**An Explicit Scheme for the Prediction of Ocean Acoustic Propagation in Three Dimensions**

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# AN EXPLICIT SCHEME FOR THE PREDICTION OF OCEAN ACOUSTIC PROPAGATION IN THREE DIMENSIONS*

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## SUMMARY

Because of excessive computation time, solving the parabolic equation in higher dimensions by means of implicit finite difference schemes seems to be impractical even if the scheme is unconditionally stable. To economize the computation time and computer storage, a stable explicit finite difference scheme is introduced for the solution of the parabolic equation of the Schrödinger type. This explicit scheme involes five spatial points and is conditionally stable by introducing an additional dissipative term. The complete theory with respect to the stability is proved. An application to a three-dimensional ocean acoustic propagation problem is included to demonstrate its validity.

## INTRODUCTION

Many physical problems result in the real application of parabolic equations. A familiar representative parabolic equation is the heat equation with real coefficients. A number of applications (other than heat conduction) arise in the area of quantum mechanics, plasm physics, optics, seismology, ocean acoustics, etc. [1], and result in a form of parabolic equation with complex coefficients. A familiar representative parabolic equation with complex coefficients is the Schrödinger equation. For discussion, the theory of a new stable explicit finite difference scheme as well as a real application are chosen to deal with the Schrödinger equation of multi-dimensions in the form

$$u_r = \sum_{\ell=1}^{m} i b_\ell u_{z_\ell z_\ell} . \tag{1}$$

For a more general expression, we can include the low order terms to give

$$u_r = \sum_{\ell=1}^{m} (i b_\ell u_{z_\ell z_\ell} + a_\ell u_{z_\ell} + c_\ell u + f_\ell). \tag{2}$$

As an application, a one-way ocean acoustic sound propagation in three dimensions is represented by

$$u_r = \frac{i}{2} k_0 (n^2(r,\theta,z)-1)u + \frac{i}{2k_0} \frac{\partial^2 u}{\partial z^2} + \frac{i}{2k_0 r^2} \frac{\partial^2 u}{\partial \theta^2} \tag{3}$$

where $k_0$ is a reference wavenumber and $n(r,\theta,z)$ is the three-dimensional index of refraction, which is defined as a ratio of a reference sound speed to a three-dimensional sound speed. Eq. (3) is in three-dimensional cylindrical coordinates [2].

A solution exists [3] for Eq. (3) that uses an unconditonally stable implicit finite difference scheme, which discretizes Eq. (3) by means of central finite differences for both z and $\theta$ derivatives. Then the Crank-Nicolson scheme is applied to formulate a large system of spared matrix. This system was solved by a Yale University [3] preconditioning sparse technique. Results, produced by the Crank-Nicolson scheme [2] are reasonably accurate. However, due to the step-by-step

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iteration to solve the system, excessive computer time was required. This motivated us to develop a more economical, stable explicit finite difference scheme. In the sections to follow, the main discussion is on the introduction of a conditionally stable explicit finite difference thoroughly examining its consistency, stability, and convergence. A theorem to describe the stability of this new scheme is developed and proved. Following the theoretical section, we use a three-dimensional acoustic wave equation arising from the application of underwater wave propagations as a test case to examine the validity of the theory. We examine the accuracy and speed of the theory by comparing it with the solution produced by the Crank-Nicolson scheme. As a physical illustration of the three-dimensional problem, a plot is included to describe intensity effects of the three-dimensional ocean wave propagation.

## A STABLE EXPLICIT SCHEME FOR HIGH DIMENSIONS

Chan, Shen, and Lee [4] discussed the solution to a model Schrödinger equation, i.e.,

$$u_r = i\, u_{zz}, \tag{4}$$

by the finite difference scheme

$$\frac{u_j^{n+1} - u_j^n}{k} = i\, \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}, \tag{5}$$

which is UNSTABLE where $k = \Delta r$, $h = \Delta z$.

As a consequence, a number of stable explicit schemes were introduced [4] to solve the parabolic equation of the Schrödinger type. In this paper, a scheme is selected for application and replaces scheme (5) by introducing a dissipative term, which is added to scheme (5) to give

$$\frac{u_j^{n+1} - u_j^n}{k} = i\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}\right) + (\alpha + i\beta)\left(\frac{u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n\; u_{j-2}^n}{h^4}\right), \tag{6}$$

where $\alpha$ and $\beta$ are determined to be $\alpha = -1/4$, $\beta = 1/4$ for least resitrictive stability condition. As a generalization of the scheme (6) to the Schrödinger equation of high order [1], consider the multi-dimensional Schrödinger equation

$$u_r = \sum_{\ell=1}^{m} i b_\ell u_{z_\ell z_\ell}, \tag{7}$$

where the b's are assumed to have the same sign. Without loss of generality, we assume $b > 0$ for $= 1, 2, \ldots, m$. We consider the natural extension of scheme (6) takes the form

$$\frac{u_j^{n+1} - u_j^n}{k} = \sum_{\ell=1}^{m} b_\ell \left[ i D_{j,\ell}^n u + (\alpha + i\beta)\, h_\ell^2 \,(D_{j,\ell}^n)^2 \right] u \,. \tag{8}$$

In Eq. (8), j represents a multi-index $(j_1, j_2, \ldots, j_m)$, $D_{j,\ell}^n$ is the second-order centered difference operator with respect to j, and $h_\ell$ is the corresponding mesh size.

<u>THEOREM</u>: If scheme (8) is used to solve Eq. (7), the scheme is stable if and only if $\alpha < 0$ and

$$k \le \min\left( \frac{-2\alpha}{\sum_{\ell=1}^{m} \frac{b_\ell}{h^2}},\; \frac{-\alpha}{[\alpha^2 + (\beta - \frac{1}{4})^2] \sum_{\ell=1}^{m} \frac{b_\ell}{h_\ell^2}} \right). \tag{9}$$

The least restrictive stability constraint is

$$k \le \frac{1}{2 \sum_{\ell=1}^{m} \frac{b_\ell}{h_\ell^2}}$$

and is obtained when $\alpha = -1/4$ and $\beta = 1/4$.

The proof appears in its entirety in reference 1 and is outlined below.

<u>PROOF</u>: For economy in writing, define

$$r_\ell = k/h_\ell^2\,,\quad \eta_\ell = 4 \sin^2 \frac{\theta_\ell}{2}\,,\quad F_\ell = b_\ell/h_\ell^2\,.$$

The amplification factor R can be determined to be

$$R = 1 - \sum_{\ell=1}^{m} b_\ell r_\ell \eta_\ell \,[i\eta_\ell - (\alpha + i\beta)\eta_\ell^2].$$

The stability requires that $\|R\| \le 1$. After some simplification, the stability condition can be written as

$$k \le G(\eta_1, \eta_2, \ldots, \eta_m) = 2\alpha \sum_{\ell=1}^{m} F_\ell \eta_\ell^2 \Big/ \left[\alpha^2 \left(\sum_{\ell=1}^{m} F_\ell \eta_\ell^2\right)^2 + \left(\sum_{\ell=1}^{m} F_\ell \eta_\ell\, (\beta\, \eta_\ell^2 - 1)\right)^2\right].$$

Let $p = (\eta_1, \eta_2, \ldots, \eta_m)$ and define

$D = \{p;\ 0 \le \eta_\ell \le 4,\ \ell = 1, 2, \ldots, m\}$,

$D_1 = \{p;\ p \in D \text{ and } \sum_{\ell=1}^{m} F_\ell \eta_\ell \le \beta \sum_{\ell=1}^{m} F_\ell \eta_\ell^2\}$, and

$D_2 = \{p;\ p \in D \text{ and } \sum_{\ell=1}^{m} F_\ell \eta_\ell > \beta \sum_{\ell=1}^{m} F_\ell \eta_\ell^2\}$.

Clearly, $D = D_1 \cup D_2$.

We consider two cases: $\beta \geq 1/4$ and $\beta < 1/4$.

CASE I: $\beta \geq 1/4$.

$I_a$: In $D_1$,

$$\inf_{D_1} G(n_1, n_2, \ldots, n_m) = -\alpha / \left\{ 8[\alpha^2 + (\beta - \tfrac{1}{4})^2] \sum_{\ell=1}^{m} F_\ell \right\} \tag{10}$$

$I_b$: In $D_2$,

$$\inf_{D_2} G(n_1, n_2, \ldots, n_m) = \min \left( -2\alpha / \sum_{\ell=1}^{m} F_\ell \, , \, S(w^*) \right) \tag{11}$$

where

$$S(w) = -2\alpha / \left\{ \alpha^2 w^2 + \left[ \left( \sum_{\ell=1}^{m} F_\ell \right)^{1/2} - \beta w \right]^2 \right\},$$

$$\text{and } w^* = \sup_{D_2} w, \quad w = \left( \sum_{\ell=1}^{m} F_\ell n_\ell^2 \right)^{1/2}.$$

From Eqs. (10) and (11), it can be verified that for $\beta \geq 1/4$, the stability condition is

$$k \leq \min \left( -\alpha / \left\{ 8[\alpha^2 + (\beta - \tfrac{1}{4})^2] \sum_{\ell=1}^{m} F_\ell \right\}, \; -2\alpha / \sum_{\ell=1}^{m} F_\ell \right). \tag{12}$$

CASE II: $\beta < 1/4$.

Clearly $D_1$ is empty and $D = D_2$. It is seen that

$$G \geq \min \left( -2\alpha / \sum_{\ell=1}^{m} F_\ell \, , \; -\alpha / \left\{ [\alpha^2 + (\beta - \tfrac{1}{4})^2] \sum_{\ell=1}^{m} F_\ell \right\} \right). \tag{13}$$

The general stability condition is therefore (12). Clearly, we must have $\alpha < 0$. To choose $\alpha$ and $\beta$ such that the stability condition is the least restrictive, we must take $\beta = 1/4$ so that

$$k \leq \min \left( -\frac{1}{8\alpha \sum_{\ell=1}^{m} F_\ell} \, , \; -\frac{2\alpha}{\sum_{\ell=1}^{m} F_\ell} \right).$$

To maximize the right-hand side above, we take $\alpha = -1/4$, which gives

$$k \leq \frac{1}{2 \sum_{\ell=1}^{m} \frac{b_\ell}{h_\ell^2}} \text{, establishing the result.}$$

## AN APPLICATION

In the three-dimensional ocean, a class of sound wave propagation problems can be represented by a parabolic equation of the Schrödinger type [2]. For prescribed environmental conditions, an application of a three-dimensional problem in sector can be shown as in Figure 1 for its sector region of progagation, where $r_0 \leq r \leq r_m$, $0^\circ \leq \theta \leq 5^\circ$, and $0 \leq z \leq 100$m. In actual simulation the sector is taken to be $-20^\circ \leq \theta \leq 20^\circ$.

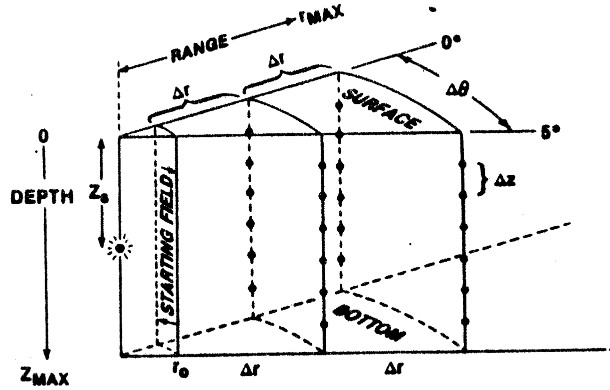


Figure 1: Sector Region of Propagation

An exact solution $u(r,\theta,z)$ has been obtained [2] and takes the form

$$u(r,\theta,z) = \sin(\Omega z)\, e^{im\theta}\, e^{i \frac{m^2}{2k_0 r}}, \tag{10}$$

which satisfies the three-dimensional parabolic equation (3).

The computation speed among explicit finite schemes [1] and an implicit finite scheme [2] using two-dimensional as well as three-dimensional examples has been examined by Chan et al. [5]. The same three-dimensional problem with known exact solution was used by Chan et al. [5] to examine, in particular, the computation speed between each explicit scheme, as described in [1], against the implicit finite difference scheme, as described in [2]. Their findings show a more favorable compuation speed for the explicit scheme than the implicit scheme. We extend their study to some three-dimensional effects using the explicit scheme, expressed by Eq. (6).

In the application, the $\Omega$ is assigned to be $\pi/100$ and the modal index m is taken to be 3. The source is placed at 50m below the surface and propagates the sound in a regular three-dimensional cylindrical region. The propagation is required to reach the maximum range at 550m where we can see three-dimensional effects. We limit the propagation to a sector of 40° (i.e., from -20° to +20°) and centered at the origin (0,0,0). For simplicity, the three-dimensional sound speed $c(r,\theta,z)$ is taken as a constant and the medium is assumed homogeneous. Initial boundary values are generated from the exact solution.

Since our numerical results produce field intensity information at all receiver depths, we can output contour plots for each angle $\theta$. Figure 2 presents a contour plot of energy flow at $\theta = 0^\circ$.

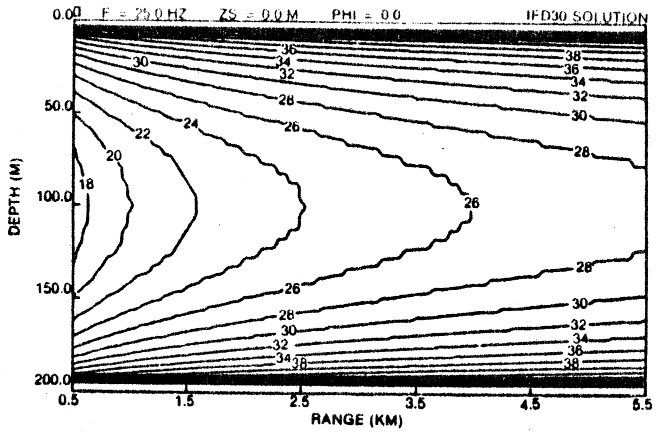


Figure 2: Contour plot of Field Intensity

The plot was produced by the existing three-dimensional Crank-Nicolson scheme in conjunction with a Yale sparse technique. The accuracy of the results have been discussed in [2]. The same calculation was performed by the explicit scheme (Eq. (8)) using the same range step size (0.001m) as used by the Crank-Nicolson scheme. The explicit scheme solution produced results very close to Crank-Nicolson's, thus, generated the same plot curves as described in Figure 2. However, the advantage of the explicit scheme is the CPU time required for the complete computation, which is approximately 3.5 times faster than the Crank-Nicoloson scheme for achieving the same accuracy.

## CONCLUSION

We have introduced an explicit finite difference scheme to solve the Schrödinger equation. This scheme was developed to be conditionally stable. Numerical results demonstrated its accuracy and agreement not only with the Crank-Nicolson scheme but also with the known exact solution. It is expected that if this scheme is implemented in a vectorized computer, its storage, implementation, and computation time advantages would become evident. We showed only one of the explicit schemes we have developed, we believe other explicit schemes we have introduced [4] may have equal advantages over implicit schemes.


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