.5) is a countably infinite subset of **FP**. (c) Moreover, $\overline{y}$ is a fixed point of $1 \cup \mathscr{P}_f$. That is, $\overline{y} = (1 \cup \mathscr{P}_f)\overline{y}$.

Remark 6.5 (Phase of a set): Recall that $\overline{y}$ specifies the phase of y, relative to metadynamics(b) (Def. 5.14).

Definition 6.6 (Selector type mapping): Given a set $D \subseteq \mathbf{FP}$, the map $f(D)$ is defined by $f(D)z = D \cap z,\ \forall z \in \mathbf{FP}$. (Compare Def. 2.5.) Then define the function $R: \mathscr{P}(\mathbf{FP}) \to \mathbf{FP}^{\mathbf{FP}}$, where $R(D) = f(D), \forall D \in \mathscr{P}(\mathbf{FP})$.

Remark 6.7: Since any subset of an **FP** set is itself an **FP** set, $f(D)$ is a self map, $(f(D): \mathbf{FP} \to \mathbf{FP})$. In particular, if $y \in \mathbf{FP}$ then $f(\tilde{y})$ is a self map. Moreover, $f(\tilde{y})x = \tilde{y} \cap x$.

The observations made in Remark 6.7 are illustrated in Fig. 6.1.

$$\mathbf{FP} \to \mathscr{P}(\mathbf{FP}) \to \mathbf{FP}^{\mathbf{FP}}$$

$$y \;\mapsto\; \tilde{y} \;\mapsto\; f(\tilde{y})$$

Figure 6.1: Illustration of the phase $\tilde{y}$ of y and the associated self map $f(\tilde{y})$

Definition 6.8 (Set induced by a mapping): If f : **FP** $\to$ **FP**, let $C(f) \subseteq \mathbf{FP}$ be the set given by $C(f) = \{ y \in \mathbf{FP} | f(\mathscr{B}y) = \mathscr{B}y \}$. (Compare Def. 2.5 and see Fig. 8.1.)

Note that $C(f(D)) = D$. (Compare Lemma 2.4.)

Remark 6.9: If $y \in \mathbf{FP}$, then $\mathscr{B}y \in \mathbf{FP}$. Moreover, if $\mathscr{O}$ is a selector and $y \in \mathbf{FP}$, then $(1 \cup \mathscr{B}\mathscr{O})y \in \mathbf{FP}$. Note that $(\mathbf{FP}, 1 \cup \mathscr{B}\mathscr{O})$ is a specialization of accretion dynamics (Def. 3.5).

Remark 6.10: Let $\mathscr{C}$ be a class. Then $\mathscr{C} \cap \mathbf{FP}$ is a subset of **FP** (axiom of comprehension).

Lemma 6.11: Let $\mathscr{C}$ be a class. Then $\forall x \in \mathbf{FP}$, $\mathscr{O}(\mathscr{C})x = f(\mathscr{C} \cap \mathbf{FP})x$.

Lemma 6.12: If $\mathscr{C}$ is a perfect class, then $\mathscr{C} \cap \mathbf{FP}$ is a finitely perfect set.

Lemma 6.13: If $D \subseteq \mathbf{FP}$ is finitely perfect, then (a) D^* (see Def. 4.10) is a perfect class and (b) $D^* \cap \mathbf{FP} = D$.

Finally we state Theorem 6.14, the main result of Sect. 6.1.

Theorem 6.14: The map $\mathscr{C} \mapsto \mathscr{C} \cap \mathbf{FP}$ is a surjection of the collection of perfect classes onto the set of finitely perfect subsets of **FP**.

Remark 6.15: **H** is a proper subset of $\mathbf{FP} \cap \mathscr{N}$.

A diagram conceptualizing the structure of **FP** is shown in Fig. 6.2. The disposition of other constructs (**H**, $\mathscr{D}$[**H**], **N**, Ω, and $\{\varnothing, \Omega\}$) in their relationship to the normal and abnormal parts of **FP** is also shown. Here[15] $\mathscr{D}[\mathbf{H}] = \{ \mathscr{D}y | y \in \mathbf{H} \}$, where $\mathscr{D}$ is the duality operator.

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[15] We use the notation $\mathscr{O}[K] = \{ \mathscr{O}y | y \in K \}$, for any operator $\mathscr{O}$ and any set K.

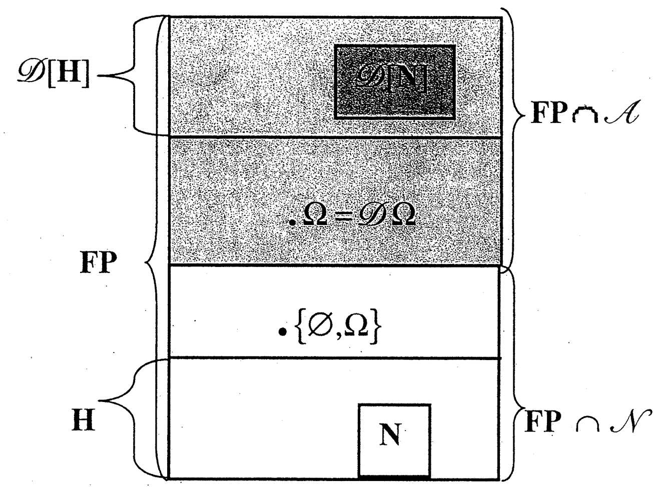

Figure 6.2: Conceptual representation of the normal and abnormal (shaded) portion of **FP**, illustrating Remark 6.15, the set $\{\varnothing,\Omega\}$ of Example 6.16,1) and other structural features

Example 6.16: 1) Take the set $x=\{\varnothing,\Omega\}\in\mathbf{FP}\cap\mathcal{N}$. A picture of *x*, shown in Fig 6.3, illustrates its non well-founded character. Note that $x\in\mathbf{H}$, since $x\in\mathcal{S}_{nwf}$.

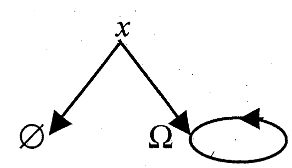

Figure 6.3: A picture of the non well-founded set $x=\{\varnothing,\Omega\}$.

2) A general example is $y=\{x_1,\cdots,x_n\}\in\mathbf{FP}\cap\mathcal{N}$, where the x_j are distinct, and $\forall j,\ |x_j|<n,\ x_j\in\mathbf{FP}$, but at least one $x_i\notin\mathbf{H}$.

6.2 Metadynamics(c)

Employing the set **FP** and its associated constructs, metadynamics(c) are introduced. The associated notions of a region of subsets of $\mathbf{FP}\cap\mathcal{N}$, of a limit point of a region and of the associated phase diagram along with their properties are developed. The main result of Sect. 6.2 is Theorem 6.27, concerning cardinal invariants. We begin by interpolating Lemma 6.17 and Def. 6.18.

Lemma 6.17: $\mathbf{FP}\cap\mathcal{N}\in\mathcal{F}$ (Def. 5.19).

Definition 6.18 (A special power set): L= $\mathscr{P}(\mathbf{FP} \cap \mathscr{N})$.

Metadynamics(c) are now specified in the following Def. 619.

Remark and Definition 6.19 (Metadynamics(c)): Lemma 5.36 and Lemma 6.17 show that $(\mathbf{L}, 1 \cup \mathscr{P}_f)$metadynamics(c) to be a sub dynamical system of ($\mathscr{S}$, $1 \cup \mathscr{P}_f$) (that is, of metadynamics(b)) on sets. This sub dynamical system $(\mathbf{L}, 1 \cup \mathscr{P}_f)$ is called metadynamics(c)). The associated equation of evolution is

$$A(r+1) = A(r) \cup \mathscr{P}_f A(r),$$

where $A(r) \subseteq (\mathbf{FP} \cap \mathscr{N})$ and r is metatime. (Compare Def. 5.5.) We note that metadynamics(c) are accretive. If $A \in \mathbf{L}$, then the phase of A relative to metadynanmics(c) is $\tilde{A}$ (Def. 5.5), the latter also an element of **L** (see Lemma 6.20).

The following Lemma 6.20 establishes a connection between the power set **L** and the limit points of metadynamics(c).

Lemma 6.20: $\mathscr{L}[\mathbf{L}] \subseteq \mathbf{L}$. (See footnote 13 and Def. 5.5.)

Next we introduce special regions of **L**, which play a central role in specifying a phase diagram for metadynamics(c).

Definition 6.21 (Regions of L):

1. Let $\mathbf{S} = \{D \subseteq \mathbf{FP} \cap \mathscr{N} | (1 \cup \mathscr{P}_f) D = D\} = \mathbf{L} \cap \mathscr{F}$
2. Region I $= \mathscr{P}_f(\mathbf{FP} \cap \mathscr{N})$.
3. Region II $= \{D \subset \mathbf{FP} \cap \mathscr{N} \| D| = \mathbf{N}$ and $D \notin \mathbf{S}\}$.
4. $\mathbf{S}_0 = \mathscr{L}[\mathscr{P}_f(\mathbf{FP} \cap \mathscr{N})]$. (See Def. 5.5.)

Note that 1. $\mathbf{FP} \cap \mathscr{N} \in \mathbf{S}$ (Lemma 6.17), 3. $\mathbf{N} \in$ Region II, and 4. $\mathbf{H} \in \mathbf{S}_0$, respectively. (Refer to Fig. 6.4.)

Prop. 6.22 and the succeeding remark specify the phase diagram of metadynamics(c). (See Remark 6.5.)

Proposition 6.22: $\mathbf{L} = \bigcup_{D \in \mathbf{S}} \mathscr{I}(D)$. Moreover this union is disjoint.

Question: If $D \in \mathbf{S}$, what is $|\mathscr{I}(D)|$?

Remark and Definition 6.23 (Phase diagram of metadynamics(c)): The decomposition in Prop. 6.22 specifies the phase diagram of metadynamics(c) as a sub partition of metadynamics(b). (See Prop. 5.23 and Def. 5.24.)

The limit point of a region of **L** is introduced in the following Def. 6.24.

Definition 6.24 (Limit point of a region of L): D is a limit point of a region of **L** if there exists a set A, which is an element of that region, such that $D = \tilde{A}$ (Def. 5.5).

The following Prop. 6.25 relates the region **S** and the set of Region II limit points.

Proposition 6.25: $\mathrm{S} = \{$ Region II limit points $\}$.

We now state Lemma 6.26 (a special case of Lemma 5.35), which we will use to specify cardinal invariants of regions of **L**, the result of Theorem 6.27 that follows.

Lemma 6.26: If $D \in \mathbf{S}$, $E \subseteq \mathbf{FP}$ and $\mathscr{P}_f D \subseteq E \subseteq D$, then $E \in \mathbf{S}$.

Theorem 6.27: a) $|\mathscr{P}(\mathbf{FP} \cap \mathcal{N})| = |\mathbf{L}| = \mathbf{2}^{\aleph}$
b) $|\text{Region } \mathbf{I}| = \mathbf{N}$.
c) $|\mathbf{S}| = \mathbf{2}^{\aleph}$; $|\mathbf{S}_0| = \mathbf{N}$.
d) $|$ Region II $| = |\{$ Region II limit points $\}| = 2^{\mathbf{N}}$.

Corollary 6.28: Metadynamics(c) generates uncountably many phases (Theorem 6.27,c).

Fig. 6.4 illustrates aspects of Def. 6.21, Def. 6.24 and Theorem 6.27 concerning the set **L**.

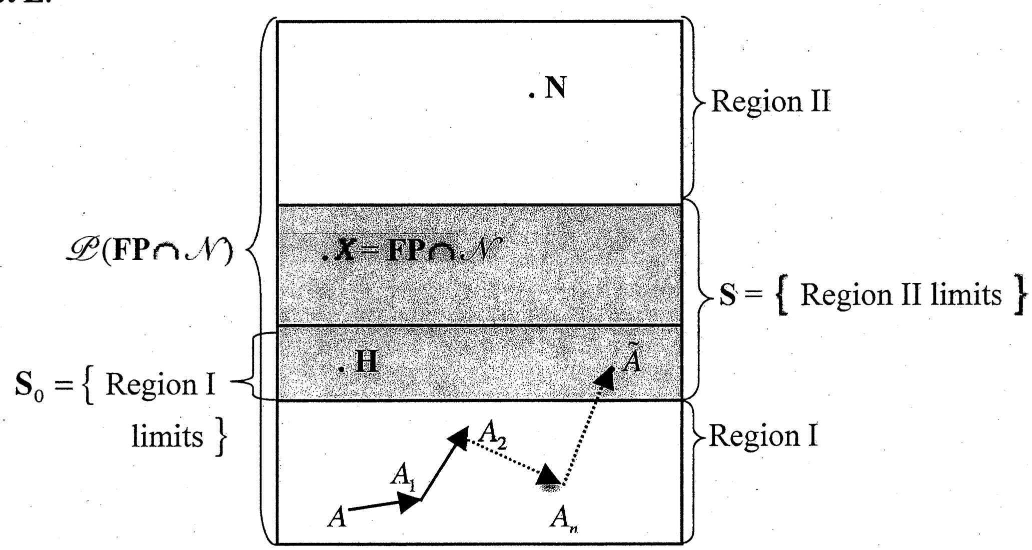

Figure 6.4: Conceptual representation of the set **L** illustrating aspects of metadynamics(c), $(\mathbf{L}, 1 \cup \mathscr{P}_f)$. Note the points in Region I denote finite sets, while all other points denote infinite sets. Recall that $\tilde{A}$ is the phase of A

7. Restrictions of consciousness operators to FP

Consciousness operators (Sect. 3.1) have a fundamental correspondence to limit points of both metadynamics(a) and metadynamics(c). By restricting consciousness operators to **FP** (recall the relevance of **FP** to neural network modeling), we shall formulate a study of this correspondence.

Choose a consciousness operator $\mathscr{K}$, and let $E = \mathbf{FP} \cap \mathscr{C}(\mathscr{K}) = \mathscr{K}\,\mathbf{FP}$ (Lemma 2.6). Then E is a set (axiom of comprehension). Moreover $E \in \mathscr{P}(\mathbf{FP})$. Results regarding the association in question are given in the following Prop. 7.1 and Example 7.2.

Proposition 7.1: $\mathscr{K}\,|\mathbf{FP} = f(E)$ (see Def. 6.6).

Example 7.2: 1. $\mathscr{W}|\mathbf{FP} = f(\mathbf{H})$, since $\mathscr{C}(\mathscr{W}) = \mathscr{S}_{wf}$ and $\mathscr{S}_{wf} \cap \mathbf{FP} = \mathbf{H}$.

2. $\mathscr{R}|\mathbf{FP} = f(\mathbf{FP} \cap \mathscr{N})$, since $\mathscr{C}(\mathscr{R}) = \mathscr{N}$.

Let D be a finitely perfect subset of $\mathbf{FP} \cap \mathscr{N}$. That is, let D be a fixed point of metadynamics(c) (Def. 6.19). Recall that D is infinite (Remark 5.3). By Remark 6.15, $D = \mathbf{H}$ and $D = \mathbf{FP} \cap D = \mathscr{N}$ furnish distinct examples of finitely perfect subsets of **FP**.

Associations of consciousness operators to metadynamics(c) are the subjects of following Remark 7.3 and Remark 7.4.

Remark 7.3: For any limit point D of metadynamics(c), we can construct the limit point D^* of metadynamics(a). Moreover, $D^* \cap \mathbf{FP} = D$ (Lemma 6.13). Since the class D^* is perfect, it is a proper class (Lemma 3.5). In particular, $D^* \neq D$, since D is a set.

Remark 7.4: If $D \in \mathbf{S}$ (Def. 6.1), then $D \subseteq \mathscr{N}$. Therefore D^* is a subclass of $\mathscr{N}$. Then from Theorem 3.11, we see that $\mathscr{K} = \mathscr{O}(D^*)$ is a particular consciousness operator.

The following Def. 7.5 and Theorem 7.6 deal with operators of the form $\mathscr{O}(D^*)$.

Definition 7.5 (A pair of particular metasets of consciousness operators):

1. $\mathbf{K} = \{\, \mathscr{O}(D^*) | D \in \mathbf{S}\}$.
2. $\mathbf{K}_0 = \{\, \mathscr{O}(D^*) | D \in \mathbf{S}_0\}$.

The cardinal invariants of $\mathbf{K}$ and $\mathbf{K}_0$ are the subject of the following Prop. 7.6.

Proposition 7.6: $|\mathbf{K}| = 2^{\mathbf{N}}$ and $|\mathbf{K}_0| = \mathbf{N}$.

Relationships between consciousness operators and **FP** are developed in the following Theorem 7.7.

Theorem 7.7: a) If $A \subseteq \mathbf{FP} \cap \mathcal{N}$, then $A^* \subset \mathcal{N}$, and $\mathcal{O}(A^*)$ is a consciousness operator.

b) If $D \in \mathbf{S}$, $\mathcal{O}(D^*)|\mathbf{FP} = f(D)$.

c) If $y \in \mathbf{FP}$, then $\tilde{y} \cap \mathcal{N} \in \mathbf{S}$ and $\mathcal{O}((\tilde{y} \cap \mathcal{N})^*)$ is a consciousness operator.

d) If $\mathcal{K}$ is a consciousness operator, and $E = \mathbf{FP} \cap \mathcal{C}(\mathcal{K})$, then $E \in \mathbf{S}$ and $\forall x \in \mathbf{FP}, \mathcal{K}x = \mathcal{O}(E^*)x$.

We illustrate aspects of Theorem 7.7 in Fig. 7.1.

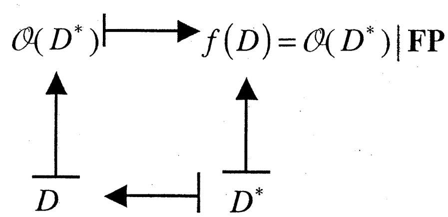

Figure 7.1: Illustration of Theorem 7.6

A connection between fixed points of metadynamoics(b) and consciousness operators is the subject of the following Prop. 7.8.

Proposition 7.8: The set $\mathbf{S}$ of subsets of $\mathbf{FP} \cap \mathcal{N}$ that are fixed points of $1 \cup \mathcal{P}_f$ is in one-to-one correspondence with the collection, $\{(\mathcal{K} \mid \mathbf{FP}) | \mathcal{K} \in \textit{Consciousness Operators}\}$. In particular, this collection is a set.

Connections of consciousness operators to $\mathcal{R}$, $\mathcal{W}$, $\mathbf{S}$ and limit points of metadynamics(a) is the subject of the following Remark 7.9.

Remark 7.9: Let $D \in \mathbf{S}$, and let $\mathcal{K} = \mathcal{O}(D^*)$. If $D = \mathbf{H}$, then $\mathcal{K} \mid \mathbf{FP} = \mathcal{W} \mid \mathbf{FP}$, and if $D = \mathbf{FP} \cap \mathcal{N}$, then $\mathcal{K} \mid \mathbf{FP} = \mathcal{R} \mid \mathbf{FP}$. $\mathbf{H}$ is the least fixed point in $\mathbf{S}$ and $\mathbf{FP} \cap \mathcal{N}$ is the greatest fixed point in $\mathbf{S}$.

8. Transfinite renormalization of a selector, restrictions of a consciousness operator

Recall remarks on the terminology of renormalization in the beginning of Sect. 4.2. We define the renormalization of a selector that depends on the mapping $\mathcal{C}(\mathcal{O}) \mapsto \mathcal{C}^*(\mathcal{O})$, a transfinite process (Def. 4.10). A diagram (in Fig 8.1) illustrating this renormalization and related constructs is given. Following that a commutative diagram (in Fig. 8.2) relating metadynamics(a) and metadynamics(c) along with related constructs is given

We begin with the definition of transfinite renormalization.

Definition 8.1 (Transfinite renormalization of a selector): The transfinite renormalization of a selector $\mathcal{O}$ is given by the following sequence of mappings.

$$\mathcal{O} \overset{Def.2.5}{\longmapsto} \mathcal{C}(\mathcal{O}) \overset{Def.4.10}{\longmapsto} \mathcal{C}^*(\mathcal{O}) \overset{Cor.4.16}{\longmapsto} \mathcal{K}(\mathcal{C}^*(\mathcal{O}) \cap \mathcal{N})$$

Definition 8.2 (Construction metaoperator Γ and phase metaoperator Σ): Let $\Gamma : f \mapsto \mathcal{K}\left(C(f)^*\right)$, and let $\Sigma : D \mapsto D^*$.

Fig. 8.1 illustrates constructs used in Def. 8.1 and Def. 8.2 as well as the relationships of those constructs to consciousness operators. In particular, I_S and I_C are the inclusion mappings between the metaclasses indicated in the figure.

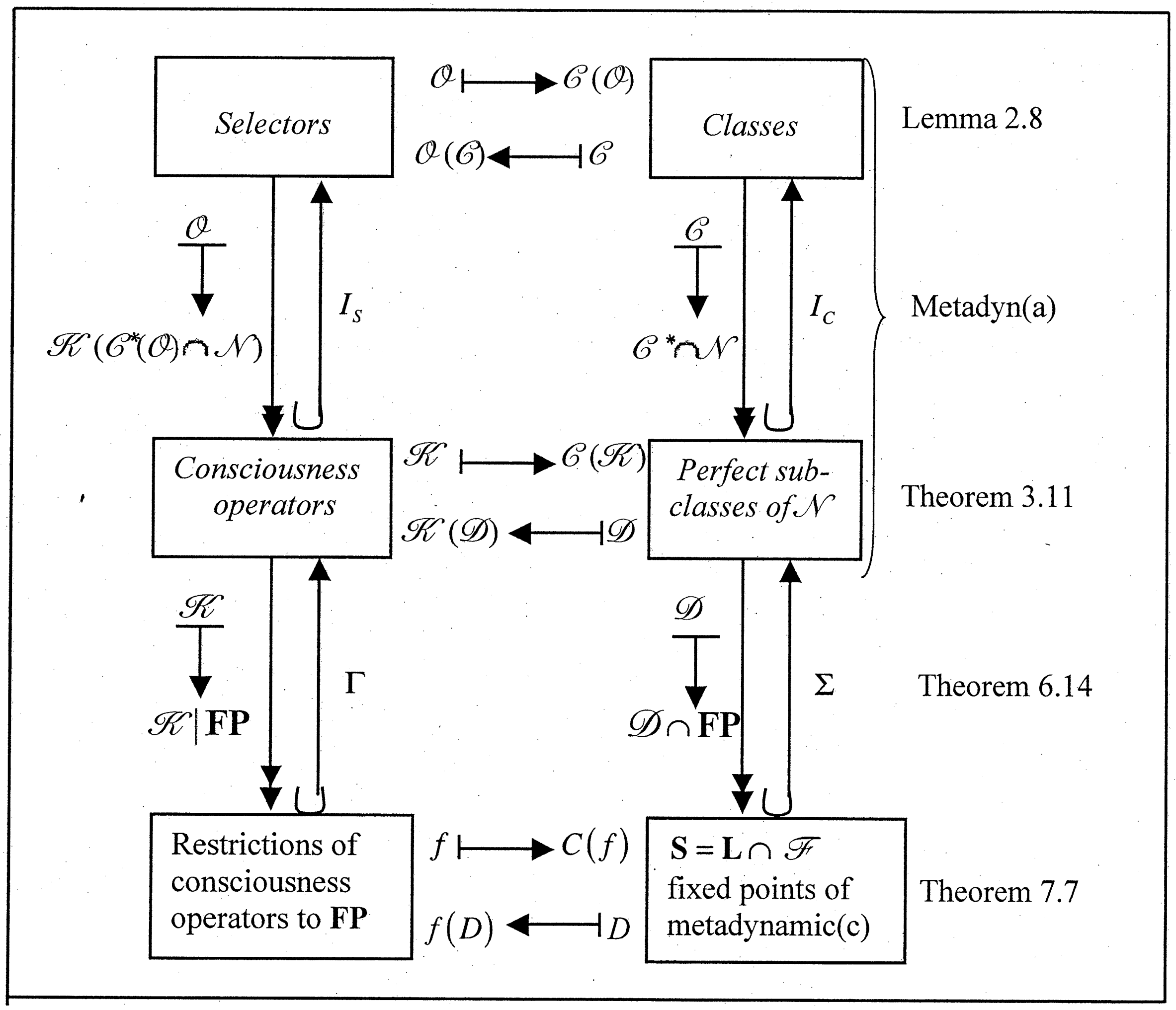

Figure 8.1: A diagram (which can be shown to be commutative) illustrating the transfinite renormalization of a selector (upper portion) and the restriction of a consciousness operator to **FP** (lower portion). (Recall that **S** is specified in Def. 6.21.)

The diagram in Fig. 8.2 (which can be shown to be commutative) illustrates relationships between transfinite limits in metadynamics(a) (namely the $\mathscr{C}^*$) and ordinary limits in metadynamics(c) (namely, the $\tilde{D}$).

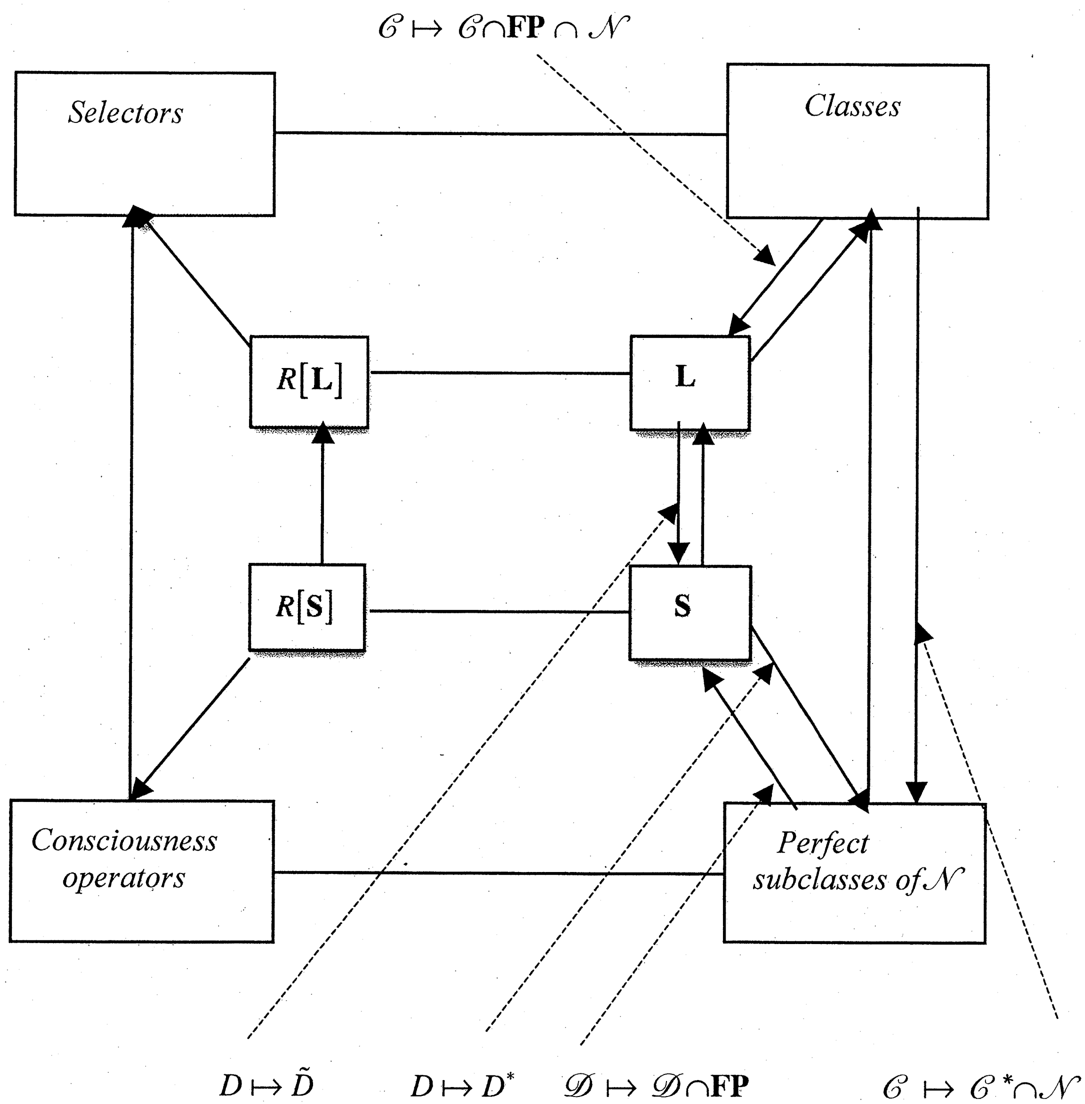

Figure 8.2: A commutative diagram illustrating transfinite limits in metadynamics(a) and ordinary limits in metadynamics(c). Horizontal maps represent bijections

9. Qualia, the quale set and the quale set operator $\mathscr{Q}$

We connect developments of this work into a coherent picture by application to the notions of experience and awareness of MZ1 and MZ2. This is done by introduction of the construct of a quale. The collection $\mathscr{Q}(x)$ of all qualia of a set x is shown to be a set.

Connections of the associated quale operator $\mathcal{Q}$ to **FP**[16] are made along with semantic observations relating $\mathcal{Q}$ and awareness. We shall see that $\mathcal{Q}(x)$ and $\mathcal{Q}$ are formal and rigorous realizations of the consciousness thesis enunciated in Def. 3.16 of MZ1.

9.1 The Quale operator.

In this section, we introduce and formalize the constructs of a quale, the collection $\mathcal{Q}(x)$ of qualia and the quale set operator $\mathcal{Q}$. Then we derive a number of their properties, which include associations to many aspects of the preceding developments.

Definition 9.1 (Quale, collection $\mathcal{Q}(x)$ of qualia): An experience is (modeled by) a set x. A quale of that experience is any set of the form $\mathcal{K}x$, where $\mathcal{K}$ is a consciousness operator[17]. The collection of qualia is given by[18] "$\mathcal{Q}(x)=\{\mathcal{K}x| \mathcal{K}\in$ Consciousness operators $\}$". Note that since $\mathcal{R}$ is a consciousness operator, $\mathcal{R}x \in \mathcal{Q}(x)$ (compare the consciousness thesis of Def. 3.16 in MZ1).

A semantic comment associated to Def. 9.1 is given in the following Observation 9.2, while a corresponding syntactical comment is given in Remark 9.3.

Observation 9.2: Referring to Cor. 3.12, we see that an awareness of an awareness is an awareness of a (primary) experience.

Remark 9.3: For each consciousness operator $\mathcal{K}$ and for each set $x \in \mathcal{S}$, $\mathcal{K}x \in \mathcal{P}(x) \cap \mathcal{N}$' (Def. 3.3).

The following proposition shows a fundamental connection between qualia and illusions (Fig. 4.1) as well as the relevance of metadynamics(a) (Def. 4.3), including especially its limit points (Def. 4.10), to qualia.

Proposition 9.4: $\forall x \in \mathcal{S}$,

a) $\mathcal{Q}(x)=\{y\subseteq x|\exists w\subseteq \mathcal{R}x$ such that $w^{*}\cap x=y\}$.

b) $\mathcal{Q}(x)=\{\mathcal{K}\langle w\rangle x|w\in\mathcal{P}\mathcal{R}x\}$. (See Def. 4.19.)

c) $\mathcal{Q}(x)=\{w^{*}\cap x|w\in\mathcal{P}\mathcal{R}x\}$.

d) $\mathcal{Q}(x)=\{\mathcal{A}\cap x \mid \mathcal{A}\in Phase(\mathcal{P}\mathcal{R}x)\}$. (See Def. 4.17.)

e) $\mathcal{Q}(x)=\{y\subseteq\mathcal{R}x|y^{*}\cap x=y\}$.

---

[16] Recall that the set of finitely pictured sets, **FP**, was introduced in Sect. 6 to establish a bridge between set theoretic developments and neural network models.

[17] In MZ1, an experience is called a primary experience and a quale is called an awareness.

[18] The quotation marks emphasize that what is given is a meta description of $\mathcal{Q}(x)$. The parity of consciousness operators in this definition points toward a solution to the issue of the choice of $\mathcal{K}$

Remark 9.5: Let $y = x \cap w^*$, where $y \in \mathcal{Q}(x)$ and where $w \in \mathcal{PR}x$. Then $w \subseteq y \subseteq \mathcal{R}x \subseteq x$. From Prop. 9.4 (a), we have analogously that $\mathcal{S}_{wf} \subseteq w^* \subseteq \mathcal{N} \subseteq \mathcal{S}$.

Remark 9.6: Let $A \in \mathcal{S}$. Then $\{\mathcal{K}\langle w\rangle | w \in A\}$ (see Def. 4.19) is a metaset[19] of consciousness operators indexed by the set A.

A formal specification of the operator $\mathcal{Q}$ as well as of $\mathcal{Q}$(x) as a set is provided by the following Cor. 9.7 and Def. 9.8.

Corollary 9.7: $\mathcal{Q}(x)$ is a set.

Definition 9.8. (Quale set, quale set operator): $\mathcal{Q}(x)$ will be called the quale set of x (the set of all qualia of x). $\mathcal{Q}$ will be called the quale set operator.

The following Lemma 9.9 and Remarks 9.10 and 9.12 establish additional properties of $\mathcal{Q}$ and $\mathcal{Q}$(x), some demonstrating a relationship of $\mathcal{Q}$ to the operators $\mathcal{B}$, $\mathcal{P}$, $\mathcal{R}$, and $\mathcal{W}$ as well as to a generic consciousness operator $\mathcal{K}$.

Lemma 9.9:

a) If $x \in \mathcal{S}_{wf}$, then $\mathcal{Q}(x) = \{x\}$.
b) $\mathcal{Q}$ is not an isotonic operator.
c) $\mathcal{Q}(\mathcal{R}y) = \mathcal{Q}(y), \forall y \in \mathcal{S}$.
d) $\mathcal{Q} = \mathcal{QR}$.
e) $\mathcal{RQ} = \mathcal{Q}$.
f) $\mathcal{QW} = \mathcal{BW} = \mathcal{WQ}$.
g) $\mathcal{Q} \subseteq \mathcal{RP}$.
h) $\mathcal{QK} = \mathcal{Q} \cap \mathcal{PK}$, where $\mathcal{K}$ is any consciousness operator.
i) $\mathcal{BK} \subseteq \mathcal{Q}$.

Remark 9.10: Suppose $y \in \mathcal{S}$, $\mathcal{R}y \neq y$, and $\mathcal{R}y \in \mathcal{S}_{wf}$. Then $\mathcal{Q}(y) = \{\mathcal{R}y\} \neq \{y\}$. We see that $\mathcal{Q}$ is a nontrivial variant of the brace operator, $\mathcal{B}$ ($\forall x$, $\mathcal{B}x = \{x\}$).

Example 9.11: The set $y = \{\varnothing, \Omega\}$ satisfies the hypotheses of Remark 9.10, and so validates the remark.

Remark 9.12: $\mathcal{Q}(x) \cap x = \varnothing$. This observation is directly connected to the consciousness axiom of removal (see the consciousness axioms in the appendix). There is a connection of it as well to the descriptions of satellites in Def. 9.23 and Fig. 9.3.

The following Cor. 9.13 confirms the semantic quality that the common aspect of two awarenesses is itself an awareness.

Corollary 9.13: y_1 and y_2 being elements of $\mathcal{Q}(x) \Rightarrow y_1 \cap y_2 \in \mathcal{Q}(x)$.

---

[19] A metaset is a metaclass that is indexed by a set.

The next result, Prop. 9.14, connects qualia to basins and phases of metadynamics(a).

Proposition 9.14:

a) If $y \in \mathscr{Q}(x)$, then $y \neq x \Rightarrow y^* \neq x^*$.

b) Let y_1 and y_2 are elements of $\mathscr{Q}(x)$. Then $y_1 \neq y_2 \Leftrightarrow y_1^* \neq y_2^*$.

c) Fix sets A and B. Then for at most one $Y \in \mathscr{Q}(A)$, $Y^* = B^*$. Moreover, $B \cap A \subseteq Y$.

Note that Part a) of Prop. 9.14 asserts that if $y \in \mathscr{Q}(x)$ and $y \neq x$, then y and x are in different basins. That is, they have different phases. This forecasts the phenomenon of a "phase transition" for qualia.

Now we introduce the formal definitions of the supremum and the maximum of a set.

Definition 9.15 (Supremum and maximum of a set): $\sup X = \mathscr{U}X$ is the least set S such that $\forall x \in X, x \subseteq S$. If $\sup X \in X$, $\max X = \sup X$. If $\sup X \notin X$, X has no maximum.

Quale are associated to many of the constructs already specified ($\mathscr{N}, \mathscr{P}, \mathscr{V}, \mathscr{K}, \mathscr{K}\langle B\rangle$, *). The following Prop. 9.16 identifies a collection of these associations.

Proposition 9.16: Let the sets $A \subseteq \mathscr{N}$ and $B \subseteq \mathscr{N}$ be specified. Then

a) $\exists Y \in \mathscr{Q}(A)$ such that $Y^* = B^* \Leftrightarrow (A \cap B^*)^* = B^* \Leftrightarrow \mathscr{P}A \cap \mathscr{V}(B^*) \neq \varnothing$ (Def. 4.23).

b) Suppose there exists a $Y \in \mathscr{Q}(A)$ such that $Y^* = B^*$. Then

1. Y is unique,
2. $Y = A \cap B^* = \mathscr{K}\langle B\rangle A$ (Def. 4.19),
3. $Y = \sup(\mathscr{P}A \cap \mathscr{V}(B^*))$,
4. $Y \in \mathscr{P}A \cap \mathscr{V}(B^*)$,
5. $Y = \max(\mathscr{P}A \cap \mathscr{V}(B^*))$,
6. $\forall W \in \mathscr{P}A \cap \mathscr{V}(B^*)$, $Y = A \cap W^*$,
7. $\mathscr{Q}(A) \cap \mathscr{V}(B^*) = \{\mathscr{K}\langle B\rangle A\}$.

In the following Def. 9.17 we specify a special quale (awareness) associated to a pair of experiences. The connection of this quale to the two classes $\mathscr{Q}A$ and $\mathscr{V}(\mathscr{B}^*)$ is given by Prop. 9.16,vii.

Definition 9.17 (Quale associated to a pair of experiences): $\mathscr{K}\langle B\rangle A$ will be called the quale associated to the pair of experiences A and B. (See Def. 4.19.) (For a semantic interpretation of this remark, see footnote 9.)

Remark 9.18: Suppose $\mathscr{P}A \cap \mathscr{V}(B^*) \neq \varnothing$. Then the awareness specified by $\mathscr{K}\langle B\rangle A$ is characterized by the unique point of intersection of the two classes $\mathscr{Q}A$ and $\mathscr{V}(B^*)$ (see Fig. 9.1).

$\mathscr{K}\langle B\rangle A$, property (vii) of Prop. 9.16, Remark 9.18, along with other features are conceptualized in Fig. 9.1.

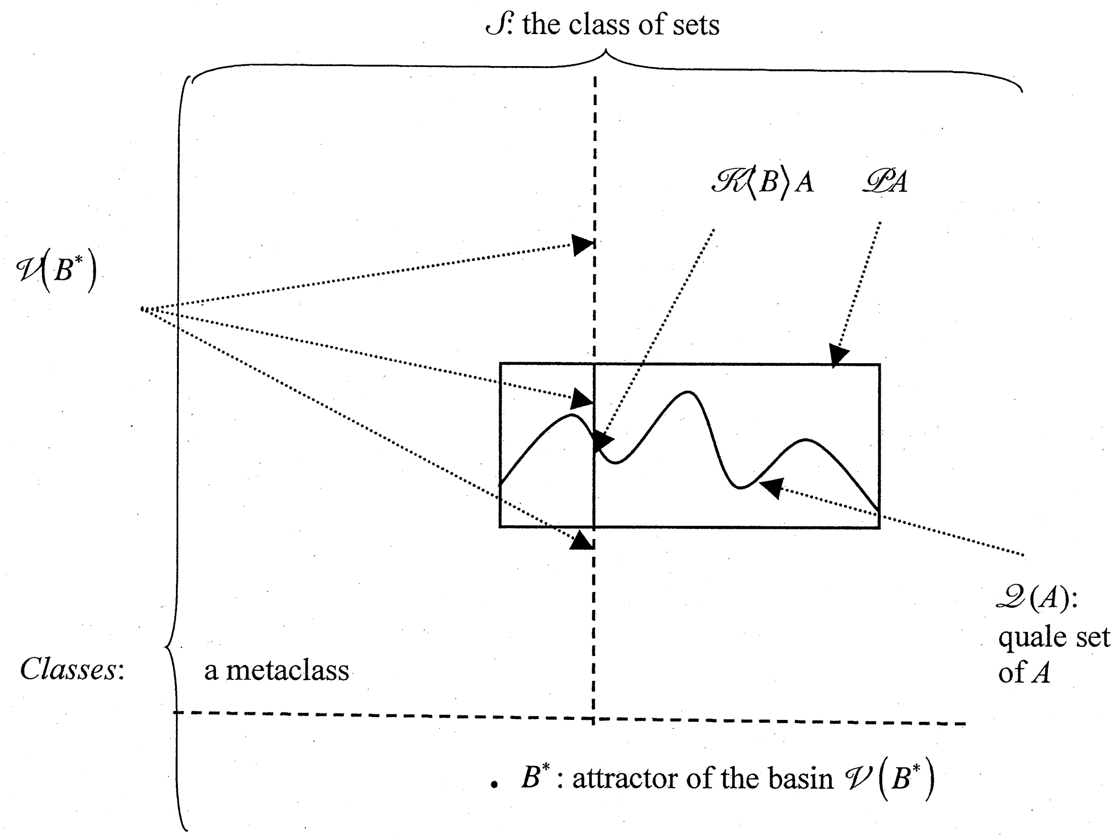

Figure 9.1: Illustration of constructs occurring in Prop. 9.16, namely, illusion ($\mathcal{V}(B^*)$), quale set ($\mathcal{Q}(A)$) and awareness ($\mathscr{K}\langle B\rangle A$)

9.2 Specialization of qualia to FP

In this section we connect qualia and certain related constructs to **FP,** establishing thereby their relationship to the modeling of neural network brain circuitry. The notion that a quale generates a satellite of experience is developed. We begin by interpolating the following Lemma 9.19 that associates limit points of metadynamics(a) to **FP**.

Lemma 9.19: Let y_1 and $y_2 \in \mathbf{FP}$, and let $\mathcal{C} = (\tilde{y})^*$ (see Def. 5.5). Then

a) $\mathcal{C} \cap \mathbf{FP} = \tilde{y}$.

b) $\mathcal{C} \doteq y^*$.

c) $y^* \cap x = \tilde{y} \cap x$.

The following Prop. 9.20 and Remark 9.21 list various relationships of qualia to **FP**.

Proposition 9.20: Let $x \in \mathbf{FP}$; then

1. If $y \in \mathbf{FP}$, and $\mathscr{K}$ is a consciousness operator, then $\mathscr{K}\langle y\rangle x \in \mathbf{FP}$ and $\mathscr{K}\langle y\rangle x = \tilde{y} \cap x \cap \mathscr{N}$ (Def. 5.5).
2. $\mathscr{Q}(x) \in \mathbf{FP}$.
3. $\mathscr{Q}(x) = \{D \cap x | D \in \mathbf{S}\} = \{\mathscr{K}x | \mathscr{K} \in \mathbf{K}\}$. (See Def. 6.21 and Def. 7.5.)
4. $\mathscr{Q}(x) = \{y \subseteq x | \exists w \subseteq \mathscr{R}x$ such that $\tilde{w} \cap x = y\}$.
5. $\mathscr{Q}(x) = \{D \cap x | D \in \mathbf{S}_0\} = \{\mathscr{K}x | \mathscr{K} \in \mathbf{K}_0\}$. (See Def. 6.21 and Def. 7.5.)
6. $\mathscr{Q}(x) = \{y \subseteq \mathscr{R}x |\ \tilde{y} \cap x = y\}$.
7. $\mathscr{Q}(x) = \{D \cap x | D \in \mathscr{L}[\mathscr{P}\mathscr{R}x]\}$.

The connection of qualia to **FP** is further emphasized in the following Remark 9.21.

Remark 9.21: The appearance of $\mathbf{S}, \mathbf{S}_0, \mathbf{K}$ and $\mathbf{K}_0$ in Prop. 9.20 shows the relevance of metadynamics(c) to qualia. Likewise the appearance of $\tilde{y}$ and $\mathscr{L}$ in the proposition shows the relevance of metadynamics(b) to qualia.

In future work, we shall address the computation of $\mathscr{Q}(A)$ when A is given. A flowchart of the computation for the operator $\mathscr{Q}|\mathbf{FP}$ is shown in Fig. 9.2.

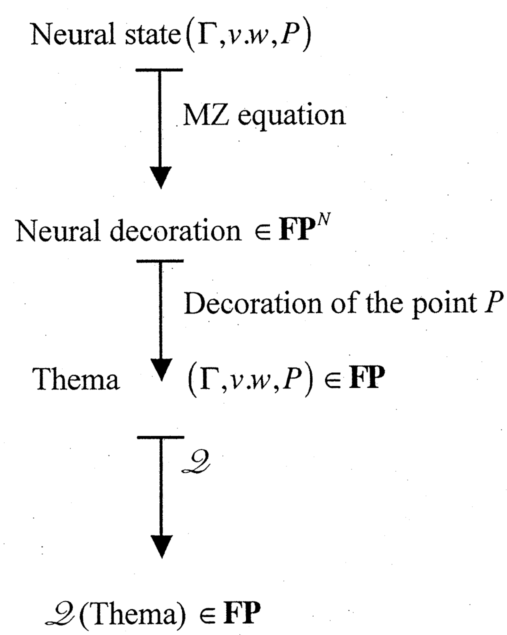

Figure 9.2: Flowchart of the computation of a collection of awarenesses[20]

[20] See MZ2, especially Table 2.2 for the definition of the terms used in Fig. 9.2.

Remark and question 9.22: Let $x \in \mathbf{FP}$. Prop. 9.20,6 implies that $\mathcal{Q}(x) = \{ y \subseteq \mathcal{R}x \mid \forall n \in \mathbf{N}, x \cap (1 \cup \mathcal{P})^n y = y \}$, which, *in principle,* shows that determination of $\mathcal{Q}(x)$ requires an infinite number of computational steps. An element of **FP** has a finite picture. So we ask, *is there a halting computation that determines such a picture?*

The semantic notion of the *concept* of an awareness, a so-called satellite of an experience, is introduced in the following Discussion 9.23 and Def. 9.24.

Discussion 9.23: Let $\boldsymbol{x} \in \mathbf{FP}$ be specified. Then $\mathcal{Q}(x)$ as a set in **FP** is also finite. Let $|\mathcal{Q}(x)| = r$ for some natural number *r*. Then we may write $\mathcal{Q}(x) = \{y_1, \cdots, y_r\}$ $= \mathcal{B}\ y_1 \cup \cdots \cup \mathcal{B}\ y_r$ where the y_j are distinct elements of $\mathcal{Q}(x)$. Now apply the brace operator to the y_j. Note that $\forall j$, $\mathcal{B} y_j \cap x = \varnothing$ (Remark 9.12), and, $\forall j, k,\ j \neq k$, $\mathcal{B} y_j \cap \mathcal{B} y_k = \varnothing$, . Recalling our semantics, we have *x* is a primary experience, *y* is an awareness, and $\mathcal{B}y$ is a concept of an awareness (see MZ1, Table 5.1). The awareness $\mathcal{B}y$ is not an element of *x*, and the $\mathcal{B} y_j$ are distinct. $\mathcal{K}x$ being an awareness of *x* and $\mathcal{B}\mathcal{K}x$ being the concept of the awareness of *x* motivates the description of $\mathcal{B}\mathcal{K}x$ as a satellite of x (follower) a notion formalized in the following Def. 9.24.

Definition 9.24 (Satellite): Given $x \in \mathbf{FP}$, a satellite is any set of the form $\mathcal{B}\mathcal{K}x$ for $\mathcal{K}$ a consciousness operator. Alternatively, $z = \mathcal{B}y$ is a satellite of *x* if $y \in \mathcal{Q}(x)$. Each singleton $\mathcal{B} y_j$, $j = 1, \ldots, r$ is a satellite of *x*. The satellite property is conceptualized in Fig. 9.3.

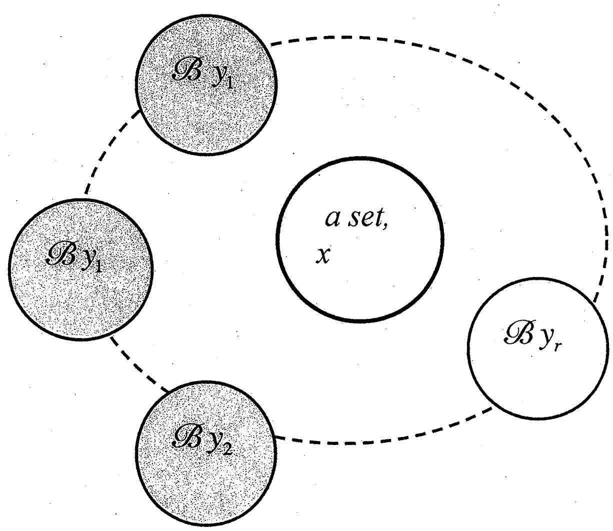

Figure 9.3: Experience *x* and its satellites (concepts associated to *x*)

Remark 9.25: The relation $x \cup \mathcal{Q}(x) = x \cup \mathcal{B}\, y_1 \cup \cdots \cup \mathcal{B}\, y_r$, shows that $1 \cup \mathcal{Q}$ is a non trivial variant of Von Neumann's successor operator. Note also that $(1 \cup \mathcal{Q}, \mathbf{FP})$ defines an accretion dynamics, the study of which we defer to future work.

10. Semantics

To critique the several dynamical systems constructed in this work, we assemble them into a hierarchy shown in Table 10.1. The expression of analogies between mental (set theoretic) and physical (material), in particular, those analogies concerning renormalization and phase diagrams gives context to a relationship between RG flows and consciousness. These analogies are displayed in Table 10.2. The syntactic developments of our theory are supplied with semantic interpretations. These are collected in the Syntax-Semantic Dictionary shown in Table 10.3, as well as in the list of Supplementary Semantic Interpretations of selected syntactical results that is displayed in Table 10.4. We conclude with a review of some philosophical interpretations of infinite regress, open questions and comments on future work.

10.1 The hierarchy of dynamical systems, comparison of mental and physical constructs

The McCulloch-Pitts system of equations for the propagation of neuronal information (modeled as a binary valued voltage) was studied at length in MZ2. That system of equations was augmented by a hierarchy of auxiliary dynamical systems (for neuronal activity, intrinsic data and memes) leading to a dynamical system for qualia. We now refine and extend those ideas by assembling what we call a hierarchy of three layers of dynamical systems of which those already developed in MZ2 collectively represent constituants of the first layer. The constructs appearing in the remaining two layers of the hierarchy are extracted variously from Sects. 3, 4, 5 and 6. In ascending order these three layers are called 1) mental activity dynamics, 2) accretion dynamics, and 3) metadynamics(a), (a'), (b) and (c), respectively. Each layer will have an intrinsic concept of a time-like variable. Since the first layer models physical systems, the time, labeled t in that layer, may be viewed as a representation of real (physical) time, but since the dynamics of the succeeding layers are not physical (are Platonic), the succeeding time-like variables are likewise not physical. These Platonic time variables are labeled s and r in ascending order in the hierarchy and are denoted as accretion time and metatime, respectively. The hierarchy is shown in Table 10.1.

A comparison of the mental and physical constructs of dynamics and renormalization is displayed in Table 10.2. The analogies shown give content to the relationship between renormalization and consciousness.

Remark 10.1: While RG theory is prominently applied to quantum mechanics, we stress that our use of it, as highlighted in Table 10.2, is developed out of classical mechanics, to which, of course, RG theory is also applied (Goldenfeld, 1992).

The Hierarchy of Dynamical Systems

1. **Mental activity dynamics:** Two examples of mental activity dynamics are a) voltage dynamics and b) memetic dynamics. The associated evolution equations are

   a) $v(a,t+1)=h_\theta\left(\sum_{\{\alpha|f(\alpha)=a\}} w(\alpha)v(s(\alpha),t)\right)$, (see (7.1) in MZ2),

   and

   b) $d(a,t+1)=(\mathcal{A}_\Gamma T_{\Gamma,w,\theta} \mathcal{W} d(t))(a)$, (see (9.17) in MZ2).

   Equation a) is an abstraction of McCulloch-Pitts dynamics for a model neuron.

2. **Accretion dynamics:** These dynamics are specified by the pair $(\mathcal{S}, 1\cup\mathcal{B}\mathcal{O})$ where $\mathcal{O}$ is a selector (Def. 3.15). With s is as accretion time, the associated evolution equation is
$$x(s+1)=(1\cup\mathcal{B}\,\mathcal{O}\,)x(s),\ \ s\in\mathbf{N}.$$

3. **Metadynamics:** There are four types of metadynamics, a), a'), b, and c) in each of which r is metatime. These are dynamics on *Classes,* on *Selectors,* on the class $\mathcal{S}$ and on the set **L** (Def. 6.18), respectively

   a) Metadynamics(a): $(Classes, 1\cup\mathcal{P})$ (Def. 4.3). The associated evolution equation is
$$\mathcal{A}(r+1)=(1\cup\mathcal{P})\mathcal{A}(r).$$

   a') Metadynamics(a'): $(Selectors, \mathcal{X})$ (Def. 4.29). The associated evolution equation, $\mathcal{O}(r+1)=\mathcal{X}(\mathcal{O}(r))$ is
$$\mathcal{O}(r+1)=\mathcal{O}((1\cup\mathcal{P}\,)\mathcal{C}(\,\mathcal{O}(r))).$$

   b) Metadynamics(b): $(\mathcal{S}, 1\cup\mathcal{P}_f)$ (Def. 5.4). The associated evolution

   equation is
$$A(r+1)=(1\cup\mathcal{P}_f)A(r).$$

   c) Metadynamics(c): $(\mathcal{P}(\mathbf{FP}\cap\mathcal{N}), 1\cup\mathcal{P}_f)$ (Def. 6.19). The associated evolution equation is the same as in b), except $A(r)\subseteq(\mathbf{FP}\cap\mathcal{N})$.

Table 10.1: The hierarchy of dynamical and metadynamical systems

Mental RG Theory	Physical RG Theory
1. Memetic dynamics See (7.1) and (9.17) in MZ2	**1. Differential dynamics**, such as found in classical mechanics
2. Accretion dynamics $(\mathcal{S}, 1 \cup \mathcal{B}\mathcal{O})$ $x(s+1) = x(s) \cup \{\mathcal{O}x(s)\}$ Here s is Platonic time.	**2. Hamiltonian dynamics** (Phase space, Hamiltonian flow) $\frac{dx(s)}{ds} = \{H, x(s)\}_{Poisson}$ Here s is real time.
3. Metadynamics $(\mathit{Selectors}, \mathcal{X})$ where $\mathcal{X}$ is the RG metaoperator $\mathcal{O}(r+1) = \mathcal{X}(\mathcal{O}(r))$	**3. RG dynamics** (Space of Hamiltonians, T) where T is the RG transformation $\mathrm{H}(r+1) = \mathrm{TH}(r)$
Fixed points of $\mathcal{X}$ and, in particular, those that are consciousness operators.	Fixed points of[21] T
$(\mathit{Selectors}, \mathcal{X})$ yields a phase diagram	(Space of Hamiltonians, T) yields a phase diagram
$\mathit{Selectors} = \bigcup_{\mathcal{O}: \mathcal{X}(\mathcal{O}) = \mathcal{O}} \text{Basin}(\mathcal{O})$. a union over RG fixed points $\mathcal{O}$	$\{\text{ Hamiltonians }\} = \bigcup_{H:TH=H} \text{Basin(H)}$, a union over RG fixed points H

Table 10.2: Mental and physical constructs of dynamics and the RG flows

10.2 Semantic interpretations

We recall the translations of set theoretic constructs into semantic language that were exhibited in Tables 3.2, 3.3, 3.4, and 5.1 in MZ1 and in Tables 2.2, 12.1 and 12.2 in MZ2. We continue this practice with the Syntax – Semantics Dictionary shown in Table 10.3. The information displayed in all of these syntax-semantic tables connects the mathematical development both to philosophical issues and to applications pertaining to the mind and consciousness.

[21] See Goldenfeld, 1992 for a discussion of various examples of RG transformations in physi

Syntax	Semantics
Dynamical system (evolution equation)	System expressing change (rule of change)
Fixed point	An element immune to change
Limit point	Ultimate target of a developing trajectory
Basin of attraction, phase	Collection of elements with a common target
Phase diagram	Decomposition of the system into basins
Set	Primary experience, information, awareness
Power set	All experiences subordinate to a primary exper.
Class of all sets $\mathcal{S}$	Collection of all experiences
$\mathcal{S}_{ord}$ (a proper class)	Transfinite timeline
Dynamics of sets	Transfinite succession of experience
Phase of a set	Phase of information
Subclass $\mathcal{C} \subseteq \mathcal{S}$ (Selector $\mathcal{O}(\mathcal{C})$)	A selection of experiences
Selectors	The collection of all selections of experiences: the preconscious mind
RG metaoperator $\mathcal{X}$ (metadyn(a'))	Platonic dynamics of the preconscious mind
RG semigroup, $\{ \mathcal{X}^{\alpha} \| \alpha \in \mathcal{S}_{ord} \}$	Transfinite dynamics of the preconscious mind
$\lim_{\alpha \in \mathcal{S}_{ord}} \mathcal{X}^{\alpha} (\mathcal{O} \cap \mathcal{R})$, transfinite limit of renormalization yielding a consc. op.	Consciousness induced by a selection of experiences
$\mathcal{K}x$	An awareness of a primary experience
$\mathcal{Q}(x)$	Collection of awareness of a primary experience
Metadynamics(b)	A canonical dynamics on experiences
Finitely pictured set	Neural decoration, theme of a meme
FP	Collection of all neuronal decorations (of all memetic themes)
Metadynamics(c)	Dynamics of selections of neural decorations

Table 10.3: The Syntax – Semantics Dictionary

In Table 10.4 we list supplementary semantic interpretations of a sampling of syntactical results.

Supplementary Semantic Interpretations
Sect. 2: Applying an operator transforms an experience to a new experience.
Def. 3.15: An accretive dynamical system generates a hierarchy of experience.
Example 3.16: The ordinals form a hierarchy of experience.
Remark 3.18: Experience (instantiated as a set) is unlimited in its growth capacity.
Fig. 4.1: The trajectory generated by metadynamics(a) constitutes a hierarchy of experience. The corresponding transfinite limit point is an illusion (Fig. 4.1, footnote 9) which itself is an experience.
Def. 4.3: Metadynamics(a) generates a hierarchy of selections of experience.
Remark 4.20: Experience leads to knowledge. Compare Kant, 1781 and footnote 10.
Def. 4.26: Two illusions are psychically equivalent if they correspond to the same phase.
Sect. 7: An awareness field may be generated from a neural state and a consciousness operator.
Def. 8.1: Transfinite renormalization is the syntactic formalization of the semantic process of infinite regress that is taken to characterize the development of consciousness from the unconscious.
Observation 9.2 and Lemma 9.9,h: An awareness of an awareness is an awareness.
Lemma 9.9 a): If x is well-founded, then $\mathscr{Q}(x)$ is the concept of x.
Remark 9.12: An experience and its associated quale are in different realms.
Cor. 9.13: The common aspect of a pair of awarenesses is an awareness.
Prop. 9.14,b: Distinct awarenesses of a common experience give rise to distinct Phases. Equivalently, equal phases give rise to a common awareness.
Prop. 9.14 c): Fix an experience A and a phase B^*. Then for at most one awareness Y of A is the phase of that awareness equal to the phase B^*.
Prop. 9.20, parts 1 and 2: The quale of an experience can be reified in a neural net.
Consciousness Axiom c): Awareness is removed from experience.
Fig. 9.3, Discussion 9.23: Experience generates multiple concepts.
Remark and question 9.22: If x is a thema, can a finite machine compute $\mathscr{Q}$(x) in finite time?

Table 10.4: Semantic interpretations of syntactical results

10.3 Concluding comments

The philosophical study of consciousness contains the construct of the homunculus (Watson, Berry, 2003), a hypothetical mental agency that observes the neuronal activity in the brain and so provides awareness, intentionality, etc. In fact the homunculus is no explanation at all of consciousness, since the question, "Who watches the homunculus? ", gives rise to an infinite regress. Consider this issue of infinite regress in the context of the information processing that the metadynamics of transfinite renormalization manifests. These dynamics produce limit points (fixed points), novel constructs that lead to semantic interpretation based on awareness and concepts of awareness, namely, on the qualia in $\mathscr{Q}(x)$. See Fig. 9.3 also.

The transfinite limit points offer both resolution of the issue of infinite regress, and by producing phases (basins) they offer the view that consciousness is associated to a change of phase of information, namely the information processed in neural networks as modeled by the McCulloch-Pitts dynamics, for example. The renormalization dynamics for metadynamics(b) in the countable realm, with its own limit points and basins, offer a change of phase of information interpretation of consciousness on the countable level. From Prop. 9.14 and the ensuing change of phase remark, we see that given an experience (a set x) and given distinct awarenesses of that experience, the different awarenesses are in different phases. Also if awareness (a set of the form $\mathscr{K}x$) is different from experience, then the phase of awareness is different from the phase of experience.

Do these limit points, basins and phases (of information) have a physical reality or must we consider them to be Platonic constructs only? To some extent this depends upon one's view of consciousness as a physical or as a virtual process. However even in the latter case, the renormalization dynamics/metadynamics produce a portrayal of consciousness by means of a model for which we expect to develop computational addenda in future work (see Remark 9.22). We expect such computation to lead to the reification of the Platonic aspects of the theory.

Appendix

Glossary

Ω – The Quine atom (specified by $\Omega = \{\Omega\}$) (Example 3.14)
$\mathscr{K}\langle y \rangle$ – A consciousness operator depending on a set (an experience) y (Def. 4.1)
$\mathscr{K}\langle y \rangle x$ – Quale (an awareness) associated to a pair of experiences x *and* y (Def. 4.19 and Def. 9.17)
N – The set of natural numbers
H – The set of hereditarily finite sets (Remark 5.6 and H-J[22], Chap. 6)
FP – The set of finitely pictured sets (Def. 6.1)
$f(D)$ – A selector type self map of **FP** (Def. 6.6)
R – A standard set theory map (Def. 6.6)
C(*f*) – A special set induced by the mapping *f* (Def. 6.8)
L – The power set $\mathscr{P}(\mathbf{FP} \cap \mathscr{N})$ (Def. 6.18)
S – $\mathbf{L} \cap \mathscr{F}$ (Def. 5.19 and Def. 6.21)
* and ** – Superscripts indicating a transfinite limit of a class (Def. 4.10) and of a selector (Def. 4.34), resp.
$\sim$ – The superscript $\sim$ indicates a limit point (in the countable realm) of metadynamics(b) (Def. 5.5)

[22] Hereafter the citation Hrbacek, Jech, 1999 will be denoted H-J.

Glossary of operators and classes

$\mathscr{A}$ – Class of abnormal sets, also as a generic symbol for a class (Remark 3.8)
$\mathscr{B}$ – Brace operator - $\mathscr{B}x = \{x\}$ is the set whose only element is x (Def. 3.5)
$\mathscr{C}$ – Generic symbol for a class (Def. 3.2)
$\mathscr{D}$ – Duality operator ($\mathscr{D}\, x = \{\ \mathscr{D}\, x, x\}$) (Fig. 6.2), also a generic symbol for a class
$\mathscr{E}$ – Elimination operator - $\mathscr{E}x = \varnothing$ Prop. 9.3)
$\mathscr{F}$ – Class of fixed points of metadynamics(b) (Def. 5.19)
$\mathscr{G}$ – Class of fixed point of metadynamics(a) (Def. 4.21)
$\mathscr{H}$ – Class of basins of fixed points in $\mathscr{F}$ (Def. 5.19)
$\mathscr{I}$ – Identity operator, more usually written as 1 (Def. 3.15)
$\mathscr{J}$ – Basin of attraction operator for metadynamics(b) (Def. 5.16)
$\mathscr{K}$ – Generic symbol of a consciousness operator (Lemma 3.1)
$\mathscr{L}$ – Limit point operator (Def. 5.5)
$\mathscr{M}$ – Selector associated to $\mathscr{N}$'(Example 3.6). Also an awareness map (Remark 4.20)
$\mathscr{N}$ – Class of normal sets - Sets that do not have themselves as an element (Example 3.6)
$\mathscr{O}$ – Generic symbol for an operator (Def. 2.1)
$\mathscr{P}$ – Power set operator (Def. 4.1 and Footnote 10)
$\mathscr{P}_f$ – Set of finite subsets operator (Def. 5.1)
$\mathscr{Q}$ – Quale set operator (Def. 9.8)
$\mathscr{R}$ – Russell operator - $\mathscr{R}A = \{x \in A \mid x \notin x\}$ (Example 3.6)
$\mathscr{S}$ – Class of sets - Sets satisfying the Zermelo-Frankel-Aczel axioms (Remark 2.4)
$\mathscr{S}_{ord}$ – Class of ordinals (finite and infinite) (Theorem 4.6)
$\mathscr{S}_{wf}$ – Class of well-founded sets (Example 3.6)
$\mathscr{S}_{nwf}$ – Class of non well-founded sets (Aczel, 1988 and Remark 2.4)
$\mathscr{T}$ – Generic dynamics transformation (Footnote 8)
$\mathscr{U}$ – Monadic union operator (Prop. 5.3)
$\mathscr{W}$ – Well-foundedness operator - Takes a set into the subset of its well-founded elements (Example 3.6)
$\mathscr{Y}$ – Structure operator associated to metadynamics(b) (Def. 5.29)
$\mathscr{Z}$ – Generic symbol for a selector (Lemma 2.8)

Glossary of metaoperators

$\mathscr{P}$ – Power class metsoperator (Def. 4.1)
$\mathscr{T}$ – Generic metaoperator on classes (Footnote 6)
$\mathscr{U}$ – Metaoperator on classes (Prop. 5.21and Aczel, 1988)
$\mathscr{V}$ – Basin of attraction metaoperator for metadynamics(a) (Def. 4.23)
$\mathscr{X}$ – Renormalization group metaoperator - Def. 4.29
Γ – Construction metaoperator (Def. 8.2)
Φ – Extension metaoperator (Def. 4.7)
Σ – Phase metaoperator (Def. 8.2)

Consciousness Axioms

	Axiom	Semantic interpretation of the axiom	Name of Axiom
a)	$\forall x$, $\mathcal{K}\, x \subseteq x$	Experience generates its own awareness	Generation
b)	$\forall x$, $x \notin \mathcal{K}\, x$	Awareness does not generate the primary experience[23]	Irreversibility
c)	$\forall x$, $\mathcal{K} x \notin x$	Awareness is removed from experience	Removal
d)	$\forall x, y$, if $x \subseteq y$, then $\mathcal{K}\, x = x \cap \mathcal{K} y$	Awareness of a sub-experience is determined by the sub-experience and awareness of the primary experience	Selection

Table A1: Axioms for a consciousness operator

Proofs

Sect. 2

Lemma 2.6: First part: Follows from Def. 2.3 and Def. 2.5. Second part: By Def. 2.3, $\exists$ classes $\mathcal{A}_1$ and $\mathcal{A}_2$, such that $\forall x$, $\mathcal{O}_1\, x = \mathcal{A}_1 \cap x$ and $\mathcal{O}_2\, x = \mathcal{A}_2 \cap x$. Thus $\mathcal{O}_1 \mathcal{O}_2 x = \mathcal{A}_1 \cap (\mathcal{A}_2 \cap x) = (\mathcal{A}_1 \cap \mathcal{A}_2) \cap x = \mathcal{A}_2 \cap (\mathcal{A}_1 \cap x) = \mathcal{O}_2 \mathcal{O}_1\, x$. Next, $(\mathcal{O}_1 \cap \mathcal{O}_2)x = \mathcal{O}_1\, x \cap \mathcal{O}_2\, x = (\mathcal{A}_1 \cap x) \cap (\mathcal{A}_2 \cap x) = (\mathcal{A}_1 \cap \mathcal{A}_2) \cap x = \mathcal{O}_1 \mathcal{O}_2\, x$.

Proposition 2.7: Lemma 2.6 establishes the semi-group property and the commutativity. The rest is straight-forward.

Lemma 2.8: First part: Since $\mathcal{Z}$ is a selector, $\mathcal{Z} x = \mathcal{C}(\mathcal{Z}) \cap x$ (Lemma 2.6). This gives $\mathcal{Z} y = \mathcal{C}(\mathcal{Z}) \cap y = \mathcal{O}(\mathcal{C}(\mathcal{Z}))y$, the last by Def. 2.5. Therefore $\mathcal{Z} = \mathcal{O}(\mathcal{C}(\mathcal{Z}))$. Second part: $\forall x$, $\mathcal{O}(\mathcal{D})\, x = \mathcal{D} \cap x$ (Def. 2.5). Then $\mathcal{C}(\mathcal{O}(\mathcal{D})) = \{y \mid \mathcal{O}(\mathcal{D})\mathcal{B}\, y = \mathcal{B}\, y\} = \mathcal{D}$.

[23] This axiom characterizes the feature of the mind, that for example, one can think about music but can not hear it without that music being presented by an external source. That is, music can not be heard without music as an experience being present.

Sect. 3

Lemma 3.1: Suppose to the contrary that $\exists A \subseteq \mathcal{C}(\mathcal{K})$ such that $A \notin \mathcal{C}(\mathcal{K})$. Then by definition of $\mathcal{C}(\mathcal{K})$, we have $\mathcal{K}\mathcal{B}A \neq \mathcal{B}A = \{A\}$. Then (i) $\mathcal{K}\mathcal{B}A = \varnothing$, since $\mathcal{K}$ is a selector. Now let $B = A \cup \mathcal{B}A$. Then $\mathcal{K}B = \mathcal{K}A \cup \mathcal{K}\mathcal{B}A$, also since $\mathcal{K}$ is a selector. Then using (i), $\mathcal{K}B = \mathcal{K}A$. Now since $A \subseteq \mathcal{C}(\mathcal{K})$, we have $\mathcal{K}A = A$. Combining we conclude that (ii) $\mathcal{K}B = A$. Combining (ii) with $A \in B$, the latter following from the definition of B, we conclude that $\mathcal{K}B \in B$. This contradicts the property, $\forall x,\ \mathcal{K}x \notin x$, of the consciousness operator $\mathcal{K}$.

Lemma 3.4: Part a follows from Def. 3.3. Part b follows from Def. 3.2 and Def. 3.3. Part c) If $\mathcal{D} \subseteq \mathcal{C}$, then $\mathcal{D}' \subseteq \mathcal{C}'$ by Def. 3.3. If $\mathcal{C}$ is perfect, then $\mathcal{C}' \subseteq \mathcal{C}$, and hence $(\mathcal{C}')' \subseteq \mathcal{C}'$. Therefore $\mathcal{C}'$ is perfect.

Lemma 3.5: Part a): Suppose $\mathcal{C} = \boldsymbol{D}$ for some non empty set D. Since $\mathcal{C}$ is perfect by hypothesis, $\mathcal{P}D \subseteq D$. This contradicts Cantor's theorem, which asserts that $|\mathcal{P}D| > |D|$. Part b): Given $A \subseteq C_1 \cap C_2$, we have that A is a subset of both C_1 and C_2. The latter two sets being perfect implies that A is an element of each of them and so also of $C_1 \cap C_2$.

Examples 3.6: 1. See Def. 2.2. 2. Suppose $A \subseteq \mathcal{N}$. To show that $A \in \mathcal{N}$, suppose to the contrary that $A \in A$. Then A is an abnormal element of A, hence $A \not\subset \mathcal{N}$. Thus $A \notin A$, and so $A \in \mathcal{N}$. 3. Let $A \subseteq \mathcal{S}_{wf}$ and suppose to the contrary that $A \notin \mathcal{S}_{wf}$ Then there exists an infinite descent $A \ni x_1 \ni x_2 \ni \cdots \ni x_n \ni \cdots$, where each $x_i \in \mathcal{S}$. But then x_1 is non-well-founded. However, $x_1 \in A \subseteq \mathcal{S}_{wf}$, a contradiction. 4. $\mathcal{N}'$ is perfect by Lemma 3.4,c. Next we note that $\mathcal{S}_{wf} \subset \mathcal{N}' \subset \mathcal{N} \subset \mathcal{S}$. We exhibit specific sets that demonstrate these proper inclusions. $\{\varnothing,\Omega\} \in \mathcal{N}$, but $\{\varnothing,\Omega\} \notin \mathcal{N}'$. So $\mathcal{N}' \subset \mathcal{N}$. $\Omega \in \mathcal{S}$, but $\Omega \in \mathcal{N}$. So $\mathcal{N} \subset \mathcal{S}$. $\{\{\varnothing,\Omega\}\} \in \mathcal{N}'$, but $\{\{\varnothing,\Omega\}\} \notin \mathcal{S}_{wf}$. So $\mathcal{S}_{wf} \subset \mathcal{N}'$.

Lemma 3.7: Let $\mathcal{C}$ be a perfect class. Then with $\varnothing \subseteq \mathcal{C}$, we have $\varnothing \in \mathcal{C}$. Then $\varnothing^* \subseteq \mathcal{C}$. Theorem 4.14 implies that $\mathcal{C}^* = \mathcal{C}$ since $\mathcal{C}$ is perfect. Then $\varnothing^* \in \mathcal{C}$. Then appealing to Example 4.13,a, we conclude that $\mathcal{S}_{wf} \subseteq \mathcal{C}$.

Remark 3.8: First part: A specific set w where $w \in \mathcal{A}$, but $w \notin w$, is $w = \{\ \mathcal{D}0, \mathcal{D}1\}$. Second part: $\mathbf{N} \notin \mathbf{N}$, since $\mathbf{N}$ is finite but $\mathbf{N}$ itself is not finite.

Lemma 3.9: Suppose $y \in \mathcal{K}x$. Then (i) $\mathcal{K}\mathcal{B}y = \mathcal{B}y$ (using Def. 2.5, namely, ii) $\forall x, \mathcal{K}x = \{y \in x \mid \mathcal{K}\mathcal{B}\ y = \mathcal{B}y\}$ along with Lemma 2.6). Now suppose to the contrary that $y \in y$. Then $\mathcal{B}y \subseteq y$. This gives $\mathcal{K}\mathcal{B}y \subseteq \mathcal{K}y$, since $\mathcal{K}$ is a selector. Combining this with (i) gives $\mathcal{B}y \subseteq \mathcal{K}y$, from which we get $y \in \mathcal{K}y$. This contradicts consciousness operators axiom b). Then we have $y \notin y$, which when combined with $y \in x$ (the latter following from (ii)) gives $y \in \mathscr{R}x$.

Theorem 3.11: First part: To show $\mathscr{K}(\mathscr{C})$ is a consciousness operator, we check the consciousness axioms a, b, c and d.
Axiom d: (Note axiom d implies axiom a.) $\mathscr{K}(\mathscr{C})$ is a selector by construction.
Axiom b): $C \subseteq \mathscr{N} \Rightarrow$ (i) $\mathscr{K}(\mathscr{C}) \subseteq \mathscr{K}(\mathscr{N}) = \mathscr{R}$, the last since $\mathscr{R}x = \mathscr{N} \cap x$ (Def. 2.5). Now $x \notin \mathscr{R}x$ (MZ1). Then using (i), we conclude that $x \notin \mathscr{K}(\mathscr{C})x$.
Axiom c: To show: $\forall x, \mathscr{K}(\mathscr{C})x \notin x$, equivalently (Def. 2.5) that $\mathscr{C} \cap x \notin x$. (Note that $\mathscr{C} \cap x$ is a set by the Axiom of Comprehension.) Since $\mathscr{C}$ is perfect, (ii) $\mathscr{C} \cap x \in \mathscr{C}$. Now suppose to the contrary that for some set A, (iii) $\mathscr{C} \cap A \in A$. Then combining (ii) and (iii) gives $(\mathscr{C} \cap A) \in \mathscr{C} \cap A$. However $\mathscr{C} \cap A \subseteq \mathscr{N}$, since we have assumed that $\mathscr{C} \subseteq \mathscr{N}$. Then $\mathscr{C} \cap A \in \mathscr{N}$, since $\mathscr{N}$ is perfect (Example 3.6, 2). Then from the definition of $\mathscr{N}$, we obtain $\mathscr{C} \cap A \notin \mathscr{C} \cap A$, a contradiction.
Second part: Since $\mathscr{K}$ is a consciousness operator, $\mathscr{C}(\mathscr{K})$ is perfect (Lemma 3.1). Moreover, $\mathscr{K} \subseteq \mathscr{R}$ (Lemma 3.9). Hence $\mathscr{C}(\mathscr{K}) \subseteq \mathscr{C}(\mathscr{R}) = \mathscr{N}$.

Corollary 3.12: Since each $\mathscr{K}_i$, $i = 1,2$ is a consciousness operator, each $\mathscr{C}(\mathscr{K}_i)$, $i = 1,2$ is a perfect class (Lemma 3.5). Moreover, (i) $\mathscr{C}(\mathscr{K}_i) \subseteq \mathscr{N}$, $i = 1,2$. Now $\mathscr{K}_1\mathscr{K}_2 = (\mathscr{C}(\mathscr{K}_1) \cap \mathscr{C}(\mathscr{K}_2)) \cap x = (\mathscr{C}(\mathscr{K}_1) \cap \mathscr{C}(\mathscr{K}_2)) \cap x$. Now $\mathscr{C}(\mathscr{K}_1) \cap \mathscr{C}(\mathscr{K}_2)$ is a perfect class (Lemma 3.5). Moreover (i) implies $\mathscr{C}(\mathscr{K}_1) \cap \mathscr{C}(\mathscr{K}_2) \subseteq \mathscr{N}$. Then $\mathscr{K}_1\mathscr{K}_2$ is a consciousness operator (Theorem 3.11).

Proposition 3.13: If $\mathscr{K}$ is a consciousness operator, $\mathscr{W} \subseteq \mathscr{K} \subseteq \mathscr{R}$ (Example 3.6, Lemma 3.7 and Lemma 3.9). This and Cor. 3.12 give $\mathscr{R}\mathscr{K} = \mathscr{K}\mathscr{R} = \mathscr{K} \cap \mathscr{R} = \mathscr{K}$ and $\mathscr{W}\mathscr{K} = \mathscr{K}\mathscr{W} = \mathscr{K} \cap \mathscr{W} = \mathscr{W}$.

Example 3.14: Note that $\{\varnothing,\Omega\} \in \mathscr{N} - \mathscr{S}_{wf}$, so that $\mathscr{S}_{wf} \subset \mathscr{D} \subset \mathscr{N}$. Then $\mathscr{O}(\mathscr{S}_{wf}) \subset \mathscr{O}(\mathscr{D}) \subset \mathscr{O}(\mathscr{N})$. From this $\mathscr{W} \subset \mathscr{O}(\mathscr{D}) \subset \mathscr{R}$ follows (Example 3.6 *ff*). We shall show that $\mathscr{D}$ is not perfect, demonstrating that $\mathscr{O}(\mathscr{D})$ is not a consciousness operator. Then take $w = \{\{\varnothing,\Omega\}\}$. w is neither an element of $\mathscr{S}_{wf}$ nor of itself. Then $w \notin \mathscr{D}$. However $w \subseteq \mathscr{D}$.

Remark 3.16: Suppose A is a fixed point of $1 \cup \mathscr{B}$. That is $(1 \cup \mathscr{B})A = A$. Then $A = A \cup \mathscr{B}A$, which implies that $A \subseteq \mathscr{B}A$. Then either $A = \mathscr{B}A$, in which case $A = \Omega$. or $A = \varnothing$. However $(1 \cup \mathscr{B})\varnothing \neq \varnothing$, so that the only fixed point of $1 \cup \mathscr{B}$ is Ω.

Sect. 4

Remark 4.4: From Lemma 3.4, Def. 4.1 and Def. 4.2, we have $\mathscr{C}$ is perfect $\Leftrightarrow \mathscr{C}' \subset \mathscr{C}$ $\Leftrightarrow \mathscr{C} \cup \mathscr{C}' = \mathscr{C} \Leftrightarrow (1 \cup \mathscr{P})\mathscr{C} = \mathscr{C}$.

Theorem 4.5: Referring to Remark 4.4, Theorem 4.5 is a restatement of Theorem 3.11.

Theorem 4.6: See Moschovakis, 2005.

Lemma 4.8: Part a: Since $\mathscr{A}$ is a class, $\Phi_\alpha \mathscr{A}$ is a union of a class of sets (Def. 4.7) and hence is a class. Thus Φ_α is a metaoperator. Since $\mathscr{A} \subseteq \mathscr{B} \Rightarrow \mathscr{P}\mathscr{A} \subseteq \mathscr{P}\mathscr{B}$, we have $\Phi_\alpha \mathscr{A} \subseteq \Phi_\alpha \mathscr{B}$. (Def. 4.7).

Part b: Note $\forall \alpha \in \mathcal{S}_{ord}$, $(1 \cup \mathscr{P})^\alpha$ is isotone. Then $\forall x \in \mathcal{S}$, $\Phi_\alpha x = \bigcup_{w \in \mathscr{P}x} (1 \cup \mathscr{P})^\alpha w$ $= \bigcup_{w \subseteq x} (1 \cup \mathscr{P})^\alpha w = (1 \cup \mathscr{P})^\alpha x$. The first equality following from Def. 4.7 and the second from the isotonicity of $1 \cup \mathscr{P}^\alpha$.

Part c: $\Phi_\beta = \bigcup_{w \in \mathscr{P}\mathscr{A}} (1 \cup \mathscr{P})^\beta w = \bigcup_{w \in \mathscr{P}\mathscr{A}} \bigcup_{\alpha < \beta} (1 \cup \mathscr{P})^\alpha w =$ $\bigcup_{\alpha<\beta} \bigcup_{w \in \mathscr{P}\mathscr{A}} (1 \cup \mathscr{P})^\alpha w = \bigcup_{\alpha<\beta} \Phi_\alpha$. The first equality follows from Def. 4.7, the second from Thm. 4.6 and the second axiom of transfinite induction and the last from Def. 4.7.

Part d: Using Def. 4.7 and part b) of Lemma 4.8, we have $\Phi_\alpha = \bigcup_{w \in \mathscr{P}\mathscr{A}} \Phi_\alpha(w)$. Next $(1 \cup \mathscr{P})\Phi_\alpha \mathscr{A} = \bigcup_{w \in \mathscr{P}\mathscr{A}} (1 \cup \mathscr{P})\Phi_\alpha w = \bigcup_{w \in \mathscr{P}\mathscr{A}} (1 \cup \mathscr{P})(1 \cup \mathscr{P})^\alpha w = \bigcup_{w \in \mathscr{P}\mathscr{A}} (1 \cup \mathscr{P})^{\alpha+1} w$ $= \Phi_{\alpha+1} \mathscr{A}$. The first equality follows from a result of Aczel, 1988, Chap. 6, the second from Part b) of Lemma 4.8, the third from Theorem 4.6 and the last from Def. 4.7.

Proposition 4.11: (i) $\mathscr{A}^* = \bigcup_{\alpha \in \mathcal{S}_{ord}} \bigcup_{w \in \mathscr{P}\mathscr{Q}} (1 \cup \mathscr{P})^\alpha w = \bigcup_{(\alpha, w) \in \mathcal{S}_{ord} \times \mathscr{P}\mathscr{A}} (1 \cup \mathscr{P})^\alpha w$.

The first equality follows from Def. 4.7, Def. 4.9 and Def. 4.10, and the second by taking the Cartesian product of classes. The last term in (i) is a union of classes, hence a class.

Lemma 4.12: Follows from Def. 4.9, Def. 4.10 and Lemma 4.8,a.

Example 4.13: Part a: H-J, Chap.14. Part b: Follows from Part a: since $\varnothing \subseteq \mathcal{S}_{ord} \subseteq \mathcal{S}_{wf}$.

Theorem 4.14: Part 1:It suffices to show that $\mathscr{P}(\mathscr{A}^*) \subseteq \mathscr{A}^*$. Suppose $y \in \mathscr{P}(\mathscr{A}^*)$, that is, $y \subseteq \mathscr{A}^*$. Then $\forall x \in y$, $x \in \mathscr{A}^*$. Then by definition of $\mathscr{A}^*$, $\exists \beta$ such that $x \in (1 \cup \mathscr{P})^\beta \mathscr{A}$. Let $\beta(x)$ be the least ordinal α such that $x \in (1 \cup \mathscr{P})^\alpha \mathscr{A}$. Then $x \in (1 \cup \mathscr{P})^{\beta(x)} \mathscr{A}$. Let $\gamma = \bigcup_{x \in y} \beta(x)$, so $\gamma = \sup_{x \in y} \beta(x) \in \mathcal{S}_{ord}$. Now $\forall x \in y, x \in (1 \cup \mathscr{P})^\gamma \mathscr{A}$ so $y \subseteq (1 \cup \mathscr{P})^\gamma \mathscr{A}$. Hence $y \in \mathscr{P}(1 \cup \mathscr{P})^\gamma \mathscr{A} \subseteq (1 \cup \mathscr{P})\,(1 \cup \mathscr{P})^\gamma \mathscr{A}$ $= (1 \cup \mathscr{P})^{\gamma+1} \mathscr{A} \subseteq \mathscr{A}$.

Part 2: Follows from Def. 4.10, since if $1 \cup \mathscr{P}$ fixes $\mathscr{A}$, any power of $1 \cup \mathscr{P}$ fixes $\mathscr{A}$.
Part 3: See Remark 4.4.

Corollary 4.15: This follows from Part 1 of Theorem 4.14.

Corollary 4.16: First part: If $\mathscr{O}$ is any selector, then $\mathscr{C}(\mathscr{O})$ is a class and by Theorem 4.14, $\mathscr{C}^*(\mathscr{O})$ is a perfect class. From Example 3.6, 2, we know that $\mathscr{N}$ is a perfect class. So by Lemma 3.5b, $\mathscr{C}^*(\mathscr{O}) \cap \mathscr{N}$ is a perfect subclass of $\mathscr{N}$. Then $\mathscr{K}(\mathscr{C}(\mathscr{O}) \cap \mathscr{N})$ is consciousness operator by Theorem 3.11.
Second part: Suppose $\mathscr{O}$ is a consciousness operator, then by Lemma 3.1, $\mathscr{C}(\mathscr{O})$ is a perfect class, and by Lemma 3.8, $\mathscr{O} \subseteq \mathscr{R}$. Then by Theorem 4.14, $\mathscr{C}^*(\mathscr{O}) = \mathscr{C}(\mathscr{O})$, and Def. 2.5, $\mathscr{C}(\mathscr{O}) \subseteq \mathscr{N}$. Then $\mathscr{K}(\mathscr{C}^*(\mathscr{O}) \cap \mathscr{N}) = \mathscr{K}(\mathscr{C}(\mathscr{O}) \cap \mathscr{N}) = \mathscr{K}(\mathscr{C}(\mathscr{O})) = \mathscr{O}$ by Lemma 2.8.

Corollary 4.22: If $\mathscr{A}$ is a class, then $\mathscr{A}^*$ is a fixed point of metadynamics(a) (Theorem 4.14), and $\mathscr{A}$ is in the basin of attraction of $\mathscr{A}^*$(Def. 4.21). Since $\mathscr{A} \to \mathscr{A}^*$ is a well defined metaoperator, the union is disjoint.

Question 4.24: The necessary condition follows from Theorem 4.14, since $\mathscr{V}(\mathscr{A}) \neq \varnothing$ implies there exists a $z \in \mathscr{S}$ such that $z^* = \mathscr{C}$.

Lemma 4.25: If $z \in \mathscr{V}(W^*)$, then $z^* = W^*$ by Def. 4.23 of $\mathscr{V}$. Hence $z \subseteq W^*$. However W^* is perfect, hence $z \in W^*$. Then $\mathscr{V}(W^*) \subseteq W^*$. If $z \in \mathscr{V}(W^*)$, then the transfinite trajectory of z is a subclass of $\mathscr{V}(W^*)$, and this subclass being indexed by $\mathscr{S}_{ord}$ is proper.

Remark 4.27: Part 1: Follows from the Def. 4.23 for $\mathscr{V}$ and Def. 4.10 for *. Part 2: See Example 4.23,a. Part 3: If $\varnothing^* \subset W^*$, then $\mathscr{V}(\varnothing^*) \cap \mathscr{V}(W^*) = \varnothing$. Hence $\mathscr{V}(W^*) \subset W^*$.

Proposition 4.28: Use Def. 4.21 and Def. 4.23.

Lemma 4.30: Part a: Follows from Fig. 4.2. b) follows from Def. 4.29 and Lemma 2.8.

Corollary 4.32: Follows from Theorem 4.5 and Lemma 4.30.

Lemma 4.35: First note that $\mathscr{C} \subseteq \mathscr{D} \Rightarrow (1 \cup \mathscr{P})\, \mathscr{C} \subseteq (1 \cup \mathscr{P})\, \mathscr{D}$. Then use transfinite induction to establish that $(1 \cup \mathscr{P})^{\alpha}\, \mathscr{C} \subseteq (1 \cup \mathscr{P})^{\alpha}\, \mathscr{D},\ \forall \alpha \in \mathscr{S}_{ord}$. Finally take the transfinite union to complete the proof.

Example 4.36: $\mathscr{O} \subseteq \mathscr{W} \Rightarrow \mathscr{O}^{**} \subseteq \mathscr{W}^{**}$ (Lemma 4.35). (i) $\mathscr{C}^*(\mathscr{W}) = \mathscr{S}^*_{wf} = \mathscr{S}_{wf}$ (Example 3.6). (ii) $\mathscr{W}^{**} = \mathscr{O}(\mathscr{C}^*(\mathscr{W})) = \mathscr{O}(\mathscr{S}_{wf}) = \mathscr{W}$ (the first by Lemma 4.35 and the last by Example 3.6). Then (i) and (ii) give (iii) $\mathcal{O}^{**} \subseteq \mathcal{W}$. Then $\mathcal{O}^{**} = \mathcal{O}(\mathcal{C}^*(\mathcal{O}))$ (Lemma 4.35). However $\mathcal{C}^*(\mathcal{O})$ is perfect (Theorem 4.14). Hence $\mathcal{S}_{wf} \subseteq \mathcal{C}^*(\mathcal{O})$ (Lemma 3.7). Therefore (iv) $\mathcal{O}(\mathcal{S}_{wf}) \subseteq \mathcal{O}(\mathcal{C}^*(\mathcal{O}))$ by the isotonicity of $\mathcal{O}$ (Def 2.5). Using Example 3.6 and Lemma 4.35, we can write (iv) as (v) $\mathcal{W} \subseteq \mathcal{O}^{**}$. Finally (iii) and (v) yield $\mathcal{O}^{**} = \mathcal{W}$.

Proposition 4.38: Part a: (i) $\mathcal{Z}^{**}= \mathcal{O}(\mathcal{C}^*(\mathcal{Z}))$ (from Lemma 4.35), and from Def. 4.29, (ii) $\mathcal{X}(\mathcal{Z}^{**}) = \mathcal{O}((1\cup\mathcal{P})\mathcal{C}(\mathcal{Z}^{**}))$. Applying $\mathcal{C}$ to (i) gives
(iii) $\mathcal{C}(\mathcal{Z}^{**}) = \mathcal{C}(\mathcal{O}(\mathcal{C}^*(\mathcal{Z}))) = \mathcal{C}^*(\mathcal{Z})$), the latter by Lemma 2.8. Now combining (ii) and (iii) gives (iv) $\mathcal{X}(\mathcal{Z}^{**}) = \mathcal{O}((1\cup\mathcal{P})\mathcal{C}^*(\mathcal{Z})) = \mathcal{O}(\mathcal{C}^*(\mathcal{Z}))$, the last by Theorem 4.14. Finally (i) and (iv) give $\mathcal{X}(\mathcal{Z}^{**}) = \mathcal{Z}^{**}$.
Part b: Combining the hypothesis and Def. 4.29, we have (i) $\mathcal{Z}= \mathcal{O}((1\cup\mathcal{P})\mathcal{C}(\mathcal{Z}))$. Now applying to (i) and using Lemma 2.8, we get $\mathcal{C}(\mathcal{Z}) = ((1\cup\mathcal{P})\mathcal{C}(\mathcal{Z}))$. Hence by Theorem 4.14, $\mathcal{C}(\mathcal{Z})^* = \mathcal{C}^*(\mathcal{Z}) = \mathcal{C}(\mathcal{Z})$. Then $\mathcal{O}(\mathcal{C}^*(\mathcal{Z})) = \mathcal{Z}$ by Lemma 2.8. Finally we conclude that $\mathcal{Z}^{**}= \mathcal{Z}$ by Lemma 4.35.
Part c: From Def. 4.34, $(\mathcal{X}^{\beta}(\mathcal{Z}))^{**} = \bigcup_{\alpha\in\mathcal{S}_{ord}} \mathcal{X}^{\alpha}(\mathcal{X}^{\beta}(\mathcal{Z})) = \bigcup_{\alpha\in\mathcal{S}_{ord}} \mathcal{X}^{\beta+\alpha}(\mathcal{Z})$,
the last from axiom 1 of transfinite induction (Sect. 4.1). Then
$\mathcal{X}^{\beta}(\mathcal{Z}))^{**} = \bigcup_{\gamma\geq\beta,\ \gamma\in\mathcal{S}_{ord}} \mathcal{X}^{\gamma}(\mathcal{Z}) = \bigcup_{\gamma\in\mathcal{S}_{ord}} \mathcal{X}^{\gamma}(\mathcal{Z}) = \mathcal{Z}^{**}$, the penultimate equality follows from a generalization of Lemma 4.30,a.

Corollary 4.39: Referring to Example 4.31, we see that there is more than one fixed point, and so, there is more than one basin. The result follows since a transfinite trajectory is limited to one basin (Prop. 4.28,c).

Corollary 4.40: Using Prop. 4.38, $\mathcal{X}((\mathcal{O}\cap\mathcal{R})^{**}) = (\mathcal{O}\cap\mathcal{R})^{**}$. Now using Lemma 4.35, we have $\mathcal{O}\subseteq\mathcal{R} \Rightarrow (\mathcal{O}\cap\mathcal{R})^{**} \subseteq \mathcal{R}^{**}=\mathcal{R}$, the last by Example 4.31 and Prop. 4.38. Then $(\mathcal{O}\cap\mathcal{R})^{**}$ is a consciousness operator, by Cor. 4.32.

Proposition 4.41: Part a: If $\mathcal{Z}$ is a selector, then $\mathcal{Z}^{**}$ is a fixed point of $\mathcal{X}$ (Prop. 4.38). Thus $\mathcal{Z}$ is in the basin of attraction if a fixed point $\mathcal{O}= \mathcal{Z}^{**}$.
Part b: Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are fixed points of $\mathcal{X}$, and suppose $\exists\,\mathcal{Z}$ such that $\mathcal{Z}$ is in the basins of both $\mathcal{C}_1$ and $\mathcal{O}_2$. Then $\mathcal{Z}^{**} = \mathcal{O}_1$, and $\mathcal{Z}^{**} = \mathcal{O}_2$. Hence $\mathcal{O}_1 = \mathcal{O}_2$.

Sect. 5

Remark 5.3: Part a: If D is finite, then $|\mathcal{P}_f D| > |D|$ in which case $\mathcal{P}_f D \subseteq D$.
Part b: Follows from the equation $\mathcal{U}\mathcal{P}_f = 1$.

Remark 5.6: $A \supseteq B \Rightarrow (1\cup\mathcal{P}_f)A \supseteq (1\cup\mathcal{P}_f)B$, using Def. 5.1. Then $\forall n$, $(1\cup\mathcal{P}_f)^n A \supseteq (1\cup\mathcal{P}_f)^n B$, so that $\tilde{A} \supseteq \tilde{B}$ by Def. 5.5.

Lemma 5.7: Let $w \in \mathscr{P}_f \tilde{A}$. Then for some $r \in \mathbf{N}$, $w = \{x_1, \cdots, x_r\}$, where each $x_i \in \tilde{A}$. From Def. 5.5 for $\tilde{A}$, we see that $\forall i,\ \exists k(i) \in \mathbf{N}$ such that $x_i \in (1 \cup \mathscr{P}_f)^{k(i)} A$. Setting $j = \sup_{1 \le i \le r} k(i)$, we have $\forall i,\ x_i \in (1 \cup \mathscr{P}_f)^j A$. Hence $w \subseteq (1 \cup \mathscr{P}_f)^j A$. Then since w is finite, we have $w \in \mathscr{P}_f (1 \cup \mathscr{P}_f)^j A$. Then $w \in (1 \cup \mathscr{P}_f)^{j+1} A \subseteq \tilde{A}$. Therefore $w \in \tilde{A}$.

Remark 5.8: Follows from Lemma 5.7.

Lemma 5.10: $(1 \cup \mathscr{P}_f) A = A \Rightarrow \forall n,\ (1 \cup \mathscr{P}_f)^n A = A$. Then $\tilde{A} = \bigcup_{n \in \mathbf{N}} (1 \cup \mathscr{P}_f)^n A = A$ (Def. 5.5).

Example 5.11: Part a: Lemma 5.5 implies that $\tilde{\varnothing}$ is a fixed point. Part b: Let $w = \{x_1, \cdots, x_r\}$, $r \in \mathbf{N}$ where each $x_i \in \mathbf{FP}$. Then by Def. 6.1 of **FP**, w has a finite picture. c) Let $w = \{x_1, \cdots, x_r\}$, where each $x_i \in \mathbf{FP} \cap \mathscr{N}$. Then $w \in \mathbf{FP}$. Now $\forall i, x_i \in \mathscr{N} \Rightarrow x \in \mathscr{N}$, because $\mathscr{N}$ is perfect (Example 3.6,2). Therefore $x \in \mathbf{FP} \cap \mathscr{N}$.

Proposition 5.12: Part a: follows from Lemma 5.7. Part b: follows from Def. 5.9 and Lemma 5.7. Part c: follows from Def. 5.2 and Def. 5.9. Part d: Part b gives $(1 \cup \mathscr{P}_f)\tilde{A} = \tilde{A}$. Then by induction we can show that $(1 \cup \mathscr{P}_f)^n \tilde{A} = \tilde{A},\ \forall n$. Therefore $\tilde{\tilde{A}} = \bigcup_{n \in \mathbf{N}} (1 \cup \mathscr{P}_f)^n \tilde{A} = \tilde{A}$.

Interpretation 5.13: Follows from Def. 5.5 and Prop. 5.12,b.

Proposition 5.15: First assertion: Suppose $\tilde{A} = F$. Then $A \subseteq F$ since $A \subseteq \tilde{A}$. Moreover $\{A | \tilde{A} = F\} = \{A \subseteq F | \tilde{A} = F\}$, the latter being a set by the axiom of comprehension. Second assertion: In fact we shall show that $\exists x \in F$ such that if $B = F - \{x\}$, then $\tilde{B} = F$. Suppose not, then $\forall x \in F,\ (F - \{x\})^{\sim} = (F - \{x\})$. Then Prop 5.12,a gives $\forall x \in F$, $\mathscr{P}_f(F - \{x\}) \subseteq (F - \{x\})$. Then (i) $\bigcap_{x \in F} \mathscr{P}_f(F - \{x\}) \subseteq \bigcap_{x \in F} (F - \{x\})$ (H-J, Chap. 2.) Also (ii) $\mathscr{P}_f\left(\bigcap_{x \in F}(F - \{x\})\right) = \bigcap_{x \in F} \mathscr{P}_f(F - \{x\})$ by the definition of $\mathscr{P}_f$. Now inserting the relation $\bigcap_{x \in F} \mathscr{P}_f(F - \{x\}) = \varnothing$ into (i) and (ii) gives $\mathscr{P}_f \varnothing \subseteq \varnothing$, a contradiction.

Remark 5.16: Part a: $\mathscr{J}(Y) \neq \varnothing \Rightarrow \exists$ a set A such that (i) $\tilde{A} = Y$. Then using Prop. 5.12, we have (ii) $(1 \cup \mathscr{P}_f) Y = (1 \cup \mathscr{P}_f)\tilde{A} = \tilde{A}$. Combinig (i) and (ii) gives

$(1\cup\mathscr{P}_f)Y=Y$. Part b: $\mathscr{J}(Y)\subseteq\mathscr{P}Y$ follows from Remark 5.6,c. This is a restatement of Prop. 5.15, since $\tilde{A}\neq A$ implies that A is not a fixed point.

Proposition 5.21: Let (i) $W=\mathscr{J}(F)$ for some $F\in\mathscr{F}$. Then by the definition of $\mathscr{J}$ (Def. 5.16), (ii) $W\cap\mathscr{F}=\{F\}$. Then applying $\mathscr{U}$ to (ii), we get $\mathscr{U}(W\cap\mathscr{F})=F$, which we write as (iii) $F=\mathscr{U}\mathscr{O}(\mathscr{F})W$ (Def. 2.5). Hence the relation (i) is invertible, and so, (iii), gives $\mathscr{J}^{-1}=\mathscr{U}\mathscr{O}(\mathscr{F})$.

Corollary 5.22: Follows directly from Prop. 5.12,b and Lemma 5.10.

Proposition 5.23: If $A\in\mathscr{S}$, then $A\in\mathscr{J}(\tilde{A})$ (Def. 5.16). To show that $F_1\neq F_2\Rightarrow\mathscr{J}(F_1)\cap\mathscr{J}(F_2)=\varnothing$, suppose $A\in\mathscr{J}(F_1)\cap\mathscr{J}(F_2)$. Then $\tilde{A}=F_1$ and $\tilde{A}=F_2$. Since $\tilde{A}$ is well defined, we conclude that $F_1=F_2$.

Corollary 5.25: The identity of the classes follows from Prop. 5.12 and Lemma 5.10. To show that $\mathscr{F}$ is a proper class. Suppose to the contrary that $\mathscr{F}=I$ for some set I. Then by Prop. 5.23, $\mathscr{S}=\bigcup_{F\in I}\mathscr{J}(F)=\bigcup_{F\in I}\{A\in\mathscr{S}\,|\,\tilde{A}=F\}$. Now by the axiom of replacement (H-J, Chap. 6), there exists a function g such that $\operatorname{dom} g=I$ and such that $\forall F\in I$, $g(F)=\{A\in\mathscr{S}\,|\,\tilde{A}=F\}$. Then $\mathscr{S}=\bigcup_{F\in I}g(F)$, which is a set (H-J, Chap. 2), a contradiction.

Proposition 5.26: Part i: Follows directly from Def. 5.5. Part ii: Apply Prop. 5.12 to $\mathscr{L}A=\tilde{A}$. Part iii: $\mathscr{L}A=\sup_{n\in\mathbf{N}}(1\cup\mathscr{P}_f)^nA$ (Def. 5.5). (The supremum over a collection of sets is specified in Def. 9.15.) Then $\mathscr{L}(1\cup\mathscr{P}_f)A=\sup_{n\in\mathbf{N}}(1\cup\mathscr{P}_f)^n(1\cup\mathscr{P}_f)A$ $=\sup_{n\in\mathbf{N}}(1\cup\mathscr{P}_f)^{n+1}A$. Now since $(1\cup\mathscr{P}_f)B\supseteq B,\forall B$, then
$(1\cup\mathscr{P}_f)A=A\cup\sup_{n\in\mathbf{N}}(1\cup\mathscr{P}_f)^{n+1}A=\sup_{m\in\mathbf{N}}(1\cup\mathscr{P}_f)^mA=\mathscr{L}A$.
Part iv: $A\subseteq B\Rightarrow(1\cup\mathscr{P}_f)A\subseteq(1\cup\mathscr{P}_f)B$. Then by induction, $\forall n,(1\cup\mathscr{P}_f)^nA\subseteq(1\cup\mathscr{P}_f)^nB$. Then $\sup_{n\in\mathbf{N}}(1\cup\mathscr{P}_f)^nA\subseteq\sup_{n\in\mathbf{N}}(1\cup\mathscr{P}_f)^nB$.
Part v: $\forall n, A\subseteq(1\cup\mathscr{P}_f)^nA$. Then $A\subseteq\mathscr{L}A$, and so, $1\subseteq\mathscr{L}$. Now using Prop. 5.12, we have $\mathscr{L}A=\mathscr{L}(\mathscr{L}A)=\tilde{\tilde{A}}=\tilde{A}=\mathscr{L}A$. Therefore $\mathscr{L}^2=\mathscr{L}$.
Part vi: Use Def. 5.5 for $\mathscr{L}$ and Def. 5.15 for $\mathscr{J}$. Part vii: $(\mathscr{L}(A))^{\sim}=\mathscr{L}A$ (Prop. 5.12). Therefore $\mathscr{L}(A)\in\mathscr{J}(\mathscr{L}(A))$. Part viii: The last relation in Part vii implies $\{\mathscr{L}(A)\}\subseteq\mathscr{J}(\mathscr{L}(A))\Rightarrow\mathscr{B}\mathscr{L}A\subseteq\mathscr{J}\mathscr{L}A$. Part ix: Follows using Defs. 5.5 and 5.15.

Proposition 5.27: Let $A \in \mathcal{S}$, and consider the class $\mathcal{C}_A = \{F \in \mathcal{F} | A \subseteq F\}$. From Remark 5.6, we see that $\forall F \in \mathcal{C}_A$, we have $\mathcal{L}A \subseteq \mathcal{L}F = F$, the inclusion following from Prop. 5.16,iv and the equality from Lemma 5.10. Hence $\forall F \in \mathcal{C}_A$, we have $\tilde{A} \subseteq F$. Moreover, $\tilde{A} \in \mathcal{C}_A$, since $\tilde{A} \in \mathcal{F}$ (Prop. 5.12), and $A \subseteq \tilde{A}$ (Prop 5.26,v). Thus $\tilde{A}$ is the least element (relative to inclusion) in $\mathcal{C}_A$. Then $\tilde{A} = \bigcap_{F \in \mathcal{C}_A} F$.

Proposition 5.28: Part a: H-J, Chap. 6. Part b: H-J, Chap. 8. Part c: Follows from $F - \mathcal{P}_f(F) \subseteq F$. Part d: Apply (a) to $\mathcal{J}(F) \subseteq \mathcal{P}(A)$. Part e: Follows from Part a, since $\mathcal{P}(F) - \mathcal{J}(F) \subseteq \mathcal{P}(F)$. Part f: Follows by noting that $\mathcal{F} \cap \mathcal{P}(F) \subseteq \mathcal{P}(F)$.

Remark 5.30: Part a: $A \subseteq \mathcal{P}A$, but $A \not\subset \mathcal{Y}A$ (Def. 5.29). Part b: Let $A \in \mathcal{S}$. Then from Def. 5.29, $\mathcal{Y}A = \mathcal{F} \cap \{B | B \subset A\} = \mathcal{F} \cap (\mathcal{P}A - \mathcal{B}A) = \mathcal{F} \cap (\mathcal{P} - \mathcal{B})A$ $= \mathcal{O}(\mathcal{F})(\mathcal{P} - \mathcal{B})A$ (Def. 2.4), Therefore $\mathcal{Y} = \mathcal{O}(\mathcal{F})(\mathcal{P} - \mathcal{B})$.

Lemma 5.31: Since $\mathcal{Y}F$ is a set (Def. 5.29) and $\mathcal{J}$ is an operator, the restriction of $\mathcal{J}$ to $\mathcal{Y}F$ is a function (axiom of replacement). The result now follows from H-J, Chap. 2.

Theorem 5.32: Part a: If both $G \in \mathcal{F}$ and $G \in \mathcal{P}F$, then (i) $\mathcal{J}G \subseteq \mathcal{P}F$, since if $W \in \mathcal{J}G$, then $\tilde{W} = G$ (Def. 5.16), and hence $W \subseteq G$ (Def. 5.5). Since $G \in \mathcal{P}F$, then $G \subseteq F$. Combining, we conclude that if $W \in \mathcal{J}(G)$, then $W \subseteq F$.

We now show (ii) $\bigcup_{\{G \in \mathcal{F} | G \in \mathcal{P}F\}} \mathcal{J}(G) = \mathcal{P}F$. This union is a set (Lemma 5.31).

Then from (i), we have $\bigcup_{\{G \in \mathcal{F} | G \in \mathcal{P}F\}} \mathcal{J}(G) \subseteq \mathcal{P}F$. Next suppose that $V \in \mathcal{P}F$, and let $H = \tilde{V}$. Then $H \in \mathcal{F}$ (Prop. 5.12). Now $V \in \mathcal{P}F \Rightarrow V \subseteq F \Rightarrow \tilde{V} \subseteq \tilde{F} = F \Rightarrow H \subseteq F$. Hence $H \in \mathcal{F}$ and $H \in \mathcal{P}F$. Finally, $V \in \mathcal{J}(\tilde{V}) = \mathcal{J}(H)$ by the definition of $\mathcal{J}$. We conclude $V \in \bigcup_{\{G \in \mathcal{F} | G \in \mathcal{P}F\}} \mathcal{J}(G)$, since H is one of these $G's$, demonstrating (2).

Note that (ii) is a disjoint union (Prop. 5.23), and so, $\mathcal{P}F - \bigcup_{G \in \mathcal{Y}F} \mathcal{J}(G) = \mathcal{J}(F)$.

Part b: $\mathcal{J}(G) \subseteq \mathcal{P}G$ (Remark 5.16,b), $\forall G \in \mathcal{Y}F$ (Def. 5.29). Then $\bigcup_{G \in \mathcal{Y}F} \mathcal{P}(G)$ is a set by the axiom of replacement. Moreover this set includes

$\bigcup_{G \in \mathcal{Y}F} \mathcal{J}(G)$. Hence $\mathcal{P}F - \bigcup_{G \in \mathcal{Y}F} \mathcal{P}G \subseteq \mathcal{P}F - \bigcup_{G \in \mathcal{Y}F} \mathcal{J}(G) = \mathcal{J}(F)$, the equality following from Part a of the theorem.

Next we shall show that (iv) $\mathscr{J}(F) \subseteq \mathscr{P}F - \bigcup_{G \in \mathscr{Y}F} \mathscr{P}G$. Suppose $W \in \mathscr{J}(F)$. Then since $\mathscr{J}(F) \subseteq \mathscr{P}F$ (Remark 5.17), we have $W \subseteq F$, and so $W \in \mathscr{P}F$. Suppose now that $G \in \mathscr{Y}F$. That is, $G \in \mathscr{F}$ and $G \subset F$. Then using (ii), $\mathscr{P}G = \bigcup_{K \in \mathscr{Y}G} \mathscr{J}(K)$. If $K \in \mathscr{F}$ and $K \subseteq G$, then $K \neq F$ (since $G \subset F$). Also $W \notin \mathscr{J}(K)$, since $\mathscr{J}(F) \cap \mathscr{J}(K) = \varnothing$ (Prop. 5.23). Thus $W \notin \mathscr{P}G, \forall G \in \mathscr{Y}F$. This concludes the proof of (iv), which when combined with (iii) establishes the proof of Part b.

Remark 5.34: Follows from Fig. 5.1.

Remark 5.35: Follows from Remark 5.6 and Fig. 5.1.

Lemma 5.35: $\mathscr{P}_f A \subseteq \mathscr{P}_f F$, since $\mathscr{P}_f$ is isotone. By hypothesis, $\mathscr{P}_f A \subseteq \mathscr{P}_f F \subseteq A$. Therefore $\mathscr{P}_f A \subseteq A$, and so, $(1 \cup \mathscr{P}_f)A = A$.

Lemma 5.36: $G \in \mathscr{F}$ means $(1 \cup \mathscr{P}_f)G = G$. $V \subseteq G \Rightarrow (1 \cup \mathscr{P}_f)V \subseteq (1 \cup \mathscr{P}_f)G$, since both 1 and $\mathscr{P}_f$ are isotone. Thus $V \in \mathscr{P}G \Rightarrow (1 \cup \mathscr{P}_f)V \in \mathscr{P}G$.

Proposition 5.37: $\mathscr{S} = \bigcup_{F \in \mathscr{F}} \mathscr{J}(F)$ (Prop. 5.23). Then since $\mathscr{P}G \subseteq \mathscr{S}$, we have (i) $\mathscr{P}G = \bigcup_{F \in \mathscr{F}} \mathscr{P}G \cap \mathscr{J}(F)$. Now suppose that for $F \in \mathscr{F}$, $\mathscr{P}G \cap \mathscr{J}(F) \neq \varnothing$. Then $\exists W \in \mathscr{J}(F)$, so that $W \subseteq G$. Therefore $\tilde{W} \subseteq \tilde{G}$, since the tilda is isotone. However $\tilde{G} = G$, since $G \in \mathscr{F}$. Also $\tilde{W} = F$ (Def. 5.14). Hence $F \subseteq G$. Therefore (ii) $F \in \mathscr{F} \cap \mathscr{P}G$. Now suppose that $H \in \mathscr{F} \cap \mathscr{P}G$; then $H \subseteq G$.
If $W \in \mathscr{J}(H)$, then $W \subseteq H$ since $\tilde{W} = H$ (Def. 5.13) and $W \subseteq \tilde{W}$ (Prop. 5.26,v). Thus if $W \in \mathscr{J}(H)$, then $W \subseteq H \subseteq G$, and hence $W \in \mathscr{P}G$. We conclude that if (iii) $H \in \mathscr{F} \cap \mathscr{P}G$, then $\mathscr{J}(H) \subseteq \mathscr{P}G$ and so, $\mathscr{P}G \cap \mathscr{J}(H) = \mathscr{J}(H)$.
Using (i) and (ii), we have (iv) $\mathscr{P}G = \bigcup_{F \in \mathscr{F} \cap \mathscr{P}G} \mathscr{P}G \cap \mathscr{J}(F)$. Using (iii), we can simplify (iv) further to get $\mathscr{P}G = \bigcup_{H \in \mathscr{F} \cap \mathscr{P}G} \mathscr{J}(H)$, completing the proof.

Sect. 6

Proposition 6.2: See Aczel, 1988, Chap. 3 and Moschovakis, 2005.

Remark 6.3: $\mathscr{P}_f\mathbf{FP} \subseteq \mathbf{FP}$, hence $(1 \cup \mathscr{P}_f)\mathbf{FP} = \mathbf{FP}$.

Proposition 6.4: Part a: $y \in \mathbf{FP} \Rightarrow \forall z \in y,\ z \in \mathbf{FP}$. Also $y \in \mathbf{FP}$ implies that y is finite by definition.
Part b: From Def. 6.1, we conclude that $y \in \mathbf{FP} \Rightarrow y \subseteq \mathbf{FP}$. We can show by induction (Remark 6.3) that $\forall n \in \mathbf{N},\ (1 \cup \mathscr{P}_f)^n y \subseteq \mathbf{FP}$. Then as a union of **FP** subsets (Def. 5.5), $\tilde{y} \subseteq \mathbf{FP}$. **FP** being countable implies that $\tilde{y}$ is countable. Moreover $\tilde{y}$ is infinite, since it is finitely perfect (Remark 5.3). Thus $\tilde{y}$ is countably infinite.
Part c: Prop. 5.11.

Remark 6.7: See the definitions.

Remark 6.9: $\mathcal{O}$ being a selector and $y \in \mathbf{FP} \Rightarrow \mathcal{O}y \in \mathbf{FP}$, since $\mathcal{O}y$ is a subset of y.

Remark 6.10: Axiom of comprehension.

Lemma 6.11: $x \in \mathbf{FP} \Rightarrow x \subseteq \mathbf{FP} \Rightarrow \mathbf{FP} \cap x = x$. Also $\mathcal{O}(\mathcal{C})x = \mathcal{C} \cap x$ (Def. 2.4). Combining the last two equations gives $\mathcal{O}(\mathcal{C})x = \mathcal{C} \cap (\mathbf{FP} \cap x) =$ $(\mathcal{C} \cap \mathbf{FP}) \cap x = f(\mathcal{C} \cap \mathbf{FP})x$ (the last using Def. 6.6 and Remark 6.10).

Lemma 6.12: Suppose A is a finite subset of $\mathcal{C} \cap \mathbf{FP}$. Then it is also a subset of each of $\mathcal{C}$ and **FP**. Since $\mathcal{C}$ is perfect, $A \in \mathcal{C}$. Since **FP** is finitely perfect, $A \in \mathbf{FP}$. Taking these two observations together gives $A \in \mathcal{C} \cap \mathbf{FP}$.

Lemma 6.13: Part a: This follows from Theorem 4.14.
Part b: Using $D \subseteq \mathbf{FP}$ and $D \subseteq D^*$ (Lemma 4.12) gives $D \subseteq D^* \cap \mathbf{FP}$. We also have $D^* = \bigcup_{\alpha \in \mathcal{S}_{ord}} (1 \cup \mathscr{P})^\alpha D$ (Def 4.10). Now $D^* \cap \mathbf{FP} = \mathbf{FP} \cap D^*$, the right member being a set by the axiom of comprehension. Next note $\mathbf{FP} \cap D^* = \bigcup_{\alpha \in \mathcal{S}_{ord}} \left(\mathbf{FP} \cap (1 \cup \mathscr{P})^\alpha D\right)$.
Then there exists $\beta \in \mathcal{S}_{ord}$ such that (i) $\mathbf{FP} \cap D^* = \mathbf{FP} \cap (1 \cup \mathscr{P})^\beta D$. (Compare with the proof of Theorem 4.14.) The right hand side of (i) is equal to $\mathbf{FP} \cap (1 \cup \mathscr{P}_f)^\beta D$ (proof by induction on β). Further, $\mathbf{FP} \cap (1 \cup \mathscr{P}_f)^\beta D = \mathbf{FP} \cap D$, since D is a fixed point of $1 \cup \mathscr{P}_f$. Finally, $\mathbf{FP} \cap D = D$, since $D \subseteq \mathbf{FP}$ by hypothesis.

Theorem 6.14: Follows from Lemma 6.12 (which gives the well-definedness) and Lemma 6.13 (which gives the surjectivity).

Remark 6.15: See Example 6.16,i.

Lemma 6.17: $\mathscr{P}_f$ is isotone implies that $\mathscr{P}_f(\mathbf{FP} \cap \mathscr{N}) \subseteq \mathscr{P}_f\mathbf{FP} \cap \mathscr{P}_f\mathscr{N}$. (Def. 5.1). Also $\mathscr{P}_f\mathscr{N} \subseteq \mathscr{P}\mathscr{N}$, by definition. Hence $\mathscr{P}_f(\mathbf{FP} \cap \mathscr{N}) \subseteq \mathscr{P}_f\mathbf{FP} \cap \mathscr{P}\mathscr{N}$ $\subseteq \mathbf{FP} \cap \mathscr{N}$, since $\mathbf{FP}$ is finitely perfect and $\mathscr{N}$ is perfect.

Lemma 6.20: $D \in \mathbf{L} \Rightarrow (1 \cup \mathscr{P}_f)D \in \mathbf{L}$ (Remark 6.19). We can show by induction that $\forall n \in \mathbf{N}, (1 \cup \mathscr{P}_f)^n D \in \mathbf{L}$. Now $\mathscr{L}D = \bigcup_{n \in \mathbf{N}} (1 \cup \mathscr{P}_f)^n D$ (Def. 5.5). Moreover, $\mathbf{L} = \mathscr{P}(\mathbf{FP} \cap \mathscr{N})$, being a power set, is closed under indexed unions (H-J, Chap. 2). Therefore $\mathscr{L}D \in \mathbf{L}$.

Proposition 6.22: Special case of Prop. 5.37.

Proposition 6.25: Suppose $D \in \mathbf{S}$. We know that $|D| = \mathbf{N}$, since $\tilde{D} = D$ (Remark 5.3). We claim that $\exists\, d \in D$ such that $A = D - \{d\} \notin \mathbf{S}$ (refer to the proof of Prop. 5.15). Now $|A| = |D| = \mathbf{N}$. Then since A is not a fixed point, $A \in$ Region II (Def 6.21,3). Now $A \subseteq D \Rightarrow \tilde{A} \subseteq \tilde{D} \Rightarrow \tilde{A} \subseteq D$. However $\tilde{A} \neq A$, since $A \notin \mathbf{S}$ (that is, A is not a fixed point). We conclude that $A \subset \tilde{A} \subseteq D$ (Prop. 5.26,v). Since $A = D - \{d\}$, we have $\tilde{A} = D$. That is, D is a limit point of an element of Region II. This completes the proof.

Lemma 6.26: This is a specialization of Lemma 5.35.

Theorem 6.27: Part a: $\mathbf{H} \subseteq \mathbf{FP} \cap \mathscr{N} \subseteq \mathbf{FP}$. Therefore $|\mathbf{H}| \leq |\mathscr{R}\mathbf{FP}| \leq |\mathbf{FP}|$. Then since $|\mathbf{H}| = |\mathbf{FP}| = \mathbf{N}$, $|\mathscr{R}\mathbf{FP}| = \mathbf{N}$, and so $|\mathscr{P}\mathscr{R}\mathbf{FP}| = 2^{\mathbf{N}}$.
Part b: $|\mathscr{R}\mathbf{FP}| = \mathbf{N} \Rightarrow |\mathscr{P}_f\mathscr{R}\mathbf{FP}| = \mathbf{N}$ (H-J, Chap. 3).

Part c: First part (to show $|\mathbf{S}| = 2^{\mathbf{N}}$): Let $X = \mathbf{FP} \cap \mathscr{N} = \mathscr{R}\mathbf{FP}$, then $X \in \mathbf{S}$. (Note that X is the maximal element of $\mathbf{S}$ (Def. 6.21,1).) If $E \subseteq \mathbf{FP}$, and E satisfies $\mathscr{P}_f X \subseteq E \subseteq X$, then $E \in \mathbf{S}$ (setting $D = X$ in Lemma 6.26). We claim that $|J| = 2^{\mathbf{N}}$, where J is the set $J = \{E \subseteq X | \mathscr{P}_f X \subseteq E\}$. Assuming the claim, we have $J \subseteq \mathbf{S} \subseteq \mathbf{L}$, and so, $|J| \leq |\mathbf{S}| \leq |\mathbf{L}|$. However, $|\mathbf{L}| = 2^{\mathbf{N}}$ and $|J| = 2^{\mathbf{N}}$. Then by the Cantor-Bernstein Theorem (H-J, Chap. 4), we conclude that $|\mathbf{S}| = 2^{\mathbf{N}}$, demonstrating the first part.

Proof of claim ($|J| = 2^{\mathbf{N}}$): If $\mathscr{P}_f X \subseteq E \subseteq X$, then $E - \mathscr{P}_f X \subseteq X - \mathscr{P}_f X$. Conversely, if $D \subseteq X - \mathscr{P}_f X$, then $\mathscr{P}_f X \subseteq D \cup \mathscr{P}_f X \subseteq X$. Then J is in one to one correspondence with $\mathscr{P}(X - \mathscr{P}_f X)$. Therefore, $|J| = 2$ raised to the power $|X - \mathscr{P}_f X|$. Thus we need only show that $|X - \mathscr{P}_f X| = \mathbf{N}$. From Part a, we have $|X| = \mathbf{N}$, which implies that $|X - \mathscr{P}_f X| \leq \mathbf{N}$. Then consider the set $C = \{\{n, \Omega\} | n \in \mathbf{N}\}$. ($C$ is a set because it is the range of the function, $n \mapsto \{n,\Omega\}$.) If (i) $m \neq n$, then $\{m,\Omega\} \neq \{n,\Omega\}$ (so that there are an infinite number of such pairs). We now demonstrate the auxiliary property that $\forall n,\ \{n,\Omega\} \in X - \mathscr{P}_f X$.

Demonstration: For a fixed n, (ii) $\{n,\Omega\} \in \mathbf{FP}$. Note that (iii) $\{n,\Omega\} \notin \{n,\Omega\}$, so that $\{n,\Omega\} \in \mathscr{N}$. Then combining (ii) and (iii), we have $\{n,\Omega\} \in \mathbf{FP} \cap \mathscr{N} = X$. Continuing, if $z \in \mathscr{P}_f X$, then $\forall w \in z, w \in \mathscr{N}$. Therefore $\{n,\Omega\} \notin \mathscr{P}_f X$ (since $\Omega \in \mathscr{A}$).

Continuing with the proof of the claim: Using (i), we have $|C| = |\{\{n,\Omega\} | n \in \mathbf{N}\}| = \mathbf{N}$. However $C \subseteq X - \mathscr{P}_f X \subseteq X$, and therefore $|X - \mathscr{P}_f X| = \mathbf{N}$, completing the proof of the claim.

Second part (to show $|\mathbf{S_0}| = \mathbf{N}$): $|\mathscr{P}_f X| = \mathbf{N}$, since both $\mathbf{H} \subseteq \mathscr{P}_f X \subseteq X$ and $|\mathbf{H}| = |\mathbf{X}| = \mathbf{N}$. Also $\mathbf{S}_0 = \mathscr{L}[\mathscr{P}_f X]$ (Def. 6.21). Then $|\mathbf{S_0}| \leq \mathbf{N}$. We shall now construct a proper subset $Q \in \mathbf{S}_0$ such that $|Q| = \mathbf{N}$, completing the proof.

Construction: Recall that $\mathscr{P}_f X \subset X$. Then $\forall n,\ \mathscr{P}_f^{n+1} X \subset \mathscr{P}_f^{n} X$, since $\mathscr{P}_f$ is isotone (Def. 5.1) and one to one (Remark 5.3,b). Let $X_n \overset{def}{=} \mathscr{P}_f^{n} X$. Then $X_n \subset X$. Moreover, (iv) $X_{n+1} \subset X_n,\ \forall n$. Next $\forall n$, choose a (v) $y_n \in X_n - X_{n+1}$. Now set $Q = \{\tilde{y}_n | n \geq 1\}$. We shall show that if $m > n$, then $\tilde{y}_m \neq \tilde{y}_n$, which completes the proof that $|Q| = \mathbf{N}$.

Note that (vi) $X_n = \mathscr{P}_f X_{n-1}$ and so from (iv), X_{n-1} is finitely perfect. Therefore $y_n \in X_n \Rightarrow y_n \subseteq X_{n-1}$. Then using (v), we have that $y_n \subseteq X_{n-1} - X_n$ if $n \geq 1$. Next since X_{n-1} is finitely perfect, (v) and (vi) show that $\tilde{X}_{n-1} = X_{n-1}$ if $n \geq 1$. Hence $y_n \subseteq X_{n-1} \Rightarrow \tilde{y}_n \subseteq X_{n-1}$. On the other hand, $y_n \subseteq \tilde{y}_n$ and $y_n \not\subset X_n$. Therefore $\tilde{y}_n \not\subset X_n$. Hence, $\forall n \geq 1,\ \tilde{y}_n \subseteq X_{n-1} - X_n$. Now suppose that $m > n$. Then $m - 1 \geq n$ and (vii) $X_{m-1} \subseteq X_n$. Moreover, using (vi) and (vii), we have $\tilde{y}_n \not\subset X_{m-1}$. Hence $\tilde{y}_n \notin \mathscr{P} X_{m-1}$. On the other hand, $\tilde{y}_m \subseteq X_{m-1}$. Hence $\tilde{y}_m \in \mathscr{P} X_{m-1}$. We conclude that $\tilde{y}_n \neq \tilde{y}_m$. This completes the proof of Part c.

Part d: We have $|\{$ Region II limit points $\}| = |\mathbf{S}|$ (Prop. 6.25). Now $\mathscr{L}\,|$Region II is a surjective map of Region II to $\mathbf{S}$ (Prop. 6.25). In fact, $\mathscr{L}\,|$Region II is a function by the axiom of replacement. By the axiom of choice, $|\mathbf{S}| \leq |$ Region II $|$ (H-J, Chap. 8) Moreover, $|$ Region II $| \leq |\mathbf{L}|$, since Region II $\subseteq \mathbf{L}$. However, $|\mathbf{S}| = |\mathbf{L}|$ (Parts a and c). Then the Cantor-Bernstein Theorem implies that $|$ Region II $| = |\mathbf{S}| = 2^{\mathbf{N}}$ (using Part c).

Corollary 6.28: Follows from Theorem 6.27,c.

Sect. 7

Proposition 7.1: Special case of Lemma 6.11.

Proposition 7.6: Both parts follow from Remark 2.8, Lemma 6.13 and Theorem 6.27,c.

Theorem 7.7: Part a: First assertion: $\mathcal{N}$ is perfect (Example 3.6), and so, $\mathcal{N}^* = \mathcal{N}$ (Theorem 4.14). Now $\mathcal{A} \subseteq \mathcal{N} \Rightarrow \mathcal{A}^* \subseteq \mathcal{N}^*$. Hence $\mathcal{A}^* \subseteq \mathcal{N}$. The second assertion follows from Theorem 3.11. Part b: Follows from Lemma 6.13 and Prop 7.1.
Part c: Follows from b and that $y \in \mathbf{FP} \Rightarrow \tilde{y} \subseteq \mathbf{FP}$ (Remark 6.4,b). Part d: From Prop. 7.1, (i) $\forall x \in \mathbf{FP}, \mathscr{K}x = f(E)x = E \cap x$, where $E = \mathbf{FP} \cap \mathcal{C}(\mathscr{K})$. Since $\mathscr{K}$ is a consciousness operator, $\mathcal{C}(\mathscr{K})$ is perfect, and $\mathcal{C}(\mathscr{K}) \subseteq \mathcal{N}$ (Theorem 3.11). We conclude that E is finitely perfect (Lemma 6.12), and we also note that $\mathcal{C}(\mathscr{K}) \subseteq \mathcal{N} \Rightarrow E \subseteq \mathcal{N}$, since $E \subseteq \mathcal{C}(\mathscr{K})$. Then $E \subseteq \mathbf{FP} \cap \mathcal{N}$, and $(1 \cup \mathscr{P}_f)E = E$ (Def. 5.2). Hence $E \in \mathbf{S}$ (Def. 6.21,1). Hence using Part b, $\forall x \in \mathbf{FP}$, we have $\mathcal{O}(E^*)x = E \cap x = \mathscr{K}x$, the last equality following from (i).

Proposition 7.8: First part: If $D \in \mathbf{S}$, $\mathcal{O}(D^*)$ is a consciousness operator (Remark 7.4), and $\mathcal{O}(D^*)|\mathbf{FP} = f(D)$ (Theorem 7.7,b). Moreover, the map $D \mapsto f(D)$ is one to one, since $D \subseteq \mathbf{FP}$ (see the note following Def. 6.8). Hence the mapping $D \mapsto \mathcal{O}(D^*)|\mathbf{FP}$ is one to one. Now let $E = \mathcal{C}(\mathscr{K}) \cap \mathbf{FP}$ where $\mathscr{K}$ is a consciousness operator. Theorem 7.7,d gives $E \in \mathbf{S}$ and $\mathscr{K}|\mathbf{FP} = \mathcal{O}(E^*)|\mathbf{FP}$. Hence the map $D \mapsto \mathcal{O}(D^*)$ from $\mathbf{S}$ to "$\{(\mathscr{K}|\mathbf{FP}) | \mathscr{K} \in \mathit{Consciousness\ Operators}\}$" is one-to-one and onto.

Second part: From the first part, we conclude that the collection "$\{(\mathscr{K}|\mathbf{FP}) | \mathscr{K} \in \mathit{Consciousness\ Operators}\}$" $= \{f(D) | D \in \mathbf{S}\} = R[\mathbf{S}]$. (For the last equality, refer to Def. 6.6 and H-J, Chap. 2.

Remark 7.9: Follows from Theorem 7.7,b and Example 7.2.

Sect. 9

Remark 9.3: $\forall x \in \mathcal{S}, \mathscr{K}x \subseteq x$ (consciousness axioms). Therefore (i) $\mathscr{K}x \in \mathscr{P}x$. Now $\mathscr{K} \subseteq \mathscr{R}$ (Lemma 3.9), and $\mathscr{R}\mathscr{K} = \mathscr{R} \cap \mathscr{K} = \mathscr{K}$ (Cor. 3.12). Therefore $\mathscr{R}\mathscr{K}x = \mathscr{K}x$. That is, any element $y \in \mathscr{K}x$ is in $\mathcal{N}$. Therefore (ii) $\mathscr{K}x \in \mathcal{N}'$ (Def. 3.3). Combining (i) and (ii) completes the proof.

Proposition 9.4: Parts a, b, c and d follow directly from Part e. Part e: If $y \in \mathcal{Q}(x)$, there exists a consciousness operator $\mathscr{K}$ such that $y = \mathscr{K}x$. That is, there exists a perfect class $\mathcal{C} \subseteq \mathcal{N}$ such that (i) $y = \mathcal{C} \cap x$ (Theorem 3.11). Now since $y \subseteq x$ and

$y \subseteq y^*$ then (ii) $y \subseteq y^* \cap x$. Then using (i), we have (iii) $y^* \cap x = (\mathcal{C} \cap x)^* \cap x$. Observe that $(\mathcal{C} \cap x)^* \subseteq \mathcal{C}^* \cap x^*$ (isotonicity of *). However $\mathcal{C}^* = \mathcal{C}$, since $\mathcal{C}$ is perfect (Theorem 4.14). Therefore (iv) $(\mathcal{C} \cap x)^* \subseteq \mathcal{C} \cap x^*$. Now combining (iii) and (iv) gives $y^* \cap x \subseteq \mathcal{C} \cap x^* \cap x = \mathcal{C} \cap x$ (since $x \subseteq x^*$). Then using (i), we get (v) $y^* \cap x \subseteq y$. Now using (ii) and (v), we conclude that $y^* \cap x = y$. Using $\mathcal{C} \subseteq \mathcal{N}$ and (i), we get $y \subseteq \mathcal{N} \cap x = \mathcal{R}x$ (Example 3.6,2). We now know that $\mathcal{Q}(x) \subseteq \{y \subseteq \mathcal{R}x \mid y^* \cap x = y\}$. Next we show that if $y \subseteq \mathcal{R}x$, and $y^* \cap x = y$, then $y \in \mathcal{Q}(x)$. Since $y \subseteq \mathcal{R}x$, then $y \subseteq \mathcal{N}$. Now setting $\mathcal{D} = y^*$, we have $y^* \subseteq \mathcal{N}$, and by Theorem 4.14, y^* is perfect. Therefore $\mathcal{O}(\mathcal{D})$ is a consciousness operator (Theorem 3.11) and $\mathcal{O}(\mathcal{D})x = y$. Hence $y \in \mathcal{Q}(x)$ (Def. 4.1), completing the proof of Part e.

Remark 9.5: Combining the hypothesis $w \subseteq x$ with $w \subseteq w^*$, we have $w \subseteq x \cap w^* = y$ (by hypothesis). $y \subseteq \mathcal{R}x$ follows from Prop. 9.4,c.

Corollary 9.7: Let $\mathcal{Z}_X$ be the operator defined by $w \mapsto x \cap w^*$. $\mathcal{Z}_X | \mathcal{P}\mathcal{R}x$ is a function (axiom of replacement). Now Prop 9.4,c may be rewritten as $\mathcal{Q}(x) = \mathcal{Z}_X[\mathcal{P}\mathcal{R}x]$ (see footnote 13), showing that $\mathcal{Q}(x)$ is a set.

Lemma 9.9: Part a: If $x \in \mathcal{S}_{wf}$, then $x \subseteq \mathcal{S}_{wf}$ (H-J, Chap. 14), and $\mathcal{R}x \subseteq x$. Therefore if $w \subseteq \mathcal{R}x$, then $w \subseteq \mathcal{S}_{wf}$. Hence $w^* = \mathcal{S}_{wf}$ (Lemma 3.7, Remark 4.4 and Theorem 4.14). Now $x \subseteq \mathcal{S}_{wf}$ and $w^* = \mathcal{S}_{wf} \Rightarrow x \cap w^* = x$. Hence $\mathcal{Q}(x) = \{x\}$ (Prop. 9.4,c).
Part b: Since $\mathcal{B}$ is not isotone, Part a implies that $\mathcal{Q}$ is not isotone.
Part c: $\mathcal{Q}(y) = \{y \cap w^* \mid w \subseteq \mathcal{R}y\} = \{y \cap w^* \mid w \subseteq \mathcal{R}(\mathcal{R}y)\}$, since $\mathcal{R}^2 = \mathcal{R}$. Now $w \subseteq \mathcal{R}y \Rightarrow w \subseteq y \cap \mathcal{N} = w \subseteq \mathcal{N} \Rightarrow w^* \subseteq \mathcal{N}$ (since $\mathcal{N}^* = \mathcal{N}$, that is, since $\mathcal{N}$ is perfect). Hence $w \subseteq \mathcal{R}y \Rightarrow y \cap w^* = y \cap (w^* \cap \mathcal{N}) = (y \cap \mathcal{N}) \cap w^* = \mathcal{R}y \cap w^*$. We conclude that $\mathcal{Q}(y) = \{\mathcal{R}y \cap w^* \mid w \subseteq \mathcal{R}(\mathcal{R}y)\} = \mathcal{Q}(\mathcal{R}y)$ (Prop. 9.4,c).
Part d: Follows from Part c.
Part e: (i) $y \in \mathcal{Q}(x) \Rightarrow y \subseteq \mathcal{R}x$ (Prop. 9.4 e) $\Rightarrow \forall z \in y,\ z \in \mathcal{N}$ (by definition of $\mathcal{R}$) $\Rightarrow$ (ii) $y \in \mathcal{N}$ (Example 3.6,2) $\Rightarrow y \in \mathcal{N} \cap \mathcal{Q}(x)$ (combining (i) and (ii)) $\Rightarrow y \in \mathcal{R}(\mathcal{Q}(x))$.
Part f: The first equality follows from Part a. For the second, we shall show that $\forall x \in$, $\mathcal{W}\mathcal{Q}(x) = \mathcal{B}\mathcal{W}x$. Suppose then that $y \in \mathcal{W}\mathcal{Q}(x)$. Then (i) $y \in \mathcal{S}_{wf} \cap \mathcal{Q}(x)$ (Example 3.6). Then since $y \in \mathcal{S}_{wf}$ (from (i)), it follows that $y \subseteq \mathcal{S}_{wf}$ (H-J, Chap. 14). Hence $\varnothing \subseteq y \subseteq \mathcal{S}_{wf}$. Then the isotonicity of $*$ (Lemma 4.12) yields (ii) $\varnothing^* \subseteq y^* \subseteq \mathcal{S}^*_{wf}$. Now using $\varnothing^* = \mathcal{S}_{wf}$ (Example 4.13) and $\mathcal{S}^*_{wf} = \mathcal{S}_{wf}$ (Example 3.6 and Theorem 4.14)

in (ii), we conclude that $\mathcal{S}_{wf} \subseteq y^* \subseteq \mathcal{S}_{wf}$. Hence (iii) $y^* = \mathcal{S}_{wf}$. Using (i), we have (iv) $y \in \mathcal{Q}(x)$, from which we have (v) $y^* \cap x = y$ (Prop. 9.4,e). (ii) and (v) imply (vi) $y = \mathcal{S}_{wf} \cap x = \mathcal{W}x$ (refer to Example 3.6 for the latter equality). Conversely, (vii) $\mathcal{W}x \in \mathcal{W}\mathcal{Q}(x)$ (Def. 9.1 and the well-foundedness of $\mathcal{W}x$). Combining (vi) and (vii) yields $\mathcal{W}\mathcal{Q}(x) = \{\mathcal{W}x\}$, concluding the proof.

Part g: Prop. 9.4 tells us that $\forall y \in \mathcal{Q}(x),\ y \subseteq \mathcal{R}x$. Hence $y \in \mathcal{P}\mathcal{R}x$. Thus $\forall x,\ \mathcal{Q}(x) \subseteq \mathcal{P}\mathcal{R}x$. Hence $\mathcal{Q} \subseteq \mathcal{P}\mathcal{R}$.

Part h: Def. 9.1 $\Rightarrow$ (i) $\mathcal{Q}(\mathcal{K}x) = \{\mathcal{K}_1\mathcal{K}x \mid \mathcal{K}_1$ is a consciousness operator $\} \subseteq \mathcal{Q}(x)$, since $\mathcal{K}_1\mathcal{K}$ is also a consciousness operator (Cor. 3.12).

Also (ii) $\mathcal{Q}(\mathcal{K}x) \subseteq \mathcal{P}(\mathcal{K}x)$ (Lemma 9.9,g). Combining (i) and (ii), we have $\mathcal{Q}\mathcal{K} \subseteq \mathcal{Q} \cap \mathcal{P}\mathcal{K}$. Next suppose that (iii) $y \in \mathcal{Q}(x) \cap \mathcal{P}\mathcal{K}x$. The first part in the intersection in (iii) gives (iv) $y = \mathcal{K}_2 x$, for some consciousness operator $\mathcal{K}_2$, while the second part gives (v) $y \subseteq \mathcal{K}x = \mathcal{C}(\mathcal{K}) \cap x \subseteq \mathcal{C}(\mathcal{K})$, the equality here following from Lemma 2.6 since $\mathcal{K}$ is a selector (consciousness axiom d). Now (vi) $\mathcal{K}y = \mathcal{C}(\mathcal{K}) \cap y$ (Def. 2.6). Combining (v) and (vi) gives $y = \mathcal{K}y$. Then combining (iv) and (vi) gives $y = \mathcal{K}\mathcal{K}_2 x = \mathcal{K}_2 \mathcal{K}x \in \mathcal{Q}(\mathcal{K}x)$ (Def. 9.1). Therefore $\mathcal{Q}(x) \cap \mathcal{P}\mathcal{K}x \subseteq \mathcal{Q}(\mathcal{K}x)$.

Part i: $y \in \mathcal{Q}(x) \Leftrightarrow y = \mathcal{K}x$ for some consciousness operator $\mathcal{K}$ (Def. 9.1). Then $\forall \mathcal{K},\ \mathcal{K}x \in \mathcal{Q}(x)$, and hence $\mathcal{B}\mathcal{K}x \subseteq \mathcal{Q}(x)$.

Remark 9.10: $\mathcal{Q}(y) = \mathcal{Q}(\mathcal{R}y) = \{\mathcal{R}y\} \neq \{y\}$. The first equality follows from Lemma 9.9,c), the second follows from Lemma 9.9,a since $\mathcal{R}y \in \mathcal{S}_{wf}$. The inequation follows from $\mathcal{R}y \neq y$.

Corollary 9.13: Let $y_1 = \mathcal{K}_1 x$ and let $y_2 = \mathcal{K}_2 x$. Then using Cor. 3.12, we conclude that $y_1 \cap y_2 = (\mathcal{K}_1 \cap \mathcal{K}_2)x \in \mathcal{Q}(x)$.

Proposition 9.14: Part a: We have (i) $y^* \cap x = y$ (Prop. 9.4,e). Since $x \subseteq x^*$, we have (ii) $x^* \cap x = x$. Applying $y \neq x$ to (i) and (ii) gives $y^* \neq x^*$. Part b: We have $y_1^* \cap x = y_1$ and $y_2^* \cap x = y_2$ (Prop. 9.4,e). Therefore $y_1 \neq y_2 \Rightarrow y_1^* \neq y_2^*$. Part c: Suppose $Y \in \mathcal{Q}(A)$, and $Y^* = B^*$. Then $Y = B^* \cap A$ (Prop. 9.4,e). Moreover $B \cap A \subseteq Y$ since $B \subseteq B^*$.

Proposition 9.16: Part a: First assertion implies second: $\exists Y \in \mathcal{Q}(A)$ such that $Y^* = B^* \Rightarrow Y = A \cap Y^*$ and $Y^* = B^*$ (Prop. 9.4,e). This in turn implies that $Y = A \cap B^*$ and $(A \cap B^*)^* = B^*$; Second assertion implies third: Suppose $(A \cap B^*)^* = B^*$, so that

$A \cap B^* \in \mathscr{V}(B^*)$ (Def. 4.23). Hence $\mathscr{P}A \cap \mathscr{V}(B^*) \neq \varnothing$, since $A \cap B^* \in \mathscr{P}A$;
Third assertion implies first: Suppose $\mathscr{P}A \cap \mathscr{V}(B^*) \neq \varnothing$. Then $\exists W \subseteq A$ such that (i) $W^* = B^*$ (Def. 4.23). Let $Y = A \cap B^*$. Now (i) implies $Y = A \cap W^*$, and since $W \subseteq A \subseteq \mathscr{N}$, (using the hypotheses on A) we have $W \subseteq \mathscr{R}A$. Then using Prop. 9.4,a, we have $A \cap W^* \in \mathscr{Q}(A)$. There remains to show (ii) $Y^* = B^*$, which we shall do in two parts, I and II.
Part I: $W \subseteq A$ and $W \subseteq W^* \Rightarrow W \subseteq A \cap W^* = Y$ (using (i)). This in turn implies that $W^* \subseteq Y^*$ (isotonicity of *), and so that $B^* \subseteq Y^*$ (using (i)).
Part II: $Y^* = (A \cap W^*)^* \subseteq A^* \cap W^{**}$ (isotonicity of * (Lemma 4.12)). Thus $y \in A^* \cap W^*$ (since W^* is a fixed point). However, $W \subseteq A \Rightarrow W^* \subseteq A^*$ (isotonicity of *). Hence, $Y^* \subseteq W^* = B^*$ (the equality coming from (i)). Comparing Part I and Part II, we conclude that $Y^* = B^*$.
Part b: (1) Uniqueness of Y follows from Prop. 9.14,c. (2) $Y \in \mathscr{Q}(A) \Rightarrow Y = A \cap Y^*$ (Prop. 9.4,e). Since $Y^* = B^*$ (by hypothesis), we have $Y = A \cap B^*$, and then from Def. 4.19, we have that $Y = \mathscr{K}\langle B \rangle A$.
(3), (4), (5): Suppose $W \in \mathscr{P}A \cap \mathscr{V}(B^*)$, then $W \subseteq A$ and $W^* = B^*$. Moreover, $W \subseteq W^* = B^*$, and so, using (2), $W \subseteq A \cap B^* = Y$. Note that $Y \in \mathscr{P}A \cap \mathscr{V}(B^*)$, since both $Y \in \mathscr{Q}(A) \subseteq \mathscr{P}A$ (Prop. 9.9, g) and $Y \in \mathscr{V}(B^*)$ (the latter following from the hypothesis $Y^* = B^*$). (6) Follows from (2) and Def. 4.23. (7) is a restatement of (1) and (2) combined.

Lemma 9.19: Part a: Since $y \in \mathbf{FP}$, then by Remark 6.4, $\tilde{y} \subseteq \mathbf{FP}$ and $\tilde{y}$ is finitely perfect (Def. 5.2). Applying Lemma 6.13 to $\tilde{y}$, we get $(\tilde{y})^* \cap \mathbf{FP} = \tilde{y}$. Hence by hypothesis on $\mathscr{C}$, $\mathscr{C} \cap \mathbf{FP} = \tilde{y}$. Part b: Since $y \subseteq \mathbf{FP}$, we can show by induction that $\forall n,\ (1 \cup \mathscr{P}_f)^n y = (1 \cup \mathscr{P})^n y$. Now taking the supremum of both sides of the last equation gives $\tilde{y} = (1 \cup \mathscr{P})^\omega y$. Then with $\alpha \in \mathscr{S}_{ord}$, we have $(1 \cup \mathscr{P})^\alpha \tilde{y} = (1 \cup \mathscr{P})^\alpha (1 \cup \mathscr{P})^\omega y = (1 \cup \mathscr{P})^{\alpha+\omega} y$ (the last equality by transfinite induction (Theorem 4.6)). Then (iii) $\sup_{\alpha \in \mathscr{S}_{ord}} (1 \cup \mathscr{P})^\alpha \tilde{y} = \sup_{\alpha \in \mathscr{S}_{ord}} (1 \cup \mathscr{P})^{\alpha+\omega} y$. The left member of (iii) equals $\mathscr{C}$ and because the operator $1 \cup \mathscr{P}$ is accretive, the right member of (iii) equals y^*, completing the proof of Part b.
Part c: Apply Lemma 6.11 to $\mathscr{C} = (\tilde{y})^*$.

Proposition 9.20: Part1: By hypothesis x and $y \in \mathbf{FP}$; thus $\mathscr{K}\langle y \rangle x \subseteq x$, and $\mathscr{K}\langle y \rangle x \in \mathbf{FP}$. Moreover, $\mathscr{K}\langle y \rangle x = y^* \cap x \cap \mathscr{N}$ (Def. 4.19). Then $y^* \cap x = \tilde{y} \cap x$ (Lemma 9.19,c).

Part 2: Since $x \in \mathbf{FP}$, then $\mathscr{P}x \in \mathbf{FP}$. Then since $\mathscr{Q}(x) \subseteq \mathscr{P}x$ (Lemma 9.9,g), we deduce that $\mathscr{Q}(x) \in \mathbf{FP}$. Part 3. Follows from the Def. 9.1 of $\mathscr{Q}(x)$, Def 7.5,1 and Theorem 7.7,d. Part 4. Follows from Prop. 9.4,c and Lemma 9.19,c. Part 5: Follows from Part 4, Def. 6.21 of $\mathbf{S_0}$, and Def. 7.5 of $\mathbf{K_0}$. Part 6: Follows from Prop. 9.4,e and Lemma 9.19,c. Part 7. This is a restatement of Part 6 (see footnote 13).

Remark 9.22: Since $x \in \mathbf{FP}$, we can write $\mathscr{Q}(x)$ as $\mathscr{Q}(x) = \{y \subseteq \mathscr{R}x | P(x,y)\}$, where the proposition $P(x,y)$ is "$y \subseteq x$ and $\tilde{y} \cap x = y$". (See Prop. 9.20,6 and recall that $\mathscr{R}$ is a selector.) The claim, $P(x,y) \Leftrightarrow \forall n \in \mathbf{N},\ x \cap (1 \cup \mathscr{P})^n y = y$, completes the proof of the remark.

Proof of claim: We have (i) $\tilde{y} = (1 \cup \mathscr{P}_f)^\omega y$ (Def. 5.5), (ii) $y \subseteq x \Rightarrow x \cap y = y$ and (iii) $\forall n \in \mathbf{N},\ y \subseteq (1 \cup \mathscr{P}_f)^n y$ (since metadynamics(b) is accretive). (iii) implies (iv) $x \cap y \subseteq (1 \cup \mathscr{P}_f)^n y \subseteq x \cap (1 \cup \mathscr{P}_f)^\omega y$. Now note that (i), (ii) and (iv) give (v) $y \subseteq x \cap (1 \cup \mathscr{P}_f)^n y \subseteq x \cap \tilde{y}$. Hence from (v), we have $P(x,y) \Leftrightarrow \forall n \in \mathbf{N},\ x \cap (1 \cup \mathscr{P}_f)^n y = y$. Now both $x \in \mathbf{FP}$ and $y \subseteq x \Rightarrow y \in \mathbf{FP}$. This in turn implies that $\forall n \in \mathbf{N}$, $(1 \cup \mathscr{P}_f)^n y = (1 \cup \mathscr{P})^n y$.

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