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MATHEMATICAL FOUNDATIONS FOR FAST ALGORITHMS FOR THE BIHARMONIC EQUATION

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YALEU/DCS/TR-765
February 1990

A part of this work was supported by ONR grant N00014-89-J-1906.

ACKNOWLEDGEMENTS

I want to thank Profs. Vladimir Rokhlin and Leslie Greengard for their valuable advice, and the Department of Computer Science of Yale University and Prof. Martin Schultz for making resources available. I also want to express my thanks to Donka F. Farkas.

CONTENTS

CHAPTER 1

INTRODUCTION

Numerous problems in physics and engineering can be formulated in terms of the biharmonic equation

$$\Delta\Delta u = f \tag{1.1}$$

where Δ is the Laplace operator and u is subject to one of several sets of boundary conditions. Examples include the study of a clamped or supported plate in elasticity theory (see for example [14, 22, 23]), and steady Stokes flow, i.e. the steady motion of an incompressible fluid in the infinite viscosity limit (see for example [13, 29]).

We will restrict our attention to the homogeneous equation ($f = 0$ in (1.1)). The reason for this is twofold. Such problems are of physical interest in their own right, corresponding essentially to the absence of interior "load" in elasticity or "force" in Stokes flow. Moreover, the inhomogeneous problem can always be reduced to a homogeneous equation with different boundary conditions by standard techniques.

In this dissertation, we develop mathematical tools necessary for the construction of efficient and robust numerical algorithms for the solution of several boundary value problems for the biharmonic equation (see Section 2.1). We reduce each of these problems to a second kind integral (or in some cases integro-differential) equation on the boundary of the region by means of a local analysis of the singularity of the fundamental solution and its derivatives. The algorithms

for solving the boundary value problems based on these integral equations are of order $O(N + P)$, where N is the number of points in the discretization of the boundary, and P is the number of points in the domain at which the solution is to be computed. The derivation for all of these equations falls into a single unifying framework. Furthermore, emphasis has been placed on obtaining integral equations with smooth kernels whenever possible, which allows for much higher orders of convergence of the resulting numerical algorithms.

Most mathematical literature on the subject is devoted to the clamped plate problem (see Section 2.1), often referred to as the Dirichlet problem. Classically, fairly strong regularity conditions have been imposed both on the region and the boundary values for which the problem is to be solved (see, for example, [16, 17, 19, 20, 22, 24, 31]), but recently, much work has been devoted to relaxing these conditions (see [1, 4, 5, 7, 11, 12, 32]). However, none of these is suitable for efficient numerical computations, and they do not suggest any general way of solving other boundary value problems.

The purpose of this dissertation is to introduce an approach to the derivation of second kind integral equations for the solution of the biharmonic equation with the following properties:

1. The kernels and integral equations are suitable for numerical computation,

2. The same heuristic method for choosing the kernels can be applied to many other boundary value problems,

3. The method of analysis extends to three and higher dimensions.

The regularity conditions which need to be imposed are fairly strong, but they are natural in the sense that they are needed not only for the proofs of the theorems, but also for achieving a high order convergence of the numerical algorithms.

Previous numerical approaches fall into one of two categories, namely finite difference/finite element methods [3, 15, 25], and integral equation methods [6, 21, 26]. There are two serious difficulties associated with the first approach.

If a grid is laid out over the domain of interest so that there are $O(N)$ points in the discretization of the boundary and $O(N^2)$ points in the discretization of the interior, then the condition number of the resulting linear system will be proportional to N^4 (as opposed to second order partial differential equations, where the condition number is $O(N^2)$). This results in a catastrophic loss of precision even for relatively small N. Furthermore, while there exist fast direct methods for the solution of such linear systems resulting from the discretization of rectangular regions [3], no such method is applicable in regions of complicated shape. When such methods are implemented, the resulting computer programs tend to require excessive amounts of CPU time and to produce results with unsatisfactory accuracy.

Integral equation methods are based on either first or second kind equations. Second kind equations have long been known to be well-posed (see for example [30]) and to result in stable numerical algorithms (see for example [2]). In particular, the linear systems resulting from their discretization have asymptotically bounded condition numbers and the boundary integral operators often have smooth kernels [21, 27].

First kind integral equations, on the other hand, are ill-posed, their analysis is more difficult, and numerical algorithms for their solution are hard to design. Historically, first kind equations have been avoided, but recently several papers have appeared showing that so long as the kernel of such an equation is sufficiently singular, it is only weakly ill-posed and can be solved numerically. Nonetheless, the analysis of such problems is detailed, the resulting linear systems have high condition numbers, and the design of efficient numerical algorithms is difficult. Therefore, we restrict our attention to second kind equations, imitating the potential theory approach to the solution of the Dirichlet and Neumann problems for the Laplace equation [9].

In this dissertation, we solve three of the more frequently encountered boundary value problems for the biharmonic equation (problems (C), (F) and (S) of Section 2.1). While the reduction of problem (C) to second kind integral equations is discussed in [1, 4, 5, 16, 17, 19, 20, 22, 24, 31], and the reduction of problem (F) to second kind integral equations is discussed in [5, 21, 22], (although

the resulting equations are different from those obtained here), it appears that a boundary integral approach to (S) has not been considered previously. The kernels in the integral equations will be smooth, with the exception of one case, where the kernel has a logarithmic singularity. This facilitates the construction of accurate and efficient numerical algorithms for the solution of the problems.

Following is an outline of the dissertation.

Chapter 2 contains the statements of the boundary value problems we solve and a general description of the solution process.

Chapter 3 contains the definition of the kernels to be used and the proof of their smoothness. It also contains the computation of the diagonal terms in the integral equations (the jump conditions of multiple layer potentials), and a description of the heuristic reasons for choosing the kernels as we do.

Chapter 4 is devoted to a study of the existence and uniqueness of the solutions of the integral (or in some cases integro-differential) equations set up in Chapter 3. In the last section we return to the boundary value problems stated in Chapter 2. We discuss the fact that the solutions of the integral equations set up in Chapter 3 yield solutions of the respective boundary value problems via the potential formulas.

Chapter 5 has two parts. In the first part we describe the numerical algorithms which have been implemented based on the theory described in the previous chapters, and the principal numerical tools needed to implement them. Several numerical results are presented to illustrate the performance of the algorithms. In the second part, we close by stating further problems of interest.

CHAPTER 2

THE BOUNDARY VALUE PROBLEMS

2.1 Statement of the the Problems

Let D be a bounded domain in $\mathbf{R}^2$ with C^k boundary ∂D where $k \geq 2$. We will look at the following three problems:

The clamped plate problem (C): given C^0 functions g_1 and g_2 on ∂D, find a continuous function u on $\overline{D}$, so that

$$\begin{array}{lll} \Delta^2 u = 0 & \text{on} & D \\ u = g_1 & \text{on} & \partial D \\ \partial u/\partial \nu = g_2 & \text{on} & \partial D. \end{array} \qquad \text{(C)}$$

The fluid dynamics problem (F): given C^0 functions g_1 and g_2 on ∂D, find a continuous function u on $\overline{D}$, so that

$$\begin{array}{lll} \Delta^2 u = 0 & \text{on} & D \\ \partial u/\partial \nu = g_1 & \text{on} & \partial D \\ \partial u/\partial \tau = g_2 & \text{on} & \partial D. \end{array} \qquad \text{(F)}$$

The supported plate problem (S): assume $k \geq 4$, let g_1 of class C^2 and g_2 of class C^0 be given on ∂D, and let $\mu \in \mathbf{R}$ be a real number. Find a continuous function u on $\overline{D}$, so that

$$\begin{array}{lll} \Delta^2 u = 0 & \text{on} & D \\ u = g_1 & \text{on} & \partial D \\ \Delta u + \mu \cdot \partial^2 u/\partial \tau^2 = g_2 & \text{on} & \partial D. \end{array} \qquad \text{(S)}$$

In (C), (F), (S), the partial derivatives with respect to ν and τ are derivatives in the normal and tangent direction at the boundary (the precise definitions will be given in Section 3.1).

Note that problem (C) can be reduced to problem (F) by taking the derivative of g_1 in the tangent direction along the boundary, and problem (F) can be reduced to problem (C) by integrating g_1 along the boundary. Nevertheless, since numerical differentiation is unstable, it is worth stating and solving problem (C) as a problem in itself. Also, it is worth stating and solving problem (F) directly rather than reducing it to problem (C), since the extra effort in obtaining a direct solution is minimal, and a direct solution to problem (F) is a useful tool towards solving related problems.

In the case of problem (S) we will consider two separate cases. First we have problem (S0) (corresponding to the case $\mu = 0$): Assume ∂D is of class C^k with $k \geq 2$, and let h_1 and h_2 be C^0 functions on ∂D; find a continuous function u on $\overline{D}$, so that

$$\begin{array}{lll} \Delta^2 u = 0 & \text{on} & D \\ u = h_1 & \text{on} & \partial D \\ \Delta u = h_2 & \text{on} & \partial D. \end{array} \qquad \text{(S0)}$$

Second, we have the general problem stated before, with $\mu \neq 0$, which will be denoted (Sμ).

From a purely mathematical point of view (not counting computational complexity), problem (S0) is equivalent to the following two Poisson equations:

$$\begin{array}{lll} \Delta u = v & \text{on} & D \\ u = h_1 & \text{on} & \partial D \end{array} \qquad \text{(P1)}$$

and

$$\begin{array}{lll} \Delta v = 0 & \text{on} & D \\ v = h_2 & \text{on} & \partial D \end{array} \qquad \text{(P2)}$$

A computational approach based on this decomposition, however, is necessarily inefficient. In order to see this, let N represent the number of points used in the discretization of the boundary ∂D, let P represent the number of points

in the domain D where the values of the solution u are to be computed, and let M represent the number of points inside D needed to solve the Poisson equation (P1) (M is of the order $O(N^2)$, while frequently P is constant). An algorithm solving the two problems (P1) and (P2) is at best of the order $O(M) = O(N^2)$ since the values of v have to be computed at M points inside D. On the other hand, it will be shown in Section 5.1 that the algorithm based on the tools developed in this dissertation is of the order $O(N + P)$. Therefore, it is of interest to solve the problem (S0) directly.

2.2 Outline of the Solution to the Problems

The approach to solving these problems is the following: we try to represent u as a multiple layer potential

$$u(P) = \int [K_1(P,Q)\sigma_1(Q) + K_2(P,Q)\sigma_2(Q)]ds(Q), \tag{2.1}$$

where $P \in D$, and $\int f(Q)ds(Q)$ represents the integral of the function $f(Q)$ along ∂D with respect to arclength. If P is in a tubular neighborhood of ∂D in D then $\partial/\partial\nu$ and $\partial/\partial\tau$ at P are well defined. For each problem, we take the appropriate derivatives of u. For example, for (F) we obtain

$$\begin{aligned}
\frac{\partial u}{\partial \nu}(P) &= \int [\frac{\partial K_1}{\partial \nu_P}(P,Q)\sigma_1(Q) + \frac{\partial K_2}{\partial \nu_P}(P,Q)\sigma_2(Q)]ds(Q) \\
\frac{\partial u}{\partial \tau}(P) &= \int [\frac{\partial K_1}{\partial \tau_P}(P,Q)\sigma_1(Q) + \frac{\partial K_2}{\partial \tau_P}(P,Q)\sigma_2(Q)]ds(Q).
\end{aligned}$$

Take the limit as $P \to P_0 \in \partial D$. On the left hand side we get $g_1(P_0)$ and $g_2(P_0)$. Denote

$$\begin{aligned}
K_{11}(P_0,Q) &= \frac{\partial K_1}{\partial \nu_P}(P_0,Q) \\
K_{12}(P_0,Q) &= \frac{\partial K_2}{\partial \nu_P}(P_0,Q) \\
K_{21}(P_0,Q) &= \frac{\partial K_1}{\partial \tau_P}(P_0,Q) \\
K_{22}(P_0,Q) &= \frac{\partial K_2}{\partial \tau_P}(P_0,Q)
\end{aligned}$$

when $P_0 \in \partial D$. Next, we find D_{11} which satisfies the jump condition

$$\lim_{P \to P_0} \int K_{11}(P,Q)\sigma_1(Q)ds(Q) = D_{11}\sigma_1(P_0) + \int K_{11}(P_0,Q)\sigma_1(Q)ds(Q). \quad (2.2)$$

Similarly we will obtain D_{12}, D_{21} and D_{22}. This leads to a system in σ_1 and σ_2

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} \sigma_1(P_0) \\ \sigma_2(P_0) \end{pmatrix} + \quad (2.3)$$
$$\int \begin{pmatrix} K_{11}(P_0,Q) & K_{12}(P_0,Q) \\ K_{21}(P_0,Q) & K_{22}(P_0,Q) \end{pmatrix} \begin{pmatrix} \sigma_1(Q) \\ \sigma_2(Q) \end{pmatrix} ds(Q) = \begin{pmatrix} g_1(P_0) \\ g_2(P_0) \end{pmatrix},$$

where the integral is on ∂D and P_0 is an arbitrary point on ∂D.

For each of the problems stated, the expression of the kernels K_1 and K_2 will be given, and K_{ij} will be computed; the operators D_{ij} will turn out to be combinations of multiplication operators, Hilbert transforms and differentiation operators with respect to arclength. With these choices, almost all the kernels K_{ij} are smooth and those which are not smooth involve only logarithmic singularities. The latter pose no computational difficulties, since there are methods for computing integrals with singularities of this type efficiently (details will be given in Section 5.1).

The resulting integro-differential equations for σ_1 and σ_2 are of the second kind, or can be reduced to equations of the second kind, so they can be solved using the Fredholm theory. They will be studied in Section 4.2. Finally, we will discuss how the potential formula (2.1) provides a solution to the boundary value problem at hand with the charges arising from the integral equation.

From the analysis of the integral or integro-differential equations we will see that, based on the theory developed in this dissertation, we can construct efficient and stable algorithms for solving problems (C), (F) and (S0). The corresponding algorithm for solving problem (Sμ), while still preferable to algorithms based on finite difference or finite element methods, is not satisfactory. Indeed, it will be shown that the condition number of the linear system arising from the integro-differential equation which yields the charges for the potential formula has the condition number increasing as a fourth power of the number of points N in the

discretization of the boundary ∂D. From that point on, because of the limited usefulness of this approach for problem (Sμ), we will skip some of the details of the proofs for problem (Sμ), and we will give all the necessary proofs for solving problems (C), (F) and (S0).

CHAPTER 3

SETUP OF THE INTEGRAL EQUATIONS

This chapter is devoted to setting up the integral equations for each of the three problems listed in Section 2.1.

3.1 Notation

Given two vectors $v = (a, b)$ and $w = (x, y)$ in $\mathbf{R}^2$, their dot product $ax + by$ will be denoted $v \cdot w$. Let D represent a bounded domain in $\mathbf{R}^2$ with C^k boundary, where $k \geq 2$. Let ∂D be its boundary. The boundary is a closed curve, and $Q = Q(s) = (x(s), y(s))$ will usually denote a point on the curve, where s is the arclength parameterization of the curve. The unit tangent vector to the curve is given by $(x'(s), y'(s))$ and it will be denoted by τ. A function f on the curve, will also be written as $f(Q)$ or $f(x(s), y(s))$ or simply $f(s)$. The derivative $f'(s)$ will also be denoted as df/ds or $df/d\tau$. The unit normal vector to the curve will be denoted ν, and the orientation of the curve will be chosen so that the normal vector points outward. In terms of x and y we have $\nu = (y', -x')$. We define the curvature as $\kappa = -x''y' + y''x'$.

The boundary ∂D of D has a tubular neighborhood T in $\overline{D}$. T is homeomorphic to $\partial D \times (-\epsilon, 0]$ (for suitable ϵ) via $(Q, t) \longrightarrow Q + t\nu(Q)$. For a function u defined on T, differentiation with respect to ν and τ make sense in T, namely

$$\frac{\partial u}{\partial \nu}(Q + t\nu(Q)) = \nu(Q) \cdot \mathrm{grad} u(Q + t\nu(Q))$$

and

$$\frac{\partial u}{\partial \tau}(Q + t\nu(Q)) = \tau(Q) \cdot \text{gradu}(Q + t\nu(Q)).$$

We will also use the notation $\partial u/\partial \nu = \partial_\nu u = u_\nu$ and $\partial u/\partial \tau = \partial_\tau u = u_\tau$, and $\partial_{\nu\nu} u = u_{\nu\nu}$, etc., for higher order derivatives.

The derivatives u_ν, u_τ, etc., on the boundary ∂D are defined as

$$\frac{\partial u}{\partial \nu}(Q) = \lim_{\substack{t \to 0 \\ t<0}} \frac{\partial u}{\partial \nu}(Q + t\nu(Q))$$

and

$$\frac{\partial u}{\partial \tau}(Q) = \lim_{\substack{t \to 0 \\ t<0}} \frac{\partial u}{\partial \tau}(Q + t\nu(Q))$$

etc.. The derivative of u on the boundary exists if the limit is uniform. This is the meaning of the various derivatives in the statements of the boundary value problems in Section 2.1.

Note that if u is a function defined on the tubular neighborhood T of ∂D in $\overline{D}$, then taking the derivative $\partial u/\partial \tau$ in the sense of the uniform limit from the interior of T to ∂D is the same as taking the derivative of $du/ds = u'(s)$ of the restriction of u to ∂D. Indeed, they are both equal to $x'(s)(\partial u/\partial x) + y'(s)(\partial u/\partial y)$. This is not true however of the second derivatives $u_{\tau\tau} = \partial^2 u/\partial \tau^2$ and $u''(s) = d^2u/ds^2$ (the difference is $x''u_x + y''u_y$).

Given two points $P = (a, b)$ and $Q = (x, y)$ in the plane, denote $r = \sqrt{(x-a)^2 + (y-b)^2}$ and denote

$$G(P, Q) = \frac{1}{8\pi} r^2 \log r$$

The function G is the Green's function for the biharmonic operator in the plane in the sense that $\Delta^2 G = -\delta$, where δ is the Dirac delta function, or more precisely, $\Delta^2_Q G(P, Q) = -\delta(P)$.

In what follows we will use integrals of the form $\int K(P, Q)\sigma(Q)ds(Q)$. This will be an integral with respect to arclength along the boundary ∂D of D. The point Q is the point "moving" along the curve, while P will be thought of as

a point in D. Often P will be thought of as being in a neighborhood of ∂D, and $P_0 \in \partial D$ will be a limiting position of P. We will simply write P instead of P_0 when there will be no danger of confusion. The kernel $K(P,Q)$ under the integral will be some differential operator applied to $G(P,Q)$. The derivatives in the differential operator will be taken with respect to the normal and tangent directions at Q and P. These derivatives will be denoted by ∂_{ν_Q}, ∂_{τ_Q}, ∂_{ν_P}, ∂_{τ_P}. In order to simplify the notation we will always drop the index Q, so $\partial_\nu = \partial_{\nu_Q}$ and $\partial_\tau = \partial_{\tau_Q}$.

Given two points $P \in D$ and $Q \in \partial D$, denote $v =$ the vector from P to Q. G can be written also as

$$G(P,Q) = \frac{1}{16\pi}(v \cdot v)\log(v \cdot v)$$

We give now a simple proposition which will be useful in the computations of Section 3.2.

Proposition 3.1 *With the notation set in this section we have*

$$(v \cdot \nu)^2 + (v \cdot \tau)^2 = v \cdot v$$

and

$$(v \cdot \nu_P)(\nu \cdot \nu_P) + (v \cdot \tau_P)(\nu \cdot \tau_P) = v \cdot \nu.$$

Proof: The proof is straightforward and it will be omitted.

3.2 The Kernels

We define the kernels K_1 and K_2 in the potential formula (2.1) for the three problems (C), (F), (S) in Section 2.1 as follows:

For problem (C) take $K_1 = G_{\nu\nu\nu} + 3G_{\nu\tau\tau}$ and $K_2 = -G_{\nu\nu} + G_{\tau\tau}$.

For problem (F) take $K_1 = -G_{\nu\nu} + G_{\tau\tau}$ and $K_2 = G_{\nu\nu\nu} + 3G_{\nu\tau\tau}$.

For problem (S) take $K_1 = (6+\mu)G_{\nu\nu\nu} + 3(2+\mu)G_{\nu\tau\tau} = 6\Delta G_\mu + \mu(G_{\nu\nu\nu} + 3G_{\nu\tau\tau})$ and $K_2 = G_\nu$.

The heuristic reasoning for choosing these particular expressions as the kernels in the potential formulas will be explained in Section 3.5. Since $G(P,Q) = (1/16\pi)(v\cdot v)\log(v\cdot v)$, all these kernels can be expressed in terms of $v\cdot v$ and $v\cdot\nu$. Using proposition 3.1 we obtain the following expressions:

For (C):

$$\begin{aligned} K_1 &= \frac{1}{\pi}\frac{(v\cdot\nu)^3}{(v\cdot v)^2} \\ K_2 &= \frac{1}{2\pi}(\frac{1}{2} - \frac{(v\cdot\nu)^2}{v\cdot v}) \end{aligned}$$

For (F):

$$\begin{aligned} K_1 &= \frac{1}{2\pi}(\frac{1}{2} - \frac{(v\cdot\nu)^2}{v\cdot v}) \\ K_2 &= \frac{1}{\pi}\frac{(v\cdot\nu)^3}{(v\cdot v)^2} \end{aligned}$$

For (S):

$$\begin{aligned} K_1 &= \frac{1}{\pi}(3\frac{v\cdot\nu}{v\cdot v} + \mu\frac{(v\cdot\nu)^3}{(v\cdot v)^2}) \\ K_2 &= \frac{1}{8\pi}(v\cdot\nu)(\log(v\cdot v)+1) \end{aligned}$$

We will give now the explicit expressions of the kernels K_{11}, K_{12}, K_{21}, K_{221} in the integral equation (2.3) for each of the three problems (C), (F), (S). Using proposition 3.1 again, we obtain the following expressions:

For (C):

$$\begin{aligned} K_{11} &= K_1 = \frac{1}{\pi}\frac{(v\cdot\nu)^3}{(v\cdot v)^2} \\ K_{12} &= K_2 = \frac{1}{2\pi}(\frac{1}{2} - \frac{(v\cdot\nu)^2}{v\cdot v}) \\ K_{21} &= (K_1)_{\nu_P} = \frac{1}{\pi}(-3\frac{(v\cdot\nu)^2(\nu\cdot\nu_P)}{(v\cdot v)^2} + 4\frac{(v\cdot\nu)^3(v\cdot\nu_P)}{(v\cdot v)^3}) \\ K_{22} &= (K_2)_{\nu_P} = \frac{1}{\pi}(\frac{(v\cdot\nu)(\nu\cdot\nu_P)}{v\cdot v} - \frac{(v\cdot\nu)^2(v\cdot\nu_P)}{(v\cdot v)^2}) \end{aligned}$$

For (F):

$$
\begin{aligned}
K_{11} &= (K_1)_{\nu_P} = \frac{1}{\pi}\left(\frac{(v\cdot\nu)(\nu\cdot\nu_P)}{v\cdot v} - \frac{(v\cdot\nu)^2(v\cdot\nu_P)}{(v\cdot v)^2}\right)\\
K_{12} &= (K_2)_{\nu_P} = \frac{1}{\pi}\left(-3\frac{(v\cdot\nu)^2(\nu\cdot\nu_P)}{(v\cdot v)^2} + 4\frac{(v\cdot\nu)^3(v\cdot\nu_P)}{(v\cdot v)^3}\right)\\
K_{21} &= (K_1)_{\tau_P} = \frac{1}{\pi}\left(\frac{(v\cdot\nu)(\nu\cdot\tau_P)}{v\cdot v} - \frac{(v\cdot\nu)^2(v\cdot\tau_P)}{(v\cdot v)^2}\right)\\
K_{22} &= (K_2)_{\tau_P} = \frac{1}{\pi}\left(-3\frac{(v\cdot\nu)^2(\nu\cdot\tau_P)}{(v\cdot v)^2} + 4\frac{(v\cdot\nu)^3(v\cdot\tau_P)}{(v\cdot v)^3}\right)
\end{aligned}
$$

For (S):

$$
\begin{aligned}
K_{11} &= K_1 = \frac{1}{\pi}\left(3\frac{v\cdot\nu}{v\cdot v} + \mu\frac{(v\cdot\nu)^3}{(v\cdot v)^2}\right)\\
K_{12} &= K_2 = \frac{1}{8\pi}(v\cdot\nu)(\log(v\cdot v)+1)\\
K_{21} &= \Delta_P K_1 + \mu(K_1)_{\tau_P\tau_P} =\\
&\quad \frac{1}{\pi}\left(12\mu\frac{v\cdot\nu + (v\cdot\nu_P)(\nu\cdot\nu_P)}{(v\cdot v)^2} + 6\mu^2\frac{(v\cdot\nu)(\nu\cdot\tau_P)^2}{(v\cdot v)^2} -\right.\\
&\quad 4(\mu^2+2\mu)\frac{(v\cdot\nu)^3}{(v\cdot v)^3} + 24\mu\frac{(v\cdot\nu)^2(v\cdot\nu_P)(\nu\cdot\nu_P)}{(v\cdot v)^3} -\\
&\quad \left. 24\mu\frac{(v\cdot\nu)(v\cdot\nu_P)^2}{(v\cdot v)^3} - 24\mu\frac{(v\cdot\nu)^3(v\cdot\nu_P)^2}{(v\cdot v)^4}\right)\\
K_{22} &= \Delta_P K_2 + \mu(K_2)_{\tau_P\tau_P} =\\
&\quad \frac{1}{2\pi}\left(\frac{1}{2}(\mu+2)\frac{v\cdot\nu}{v\cdot v} - \mu\frac{(v\cdot\nu_P)(\nu\cdot\nu_P)}{v\cdot v} + \mu\frac{(v\cdot\nu)(v\cdot\nu_P)^2}{(v\cdot v)^2}\right)
\end{aligned}
$$

Note that these formulas define two different sets of kernels. On the one hand, one has the kernels defined for $(P,Q)\in T\times\partial D$, where $T\subset D$ is the intersection of a tubular neighborhood of ∂D in $\mathbf{R}^2$ with D; on the other hand one has the kernels defined for $(P,Q)\in\partial D\times\partial D$. There are two questions which have to be answered:

1) how smooth are the kernels on $\partial D\times\partial D$?

2) what is $\lim_{P\to P_0}\int K_{ij}(P,Q)\sigma(Q)ds(Q) - \int K_{ij}(P_0,Q)\sigma(Q)ds(Q)$ (where $\sigma(Q)$ is a function on ∂D, $P\in T\subset D$, and $P_0\in\partial D$)?

These questions will be answered in the next two sections. Then we will explain the heuristics of choosing the kernels; the choice is such that the answers to these questions should be as "nice" as possible.

3.3 Smoothness

In this section we will discuss the smoothness properties of the kernels K_{ij} computed in the previous section. Let us think of them as functions of two variables $K_{ij}(P,Q)$ defined on $\partial D \times \partial D$. Strictly speaking, they are defined only for $P \neq Q$; but the result of this section is exactly the fact that the kernels (with the possible exception of some terms for which we will have explicit elementary formulas) extend to smooth functions on $\partial D \times \partial D$. Therefore, one can afford the slight abuse of terminology of saying that they are defined on $\partial D \times \partial D$.

The main result of this section is contained in theorem 3.9. Before giving the theorem, we need some definitions and propositions.

Definition 3.2 *In this section we will denote by f, f_P, g, h_1, h_2, l the following expressions:*

$$\begin{aligned} f &= \frac{v \cdot \nu}{v \cdot v} \\ f_P &= \frac{v \cdot \nu_P}{v \cdot v} \\ g &= \frac{(\nu \cdot \tau_P)^2}{v \cdot v} \\ h_1 &= \frac{v \cdot \tau}{v \cdot v} \\ h_2 &= \frac{v \cdot \nu + (v \cdot \nu_P)(\nu \cdot \nu_P) - \frac{1}{6}\kappa'(P)(v \cdot \tau)(v \cdot v)}{(v \cdot v)^2} \\ l &= (v \cdot \nu)\log(v \cdot v). \end{aligned}$$

The following propositions refer to the smoothness of these functions. Before giving the statements,we will set some notation. Assume that the boundary ∂D is parameterized with respect to arclength. Let $Q = (x(s), y(s))$ and $P = (x(t), y(t))$ be two points on the curve. Then

$$\begin{aligned} v &= (x(s) - x(t), y(s) - y(t)) \\ \nu &= (y'(s), -x'(s)) \\ \tau &= (x'(s), y'(s)) \\ \nu_P &= (y'(t), -x'(t)) \\ \tau_P &= (x'(t), y'(t)) \end{aligned}$$

Proposition 3.3 *Assume that ∂D is real analytic. Then for t fixed we have:*

a) f, g and h_2 are analytic functions in s.

b) h_1 is the product of an analytic function in s with $1/(s-t)$.

c) $l = l_1 + l_2 \cdot (s-t)^2 \log(s-t)^2$, where l_1 and l_2 are analytic functions in s.

Proof: At points where $s \neq t$ we have $v \cdot v \neq 0$, so the kernels are analytic. The difficulty is to prove analyticity at $s = t$. At these points the obstruction to analyticity comes from a possible zero in the denominator of a fraction.

Let us assume that $t = 0$ in order to simplify the notation. We can assume $x(0) = y(0) = 0$. Let us expand $x(s)$ and $y(s)$ in power series around 0:

$$x(s) = \sum_{n=1}^{\infty} a_n s^n$$

and

$$y(s) = \sum_{n=1}^{\infty} b_n s^n$$

where we denoted for simplicity $a_n = (1/n!)x^{(n)}(0)$ and $b_n = (1/n!)y^{(n)}(0)$. We can now compute x' and y' and all the dot products appearing in f, g, h_1, h_2 and l. We can compute κ' and we can easily see that $\kappa'(0) = 6(-a_3 b_1 + a_1 b_3)$. We can also see immediately that the power series expansion of $v \cdot v$ is of order 2, and the leading coefficient is 1. More precisely

$$v \cdot v = s^2 + \sum_{n=3}^{\infty} [\sum_{\substack{i+j=n \\ i,j \geq 1}} (a_i a_j + b_i b_j)] s^n.$$

In fact the sum in the expression of $v \cdot v$ is $\sum_{n=4}^{\infty}$ because the coefficient of the term s^3 is $a_1 a_2 + b_1 b_2 = 0$. We are not writing it out explicitly because we will not need this fact. Whenever necessary, we will use without specifically pointing it out that $a_1^2 + b_1^2 = 1$ and $a_1 a_2 + b_1 b_2 = 0$. Now we will prove the statements of the proposition.

a) In order to prove that f and g are analytic, we have to prove that

$$\text{order}(v \cdot \nu) \geq 2$$

and

$$\text{order}(\nu \cdot \tau_P) \geq 1.$$

This follows immediately because the coefficient of s in the expansion of $v \cdot \nu$ turns out to be 1 and the constant term in the expansion of $\nu \cdot \tau_P$ turns out to be 0. The situation is essentially the same for

$$v \cdot \nu + (v \cdot \nu_P)(\nu \cdot \nu_P) - \frac{1}{6}\kappa'(P)(v \cdot \tau)(v \cdot v).$$

It is clear that the order of this expansion is greater than or equal to 2. But it is easy to check that the coefficients of s^2 and s^3 are 0.

b) The statement about h_1 follows immediately from the fact that $v \cdot \tau$ has an expansion in a power series of order 1 with leading coefficient 1, and $v \cdot v$ has an expansion in a power series of order 2 with leading coefficient 1.

c) We have

$$(v \cdot \nu)\log(v \cdot v) = (v \cdot \nu)\log(\frac{v \cdot v}{s^2}) + (v \cdot \nu)\log(s^2) =$$

$$(v \cdot \nu)\log(\frac{v \cdot v}{s^2}) + (-a_2 b_1 + a_1 b_2)(1 + ...)s^2 \log(s^2)$$

(The dots stand for higher order terms in the power series.)

The proof of the following proposition is very similar and it will be omitted.

Proposition 3.4 *Assume that ∂D is real analytic. Then for s fixed we have:*

a) f, g and h_2 are analytic functions in t.

b) h_1 is the product of an analytic function in t with $1/(s-t)$.

c) $l = l_1 + l_2 \cdot (s-t)^2 \log(s-t)^2$ where l_1 and l_2 are analytic functions in t.

We can now easily deduce the corresponding statements for these functions viewed as functions of two variables s and t.

Proposition 3.5 *Assume that ∂D is real analytic. Then:*

a) f, g and h_2 are analytic functions in s and t.

b) h_1 is the product of an analytic function in s and t with $1/(s-t)$.

c) $l = l_1 + l_2 \cdot (s-t)^2 \log(s-t)^2$ where l_1 and l_2 are analytic functions.

Proof: Each of the functions which was proved to be analytic separately in s and t in the preceding two propositions, has a complexification, that is, we can think of s and t as complex variables. On the other hand it is clear from the proof that these functions are bounded (in fact one can just as easily deduce that they are continuous). We can use a well known theorem from complex analysis, to deduce that these analytic functions are analytic as functions of the two complex variables s and t. It follows then that they are analytic as real functions of the real variables s and t as well.

We have similar results for the case when the boundary ∂D is not real analytic but only C^k. In this case the proof is more technical. Before proceeding with the proposition and the proof for this case, we need a technical lemma which will be used in the proposition.

Definition 3.6 *Define $\mathcal{C}$ to be the set of polynomials in the unknowns a_n, b_n, $n = 1, \ldots, \infty$. For a polynomial $N \in \mathcal{C}$ we will say that m dominates N if N is an element of the set of polynomials in a_n, b_n, $n = 1, \ldots, m$. In other words, m dominates N if no a_n or b_n appears in N for $n > m$. Whenever we will want to emphasize that m dominates N, and there will be no danger of confusion, we will write $N^{[m]}$ instead of N. Define $\mathcal{P}$ to be the set of power series in s and t with coefficients in $\mathcal{C}$.*

Note that the function $f = (v \cdot \nu)/(v \cdot v)$ is the quotient of two elements in $\mathcal{P}$. The following lemma refers to the derivatives of f.

Lemma 3.7 *The derivatives of order p of f are of the form f_1/f_2, where f_1, $f_2 \in \mathcal{P}$ have order and domination of their coefficients as specified in the following formulas:*

$$f_1 = \sum_{n=2p+2}^{\infty} \sum_{i+j=n} N_{ij}^{[i+j-p]} s^i t^j$$

and

$$f_2 = [(s-t)^2 + \sum_{n=3}^{\infty} \sum_{i+j=n} M_{ij}^{[i+j-1]} s^i t^j]^{p+1}.$$

Moreover, both the denominator and the numerator are divisible by $(s-t)^{2(p+1)}$.

Proof: The proof will be by induction. Expanding in power series and substituting, immediately yields

$$f(s,t) = \frac{\sum_{n=2}^{\infty} \sum_{i+j=n} N_{ij}^{[i+j]} s^i t^j}{(s-t)^2 + \sum_{n=3}^{\infty} \sum_{i+j=n} M_{ij}^{[i+j-1]} s^i t^j},$$

which gives the desired statement for $p = 0$.

Assume the statement true for p, and prove it for $p+1$. We will differentiate f_1/f_2 with respect to s for example, in the following way:

$$\frac{\partial(f_1/f_2)}{\partial s} = \frac{\sum_{n=2p+2}^{\infty} \sum_{i+j=n} i N_{ij}^{[i+j-p]} s^{i-1} t^j}{[(s-t)^2 + \sum_{n=3}^{\infty} \sum_{i+j=n} M_{ij}(i+j-1) s^i t^j]^{p+1}} +$$

$$\frac{[\sum_{n=2p+2}^{\infty} \sum_{i+j=n} N_{ij}^{[i+j-p]} s^i t^j](-p-1)[2(s-t) + \sum_{n=3}^{\infty} \sum_{i+j=n} i N_{ij}^{[i+j-1]} s^{i-1} t^j]}{[(s-t)^2 + \sum_{n=3}^{\infty} \sum_{i+j=n} N_{ij}(i+j-1) s^i t^j]^{p+2}}.$$

We will add up the two fractions. The denominator is clearly the one expected. Let us concentrate on the numerator. It is

$$[\sum_{n=2p+2}^{\infty} \sum_{i+j=n} i N_{ij}^{[i+j-p]} s^{i-1} t^j][(s-t)^2 + \sum_{n=3}^{\infty} \sum_{i+j=n} M_{ij}^{[i+j-1]} s^i t^j] -$$

$$(p+1)[\sum_{n=2p+2}^{\infty} \sum_{i+j=n}^{\infty} N_{ij}^{[i+j-p]} s^i t^j][2(s-t) + \sum_{n=3}^{\infty} \sum_{i+j=n} i M_{ij}^{[i+j-1]} s^{i-1} t^j].$$

Multiply out the terms, regroup them, while keeping track of degrees. We obtain that this is an expression of the form

$$\sum_{n=2p+3}^{\infty} \sum_{i+j=n} L_{ij}^{[i+j-p-1]} s^i t^j.$$

This is exactly of the type we wanted to prove with the exception of the fact that apparently, the sum starts at $2p+3$ instead of $2p+4$. In reality the sum starts from $2p+4$. Indeed, we know from lemma 3.5 that $f(s,t)$ is analytic. It must be the case then that any derivative of f is analytic. Since we know that the denominator is divisible by $(s-t)^{2p+4}$, the numerator must be divisible by $(s-t)^{2p+4}$ as well. This proves the last statement of the lemma, and it also implies that the terms of degree $2p+3$ at the numerator must have coefficients 0.

In the remainder of this section, the boundary ∂D of D will be assumed to be of class C^k with $k \geq 2$, except in one place, where we will need $k \geq 4$. This place will be pointed out when this stronger assumption will be needed.

Proposition 3.8 *Let $P_0 \in \partial D$ be a point on the curve ∂D. Then, in a suitable neighborhood of P_0 in ∂D, we have (s and t are values of the parameter giving points Q and P in this neighborhood):*

a) Assume that ∂D is C^k with $k \geq 2$. Then f and g are C^{k-2}.

b) Assume that ∂D is C^k with $k \geq 4$. Then h_2 is C^{k-4}.

c) Assume that ∂D is C^k with $k \geq 2$. Then h_1 is the product of a C^{k-2} function with $1/(s-t)$.

d) Assume that ∂D is C^k with $k \geq 2$. Then $l = l_1 + l_2 \cdot (s-t)^2 \log(s-t)^2$ where l_1 is C^{k-1} and l_2 is C^{k-2}.

Idea of the proof: Since the proof is long, we will give the idea of the proof first. In order to prove smoothness of a function $f(s)$ at a given point, say at 0, we would have to prove the following: $\lim_{s\to 0} f(s)$ exists (this defines f(0)), $\lim_{s\to 0}(f(s) - f(0))/s$ exists (this defines f'(0)), $\lim_{s\to 0} f'(s) = f'(0)$, and so on. In our case, $f(s) = A(x(s), y(s), x'(s), y'(s))/B(x(s), y(s), x'(s), y'(s))$ for some polynomials A and B. We know that if x and y are analytic then f is analytic. Denote a_n and b_n, $n = 1, 2, 3, ...$, the coefficients in the series expansions of x and y. The existence of the limits and the equalities written down for f are true because of certain cancellations and relations concerning the coefficients a_n and b_n. We will prove that the relations and cancellations which give the existence and continuity of the first $k-2$ derivatives of f, involve only a_n and b_n, $n = 1, ..., k$. It follows then that instead of working with the full power series of x and y, we could work just as well with the truncated power series. In other words, we could work with the Taylor polynomial expansions of x and y of degree k. This means that we can deduce the existence and continuity of the first $k-2$ derivatives under the assumption that x and y are C^k functions.

Proof: We will start by proving the first part of a), namely, that f is C^{k-2}. The proofs of the other parts of the proposition are very similar, and we will only point out the changes needed to adapt the proof.

Use the notation in proposition 3.3. We have

$$f(s,t) = \frac{v \cdot \nu}{v \cdot v} = \frac{(x(s) - x(t))y'(s) - (y(s) - y(t))x'(s)}{(x(s) - x(t))^2 + (y(s) - y(t))^2}. \tag{3.1}$$

We will proceed by induction.

Assume that $k = 2$. In this case we have to prove that f is continuous. Use Taylor's theorem for x, y, x', y'. We have that

$$\begin{aligned} x(s) &= a_1 s + a_2(\xi_1)s^2 \\ y(s) &= b_1 s + b_2(\xi_2)s^2 \\ x'(s) &= a_1 + 2a_2(\xi_3)s \\ y'(s) &= b_1 + 2b_2(\xi_4)s, \end{aligned}$$

where $a_n(\xi) = (1/n!)x^{(n)}(\xi)$ and $b_n(\xi) = (1/n!)y^{(n)}(\xi)$ (we can assume that $x(0) = y(0) = 0$). For the moment we will assume that t is fixed. Assume $t = 0$ to simplify the notation. Note that just like in the case of power series expansions, the denominator is dominated by s^2 and the numerator (which à-priori has order 1) has no degree one term, in other words is of order 2. Explicitly, we have

$$f(s,0) = \frac{(2a_1b_2(\xi_4) - a_1b_2(\xi_2) - 2a_2(\xi_3)b_1 + a_2(\xi_1)b_1)s^2 + (...)s^3}{s^2 + (...)s^3 + (...)s^4}.$$

We see that $\lim_{s\to 0} f(s,0) = a_1b_2 - a_2b_1$. Moreover, we see that the convergence of $f(s,t)$ to its limit $f(0,t)$ as $s \to 0$ is uniform in t (indeed, it depends only on the rate at which $a_2(t+\xi)$ and $b_2(t+\xi)$ approach $a_2(t)$ and $b_2(t)$ as $\xi \to 0$; but this rate is uniform in t since x'' and y'' are continuous, and therefore uniformly continuous on the compact set ∂D). Similarly, one can see that $\lim_{t\to 0} f(s,t) = f(s,0)$ uniformly with respect to s. It follows then that f is continuous.

Let us assume the statement true for $k-1$ (with $k > 2$). In other words, if x and y are C^{k-1} then f is C^{k-3}. Assume that x and y are C^k. We want to prove that f is C^{k-2}. We already know that f is C^{k-3}. Let F represent a derivative of order $k-3$ of f. We know that F is continuous, and we have to prove that F has derivatives with respect to s and t, and that these derivatives are continuous.

If it were possible, we would differentiate $k-3$ times $f(s,t)$ in the form given in (3.1); then we we would take the Taylor polynomial expansions of x, y, x', y', ... , $x^{(k-3)}$, $y^{(k-3)}$, and express F in terms of them; then we would prove the assertions by working directly on the coefficients a_n, b_n. It is impossible to do this explicitly, but it is possible to obtain some information about the way F can be expressed in terms of the coefficients a_n, b_n. First, we need to look at the case when x and y are analytic.

It follows from the lemma 3.7 that in the analytic case F is of the form

$$F(s,t) = \frac{\sum_{n=2k-4}^{\infty} \sum_{i+j=n} N_{ij}^{[i+j-k+3]} s^i t^j}{[(s-t)^2 + \sum_{n=3}^{\infty} \sum_{i+j=n} M_{ij}^{[i+j-1]} s^i t^j]^{k-2}}. \tag{3.2}$$

All the terms of degree less than $2k-4$ which could have appeared à-priori at the numerator, would have had coefficients dominated by $k-1$; they cancelled out. We have then that

$$F(s,0) = \frac{\sum_{i=2k-4}^{\infty} N_i^{[i-k+3]} s^i}{(s^2 + \sum_{i=3}^{\infty} M_i^{[i-1]} s^i)^{k-2}} \tag{3.3}$$

(we wrote N_i and M_i for N_{i0} and M_{i0}). The lowest order term s^{2k-4} at the numerator has its coefficient $N_{2k-4}^{[k-1]}$ dominated by $k-1$. This coefficient is giving the value $F(0,0)$. The coefficient of the next term $N_{2k-3}^{[k]}$ is dominated by k. Since we are dealing with analytic functions, we know that

$$\lim_{s \to 0} \frac{F(s,0) - F(0,0)}{s} = N_{2k-3} \tag{3.4}$$

(i.e. the derivative with respect to s exists), and

$$\lim_{s \to 0} \frac{\partial F}{\partial s}(s,0) = N_{2k-3} \tag{3.5}$$

(i.e. the derivative with respect to s is continuous).

We are finally ready to give one of the key arguments of the proof. Let us come back to the assumptions of the proposition, namely, that x and y are C^k functions. Let us assume that we differentiated f the right number of times with respect to s and t to get F. Therefore we have F expressed as a fraction of two

polynomials in x, y, x', y', ... , $x^{(k-1)}$, $y^{(k-1)}$. Let us take the Taylor polynomial expansions of degree k for x, y, of degree $k-1$ for x', y', and so on. The Taylor polynomial expansion of x is identical to the Taylor series expansion of x (if x were analytic) truncated to degree k, with the exception of the term s^k which has coefficient $a_k(\xi)$ (with some ξ) instead of a_k. The Taylor polynomial expansion of x' is identical to the Taylor series expansion of x' truncated to degree $k-1$, with the exception of the term s_{k-1} which has coefficient $ka_k(\xi)$ (with a possibly different ξ) instead of ka_k. And so on.

We have the following conclusions: all the cancellations and relations which occured in the analytic case and involved only coefficients a_n, b_n, $n = 1$, ... , $k-1$, will also occur in the C^k case. Also, all the cancellations and relations which occured under $\lim_{s\to 0}$ in the analytic case, and involved only a_n, b_n, $n = 1$, ... , k, will also occur in the C^k case. Moreover these limits will be uniform with respect to t because x and y are C^k and ∂D is compact.

Therefore, in the C^k case, $F(s,t)$ can be expressed as a quotient of two polynomials in s and t. The two polynomials are of order $2k-4$. The coefficients of the terms of degree $2k-4$ are identical to the ones in (3.2) and in (3.3). The coefficients of the terms of degree $2k-3$ look like the ones in the analytic case, except that whenever we have a_k or b_k in the analytic case, we will now have $a_k(\xi)$ or $b_k(\xi)$ in the C^k case (for various values of ξ). Recall that $\lim_{s\to 0} a_k(\xi) = a_k$ and $\lim_{s\to 0} b_k(\xi) = b_k$ in all cases where ξ appears.

In order to prove that $F(s,0)$ is differentiable with respect to s at 0 (and the derivative is N_{2k-3}) we have to prove equality (3.4). But in this equality, the coefficients which are of importance are dominated by $k-1$ respectively k, and therefore the equality is true in the C^k case, just like it was true in the analytic case.

Let us now show that this derivative is continuous. From lemma 3.7 it follows that in the analytic case $\partial F/\partial s$ is of the form

$$\frac{\partial F}{\partial s}(s,0) = \frac{\sum_{i=2k-2}^{\infty} L_i^{[i-k+2]} s^i}{(s^2 + \sum_{i=3}^{\infty} M_i^{[i-1]} s^i)^{k-1}}.$$

The lowest order term s^{2k-2} at the numerator has its coefficient $M^{[k]}_{2k-2}$ dominated by k. Since the right hand side in (3.5) is also dominated by k, it follows that (3.5) is also true in the C^k case. Clearly setting $t = 0$ did not alter the generality of the argument, so we proved that $\lim_{s\to 0}(F(s,t) - F(0,t))/s$ exists, and $\lim_{s\to 0}(\partial F/\partial s)(s,t)$ has the same limit. Also it is clear that this last limit is uniform with respect to t.

By exactly the same type of argument we can prove that $\lim_{t\to 0}(F(s,t) - F(s,0))/t$ exists, $\lim_{t\to 0}(\partial F/\partial t)(s,t)$ has the same limit, and that this last limit is uniform with respect to s. Therefore F is of class C^1. Therefore f is of class C^{k-2}.

We have to prove now the smoothness of the other functions in the statement. The proof for g is similar to the one for f. The only differences are in the initial step in the induction (that is the case $k = 2$) and in the initial step of the induction in the lemma corresponding to lemma 3.7. The changes are trivial to make, and we will not do them here.

In order to prove the smoothness statement about h_1, write it locally as

$$h_1(s,t) = \frac{1}{s-t} \cdot \frac{(s-t)(v\cdot\tau)}{v\cdot v}$$

The comments made for g apply to $(s-t)(v\cdot\tau)/(v\cdot v)$ as well.

The proof for h_2 is slightly different. Just like in the case of g the differences occur when proving the initial step in the induction (that is the case $k = 4$) and the initial step in the lemma corresponding to lemma 3.7. For the case $k = 4$ we take the following expansions

$$\begin{aligned}
x(s) &= a_1 s + a_2 s^2 + a_3 s^3 + a_4(\xi_1)s^4 \\
y(s) &= b_1 s + b_2 s^2 + b_3 s^3 + b_4(\xi_2)s^4 \\
x'(s) &= a_1 + 2a_2 s + 3a_3 s^2 + 4a_4(\xi_3)s^3 \\
y'(s) &= b_1 + 2b_2 s + 3b_3 s^2 + 4b_4(\xi_4)s^3.
\end{aligned}$$

A direct computation will yield that $h_2(s,0)$ has a limit as $s \to 0$ and the convergence of $h_2(s,t)$ to $h_2(0,t)$ is uniform in t. The initial step in the lemma

corresponding to lemma 3.7 is going to be that h_2 can be written as

$$h_2(s,t) = \frac{\sum_{n=4}^{\infty} \sum_{i+j=n} N_{ij}^{[i+j]} s^i t^j}{[(s-t)^2 + \sum_{n=3}^{\infty} \sum_{i+j=n} M_{ij}(i+j-1) s^i t^j]^2}.$$

The statement of the lemma has to be changed accordingly, but the proofs of the lemma and of the smoothness of h_2 proceed just like the ones for f.

Finally let us remark that the statement about the smoothness of l is clear once we write l as we did in the proof of part c) of proposition 3.3.

Theorem 3.9

a) All K_{ij} are of class C^{k-2} except for K_{12} and K_{21} in problem (S).

b) The kernel K_{21} from problem (S) is of the form

$$K_{21} = \lambda \frac{v \cdot \nu + (v \cdot \nu_P)(\nu \cdot \nu_P)}{(v \cdot v)^2} + \phi(P,Q),$$

where λ is a constant, and $\phi \in C^{k-2}(\partial D \times \partial D)$. The expression

$$\frac{v \cdot \nu + (v \cdot \nu_P)(\nu \cdot \nu_P)}{(v \cdot v)^2}$$

has a singularity of the type $1/(s-t)$; more precisely, if $k \geq 4$ then

$$\frac{v \cdot \nu + (v \cdot \nu_P)(\nu \cdot \nu_P)}{(v \cdot v)^2} - \frac{1}{6}\kappa'(P)\frac{v \cdot \tau}{v \cdot v} \tag{3.6}$$

is of class C^{k-4}, where κ' is the derivative of the curvature of the curve considered as a function of the variable P.

c) The kernel K_{12} from problem (S) is of class C^1, and it has second order derivatives, which have a logarithmic singularity.

d) if ∂D is real analytic then all terms in all kernels are real analytic, with the exception of the singularities pointed out in b) and c).

Remark 3.10 Note that the singularity of the type $1/(s-t)$ from the kernel K_{21} from problem (S) disappears in two particular cases: one is the case when $\mu = 0$. The other is the case of μ being arbitrary and D being a disk (in this case $\kappa' = 0$). In fact, the singularity will be "pulled out from under the integral" at the expense of a term representing a Hilbert transform in the "constant terms" (or in other words the "diagonal part"). The term in the kernel left under the integral will be the one from 3.6, so the kernel will be smooth.

Proof: At points where $P \neq Q$ we have $v \cdot v \neq 0$, so the kernels are as smooth as the various dot products appearing in the expressions of K_{ij}. Therefore, if the boundary ∂D is assumed to be C^k the kernels are of the class stated in the theorem at all points (P, Q) where $P \neq Q$. In a neighborhood of (P, P) the smoothness statements follow from proposition 3.8. Indeed, the expressions of K_{ij} are sums of products of f, f_P, g, h_1, h_2, l, and "integral" expressions in v, ν, ν_P, τ, τ_P (that is, expressions which are not fractions or logarithms).

3.4 Computation of the Diagonals of the Potential Formulas

The purpose of this section is to compute explicitly the matrices

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$$

appearing in equation (2.3) for each of the three problems (C), (F), (S). More explicitly, for each K_{ij} from each of the three problems we are trying to solve, we have to compute D_{ij} which satisfies

$$D_{ij}\sigma(P_0) = \lim_{P \to P_0} \int K_{ij}(P, Q)\sigma(Q)ds(Q) - \int K_{ij}(P_0, Q)\sigma(Q)ds(Q) \qquad (3.7)$$

for any C^∞ function σ on ∂D.

Let us start by stating the results of the computations in a theorem:

Theorem 3.11 *Under the same smoothness assumptions as in the previous section, we have the following results:*

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -\kappa & 1/2 \end{pmatrix} \text{ for problem (C),}$$

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} = \begin{pmatrix} 1/2 & -\kappa \\ 0 & \frac{1}{2}\frac{d}{ds} \end{pmatrix} \text{ for problem (F),}$$

and finally, for problem (S) we have

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(6+\mu) & 0 \\ -(\mu^2+2\mu)\kappa^2 + \frac{1}{2}(\mu^2+8\mu)\frac{d^2}{ds^2} - 2\mu\kappa'\mathcal{H} & \frac{1}{2} \end{pmatrix},$$

where κ represents the curvature of ∂D (we think of it as a function on ∂D), κ' is the derivative of κ, d/ds represents the derivative operator with respect to arclength, and $\mathcal{H}$ represents the Hilbert transform of a function on ∂D (the precise definition of the Hilbert transform is given below).

Definition 3.12 *Given a function $f(Q)$ on ∂D, the Hilbert transform $\mathcal{H}f$ of f is defined by*

$$\mathcal{H}f(P) = \frac{1}{\pi} \int \frac{v \cdot \tau}{v \cdot v} f(Q) ds(Q),$$

where v and τ and the notation are the ones set in Section 3.1.

The proof of the theorem will be split in two parts. In Part 1 we will compute all the entries D_{ij} with the exception of D_{12} and D_{21} in problem (S) at points on ∂D where the curvature κ of ∂D is $\kappa \neq 0$. In Part 2 we will discuss the remaining cases.

We will work now towards proving Part 1. First we will prove some propositions.

Proposition 3.13 *If D is the disk of radius r centered at the origin then it is enough to prove relation (3.7) when the function σ is $\sigma(Q) = \cos n(s/r)$ or $\sigma(Q) = \sin n(s/r)$, with $s \in [0, 2\pi r]$, and $P_0 = (r, 0)$.*

Proof: Depending on the kernel we are looking at we have to compute (3.7) for $\sigma \in C^k$ with $k = 0, 1$, or 2. In any case linear combinations of trigonometric polynomials are dense in the space of σ's, and the operators in (3.7) are continuous, so we can assume σ to be the functions in the statement. If $P_0 = (r \cos s_0, r \sin s_0)$ we can make $s = u + rs_0$ in the integrals. Using $\cos n(s/r) = \cos n(u/r) \cos ns_0 - \sin n(u/r) \sin ns_0$ and the similar relation for $\sin n(s/r)$ we see that we can assume that the point P_0 corresponds to the value 0 of the parameter.

Proposition 3.14 *The relation (3.7) is true in the case when D is the disk (this refers to Part 1, i. e. to all cases except D_{12} and D_{21} of problem (S)).*

Proof: Because of the previous proposition we can assume that $\sigma(Q) = \cos n(s/r)$ or $\sigma(Q) = \sin n(s/r)$. Verifying (3.7) in this case is a straightforward

computation. We take

$$Q = (r\cos(s/r), r\sin(s/r))$$
$$P_0 = (r, 0)$$
$$P = (a, 0)$$

(where we think of $a < r$ and $a \to r$). We have

$$v \cdot v = r^2 - 2ar\cos(s/r) + a^2$$
$$v \cdot \nu = r - a\cos(s/r)$$
$$v \cdot \nu_P = r\cos(s/r) - a$$
$$\nu \cdot \nu_P = \cos(s/r)$$
$$v \cdot \tau_P = r\sin(s/r)$$
$$\nu \cdot \tau_P = \sin(s/r).$$

The first integral in (3.7) is of the form

$$\int_0^{2\pi} \frac{F(r, \cos(s/r), \sin(s/r), a) \cdot \cos n(s/r)}{(r^2 - 2ar\cos(s/r) + a^2)^n} ds$$

or

$$\int_0^{2\pi} \frac{F(r, \cos(s/r), \sin(s/r), a) \cdot \sin n(s/r)}{(r^2 - 2ar\cos(s/r) + a^2)^n} ds$$

where F is a polynomial, while the second integral is

$$\int_0^{2\pi} \frac{F(r, \cos(s/r), \sin(s/r), r) \cdot \cos n(s/r)}{2r^{2n}(1 - \cos(s/r)^n} ds$$

or

$$\int_0^{2\pi} \frac{F(r, \cos(s/r), \sin(s/r), r) \cdot \sin n(s/r)}{2r^{2n}(1 - \cos(s/r)^n} ds.$$

(The integrands in the last integrals are not singular as we know from the smoothness theorem in the previous section; the zero which appears at the denominator is cancelling out with zeros appearing in the numerator.) These integrals can be computed explicitly. We have to take then $\lim_{a \to r}$. The computations are tedious, but completely straightforward, and we will not reproduce them here.

Proof of Part 1: K_{ij} will refer to any of the kernels from problems (C), (F) or (S) (with the exception of D_{12} and D_{21} from (S)). We will write K instead of K_{ij} in order to simplify the notation.

There is no loss of generality in assuming that $P_0 = (x(0), y(0))$ and $x(0) = y(0) = 0$ (so $P_0 = (0,0)$). Let $Q = (x(s), y(s))$, and $P = (ty'(0), -tx'(0))$ (think of t as $t < 0$ for now; the case $t > 0$ is treated exactly the same way). We will want to compute D from

$$D\sigma(P_0) = \lim_{P \to P_0} \int_{\partial D} K(P,Q)\sigma(Q)ds(Q) - \int_{\partial D} K(P_0,Q)\sigma(Q)ds(Q). \tag{3.8}$$

Let us denote C the circle tangent to ∂D at P_0 of radius $r = 1/\kappa$ (where $\kappa = \kappa(P_0)$ represents the curvature of ∂D at P_0) on the side opposite to ν_{P_0} (this is the "best fitting circle" to ∂D at the point P_0). Let $(x_C(s), y_C(s))$ denote the parametric equations of this circle in terms of arclength, so that $s = 0$ corresponds to P_0. The mapping $(x(s), y(s)) \longmapsto (x_C(s), y_C(s))$ establishes a diffeomorphism between a neighborhood N of P_0 in ∂D and a neighborhood N_C of P_0 in C.

Given a function σ on ∂D, one can define σ_C on C, by defining it on N_C through this diffeomorphism, and extending it smoothly to C (we can always change σ_C near the boundary of N_C in C if necessary). We consider the difference

$$D_C\sigma(P_0) = \lim_{P \to P_0} \int_C K(P,Q_C)\sigma_C(Q_C)ds(Q_C) - \int_C K(P_0,Q_C)\sigma_C(Q_C)ds(Q_C) \tag{3.9}$$

In order to prove Part 1, we will prove that $D = D_C$.

We will split all integrals into integrals on N or N_C (whichever is appropriate) and the complement of N and N_C. Let us take the difference of the expressions in (3.8) and (3.9) and regroup the terms. We want to show that all the differences obtained in this way are zero in the limit. We have to look at the following differences:

$$\int_{\partial D \setminus N} K(P,Q)\sigma(Q)ds(Q) - \int_{\partial D \setminus N} K(P_0,Q)\sigma(Q)ds(Q) \tag{3.10}$$

$$\int_{C \setminus N_C} K(P,Q_C)\sigma_C(Q_C)ds(Q_C) - \int_{C \setminus N_C} K(P_0,Q_C)\sigma_C(Q_C)ds(Q_C) \tag{3.11}$$

$$\int_N K(P_0,Q)\sigma(Q)ds(Q) - \int_{N_C} K(P_0,Q_C)\sigma_C(Q_C)ds(Q_C) \qquad (3.12)$$

$$\int_N K(P,Q)\sigma(Q)ds(Q) - \int_{N_C} K(P,Q_C)\sigma_C(Q_C)ds(Q_C). \qquad (3.13)$$

It is clear that once N and N_C are fixed, the differences (3.10) and (3.11) tend to 0 in the limit as $P \to P_0$. It is easy to see that (3.12) can be made smaller than any given ϵ by choosing N and N_C small enough (indeed, one can argue for example that since K and K_C are continuous on ∂D and C, $|K(P_0,Q) - K(P_0,P_0)|$ and $|K_C(P_0,Q) - K_C(P_0,P_0)|$ can be made as small as we want).

The non-trivial part of this proposition is showing that (3.13) can be made small by choosing N and N_C small enough (independently of P). We will show this by proving that, with the proper choice of N and N_C (and independently on P approaching P_0 along the normal direction to the boundary),

$$\int_N \frac{L(P,Q)}{M(P,Q)}\sigma(Q)ds(Q) - \int_{N_C} \frac{L(P,Q_C)}{M(P,Q_C)}\sigma_C(Q)ds(Q)$$

can be made small for any of the fractions L/M appearing in any of the kernels K under consideration. (One can work directly on K or on the individual fractions; we chose to work on the individual fractions here.) The various possibilities for M are $(v\cdot v)^n$, $n = 1,2,3$. Notice that the various possibilities for L are monomials in $(v\cdot\nu)$, $(v\cdot\nu_P)$, $(\nu\cdot\nu_P)$, $(v\cdot\tau_P)$ and $(\nu\cdot\tau_P)$.

Let us recall that we took $Q = (x(s),y(s))$, $P_0 = (0,0)$, $P = (ty'(0),-tx'(0))$ $(t<0)$, and let $Q_C = x_C(s),y_C(s))$. Let us start by studying $L(P,Q)/M(P,Q)$. We have:

$$v = (x(s) - ty'(0), y(s) + tx'(0)$$

$$\nu = (y'(s), -x'(s))$$

$$\nu_P = (y'(0), -x'(0))$$

$$\tau = (x'(s), y'(s))$$

$$\tau_P = (x'(0), y'(0)).$$

Just like we did in the previous section, let us take the Taylor expansions for x, y, x', y'. We have:

$$\begin{aligned} x(s) &= a_1 s + a_2(\xi_1)s^2 \\ y(s) &= b_1 s + b_2(\xi_2)s^2 \\ x'(s) &= a_1 + 2a_2(\xi_3)s \\ y'(s) &= b_1 + 2b_2(\xi_4)s. \end{aligned}$$

In this notation $a_1 = x'(0)$ and $b_1 = y'(0)$. It follows that

$$\begin{aligned} v \cdot v &= (s^2 + t^2) + 2o_{12}s^3 + \kappa_{12}s^2 t + \gamma_1 s^4 \\ v \cdot \nu &= -t - \frac{\kappa_{12}}{2}s^2 + \kappa_{34}s^2 - 2o_{34}st + \gamma_2 s^3 \\ v \cdot \nu_P &= -t - \frac{\kappa_{12}}{2}s^2 \\ \nu \cdot \nu_P &= 1 + 2o_{34}s \\ v \cdot \tau_P &= s + o_{12}s^2 \\ \nu \cdot \tau_P &= \kappa_{34}s, \end{aligned}$$

where

$$\begin{aligned} \kappa_{12} &= 2(a_1 b_2(\xi_2) - a_2(\xi_1)b_1) \\ \kappa_{34} &= 2(a_1 b_2(\xi_4) - a_2(\xi_3)b_1) \\ o_{12} &= a_1 a_2(\xi_1) + b_1 b_2(\xi_2)) \\ o_{34} &= a_1 a_2(\xi_3) + b_1 b_2(\xi_4)) \\ \gamma_1 &= a_2(\xi_1)^2 + b_2(\xi_2)^2 \\ \gamma_2 &= a_2(\xi_1)b_2(\xi_4) - a_2(\xi_3)b_2(\xi_2). \end{aligned}$$

Note that by taking N small enough, o_{12} and o_{34} can be made as small as we want. Since this is all we need in this proof, we will drop the indexes and write simply o. Also, the values of κ_{12} and κ_{34} are close to the curvature κ at P_0, so we will drop the indexes and write $\kappa + o$ instead. We will use rules like $o + o = o$,

$g \cdot o = o$ for any bounded function g, etc.. With this notation we have

$$\begin{aligned} v \cdot v &= (s^2 + t^2) + os^3 + (\kappa + o)s^2 t + \gamma_1 s^4 \\ v \cdot \nu &= -t + (\frac{\kappa}{2} + o)s^2 + ost + \gamma_2 s^3 \\ v \cdot \nu_P &= -t - (\frac{\kappa}{2} + o)s^2 \\ \nu \cdot \nu_P &= 1 + os \\ v \cdot \tau_P &= s + os^2 \\ \nu \cdot \tau_P &= (\kappa + o)s. \end{aligned} \tag{3.14}$$

In order to make things easier let us introduce some notation.

Definition 3.15 *Let P denote the set of polynomials of the form $f = f_1 + f_2 + f_3$ where f_1 is a homogeneous polynomial of degree n in s and t with constant coefficients, f_2 is a homogeneous polynomial of degree $n+1$ in s and t with coefficients of the type $\alpha\kappa + \beta o + \gamma$, and f_3 is a polynomial of order $n+2$ in s and t with coefficients which are themselves polynomials in a_1, b_1, $a_2(\xi_i)$, $b_2(\xi_i)$. The number n will be called the order of f.*

Remark that if f and g are two elements in $\mathcal{P}$ of orders p and q, then the product fg is an element in $\mathcal{P}$ of order $p+q$. Also note that all the entries in (3.14) are elements of $\mathcal{P}$.

Let us turn our attention back to the fractions L/M. It is clear that both L and M are elements of $\mathcal{P}$. Since $M = (v \cdot v)^n$ the order of M is $2n$. By looking at the explicit expressions which are the possible values for L, we see that the order p of L satisfies $p \geq 2n-2$; (p can be $2n-2$, $2n-1$, or $2n$; the worse case for the estimates which follow is $p = 2n-2$). Explicitly,

$$\frac{L(P,Q)}{M(P,Q)} = \frac{f_1 + f_2 + f_3}{g_1 + g_2 + g_3}.$$

All this has been written for $Q \in \partial D$. We will now look at the changes which need to be made when we replace ∂D by C. Since everything we did is expressed in terms of the coefficients of the Taylor expansions of x, y, x', y', the only items which are different are the values of $a_2(\xi_i)$, $b_2(\xi_i)$. Therefore

$$\frac{L(P,Q_C)}{M(P,Q_C)} = \frac{f_1 + \tilde{f}_2 + \tilde{f}_3}{g_1 + \tilde{g}_2 + \tilde{g}_3},$$

where the coefficients of f_2 and $\tilde{f}_2$ differ by a quantity of o, the coefficients of g_2 and $\tilde{g}_2$ differ by a quantity of o, and we have similar statements for f_3 and $\tilde{f}_3$, and g_3 and $\tilde{g}_3$.

We are trying to see that

$$\int_N \left| \frac{f_1+f_2+f_3}{g_1+g_2+g_3} - \frac{f_1+\tilde{f}_2+\tilde{f}_3}{g_1+\tilde{g}_2+\tilde{g}_3} \right| |\sigma(s)| ds$$

can be made arbitrarily small by choosing N small enough. Subtracting the two fractions under the absolute value we get F/G, where $F = (F_1+F_2+F_3) \in \mathcal{P}$ is of order $p+2n$, and $G = (G_1+G_2+G_3) \in \mathcal{P}$ is of order $4n$. Clearly $F_1 = 0$, the coefficients of F_2 and F_3 are all equal to some o quantities, and $G_1 = (s^2+t^2)^{2n}$. The o quantities can be made smaller than any given δ if N is small enough. Using this and some obvious upper bounds, we get

$$\int_N \left| \frac{f_1+f_2+f_3}{g_1+g_2+g_3} - \frac{f_1+\tilde{f}_2+\tilde{f}_3}{g_1+\tilde{g}_2+\tilde{g}_3} \right| |\sigma(s)| ds \leq$$

$$C \left(\int_N \frac{|F_2|}{(s^2+t^2)^{2n}} ds + \int_N \frac{|F_3|}{(s^2+t^2)^{2n}} ds \right)$$

for suitable C. Since $p \geq 2n-2$, we have that the degree of F_3 is at least $4n$. It is easy to see that for any polynomial H of degree $\geq 4n$, the fraction $H/(s^2+t^2)^{2n}$ is bounded for s and t varying in finite intervals. Since in addition the coefficients of F_3 are o quantities, it follows that

$$\int_N \frac{|F_3|}{(s^2+t^2)^{2n}} ds < C\delta$$

with suitable C.

The degree of F_2 (which is $p+2n+1$) is at least $4n-1$. The case which is not proven by the preceding arguments, is $p = 2n-2$. In this case the degree of F_2 is $4n-1$. To get the estimate in this case we need a lemma.

Lemma 3.16 *The polynomial F_2 has no term in s^{4n-1}, or in other words, all terms in F_2 contain the factor t.*

Proof of the lemma: Let us look at $(f_1+f_2+f_3)/(g_1+g_2+g_3)$. We know from the smoothness theorem in the preceding section that this has to be a smooth function when $t = 0$. It follows from the proof of the smoothness theorem that since the denominator is of order $2n$, the numerator has to have order at least $2n$ when we set $t = 0$. It follows that f_1 (which, as we know, is a homogeneous polynomial of degree $2n-2$ in s and t) has no term in s^{2n-2} (that is, all terms have to contain t). Also, f_2 has no term in s^{2n-1}.

On the other hand, $F_2 = f_1(\tilde{g}_2 - g_2) + g_1(f_2 - \tilde{f}_2)$. It is clear then that all the terms in the homogeneous polynomial F_2 will contain t.

Let us finish the proof of Part 1 now. We have

$$\int_N \frac{|F_2|}{(s^2+t^2)^{2n}} ds \leq \int_N |o| \cdot \frac{|t|}{s^2+t^2} \cdot \left| \frac{H_1}{(s^2+t^2)^{2n-1}} \right| ds,$$

where H_1 is a polynomial of degree $4n-2$. Therefore this last integral is bounded by $C\delta \int_N t/(s^2+t^2)ds$ for suitable C. But $\int t/(s^2+t^2)ds = \arctan(s/t)$ which is a bounded function. This finishes the proof of Part 1.

Proof of Part 2: The proof for the entry D_{12} from problem (S) is very simple. In this case one can move the limit under the integral, and it follows that $D_{12} = 0$.

The proof for the entry D_{21} is more complicated. The general idea is the same, namely, we will reduce the problem to a particular case, and do the computations on that particular case. But the particular case can not be taken to be a circle any more; instead, we will take it to be an ellipse. Let C represent an ellipse tangent to the curve ∂D at P_0, so that the curvatures, and the derivatives of the curvatures at P_0 for C and ∂D are equal. We define the operator D_C by the formula (3.9), and we have to prove that $D = D_C$. We split the integrals into various pieces, and show that the differences (3.10), (3.11), (3.12), (3.13) are small in the limit when the neighborhoods of P_0 are small enough. The non-trivial part is (3.13).

Like in the smoothness theorem, we assume that the boundary ∂D is C^k with $k \geq 4$. We expand x, y, x', y' as follows:

$$x(s) = a_1 s + a_2 s^2 + a_3 s^3 + a_4(\xi_1) s^4$$

$$y(s) = b_1 s + b_2 s^2 + b_3 s^3 + b_4(\xi_2)s^4$$
$$x'(s) = a_1 + 2a_2 s + 3a_3 s^2 + 4a_4(\xi_3)s^3$$
$$y'(s) = b_1 + 2b_2 s + 3b_3 s^2 + 4b_4(\xi_4)s^3.$$

The fractions L/M in the expression of K_{21} have $M = (v \cdot v)^n$ with $n = 2, 3, 4$, and L can be $v \cdot \nu + (v \cdot \nu_P)(\nu \cdot \nu_P) - (\kappa'/6)(v \cdot \tau)(v \cdot v)$, $(v \cdot \nu)(\nu \cdot \tau_P)^2$, $(v \cdot \nu)^3$, $(v \cdot \nu)^2(v \cdot \nu_P)(\nu \cdot \nu_P)$, $(v \cdot \nu)(v \cdot \nu_P)^2$, $(v \cdot \nu)^3(v \cdot \nu_P)^2$. A straightforward computation shows that all these are polynomials in s and t of the form $f = f_1 + f_2 + f_3 + f_4$, where f_1 is a homogeneous polynomial in s and t of some degree n with constant coefficients, f_2 is a homogeneous polynomial of degree $n + 1$ in s and t with coefficients of the type $\alpha\kappa + \beta$, f_3 is a homogeneous polynomial of degree $n + 2$ in s and t with coefficients which are polynomial expressions in κ and κ', and f_4 is a polynomial of order $n + 3$ with coefficients which are polynomials in all the coefficients of the expansions of x, y, x', y'. From this point on, the proof of the fact that that (3.13) is small, is exactly the same as the proof given before, and we will skip the details.

We can assume therefore, that ∂D is an ellipse, given by $x = a\cos t$, $y = b\sin t$, and that σ is one of $\sigma(Q) = \cos nt$ and $\sigma(Q) = \sin nt$, $t \in [0, 2\pi]$. (We can not assume though that P_0 is some previously chosen point on the ellipse.) The computation of the integrals on the ellipse can be done explicitly, and we will not give the details here.

We conclude this proof by pointing out that if ∂D has points where the curvature is 0, then a continuity argument proves (3.8) in the case P_0 is isolated, or in the case when P_0 has arbitrarily close points P_1 where the curvature is not 0. If the curvature is identically 0 in a neighborhood of P_0 then one can use a straight line instead of a circle (with the obvious modifications in the proof).

Corollary 3.17 *Let u be given by*

$$u(P) = \int_{\partial D} [K_1(P, Q)\sigma_1(Q) + K_2(P, Q)\sigma_2(Q)]ds(Q)$$

where $K_1 = G_{\nu\nu\nu} + 3G_{\nu\tau\tau}$ and $K_2 = -G_{\nu\nu} + G_{\tau\tau}$ are the kernels for the problem (C). Let u and u_ν be defined in a tubular neighborhood of $\mathbf{R}^2$. Let u_- and u_+ be

the limits of u as $P \to \partial D$ from inside and from outside, respectively, and $(u_\nu)_-$ and $(u_\nu)_+$ be defined in a similar way. Then $u_- - u_+ = \sigma_1$ and $(u_\nu)_- - (u_\nu)_+ = \sigma_2$.

Proof: In theorem 3.11 we computed the values of u_- and $(u_\nu)_-$ in terms of the matrix (D_{ij}), (σ_1, σ_2), and a boundary integral. Clearly a similar statement holds for u_+ and $(u_\nu)_+$ except that the sign in front of the non-integral term is opposite, and the curvature κ is computed from the outside, so it has opposite sign. Subtracting the two relations leads to the desired result.

3.5 Heuristic Reasons for Choosing the Kernels

This section contains a brief description of the heuristic reasoning underlying the choice of kernels in the potential formulas for the various problems.

As a starting point consider Green's function $G = (1/8\pi)r^2 \log r$. Written in terms of P and Q this is $G(P,Q) = (1/16\pi)(v \cdot v)\log(v \cdot v)$ where v is the vector from P to Q. By analogy with the solutions to boundary value problems for the Laplace equation, we seek a potential formula for u in the form

$$u(P) = \int_{\partial D} [K_1(P,Q)\sigma_1(Q) + K_2(P,Q)\sigma_2(Q] ds(Q),$$

where K_1 and K_2 are of the form $\sum \alpha_{pq} \partial_\nu{}^p \partial_\tau{}^q G$, where ∂_ν and ∂_τ represent differentiation in the normal and tangent directions to ∂D at Q, p and q are non-negative integers, and α_{pq} are real numbers. One would like to choose expressions for K_1 and K_2 that would achieve two goals:

1) K_1 and K_2 should be as smooth as possible when both P and Q are on the boundary ∂D of the domain D.

2) The resulting integral equation should be as close to a second kind equation as possible. In other words, the diagonal terms should contain only "nice" operators, like multiplications with constants or intrinsic functions, Hilbert transforms, derivatives with respect to arclength, and the matrix of diagonal terms

should be close to being non-singular in the sense that its kernel should have low dimension.

Analysis of the special cases where the domain D is the upper half plane or a disk of variable radius is nearly sufficient for finding the kernels. The procedure is as follows:

1) Compute all the partial derivatives $K_{pq} = \partial_\nu{}^p \partial_\tau{}^q G$ of G up to order six (less might be enough, depending on the order of differentiation in the boundary conditions).

2) For each partial derivative K_{pq}, compute

$$\int_{\partial D} K_{pq}(P,Q)\sigma(Q)ds(Q)$$

and

$$\int_{\partial D} K_{pq}(P_0,Q)\sigma(Q)ds(Q),$$

where P is in a tubular neighborhood of ∂D in D and P_0 is on ∂D, using the test functions $\sigma = \cos ns$ and $\sigma = \sin ns$. This can be done explicitly for the special cases singled out above.

If K_1 represents a linear combination of the form $K_1 = \sum \alpha_{pq} \partial_\nu{}^p \partial_\tau{}^q G$, then denote by $\mathcal{D}_1 = \mathcal{D}(K_1)$ the operator

$$\mathcal{D}(K_1)\sigma = \lim_{P \to P_0} \int K_1(P,Q)\sigma(Q)ds(Q) - \int K_1(P_0,Q)\sigma(Q)ds(Q).$$

The computations performed thus far, enable us to write down explicit results for $\mathcal{D}_1\sigma$ in the cases $\sigma = \cos n(s-s_0)$ and $\sigma = \sin n(s-s_0)$, when the limits exist. Therefore, when the limits exist, we can "guess" the expression of the operator $\mathcal{D}_1$ for given K_1.

3) Let ∂_1 and ∂_2 be the differential operators appearing in the boundary conditions of the given boundary value problem. If K_1 and K_2 represent two linear combinations of partial derivatives of G, let M denote the 2×2 matrix of operators computed in the previous step

$$\begin{pmatrix} \mathcal{D}(\partial_1 K_1) & \mathcal{D}(\partial_1 K_2) \\ \mathcal{D}(\partial_2 K_1) & \mathcal{D}(\partial_2 K_2) \end{pmatrix}$$

Choose the linear combinations giving the kernels K_1 and K_2 in such a way that M is as close to being non-singular as possible. If several choices are possible, try to make $\partial_i K_j$ as smooth as possible.

Note that this work yields more than just the kernels: in the process of trying to find the kernels one also finds a conjecture as to what the matrix of diagonal terms is going to be.

The use of computers can be of enormous help in carrying out this guesswork. One can use a symbolic manipulation package to carry out a large part of the computations. Computations done on paper (for instance that of a partial derivative of a complicated expression) can also be verifyed by numerical experimentation. Furthermore, the computations needed to guess and verify the diagonal terms, as well as verification of smoothness, can be done numerically on some relevant examples.

CHAPTER 4

THE INTEGRAL EQUATIONS AND THEIR USE

4.1 Classical Formulas

This section contains some geometric formulas which will be necessary in the study of the integral equations.

Proposition 4.1 *Let κ represent the curvature of ∂D (think of it as a function on ∂D) and G represent Green's function (the notation is the same as in the previous chapter). Then the clamped plate potential given by the functions*

$$\sigma_1 = 2\kappa \text{ and } \sigma_2 = 1$$

is piecewise constant. More precisely if $u(P)$ is given by

$$u(P) = \int_{\partial D} [G_{\nu\nu\nu}(P,Q) + 3G_{\nu\tau\tau}(P,Q)] + 2\kappa(Q)[-G_{\nu\nu}(P,Q) + G_{\tau\tau}(P,Q)]ds(Q)$$

then

$$u(P) = \begin{cases} 1 & \text{if } P \in D \\ 1/2 & \text{if } P \in \partial D \\ 0 & \text{if } P \notin D. \end{cases}$$

Proof: The proof is easier if one uses complex plane notation. Let $z = x + iy$, $P = a$, $Q = z$, $v = z - a$. Then $\tau = z'$ and $\nu = -iz'$. We have

$$\tau = \text{Re}[(z-a)\overline{z'}]$$

$$\nu = -\text{Im}[(z-a)\overline{z'}]$$

$$\kappa = \text{Im}(z''\overline{z'}).$$

Using the explicit formulas for the kernels $G_{\nu\nu\nu}+3G_{\nu\tau\tau}$ and $-G_{\nu\nu}+G_{\tau\tau}$, we have

$$u(P)=\int[G_{\nu\nu\nu}(P,Q)+3G_{\nu\tau\tau}(P,Q)]+2\kappa(Q)[-G_{\nu\nu}(P,Q)+G_{\tau\tau}(P,Q)]ds(Q)=$$

$$\frac{1}{\pi}\int-\frac{(\mathrm{Im}[(z-a)\overline{z'}])^3}{|z-a|^4}+\left(\frac{1}{2}-\frac{(\mathrm{Im}[(z-a)\overline{z'}])^2}{|z-a|^2}\right)\mathrm{Im}(z''\overline{z'})ds.$$

Denote $w=(z-a)\overline{z'}$. Using the fact that the derivative z' is taken with respect to arclength, a simple computation shows that $|z-a|^2=|w|^2$ and $z''\overline{z'}=((\overline{w'-1})w)/|w|^2$. Substituting in the expression of $u(P)$ and computing we obtain

$$u(P)=\frac{1}{2\pi}\int\frac{\mathrm{Im}(\overline{w'}w)[(\mathrm{Re}w)^2-(\mathrm{Im}w)^2]-\mathrm{Im}w|w|^2}{|w|^4}ds.$$

Splitting into two fractions and making some obvious computations leads to

$$u(P)=\frac{1}{2\pi}\mathrm{Re}\int\frac{\mathrm{Im}(\overline{w'}w)w^2}{|w|^4}ds-\frac{1}{2\pi}\mathrm{Im}\int\frac{w}{|w|^2}ds=$$

$$\frac{1}{4\pi i}\mathrm{Re}\int-\left(\frac{w}{\overline{w}}\right)'ds-\frac{1}{2\pi}\mathrm{Im}\int\frac{1}{\overline{w}}ds.$$

The first integral of this last line is always 0 since it is the integral of an exact form on a closed curve. In the second integral replace back $w=(z-a)\overline{z'}$ to get

$$u(P)=-\frac{1}{2\pi}\mathrm{Im}\int\frac{\overline{z'}}{\overline{z-a}}ds=\frac{1}{2\pi}\mathrm{Im}\int\frac{\mathrm{d}z}{z-a}.$$

The value of this integral is 1 or $1/2$ or 0 depending on the position of a relative to D.

We will now give two generalizations of Green's formula associated with the biharmonic operator. First we need a definition:

Definition 4.2 *Let u be a C^2 function on a domain D in $\mathbf{R}^2$. Define the vector field* bigrad$u$ *relative to the (x,y) coordinates to be*

$$\mathrm{bigrad}u=(u_{xx}-u_{yy})\mathrm{i}+2u_{xy}\mathrm{j}$$

where i *and* j *are the unit tangent vectors in the x and y directions.*

Remark 4.3 This vector field is dependent on the coordinates relative to which it is defined, and on their orientation. The bilinear form defined by the dot product bigrad$u \cdot$ bigrad$v$ is independent on the coordinate system. Also note that this is a non-negative definite bilinear form.

The proof of these facts is trivial and it will be omitted.

Note that one can also talk of the bigradient as an operator, namely

$$\text{bigrad} = \text{i}(\partial_{xx} - \partial_{yy}) + \text{j}2\partial_{xy}.$$

Proposition 4.4 *Let u and v two C^4 functions on $\overline{D}$. Then*

$$\int_D \Delta u \Delta v = -\int_{\partial D} [u(\Delta v)_\nu - u_\nu \Delta v] ds + \int_D u\Delta^2 v dx dy$$

and

$$\int_D \text{bigrad}u \cdot \text{bigrad}v dx dy =$$
$$-\int_{\partial D} [u(v_{\nu\nu\nu} + 3v_{\nu\nu\tau}) + u_\nu(-v_{\nu\nu} + v_{\tau\tau})] ds + \int_D u\Delta^2 v dx dy$$

Proof: The proof of the first formula follows easily from Green's classical formula written in the following form:

$$\int_{\partial D} (uw_\nu - wu_\nu) ds = \int_D (u\Delta w - w\Delta u) dx dy.$$

Put $w = \Delta v$ to get

$$\int_{\partial D} (u\Delta v_\nu - \Delta v u_\nu) ds = \int_D (u\Delta^2 v - \Delta v \Delta u) dx dy$$

which is exactly the formula we wanted to prove.

The proof of the second formula is not difficult either but requires some computations. Write out the bigrad operator in terms of derivatives with respect to x and y, and do some obvious computations to get

$$\int_D \text{bigrad}u \cdot \text{bigrad}v dx dy = \int_D \text{grad}u_x \cdot \text{bigrad}v dx dy +$$
$$\int_D \text{grad}u_y \cdot [2v_{xy}\text{i} + (-v_{xx} + v_{yy})\text{j}] dx dy$$

where i and j are the unit vectors in the x and y directions. Now use Stokes' theorem in the form

$$\int_D \text{grad} f \cdot \mu dxdy = \int_{\partial D} f\mu \cdot \nu ds - \int_D f \text{div}\mu dxdy$$

(where f is a function, μ is a vector field, and ν is the normal vector to ∂D), and write $\nu = y'\text{i} - x'\text{j}$. The right hand side becomes

$$\int_{\partial D} u_x[(v_{xx} - v_{yy})y' - 2v_{xy}x'] + u_y[2v_{xy}y' + (v_{xx} - v_{yy})x']ds -$$

$$\int_D u_x[(v_{xx} - v_{yy})_x - (2v_{xy})_y] + u_y[2(v_{xy})_x + (-v_{xx} + v_{yy})_y]dxdy.$$

In each bracket, let us regroup terms using the fact that $\partial_\nu = y'\partial_x - x'\partial_y$ and $\partial_\tau = x'\partial_x + y'\partial_y$. We obtain

$$\int_{\partial D}[\text{grad}u \cdot \text{grad}v_\nu + \text{grad}u \cdot (-\text{i}\partial_y + \text{j}\partial_x)v_\tau]ds - \int_D \text{grad}u \cdot \text{grad}\Delta v dxdy.$$

In the first integral use the fact that $\text{i}\partial_y - \text{j}\partial_x = \nu\partial_\tau - \tau\partial_\nu$ (which is easy to verify) and write $\text{grad} = \nu\partial_\nu + \tau\partial_\tau$. In the second integral use Stokes' theorem mentioned before. We obtain

$$\int_{\partial D}[u_\nu(v_{\nu\nu} - v_{\tau\tau}) + 2u_\tau v_{\nu\tau}]ds - \int_{\partial D} u(v_{\nu\nu} + v_{\tau\tau})ds + \int_D u\Delta^2 v dxdy.$$

In the second term of the first integral use the fact that $\int_{\partial D} f_\tau g ds = -\int_{\partial D} f g_\tau ds$ (this is just integration by parts). We obtain

$$\int_D u\Delta^2 v dxdy - \int_{\partial D}[u(v_{\nu\nu\nu} + 3v_{\nu\tau\tau}) - u_\nu(v_{\nu\nu} - v_{\tau\tau})]ds.$$

This is exactly the expression we were trying to arrive to.

Definition 4.5 *Denote $\mathcal{B}(D)$ the vector space of solutions of* bigrad$u = 0$ *over a domain D. We will simply write $\mathcal{B}$ when there is no danger of confusion.*

Proposition 4.6 $\mathcal{B}$ is independent of the choice of coordinates. Every element of $\mathcal{B}$ is locally a polynomial of degree at most 2. If D is a connected domain, the vector space $\mathcal{B}$ is of dimension 4, and a basis is given by 1, x, y, $x^2 + y^2$.

Proof: It follows that u_{xxx}, u_{xxy}, u_{xyy}, u_{yyy} are all zero. Then u is locally a polynomial of degree two. It is easy to verify that the only homogeneous polynomial of degree 2 in $\mathcal{B}$ is $x^2 + y^2$. The remark is now clear.

Corollary 4.7 *Let $u \in \mathcal{B}$ be a polynomial defined over $\mathbb{R}^2$, and let*

$$\begin{aligned} K_1 &= G_{\nu\nu\nu} + 3G_{\nu\tau\tau} \\ K_2 &= -G_{\nu\nu} + G_{\tau\tau} \end{aligned}$$

where G is Green's function from Section 3.1. Then

$$\int_{\partial D} [u(Q)K_1(P,Q) + u_\nu(Q)K_2(P,Q)]ds = \begin{cases} -u(P) & \text{if } P \in D \\ 0 & \text{if } P \notin D. \end{cases}$$

Proof: The corollary follows immediately from Green's second formula.

4.2 The Rank of the Integral Equations

In this section we will study the rank of the integral equations for the boundary value problems we set out to solve. The integral equations will be considered on the appropriate L^2 spaces. With the use of the formulas in Section 4.1 the proofs are not too complicated: we will try to imitate some of the classical proofs which can be found in the mathematical literature (see for example [1, 4, 9]).

We will assume D to be a bounded domain. When D is not simply connected, we will assume $\mathbb{R}^2 \setminus D$ to have a finite number of connected components. Let E_0 be the unbounded component of $\mathbb{R}^2 \setminus D$, and let E_1, ... , E_n be the bounded components of $\mathbb{R}^2 \setminus D$.

We will look at the integral equation corresponding to problem (C).

Theorem 4.8 *Let*

$$\begin{aligned} K_1 &= G_{\nu\nu\nu} + 3G_{\nu\tau\tau} \\ K_2 &= -G_{\nu\nu} + G_{\tau\tau}. \end{aligned}$$

Then any solution of the equation

$$\begin{pmatrix} 1/2 & 0 \\ -\kappa(P) & 1/2 \end{pmatrix} \begin{pmatrix} \sigma_1(P) \\ \sigma_2(P) \end{pmatrix} + \int_{\partial D} \begin{pmatrix} K_1(P,Q) & K_2(P,Q) \\ (K_1)_{\nu_P}(P,Q) & (K_2)_{\nu_P}(P,Q) \end{pmatrix} \begin{pmatrix} \sigma_1(Q) \\ \sigma_2(Q) \end{pmatrix} ds(Q) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

satisfies $(\sigma_1, \sigma_2) = (0,0)$ *on* ∂E_0, *and* $(\sigma_1, \sigma_2) = (\alpha_i, \partial_\nu \alpha_i)$ *on* ∂E_i *for some elements* $\alpha_i \in \mathcal{B}(E_i)$. *($\mathcal{B}$ was defined in the previous section.) Conversely, any such* (σ_1, σ_2) *is a solution of this equation.*

Proof: Let σ_1 and σ_2 be solutions of the integral equation. Let

$$u(P) = \int_{\partial D} [K_1(P,Q)\sigma_1(Q) + K_2(P,Q)\sigma_2(Q)] ds(Q)$$

be the potential given by σ_1 and σ_2, which is defined for $P \in D$ and $P \in \mathbf{R}^2 \setminus D$. Let P be in a tubular neighborhood of ∂D in D, and take $\lim_{P \to \partial D} u$ and $\lim_{P \to \partial D} u_\nu$ (from inside D, non tangentially). Because of the integral equations, these are zero; because of the uniqueness theorem in the Section 4.1, $u = 0$ in D.

Now we need information about u on $\mathbf{R}^2 \setminus D$. The intermediary aim is to deduce that $\int_{\mathbf{R}^2 \setminus D} |\text{bigrad} u|^2 dxdy = 0$. For this we would like to use lemma 4.9 (immediately following the proof of the theorem) applied to u and a suitably chosen cutoff of u. The cutoff will be chosen in the following way: let ϕ be a smooth function so that $\phi(P) = 1$ for $|P| < 1$ and $\phi(P) = 0$ for $|P| > 2$. Consider the function $v(P) = \phi(P/r)u(P)$ (where r is some real number). This function is smooth, with bounded support in $\mathbb{R}^2 \setminus D$. It can be approximated by functions v which are smooth and have compact support in $\mathbf{R}^2$. Therefore we can apply lemma 4.9 to u and v and we obtain

$$\int_{\mathbf{R}^2} \text{bigrad} u \cdot \text{bigrad} v dxdy = 0.$$

Replace v by $\phi(P/r)u(P)$ and take in account that $u = 0$ on D. We have then

$$\int_{B_r \setminus D} |\text{bigrad} u|^2 dxdy + \int_{B_{2r} \setminus B_r} \text{bigrad} u \cdot \text{bigrad}(\phi u) dxdy = 0,$$

where B_r is the disk of radius r centered at the origin. We want to show that the second integral goes to zero as $r \to \infty$. Keeping in mind the explicit expressions of K_1 and K_2 in the potential formula giving u we see that as $|P| \to \infty$, $u(P)$ is bounded by a constant, the first derivatives of u are bounded by a constant times $1/|P|$, the second derivatives of u are bounded by a constant times $1/|P|^2$; also, each differentiation of $\phi(P/r)$ takes out a constant times $1/r$. Therefore the integrand in the second integral is bounded by a constant times $(1/r^2)(1/r^2) = 1/r^4$. The integral itself will be bounded then by a constant times $1/r^2$, so it goes to zero as $r \to \infty$.

From $\int_{\mathbf{R}^2 \setminus D} |\text{bigrad}u|^2 dxdy = 0$ it follows that $|\text{bigrad}u| = 0$ on $\mathbf{R}^2 \setminus D$. This proves that $u_{xx} - u_{yy} = 0$ and $u_{xy} = 0$ outside $\overline{D}$. Therefore u is a locally polynomial function in the vector space $\mathcal{B}$ over $\mathbf{R}^2 \setminus D$. Denote $\psi_0 = u$ the polynomial on E_0, and $\psi_i = u$ the polynomial on E_i. Use now the jump relation from corollary 3.17. It follows that

$$(\sigma_1, \sigma_2) = -(\psi_i, (\psi_i)_\nu)$$

on ∂E_i, for $i = 0, \ldots, n$.

Let $\alpha_i = (\psi_i - \psi_0)$. Then

$$(\sigma_1, \sigma_2) + (\alpha_i, (\alpha_i)_\nu) = (\psi_0, (\psi_0)_\nu)$$

everywhere on ∂D. Using corollary 4.7 note that for points $P \notin D$ the potential given by (σ_1, σ_2) is equal to the potential given by $-(\psi_0, (\psi_0)_\nu)$ outside D. Using corollary 4.7 again, we see that the potential given by $-(\psi_0, (\psi_0)_\nu)$ is 0 outside D. Therefore we obtained that $u(P) = 0$ outside D. Since $u = \psi_0$ on the component E_0 of $\mathbf{R}^2 \setminus D$, it follows that $\psi_0 = 0$. This proves the first statement of the theorem.

The second statement follows from the fact that the potential given by a function which is $(\alpha_i, (\alpha_i)_\nu)$ on ∂E_i and 0 on the rest of the boundary of ∂D, is 0 on D according to corollary 4.7 (we simply take E_i to be D in the corollary). Taking the appropriate derivative and the limit as P goes to the boundary proves the second statement.

Lemma 4.9 *If u is given by the potential formula*

$$u(P) = \int_{\partial D} [K_1(P,Q)\sigma_1(Q) + K_2(P,Q)\sigma_2(Q)]ds(Q)$$

and v is a smooth function with compact support, then

$$\int_{\mathbb{R}^2} \text{bigrad}u \cdot \text{bigrad}v dxdy = 0.$$

Proof: Compute:

$$\int_{\mathbb{R}^2} \text{bigrad}u \cdot \text{bigrad}v dxdy =$$

$$\int_D \text{bigrad}u \cdot \text{bigrad}v dxdy + \int_{\mathbb{R}^2 \setminus D} \text{bigrad}u \cdot \text{bigrad}v dxdy =$$

$$-\int_{\partial D} [u_-(v_{\nu\nu\nu} + 3v_{\nu\nu\tau}) + (u_\nu)_-(-v_{\nu\nu} + v_{\tau\tau})]ds + \int_D u\Delta^2 v dxdy +$$

$$\int_{\partial D} [u_+(v_{\nu\nu\nu} + 3v_{\nu\nu\tau}) + (u_\nu)_+(-v_{\nu\nu} + v_{\tau\tau})]ds + \int_{\mathbb{R}^2 \setminus D} u\Delta^2 v dxdy =$$

$$-\int_{\partial D} [(u_- - u_+)(v_{\nu\nu\nu} + 3v_{\nu\nu\tau}) + ((u_\nu)_- - (u_\nu)_+)(-v_{\nu\nu} + v_{\tau\tau})]ds + \int_{\mathbb{R}^2} u\Delta^2 v dxdy =$$

$$\int_{\partial D} [\sigma_1(v_{\nu\nu\nu} + 3v_{\nu\nu\tau}) + \sigma_2(-v_{\nu\nu} + v_{\tau\tau})]ds + \int_{\mathbb{R}^2} u\Delta^2 v dxdy$$

On the other hand, since u is a potential, the last term becomes

$$\int_{\mathbb{R}^2} u\Delta^2 v dxdy = \int_{\mathbb{R}^2} \left(\int K_1(P,Q)\sigma_1(Q) + K_2(P,Q)\sigma_2(Q) ds(Q) \right) \Delta^2 v dxdy =$$

$$\int \left[\left(\int_{\mathbb{R}^2} K_1(P,Q)\Delta^2 v dxdy \right) \sigma_1(Q) + \left(\int_{\mathbb{R}^2} K_2(P,Q)\Delta^2 v dxdy \right) \sigma_2(Q) \right] ds(Q) =$$

$$\int \left[\left(\int_{\mathbb{R}^2} (G_{\nu\nu\nu} + 3G_{\nu\tau\tau})(P,Q)\Delta^2 v dxdy \right) \sigma_1(Q) + \right.$$

$$\left. \left(\int_{\mathbb{R}^2} (-G_{\nu\nu} + G_{\tau\tau})(P,Q)\Delta^2 v dxdy \right) \sigma_2(Q) \right] ds(Q) =$$

$$-\int [(v_{\nu\nu\nu} + 3v_{\nu\tau\tau})\sigma_1(Q) + (-v_{\nu\nu} + v_{\tau\tau})\sigma_2(Q)]ds(Q).$$

Substituting this in the expression we were calculating, we get that

$$\int \text{bigrad}u \cdot \text{bigrad}v dxdy = 0.$$

This ends the proof of the lemma.

Next, we look at the integral equation corresponding to problem (F). As expected, this will turn out to be closely related to the integral equation corresponding to problem (C).

Theorem 4.10 *Assume D is simply connected and let*

$$\begin{aligned} K_1 &= -G_{\nu\nu} + G_{\tau\tau} \\ K_2 &= G_{\nu\nu\nu} + 3G_{\nu\tau\tau}. \end{aligned}$$

Let (σ_1, σ_2) be a solution of

$$\begin{pmatrix} 1/2 & -\kappa(P) \\ 0 & \frac{1}{2}\frac{d}{ds} \end{pmatrix} \begin{pmatrix} \sigma_1(P) \\ \sigma_2(P) \end{pmatrix} + \int \begin{pmatrix} (K_1)_{\nu_P}(P,Q) & (K_2)_{\nu_P}(P,Q) \\ (K_1)_{\tau_P}(P,Q) & (K_2)_{\tau_P}(P,Q) \end{pmatrix} \begin{pmatrix} \sigma_1(Q) \\ \sigma_2(Q) \end{pmatrix} ds(Q) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then $(\sigma_1, \sigma_2) = c(2\kappa, 1)$ on ∂D, where c is a locally constant function. Conversely, any such (σ_1, σ_2) is a solution of this equation.

Proof: Let us prove the first statement. There is no loss of generality in assuming that D is connected. Let (σ_1, σ_2) be a solution of the equation. Integrating the second equation with respect to s along ∂D gives

$$\frac{1}{2}\sigma_2 + \int [(-G_{\nu\nu} + G_{\tau\tau})\sigma_1 + (G_{\nu\nu\nu} + 3G_{\nu\tau\tau})\sigma_2] ds = c$$

for some constant c. Subtract from this equation

$$\int [(-G_{\nu\nu} + G_{\tau\tau})2\kappa + (G_{\nu\nu\nu} + 3G_{\nu\tau\tau})] ds = \frac{1}{2}$$

multiplied by c to get

$$\frac{1}{2}(\sigma_2 - c) + \int [(-G_{\nu\nu} + G_{\tau\tau})(\sigma_1 - 2c\kappa) + (G_{\nu\nu\nu} + 3G_{\nu\tau\tau})(\sigma_2 - c)] ds = 0.$$

Rewrite the first equation as

$$\frac{1}{2}(\sigma_1 - 2c\kappa) - \kappa(\sigma_2 - c) + \int [(-G_{\nu\nu} + G_{\tau\tau})_{\nu_P}\sigma_1 + (G_{\nu\nu\nu} + 3G_{\nu\tau\tau})_{\nu_P}\sigma_2] ds = 0$$

and subtract

$$\int [(-G_{\nu\nu} + G_{\tau\tau})_{\nu_P} 2\kappa + (G_{\nu\nu\nu} + 3G_{\nu\tau\tau})_{\nu_P}] ds = 0$$

(a consequence of proposition 4.1) to get

$$\frac{1}{2}(\sigma_1 - 2c\kappa) - \kappa(\sigma_2 - c) + \int[(-G_{\nu\nu} + G_{\tau\tau})_{\nu_P}(\sigma_1 - 2c\kappa) + (G_{\nu\nu\nu} + 3G_{\nu\tau\tau})_{\nu_P}(\sigma_2 - c)]ds = 0.$$

We obtained that $\sigma_1 - 2\kappa c$ and $\sigma_2 - c$ satisfy the system of integral equations from theorem 4.8, so $\sigma_1 - 2\kappa c = 0$ and $\sigma_2 - c = 0$ on ∂D. This proves the first statement of the theorem.

To prove the second statement, remember that according to proposition 4.1

$$\int_{\partial E_i}[G_{\nu\nu\nu}(P,Q) + 3G_{\nu\tau\tau}(P,Q)] + 2\kappa(Q)[-G_{\nu\nu}(P,Q) + G_{\tau\tau}(P,Q)]ds(Q)$$

is locally constant for $P \notin \partial D$. Assume that P is in a tubular neighborhood of ∂D in D, differentiate with respect to ν_P and with respect to τ_P, and take $\lim_{P\to\partial D}$. Using theorem 3.11 we obtain that a function which is a constant multiple of $(2\kappa, 1)$ is a solution of the integral equation.

In the context of the previous theorem we make some remarks relevant to the numerical implementation of algorithms based on the theorem.

Remark 4.11 *In the context of appropriately smooth functions, the operator T defined by*

$$T : \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \longmapsto \begin{pmatrix} \frac{1}{2} & -\kappa \\ 0 & \frac{1}{2}\frac{d}{ds} \end{pmatrix} \begin{pmatrix} \sigma_1(P) \\ \sigma_2(P) \end{pmatrix} + \int \begin{pmatrix} (K_1)_{\nu_P}(P,Q) & (K_2)_{\nu_P}(P,Q) \\ (K_1)_{\tau_P}(P,Q) & (K_2)_{\tau_P}(P,Q) \end{pmatrix} \begin{pmatrix} \sigma_1(Q) \\ \sigma_2(Q) \end{pmatrix} ds(Q)$$

has its kernel equal to the one dimensional space generated by $v = (2\kappa, 1)$, and its range is the subspace orthogonal to the one dimensional subspace generated by $w = (0, 1)$. It follows then that the operator

$$\sigma \longmapsto T\sigma + (\sigma \cdot v)w = (T + wv^t)\sigma$$

is injective and surjective.

This remark might seem useful in building the numerical algorithm. The integral equation (4.4) will correspond to a linear system. The linear system corresponding to equation (4.4) is rank-one deficient, so if the right hand side is given correctly, it has infinitely many solutions, and any solution is suitable for solving problem (F). Instead of having to solve such a system, it would be very convenient to solve the system corresponding to

$$(T + wv^t)\sigma = g$$

which has a unique solution σ, which is also suitable for solving problem (F). There are two drawbacks though: first, because of the term d/ds in the diagonal part, the condition number of the matrix is increasing linearly with the number of points on the boundary, and second, the correction $(\sigma \cdot v)w$ might cause even a matrix with constant condition number to have a condition number which increases with the number of points.

A much better approach is to deal with the differentiation separately, by multiplying the whole equation by the "inverse" of

$$\begin{pmatrix} \frac{1}{2} & -\kappa \\ 0 & \frac{1}{2}\frac{d}{ds} \end{pmatrix},$$

i. e. by the operator matrix

$$\begin{pmatrix} 2 & 4\kappa \int \dots \, ds \\ 0 & 2 \int \dots \, ds \end{pmatrix}.$$

This is not an inverse properly speaking, and it is not unique. Let us just say at this point that multiplying by the "inverse" entails computing an indefinite integral every time the integral with the kernels is computed, and computing one additional indefinite integral initially, or at the end. This will be made more precise in Section 5.1. We will also see that it can be done efficiently and it yields an algorithm with "constant condition number", without increasing the complexity of the computation.

We are now considering the integral equation corresponding to problem (S0). With the choice of kernels we made, the simplicity of the boundary value problem carries over to the integral equation.

Theorem 4.12 *Assume that D is simply connected, and let*

$$\begin{aligned} K_1 &= 6\Delta G_\nu \\ K_2 &= G_\nu. \end{aligned}$$

Then the only solution of the equation

$$\begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \sigma_1(P) \\ \sigma_2(P) \end{pmatrix} + \int \begin{pmatrix} K_1(P,Q) & K_2(P,Q) \\ \Delta_P K_1(P,Q) & \Delta_P K_2(P,Q) \end{pmatrix} \begin{pmatrix} \sigma_1(Q) \\ \sigma_2(Q) \end{pmatrix} ds(Q) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is $\sigma_1 = 0$ and $\sigma_2 = 0$.

Proof: Let us write out explicitly the expressions of the kernels under the integral. The system of integral equations is

$$\begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \sigma_1(P) \\ \sigma_2(P) \end{pmatrix} + \frac{1}{\pi} \int \begin{pmatrix} 3(v\cdot\nu)/(v\cdot v) & \frac{1}{8}(v\cdot\nu)(\log(v\cdot v)+1) \\ 0 & \frac{1}{2}(v\cdot\nu)/(v\cdot v) \end{pmatrix} \begin{pmatrix} \sigma_1(Q) \\ \sigma_2(Q) \end{pmatrix} ds(Q) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This system of integral equations for σ_1 and σ_2 decouples. The second equation

$$\frac{1}{2}\sigma_2 + \frac{1}{\pi}\int \frac{1}{2}(v\cdot\nu)/(v\cdot v)\sigma_2 ds = 0$$

is exactly the equation one obtains when trying to solve the interior Dirichlet problem

$$\begin{aligned} \Delta w = 0 \quad &\text{on} \quad D \\ w = g_2 \quad &\text{on} \quad \partial D \end{aligned}$$

by looking for w as a double layer potential (see [9] for example). We know (from [9] for example) that the only solution of this equation is the trivial one. Therefore $\sigma_2 = 0$. Then the first equation becomes the equation of the same Dirichlet problem, so $\sigma_1 = 0$.

Now we will look at the integral equation corresponding to problem (Sμ).

Remark 4.13 *Assume that D is simply connected. Let*

$$\begin{aligned} K_1 &= (6+\mu)G_{\nu\nu\nu} + 3(2+\mu)G_{\nu\tau\tau} = 6\Delta G_\nu + \mu(G_{\nu\nu\nu} + 3G_{\nu\tau\tau}) \\ K_2 &= G_\nu. \end{aligned}$$

Denote $K_{21} = \Delta_P K_1 + \mu(K_1)_{\tau_P\tau_P}$ *and* $K_{22} = \Delta_P K_2 + \mu(K_2)_{\tau_P\tau_P}$. *Consider the system*

$$\begin{pmatrix} \frac{1}{2}(6+\mu) & 0 \\ -(\mu^2+2\mu)\kappa^2 + \frac{1}{2}(\mu^2+8\mu)\frac{d^2}{ds^2} - 2\mu\kappa'\mathcal{H} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \sigma_1(P) \\ \sigma_2(P) \end{pmatrix} + \int \begin{pmatrix} K_1(P,Q) & K_2(P,Q) \\ K_{21}(P,Q) & K_{22}(P,Q) \end{pmatrix} \begin{pmatrix} \sigma_1(Q) \\ \sigma_2(Q) \end{pmatrix} ds(Q) = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \tag{4.1}$$

If $\mu \neq -6$, *the corresponding homogeneous system has* $\sigma_1 = 0$ *and* $\sigma_2 = 0$ *as the only solution.*

In the case $\mu = -6$ the integral equation is not of second kind. Numerical evidence suggests that the equation has a unique solution even in this case. It is pointless trying to prove it though, since the Fredholm theory would not apply in this case.

Remark 4.14 Note that if $\mu \neq 0$ (and $\mu \neq -6$) the condition number of the system arising from the integral equation (4.1) increases as fourth power of the number of points in the discretization of ∂D. Indeed, the diagonal part of the integral equation dominates the integral part. In the behavior of the condition number versus the number of points, a power of 2 is coming from the fact that d^2/ds^2 appears in the operator

$$\begin{pmatrix} \frac{1}{2}(6+\mu) & 0 \\ -(\mu^2+2\mu)\kappa^2 + \frac{1}{2}(\mu^2+8\mu)\frac{d^2}{ds^2} - 2\mu\kappa'\mathcal{H} & \frac{1}{2} \end{pmatrix}.$$

Another power of 2 is coming from the fact that d^2/ds^2 appears in the "inverse" operator as well. Indeed, the "inverse" operator is

$$\begin{pmatrix} \frac{2}{6+\mu} & 0 \\ -\frac{4}{6+\mu}[-(\mu^2+2\mu)\kappa^2 + \frac{1}{2}(\mu^2+8\mu)\frac{d^2}{ds^2} - 2\mu\kappa'\mathcal{H}] & 2 \end{pmatrix}.$$

We conclude that the approach of solving problem (Sμ) using this integral equation is not very useful for numerical computations. For this reason, starting with remark 4.13, we will not spend any effort in giving proofs to the statements about problem (Sμ). Strong numerical evidence suggests though that the only solution of equation (4.1) is the trivial one. Therefore, the equation has a solution for any right hand side. Numerical evidence also suggests that problem (Sμ) has a unique solution, which can be obtained via the potential formula (2.1), using the kernels given in Section 3.2 and the charges obtained from equation 4.1.

4.3 Solution to the Problems

In this section we will study the existence of solutions to the problems stated in Section 2.1.

Let us start by looking at the extent of uniqueness of the solutions of problems (C), (F) and (S0).

Proposition 4.15 *Assume that the boundary data in the three problems (C), (F) and (S0) are zero, that is, $g_1 = 0$ and $g_2 = 0$. Then the solutions for the three problems are:*

Problem (C): $u = 0$.

Problem (F): assuming that D is simply connected, $u = c$, where c is a locally constant function on D.

Problem (S0): $u = 0$.

Proof: Problem (C): Take $u = v$ in Green's first formula. It follows that $\int_D (\Delta u)^2 dxdy = 0$. Therefore $\Delta u = 0$ in D. Using the classical uniqueness theorem for the solution of $\Delta u = 0$ with $u = 0$ on the boundary one gets immediately that $u = 0$.

Problem (F): The condition $u_\tau = 0$ on ∂D implies $u = c$ on ∂D, where c is locally constant on ∂D. The function c extends naturally to D. Let $v = u - c$. Problem (F) reduces to problem (C) for v. Therefore $v = 0$ and so $u = c$ on D.

Problem (S0): $\Delta u = 0$ on ∂D, so it follows from the classical uniqueness for the Laplace equation that $\Delta u = 0$ on D. Using the uniqueness theorem again, $u = 0$ on D.

Now we will state formally the compactness of the integral operators in the integral equations.

Theorem 4.16 *For each of the four problems (C), (F), (S0) and (Sμ), the operator*

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \longmapsto \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$$

is a compact operator from $L^2(\partial D) \times L^2(\partial D)$ to itself.

Proof: This is clear because we showed that all entries in K_{ij} are either smooth, or, at worst, they contain logarithms.

We are now ready to discuss the existence of the solutions to the boundary value problems, via the corresponding integral equation and potential formula.

Theorem 4.17 *Assume D is simply connected, and let g_1 and g_2 be C^0 functions on ∂D. Let $K_1 = G_{\nu\nu\nu} + 3G_{\nu\tau\tau}$ and $K_2 = -G_{\nu\nu} + G_{\tau\tau}$. Then the solution u of*

$$\begin{array}{ll} \Delta^2 u = 0 & \text{on } D \\ u = g_1 & \text{on } \partial D \\ \partial u/\partial \nu = g_2 & \text{on } \partial D \end{array}$$

is given by

$$u(P) = \int_{\partial D} [K_1(P,Q)\sigma_1(Q) + K_2(P,Q)\sigma_2(Q)] ds(Q), \tag{4.2}$$

where σ_1 and σ_2 are solutions of

$$\begin{pmatrix} \frac{1}{2} & 0 \\ -\kappa(P) & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \sigma_1(P) \\ \sigma_2(P) \end{pmatrix} + \int_{\partial D} \begin{pmatrix} K_1(P,Q) & K_2(P,Q) \\ (K_1)_{\nu_P}(P,Q) & (K_2)_{\nu_P}(P,Q) \end{pmatrix} \begin{pmatrix} \sigma_1(Q) \\ \sigma_2(Q) \end{pmatrix} ds(Q) = \begin{pmatrix} g_1(P) \\ g_2(P) \end{pmatrix} \tag{4.3}$$

(κ is the curvature of ∂D).

Proof: The operator giving the left hand side of the integral equation is the sum of an invertible operator and a compact operator from $L^2(\partial D) \times L^2(\partial D)$ to itself. Using the Fredholm theory, we see that the integral equation has a unique solution because of theorem 4.8. Therefore, σ_1 and σ_2 exist and are in L^2. Since g_1 and g_2 are continuous, it follows that σ_1 and σ_2 are continuous (this follows for example, from proposition (3.14) in [9]). The function u is defined then on D, and it can be extended continuously to $\overline{D}$. Also, the derivative at the boundary exists. Because of the integral equation and theorem 3.11 it follows that this extension satisfies the boundary conditions.

Remark 4.18 If D is not simply connected, the situation is more complicated. Theorem 4.8 gives one main ingredient towards solving problem (C) in this case. The idea is to split the right hand side g_1 and g_2 into a part which is in the range of the integral equation operator, and a part which can be handled explicitly. We will not pursue the multiply connected case here.

Theorem 4.19 *Assume that D is simply connected and let g_1 and g_2 be C^0 functions on ∂D. Let $K_1 = -G_{\nu\nu} + G_{\tau\tau}$ and $K_2 = G_{\nu\nu\nu} + 3G_{\nu\tau\tau}$. The problem*

$$\begin{array}{lll} \Delta^2 u = 0 & \text{on} & D \\ \partial u/\partial \nu = g_1 & \text{on} & \partial D \\ \partial u/\partial \tau = g_2 & \text{on} & \partial D \end{array}$$

has solutions if and only if

$$\int_{\partial D} g_2(s)ds = 0.$$

The solution is unique up to an additive constant (on each component of D). A solution is given by

$$u(P) = \int_{\partial D} [K_1(P,Q)\sigma_1(Q) + K_2(P,Q)\sigma_2(Q)]ds(Q),$$

where σ_1 and σ_2 are solutions of

$$\begin{pmatrix} \frac{1}{2} & -\kappa \\ 0 & \frac{1}{2}\frac{d}{ds} \end{pmatrix} \begin{pmatrix} \sigma_1(P) \\ \sigma_2(P) \end{pmatrix} + \int_{\partial D} \begin{pmatrix} (K_1)_{\nu_P}(P,Q) & (K_2)_{\nu_P}(P,Q) \\ (K_1)_{\tau_P}(P,Q) & (K_2)_{\tau_P}(P,Q) \end{pmatrix} \begin{pmatrix} \sigma_1(Q) \\ \sigma_2(Q) \end{pmatrix} ds(Q) = \begin{pmatrix} g_1(P) \\ g_2(P) \end{pmatrix} \tag{4.4}$$

(κ is the curvature of ∂D).

Proof: It is clearly necessary that $\int_{\partial D} g_2 ds = 0$ because g_2 is the derivative of a function on the closed curve ∂D.

Assume now that $\int_{\partial D} g_2 ds = 0$. It follows that there is a function G_2 on ∂D so that its derivative is g_2; in other words, we can integrate the second equation term by term. The system we obtain this way is exactly the system from theorem 4.17, so it has a solution. This solution is the solution we are looking for, since the derivatives at the boundary exist and have the values prescribed by the problem. The details of this proof are the same as in the previous proof and will be omitted.

Finally, let us show how the potential formula given for problem (S) provides a solution to the boundary value problem in the case $\mu = 0$ (i. e. we solve problem (S0) via integral equations and a potential formula).

Theorem 4.20 *Let D be simply connected, and let g_1 and g_2 be C^0 functions on ∂D. Let $K_1 = 6\Delta G_\nu$ and $K_2 = G_\nu$. Then the solution u of*

$$\begin{array}{lll} \Delta^2 u = 0 & \text{on} & D \\ u = g_1 & \text{on} & \partial D \\ \Delta u = g_2 & \text{on} & \partial D \end{array}$$

is given by

$$u(P) = \int_{\partial D} [K_1(P,Q)\sigma_1(Q) + K_2(P,Q)\sigma_2(Q)]ds(Q), \tag{4.5}$$

where σ_1 and σ_2 are solutions of

$$\begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \sigma_1(P) \\ \sigma_2(P) \end{pmatrix} + \int_{\partial D} \begin{pmatrix} K_1(P,Q) & K_2(P,Q) \\ \Delta_P K_1(P,Q) & \Delta_P K_2(P,Q) \end{pmatrix} \begin{pmatrix} \sigma_1(Q) \\ \sigma_2(Q) \end{pmatrix} ds(Q) = \begin{pmatrix} g_1(P) \\ g_2(P) \end{pmatrix}.$$

The proof is identical to the one for theorem 4.17 and it will be omitted.

We will look at problem (Sμ) now.

Remark 4.21 *Let D be simply connected, let g_1 be a C^2 function and let g_2 be a C^0 function on ∂D. Let*

$$\begin{aligned} K_1 &= (6+\mu)G_{\nu\nu\nu} + 3(2+\mu)G_{\nu\tau\tau} = 6\Delta G_\nu + \mu(G_{\nu\nu\nu} + 3G_{\nu\tau\tau}) \\ K_2 &= G_\nu \end{aligned}$$

where $\mu \neq -6$. Then the solution u of

$$\begin{array}{lll} \Delta^2 u = 0 & \text{on} & D \\ u = g_1 & \text{on} & \partial D \\ \Delta u + \mu \cdot \partial^2 u/\partial \tau^2 = g_2 & \text{on} & \partial D \end{array}$$

is given by

$$u(P) = \int_{\partial D} [K_1(P,Q)\sigma_1(Q) + K_2(P,Q)\sigma_2(Q)]ds(Q), \tag{4.6}$$

where σ_1 and σ_2 are solutions of equation (4.1).

See remark 4.14 for a discussion relevant to this remark.

Idea of Proof: Let H_2 be the Sobolev space of functions in $L^2(\partial D)$ whose first and second derivatives are also in $L^2(\partial D)$. Clearly the operator

$$\begin{pmatrix} \frac{1}{2}(6+\mu) & 0 \\ -(\mu^2+2\mu)\kappa^2 + \frac{1}{2}(\mu^2+8\mu)\frac{d^2}{ds^2} - 2\mu\kappa'\mathcal{H} & \frac{1}{2} \end{pmatrix}$$

is an isomorphism from $H_2 \times L^2(\partial D)$ to itself. The operator

$$\int \begin{pmatrix} K_1(P,Q) & K_2(P,Q) \\ K_{21}(P,Q) & K_{22}(P,Q) \end{pmatrix} \begin{pmatrix} \sigma_1(Q) \\ \sigma_2(Q) \end{pmatrix} ds(Q)$$

(which is a compact operator from $L^2(\partial D) \times L^2(\partial D)$ to itself) induces a compact operator from $H_2 \times L^2(\partial D)$ to itself. Assuming the statement from remark 4.13 the proof proceeds just like the proofs of theorems 4.17 and 4.20.

The preceding remark states a method for solving problem (Sμ) for all $\mu \neq -6$. When $\mu = -6$ the integral equation is not of second kind, and we have no formal statement about it. Numerical evidence suggests that the integral equation has a unique solution even in this case, and that the solution solves problem (Sμ) through the potential formula.

CHAPTER 5

NUMERICAL TOOLS AND RESULTS

5.1 Numerical Tools

In this section we will give an outline of the algorithms which can be written based on the theorems in the preceding chapter. We will discuss briefly the stability, convergence rate and complexity of the algorithms as well as the condition number of the linear systems involved. We will also discuss some of the numerical tools which are needed in implementing the algorithms. Throughout the discussion, let us keep in mind that the aim is to find an algorithm which is stable, has a high rate of convergence, and low computational complexity. We will explain the details for problem (C) and point out the changes to be made for the other problems (see Sections 2.1 and 4.3).

The clamped plate problem (C): Let us assume that D is a bounded, connected and simply connected domain in $\mathbf{R}^2$. Given g_1 and g_2 on ∂D, we want to find u on D like in theorem 4.17. More specifically, given the curve ∂D as a set of points in $\mathbf{R}^2$, and given the values of g_1 and g_2 at these points, find the values of u at a given set of points inside D. Clearly there are two major steps according to theorem 4.17:

Step 1: find σ_1 and σ_2 from equation (4.3).

Step 2: find u from equation (4.2).

Let us start by looking at Step 1. Let N represent the number of points in the discretization of ∂D. Mathematically, Step 1 is simply solving a linear system of size $2N \times 2N$ in which the right hand side is the vector of values of g_1 and g_2, the unknowns are the entries of the vector representing the values of σ_1 and σ_2, and the matrix is $D + A$, where D is the 2×2 matrix of $N \times N$ blocks

$$\begin{pmatrix} \frac{1}{2} & 0 \\ -\kappa & \frac{1}{2} \end{pmatrix},$$

each block being a diagonal matrix, and A is the matrix corresponding to the linear operator

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \longmapsto \int_{\partial D} \begin{pmatrix} K_1(P,Q) & K_2(P,Q) \\ (K_1)_{\nu_P}(P,Q) & (K_2)_{\nu_P}(P,Q) \end{pmatrix} \begin{pmatrix} \sigma_1(Q) \\ \sigma_2(Q) \end{pmatrix} ds(Q).$$

Before continuing the discussion, let us point out that both A and D have condition numbers which are asymptotically bounded as $N \to \infty$. We continue by pointing out two substeps:

Step 1.1: Compute the curvature κ.

Step 1.2: Solve the linear system.

Step 1.1: This part of the algorithm is "easy" and "cheap". If it is not known from the input data, we have to compute the curvature κ at all points where the boundary ∂D of D is given. Assuming that the points (x, y) giving the boundary are equispaced in terms of arclength, we have to compute x', y', x'', y'', and then take $\kappa = -x''y' + y''x'$. This can be done using Fast Fourier Transforms, and it is of complexity $O(N \log(N))$. If the points (x, y) are not equispaced in terms of arclength, then one should "resample" the curve, that is, find a new set of points giving the same curve, which are equispaced in terms of arclength. This can be done in several ways, and depending on the way it is done, it might involve Fast Fourier Transforms, interpolation, and solving some non-linear equations (by Newton's method). All these are "fast" (i.e. of order $O(N \log(N))$ or less), can be made highly accurate, and they are stable. We will

not discuss this further. Let us just note that in the process of resampling one obtains x', y', x'', y'', so there would be no need of computing them separately.

Step 1.2: This is the non-trivial part of this algorithm. In order to emphasize this non-triviality let us describe shortly how a naively constructed algorithm would look like. First, one would compute the entries of matrix A. This in itself is a computation of order $O(N^2)$ since the matrix is of size $2N \times 2N$. Then, most methods for solving the system $(D+A)\sigma = g$ are of order $O(N^3)$ (or worse), but as we shall see shortly, a conjugate residual type method (to be outlined later) is of order $O(N^2)$. Our aim is to make this step of order $O(N)$.

For the sake of completeness we will give a short description of conjugate residual type methods for solving linear equations numerically. More details can be found in [8] for example.

Conjugate residual type methods: Let B be a non-singular $n \times n$ matrix. Recall that the condition number κ of B is $\kappa = ||B|| \cdot ||B^{-1}||$. Let $b \in R^n$ be a given vector. The problem we are looking at is: find $x \in R^n$ satisfying $Bx = b$. Let $(* \cdot *)$ be a positive definite bilinear form on R^n. A conjugate residual type method is a method of constructing iteratively a sequence x_i of elements in R^n with the following property: choose x_0 and compute the residual $r_0 = Bx_0 - b$. The vector x_i will be chosen so that the residual $r_i = Bx_i - b$ is minimized over the subspace of R^n generated by the vectors r_0, Br_0, B^2r_0, ... , B^ir_0 with respect to the norm given by $(* \cdot *)$.

There are various conjugate residual methods depending on the choice of the positive definite bilinear form $(* \cdot *)$. If x_1, ... , x_i and r_0, r_1, ... , r_i have been computed, then computing x_{i+1} and r_{i+1} can be done by a number of operations independent of i, which, besides some elementary operations, involve only multiplications of B with some vectors in R^n. If x represents the solution of the given equation $Bx = b$ then

$$|x_i - x| \leq c\left(\frac{\kappa - 1}{\kappa + 1}\right)^i |x_0 - x|$$

where c is a constant independent of B and x_0, and $| * |$ is the norm given by $(* \cdot *)$. Mathematically speaking one might have to go n steps in order for x_i to be

equal to x. Numerically however, one reaches $x_i = x$ within machine precision, or within the desired accuracy, after a certain number of iterations.

In typical numerical problems (like the one we are looking at) one has to solve the system $Bx = b$ several times, with increasing values of n (in our problem we will have to increase N = the number of points on the boundary, and recompute the values of the function u, to make sure that we achieve a good enough approximation of the solution of the partial differential equation). In many problems, the condition number of B increases with n, but in the problem we are looking at, the condition number of B (which is $D + A$) is asymptotically constant.

Let us continue the discussion of Step 1.2. We conclude that the number of steps in a conjugate residual type method, and therefore the number of matrix-vector multiplications needed to solve $(D+A)\sigma = g$ is bounded independently of N. Therefore the complexity of solving $(D + A)\sigma = g$ is bounded by a constant (depending on the desired accuracy) times the complexity of performing the multiplication of $(D + A)$ with a given vector v. Computing the entries of A and performing a multiplication of A with a vector, has complexity $O(N^2)$, so the complexity of this approach is $O(N^2)$.

By itself, this is an improvement over other methods of solving problem (C), since other standard methods are at best of complexity $O(N^3)$ (on regions of very particular shape). But it does not achieve our aim of solving problem (C) in $O(N)$ or $O(N \log(N))$ operations. We will point now to a method of solving $(D + A)\sigma = g$ in order $O(N)$ operations. We will use the idea of the method developed in [27] (see also [10]).

As we discussed, the complexity of solving $(D + A)\sigma = g$ is proportional to the complexity of multiplying A with a vector. Therefore, we want to be able to compute efficiently the matrix-vector multiplications corresponding to the following four integrations

$$\begin{aligned}
\sigma_1 &\longmapsto \int K_1(P,Q)\sigma_1(Q)ds(Q) \\
\sigma_2 &\longmapsto \int K_2(P,Q)\sigma_2(Q)ds(Q) \\
\sigma_1 &\longmapsto \int (K_1)_{\nu_P}(P,Q)\sigma_1(Q)ds(Q) \\
\sigma_2 &\longmapsto \int (K_2)_{\nu_P}(P,Q)\sigma_2(Q)ds(Q).
\end{aligned}$$

In [27] (see also [10]) it is explained how the matrix-vector multiplication corresponding to

$$\sigma \longmapsto \int K(P,Q)\sigma(Q)ds(Q)$$

corresponds to evaluating the field created by a set of dipoles distributed along the curve ∂D at a set of points on the curve ∂D, for the case when K represents the kernel appearing in the integral equation used to solve the Dirichlet problem for the harmonic equation $\Delta w = 0$. In the same papers the authors develop a method for evaluating the field in order $O(N)$ operations. That method can be used in our problem with the only change that some expansions in Taylor and Laurent series will be different, and some local evaluations will be different. This does not change the computational complexity of the algorithm, and therefore provides an order $O(N)$ algorithm for solving $(D + A)\sigma = g$. We will not go into further details here.

There is one thing left to be discussed in Step 1.2. In computing

$$\int_{\partial D} K(P,Q)\sigma(Q)ds(Q)$$

using the fast algorithm (where K is any of kernels we are considering and σ is a given function), we still have to perform a direct computation around every point P. This amounts to computing

$$\int_{U} K(P,Q)\sigma(Q)ds(Q),$$

where U is a neighborhood of P in ∂D. Therefore, we need to compute $K(P,P)$ at $P \in \partial D$. This is done by a simple interpolation. We point out this minor detail because it plays an important role in the discussion of the convergence rate of the algorithm.

Step 2: There is little to be discussed in this step now since it amounts to computing another integral which can be done efficiently by the method we just described.

Characteristics of the algorithm: The computational complexity and the memory requirements for this algorithm are clearly of order $O(N)$. Let us note

though that in most practical situations, the data would have to be resampled. Strictly speaking this is not part of the algorithm, but it would make the solution of the problem of order $O(N\log(N))$.

The convergence rate of the algorithm is determined by the smoothness of the curve ∂D and by the convergence rate of the interpolation formula used in computing $K(P,P)$. Indeed, the fast algorithm for evaluating the integrals corresponds to using the trapezoidal rule for computing the integrals. It is a well known classical fact that the convergence rate of the quadrature by the trapezoidal rule for periodic functions of class C^m is $m+1$. If the boundary ∂D is of class C^k, then the kernels are of class C^{k-2}. Interpolation with convergence rate p, corresponds to the fact that the smoothness of the function K is not more than $p-1$. Therefore the convergence rate of the algorithm will be $\min(k-2,p-1)+1$. This means that in practice one can easily achieve a (very appealing) convergence rate of about 7 if the boundary is smooth enough.

A last remark to be made regarding this algorithm is its stability. Indeed the integral equation leads to a linear system of (asymptotically) constant condition number (in fact the condition number is typically very reasonable; for "reasonable" curves it tends to be in the range 100.0), and all the steps of the algorithm are stable. That means that for small errors in the data, the errors in the result will be small.

The fluid dynamics problem (F): We will now discuss shortly the numerical details of problem (F). The general setup is very similar to the one for problem (C). There is one difference though, which was already pointed out in remark 4.11 and the discussion following it.

In Step 1, let us write the linear equation arising from the integral equation 4.4 as $(D+A)\sigma = g$. We know that the matrix $D+A$ is rank one deficient (in fact both D and A have the vectors $(2\kappa,1)$ and $(0,1)$ as generators for the kernels and the kernels of their transposes). If the boundary data satisfies the necessary condition for the problem to have a solution, then the right hand side g is in the image of $D+A$ (viewed as an operator). Therefore the system will

have solutions, and any solution is suitable for computing u in Step 2. Therefore we are facing the problem of finding one solution of a singular system (out of several). There are several ways of dealing with this problem; for example, one can reduce the size of the system, or one can add some terms which would make the system non-singular but would not alter the solution (see remark 4.11). But as we pointed out before, we propose a different method. We summarize its essence in the following proposition:

Proposition 5.1 *Let $\mathcal{R}$ be the set*

$$\mathcal{R} = \{(g_1, g_2) \in L^2(\partial D) \times L^2(\partial D) \text{ so that } \int_{\partial D} g_2 ds = 0\}$$

(in other words, in the notation of remark 4.11, $\mathcal{R}$ is the orthogonal of w). Let $\mathcal{I}$ be a linear operator which has the property that $(d/ds)(\mathcal{I}(f)) = f$ for any $f : \partial D \to \mathbb{R}$ with $\int_{\partial D} f = 0$. Let us consider then E to be the operator

$$E = \begin{pmatrix} 2 & 4\kappa\mathcal{I} \\ 0 & 2\mathcal{I} \end{pmatrix}.$$

We have the following:

a) $DE = I$ on $\mathcal{R}$, where I is the identity operator.

b) The system $(I + AE)\alpha = g$ has a solution, which is unique, for every g.

c) If $g \in \mathcal{R}$ and α is a solution of $(I + AE)\alpha = g$ then $\sigma = E\alpha$ is a solution of $(D + A)\sigma = g$.

Proof: a) It is clear that $DE = I$ on all elements $g = (g_1, g_2)$ for which $(d/ds)(\mathcal{I}g_2) = g_2$. This is true exactly when $g \in \mathcal{R}$.

b) According to the Fredholm theory, it is enough to check that the system $(I + AE)\alpha = 0$ has only the trivial solution. Since A takes values in $\mathcal{R}$ and $\alpha = -AE\alpha$ we see that $\alpha \in \mathcal{R}$. It follows that $DE\alpha = \alpha$, so if we denote $\sigma = E\alpha$ then σ satisfies $(D + A)\sigma = 0$. It follows from theorem 4.10 that $\sigma = cv$, where c is a constant and $v = (2\kappa, 1)$. It follows that $\alpha = 0$ because $\alpha = DE\alpha = D\sigma$ and on the other hand $D\sigma = D(cv) = 0$.

c) Since A takes values in $\mathcal{R}$ and $\alpha = -AE\alpha + g$ we see that $\alpha \in \mathcal{R}$. It follows that $DE\alpha = \alpha$, and the statement is clear.

Note that $\mathcal{I}$ is an "indefinite integral operator" along the curve ∂D. Such an operator can be constructed explicitly by taking for any given f the Fourier transform of f, multiplying each term by an appropriate coefficient depending on its index, and taking a backward Fourier transform. In terms of the discretization of the curve, all this can be done by using Fast Fourier Transforms. This means that in terms of the discretization we can construct very easily and "cheaply" the linear operator corresponding to $\mathcal{I}$. Therefore, the equation

$$(D + A)\sigma = g$$

can be replaced by

$$(I + AE)\alpha = g$$
$$\sigma = E\alpha.$$

It is clear now that the algorithm for solving problem (F) has the same structure as the one for solving problem (C), except that before each matrix application corresponding to the integrals with the kernels, one has to apply the matrix E, and that one has to apply E one more time to the solution obtained in this way. The remarks about the condition number of the system of linear equations, and the computational complexity, the rate of convergence and the stability of the algorithm made for the preceding problem apply here as well.

The supported plate problem (S): The algorithm for problem (S0) is completely analogous to the one for problem (C). There is one thing which needs to be pointed out though. One of the integrals to be computed has the kernel $G_\nu = (1/8\pi)(v\cdot\nu)(\log(v\cdot v)+1)$. As we pointed out in the theorem in Section 3.3, this is a function of class C^1, which has a second derivative which is integrable, but not continuous. It follows then that if we use the trapezoidal rule in the local computation of the fast algorithm of the type described in [27] and [10], then the rate of convergence is going to be 2. We can easily increase the rate of convergence by using the end point corrected trapezoidal rule for computing integrals as described in [28]. As it is described in [28], the reasonable weights to use are those which yield an order of convergence a little over 3. It can be

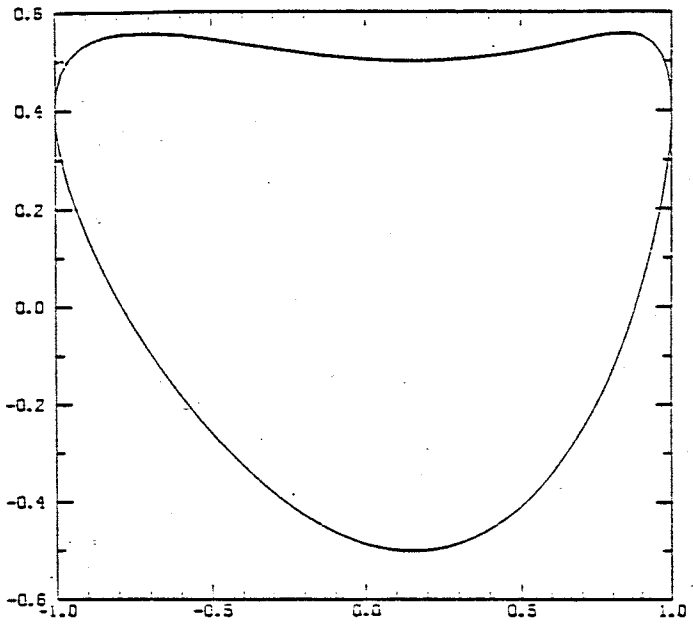

Figure 1: The Crescent (C)

We want the solution of the problem to be of the type

$$v(x,y) = \alpha \cdot [(x-x_0)^2 + (y-y_0)^2] \cdot \log[(x-x_0)^2 + (y-y_0)^2]$$

with some explicit values for α, x_0 and y_0, with the point (x_0, y_0) being outside the domain. We compute the boundary values g_1 and g_2 for the partial differential equation analytically.

For each domain we consider a finite set of points in the interior of the domain. Specificly, we take a set of 15 points spread in the subdomain which is the contraction by 1/2, centered at the origin, of the given domain. The purpose of the computer program is to compute the solution u at the given points. Then we compare the computed solution u to the exact solution v, by computing the relative L^2 norm of the difference, i. e. by computing

$$e = \frac{||u-v||}{||v||}, \tag{5.1}$$

the norm being the discrete L^2 norm on the set where u has been computed.

For each problem we start by discretizing the boundary of the domain. First, we take $n = 16$ equispaced points on the boundary of the domain. Then, we take

seen in practice that the rate of convergence with those weights is closer to 4. In any case this is quite a satisfactory rate of convergence in practical problems. The other characteristics of the algorithm are very similar to the characteristics of the previous algorithms, and we will not repeat the discussion.

For problem (Sμ) there are a few additional computational details. The computation of the derivative κ' of the curvature can be done efficiently by using Fast Fourier Transforms. They can also be used to express the second derivative operator d^2/ds^2 as a linear operator. The computation of the Hilbert transform entails the computation of an integral in the principal value sense. If this is done by using the trapezoidal formula, it would decrease the rate of convergence to 1. One can keep a high rate of convergence by recalling the well known fact that if one uses a higher order interpolation formula to compute the value of the integrand at the singular point, then one obtains correspondingly higher order convergence for the quadrature.

5.2 Numerical Examples

In this section we will present some numerical experiments. We will state some explicit problems, which we solve numerically using the algorithms described in the previous section. Our purpose is to illustrate that the condition number and the order of convergence are the ones predicted by the theory.

For each problem we consider three domains:

D is the disk of radius 1 (centered at the origin)

E is the bounded domain inside the ellipse of axes 2 and 1 (along the coordinate axes)

C is the crescent given by the equations

$$x(t) = \cos t + 0.15 \cdot \sin^2 t$$
$$y(t) = 0.5 \cdot \sin t + 0.4 \cdot \cos^2 t$$

(see figure 1).

$n = 32$, $n = 64$, $n = 128$, and in some cases $n = 256$. For each value of n we compute the values of u at the given points inside the domain, and we compute the relative L^2 error $e(n)$ defined by (5.1). For each problem and each value of n we compute the condition number $c(n)$ of the linear system corresponding to the integral equation, as explained in the previous section. We put in the third column of each table the number $q(n) = e(n)/e(n-1)$. The tables 1, 2 and 3 show the numerical results obtained from the computer programs.

Table 1: Problem (C) on Domain D

n	$c(n)$	$e(n)$	$q(n)$
16	0.141D+02	0.272D-03	
32	0.139D+02	0.104D-06	2620.783
64	0.140D+02	0.795D-09	130.331
128	0.141D+02	0.796D-11	99.926

Table 2: Problem (C) on Domain E

n	$c(n)$	$e(n)$	$q(n)$
16	0.778D+02	0.115D+00	
32	0.747D+02	0.835D-03	137.380
64	0.767D+02	0.102D-04	81.747
128	0.776D+02	0.110D-06	92.750

Table 3: Problem (C) on Domain C

n	$c(n)$	$e(n)$	$q(n)$
16	0.140D+03	0.173D+00	
32	0.253D+03	0.232D-02	74.366
64	0.310D+03	0.149D-03	15.560
128	0.359D+03	0.423D-05	35.314
256	0.359D+03	0.544D-07	77.802

The resampling of the boundary was done to an accuracy of 0.1D-6, and the interpolation formula used for computing the values of the kernels in the integral equation at points (P, P) was seventh order convergent. Therefore, the

expected values for $q(n)$ are around 128. The results for $q(n)$ obtained from the computation are slightly different because the problem is solved to the desired accuracy before we even get into the region of asymptotic behaviour.

Tables 4, 5 and 6 give the similar results for problem (F). The comments made for problem (C) apply here as well. The only difference is that because of the way the program was written, the expected values for $q(n)$ are $q(n) = 8$ for these problems. Also note that since the solution to problem (F) is not unique, the functions u and v were replaced by $u - u(0,0)$ and $v - v(0,0)$ in the computation of the error $e(n)$.

Table 4: Problem (F) on Domain D

n	$c(n)$	$e(n)$	$q(n)$
16	0.262D+01	0.344D-02	
32	0.263D+01	0.426D-03	8.066
64	0.263D+01	0.531D-04	8.022
128	0.264D+01	0.662D-05	8.017

Table 5: Problem (F) on Domain E

n	$c(n)$	$e(n)$	$q(n)$
16	0.896D+01	0.298D-01	
32	0.835D+01	0.747D-03	39.880
64	0.842D+01	0.544D-04	13.732
128	0.842D+01	0.712D-05	7.645
256	0.842D+01	0.915D-06	7.775

Table 6: Problem (F) on Domain C

n	$c(n)$	$e(n)$	$q(n)$
16	0.926D+01	0.499D-01	
32	0.217D+02	0.104D-01	4.782
64	0.179D+02	0.102D-02	10.252
128	0.168D+02	0.112D-04	90.805
256	0.169D+02	0.844D-06	13.295

Tables 7, 8 and 9 describe the numerical results for problem (S0). For this problem we used the end point corrected trapezoidal rule for computing the integral of functions involving log. The proven rate of convergence is ≥ 3. Therefore the expected values for $q(n)$ are at least 8 (asymptotically).

Table 7: Problem (S0) on Domain D

n	$c(n)$	$e(n)$	$q(n)$
16	0.115D+02	0.465D-04	
32	0.115D+02	0.213D-07	2187.196
64	0.115D+02	0.723D-09	29.424
128	0.115D+02	0.653D-10	11.070

Table 8: Problem (S0) on Domain E

n	$c(n)$	$e(n)$	$q(n)$
16	0.192D+02	0.231D-01	
32	0.187D+02	0.732D-04	316.212
64	0.187D+02	0.217D-06	337.192
128	0.187D+02	0.200D-08	108.667

Table 9: Problem (S0) on Domain C

n	$c(n)$	$e(n)$	$q(n)$
16	0.234D+02	0.397D-01	
32	0.251D+02	0.162D-03	244.651
64	0.266D+02	0.220D-05	73.723
128	0.270D+02	0.502D-07	43.840

Finally, we illustrate on two examples, that problem (Sμ) can be solved in principle, but the condition number increases very fast. We take $\mu = 1$, and we solve problem (Sμ) on C and E. Tables 10 and 11 illustrate the results. In these tables we add a column for $q_c(n) = q(n)/q(n-1)$.

Table 10: Problem (Sμ) on Domain C

n	$c(n)$	$q_c(n)$	$e(n)$	$q(n)$
16	0.112D+05		0.608D-03	
32	0.200D+06	17.783	0.244D-05	249.215
64	0.332D+07	16.581	0.113D-05	2.165
128	0.547D+08	16.472	0.809D-07	13.913

Table 11: Problem (Sμ) on Domain E

n	$c(n)$	$q_c(n)$	$e(n)$	$q(n)$
16	0.331D+05		0.104D+00	
32	0.570D+06	17.230	0.683D-03	152.628
64	0.948D+07	16.634	0.637D-05	107.067
128	0.155D+09	16.369	0.249D-06	25.644

5.3 Further Problems

This section contains some problems which can be solved using the techniques of this dissertation, as well as some problems not dealt with here which are interesting.

A) The first problem we mention is the problem of finding a better solution to problem (Sμ). This is related to the next problem mentioned here, since one approach would be to reduce problem (Sμ) to other boundary value problems. Indeed it is easy to see that problem (Sμ) can be reduced to

$$\begin{aligned} \Delta^2 u &= 0 && \text{on } D \\ u &= h_1 && \text{on } \partial D \\ \Delta u + \mu\kappa u_\nu &= h_2 && \text{on } \partial D \end{aligned}$$

where κ is the curvature of ∂D.

B) Other boundary value problems. The heuristic reasoning explained in Section 3.5 can be used to find the potential formulae and the diagonal terms for other boundary value problems as well. For example, the problem of finding u which satisfies

$$\begin{aligned} \Delta^2 u &= 0 && \text{on } D \\ \partial u/\partial \nu &= g_1 && \text{on } \partial D \\ \partial^2 u/\partial \nu^2 &= g_2 && \text{on } \partial D \end{aligned}$$

for given g_1 and g_2, can be solved by looking for u as

$$u(P) = \int_{\partial D} [(G_{\nu\nu} + 3G_{\tau\tau})(P,Q)\sigma_1(Q) + G_\nu(P,Q)\sigma_2(Q)]ds(Q),$$

for which the matrix of diagonal terms is

$$\begin{pmatrix} \frac{1}{2} & 0 \\ \frac{3}{2}\kappa & \frac{1}{2} \end{pmatrix}.$$

Numerical evidence suggests that when D is simply connected, the integral operator giving the integral equation has a kernel of dimension one, and the solution of this problem goes very much along the lines of the solution of problem (F). This problem was not included in this dissertation, since it appears to have no particular physical significance.

C) Another class of problems worth mentioning is the adjoint problems. We will explain the adjoint problem on an example. We take as a starting point the integral equation we deduced in order to solve problem (C). Let us consider the adjoint of the operator giving the integral equation, and let us consider the integral equation given by this adjoint. We obtain

$$\begin{pmatrix} \frac{1}{2} & -\kappa(Q) \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \alpha_1(Q) \\ \alpha_2(Q) \end{pmatrix} + \tag{5.2}$$

$$\int_{\partial D} \begin{pmatrix} G_{\nu\nu\nu} + 3G_{\nu\tau\tau} & (G_{\nu\nu\nu} + 3G_{\nu\tau\tau})_{\nu_P} \\ -G_{\nu\nu} + G_{\tau\tau} & (-G_{\nu\nu} + G_{\tau\tau})_{\nu_P} \end{pmatrix} (P,Q) \begin{pmatrix} \alpha_1(P) \\ \alpha_2(P) \end{pmatrix} ds(P) =$$

$$= \begin{pmatrix} g_1(Q) \\ g_2(Q) \end{pmatrix}$$

(in the notation of theorem 4.8). We will call this the adjoint integral equation. Now we will write down a boundary value problem whose solution along the lines described in Section 2.2 leads to the integral equation (5.2). An obvious choice for the potential formula is

$$u(Q) = \int_{\partial D} [G(P,Q)\alpha_1(P) + G_{\nu_P}(P,Q)\alpha_2(P)]ds(P)$$

and for the boundary value problem is

$$\begin{array}{lll} \Delta^2 u = 0 & \text{on} & \mathbf{R}^2 \setminus D \\ u_{\nu\nu\nu} + 3u_{\nu\tau\tau} = g_1 & \text{on} & \partial D \\ -u_{\nu\nu} + u_{\tau\tau} = g_2 & \text{on} & \partial D. \end{array}$$

One has to check that the diagonal terms coming from the potential formula correspond to the ones in the adjoint integral equation (this is the reason for considering the exterior problem instead of the interior one). Part of the numerical and theoretical solution of this problem comes almost for free because we have a full understanding of the integral equation. One has to determine to what extent the solution of the boundary value problem is not unique though. Some of the adjoint problems turn out to have physical significance.

D) Connected to the adjoint problems, and as a class in themselves, we mention the exterior problems. Before doing that, let us briefly discuss the biharmonicity at infinity. It is clear what the analytic condition should be: the function $u(z)$ is biharmonic at ∞ if the function $u(1/z)$ extends as a biharmonic function across the origin. It would be interesting to give the condition of biharmonicity at ∞ in terms of some growth conditions on the function and on some of its derivatives.

As for the exterior problems, one can use the same potential formulas as for the interior ones, and the integral equations differ from the corresponding ones from the interior problems by a change of sign in front of the integral. One has to redo the study of the integral equations. The results might be quite different from the ones obtained in the case of the integral equations corresponding to the interior problems. For example the dimension of the zero space of the integral equation is 4 for the exterior problems corresponding to (C) and (F). The boundary value problems themselves have to be studied as well, since there might be necessary conditions to be imposed on the boundary data, while for some problems, some part of the biharmonicity at infinity might be automatically satisfied.

E) It would be interesting to understand completely what happens when D is not simply connected. To some extent this problem is related to the preceding one, as it is suggested by theorem 4.8.

F) Other problems worth mentioning are three dimensional problems. There have been successful attempts to solve the biharmonic, and even higher order equations, in three or even higher dimensional spaces, by reducing them to second

kind integral equations (see [18, 20, 24, 32]). But none of them seems to be of much use when it comes to numerical computations. Therefore, it is legitimate to search for a method which would lead to effective numerical algorithms. The approach presented here can be generalized in principle to three (and higher) dimensional problems, but this is clearly a non-trivial task.

BIBLIOGRAPHY

[1] S. Agmon, *Multiple Layer Potentials and the Dirichlet Problem for Higher Order Elliptic Equations in the Plane*, Communications on Pure and Applied Mathematics, Vol. X (1957), pp. 179-239.

[2] K. E. Atkinson, *A Survey of Numerical Methods for the Solution of Fredholm Integral Equations of the Second Kind*, SIAM, Philadelphia, 1976.

[3] P. Bjorstad, *Numerical Solution of the Biharmonic Equation*, Ph. D. Dissertation, Stanford University, Department of Computer Science, 1980.

[4] J. Cohen and J. Gosselin, *The Dirichlet Problem for the Biharmonic Equation in a C^1 Domain in the Plane*, Indiana University Mathematics Journal, Vol. 32, Nr. 5 (1983), pp. 635-685.

[5] J. Cohen and J. Gosselin, *Adjoint Boundary Value Problems for the Biharmonic Equation on C^1 Domains in the Plane*, Arkiv for Matematik, Vol. 23, Nr. 2 (1985), pp. 217-240.

[6] M. Costabel, I. Lusikka, J. Saranen, *Comparison of Three Boundary Element Approaches for the Solution of the Clamped Plate Problem*, Boundary Elements IX, Vol. 2 (1987), pp. 19-34.

[7] B. E. J. Dahlberg, C. E. Kenig, G. C. Verchota, *The Dirichlet Problem for the Biharmonic Equation in a Lipschitz Domain*, Ann. Inst. Fourier (Grenoble), Vol. 36, Nr. 3 (1986), pp. 109-135.

[8] S. C. Eisenstat, H. C. Elman, M. H. Schultz, *Variational Iterative Methods for Nonsymmetric Systems of Linear Equations*, SIAM Journal on Numerical Analysis, Vol. 20 (1983), pp. 345-357.

[9] G. B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, 1976.

[10] L. Greengard, V. Rokhlin, *A Fast Algorithm for Particle Simulations*, Journal of Computational Physics, Vol. 73, Nr. 2 (1987), pp 325-348.

[11] C. E. Kenig, *Boundary Value Problems of Linear Elastostatics and Hydrostatics on Lipschitz Domains*, Goulaouic-Meyer-Schwartz Seminar, 1983-1984, Exp. Nr. 21, Ecole Polytechnique, Palaiseau, 1984.

[12] C. E. Kenig, *The Dirichlet Problem for the Biharmonic Equation in a Lipschitz Domain*, Bony-Sjostrand-Meyer Seminar, 1984-1985, Exp. Nr. 19, Ecole Polytechnique, Palaiseau, 1985.

[13] L. D. Landau and E. M. Lifschitz, *Fluid Mechanics*, Pergamon Press, London, 1959.

[14] L. D. Landau and E. M. Lifschitz, *Theory of Elasticity*, Pergamon Press, London, 1970.

[15] U. Langer, *Fast Iterational Method for the Solution of the First Boundary Value Problem for the Biharmonic Equation* (in Russian), Journal for Computational Mathematics and Mathematical Physics, Vol. 28, Nr, 2 (1988), pp. 209-223.

[16] G. Lauricella, *Sull'integrazione dell'equazione* $\Delta^4 U = 0$, Rendiconti Lincei, Serie V, Vol. XVI, 2 sem. (1907), pp. 373-383.

[17] G. Lauricella, *Sur l'integration de l'equation relative a l'equilibre des plaques elastiques encastrees*, Acta Mathematica, Vol. XXXII (1909), pp. 201-256.

[18] G. Lauricella, *Sull'equazione* $\Delta^{2i}U = 0$ *e su alcune estensioni delle equazioni dell'equilibrio dei corpi elastici isotropi*, Atti del IV Congresso Internazionale, Roma, 1908, Vol. III (1909), pp. 5-15.

[19] G. Lauricella, *Sull'integrazione dell'equazione* $\Delta^{2i}U = 0$ *per le aree piane*, Rendiconti Lincei, Serie V, Vol. XVIII, 2 sem. (1909), pp. 5-15.

[20] R. Marcolongo, *La teoria delle equazioni integrali e le sue appliquazioni alla Fisica-matematica*, Rendiconti Lincei, Serie V, Vol. XVI, 1 sem. (1907), pp. 742-749.

[21] A. Mayo, *The Fast Solution of Poisson's and the Biharmonic Equations on Irregular Regions*, SIAM Journal of Numerical Analysis, Vol. 21, Nr. 2 (1984), pp. 285-299.

[22] S. G. Mikhlin, *Integral Equations*, Pergamon Press, New York, 1957.

[23] N. I. Mushkelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, P. Noordhoff, Groningen, 1963.

[24] M. Nicolesco, *Les Fonctions Polyharmoniques*, Hermann et Co, Paris, 1936.

[25] P. Peisker, *On the Numerical Solution of the First Biharmonic Equation*, Mathematical Modelling and Numerical Analysis, Vol. 22, Nr. 22 (1988), pp. 655-676.

[26] G. R. Richter, *An Integral Equation Method for the Biharmonic Equation*, Advances in Computer Methods for Partial Differential Equations, R. Vichnevetsky (Editor), Publ. IMACS (AICA), 1977, pp. 41-45.

[27] V. Rokhlin, *Rapid Solution of Integral Equations of Classical Potential Theory*, Journal of Computational Physics, Vol. 60, Nr. 2 (1985), pp. 187-207.

[28] V. Rokhlin, *End-Point Corrected Trapezoidal Quadrature Rules for Singular Functions*, Research Report YALEU/DCS/RR-441, Yale University, Department of Computer Science, November 1985.

[29] R. Temam, *Navier Stokes Equations*, North-Holland, Amsterdam, 1977.

[30] F. G. Tricomi, *Integral Equations*, Dover Publications, Inc., New York, 1985.

[31] I. N. Vekua, *New Methods for Solving Elliptic Equations*, John Wiley and Sons, Inc., New York, 1967.

[32] G. Verchota, *The Dirichlet Problem for the Biharmonic Equation in C^1 Domains*, Indiana University Mathematics Journal, Vol. 36, Nr. 4 (1987), pp. 867-895.