

EVERY FINITE DISTRIBUTIVE LATTICE IS A SET OF
STABLE MATCHINGS FOR A Small STABLE MARRIAGE
INSTANCE

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Every Finite Distributive Lattice is a Set of Stable Matchings for a *Small* Stable Marriage Instance

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1 Introduction

An instance I of the stable marriage problem consists of n men and n women, each of whom has a rank-ordered preference list of the n people of the opposite sex. A marriage M is a one-one matching of the men and the women. Marriage M is said to be *unstable* if there is a man and a woman who are not matched to each other in M , but who both prefer each other to their respective mates given in M . A marriage that is not unstable is called *stable*. The fundamental theorem [GS] states that there is a stable marriage for any problem instance I .

We say that marriage M *weakly dominates* marriage M' if no man prefers his mate in M' to his mate in M . For a given instance I of the stable marriage problem, let $S(I)$ be the set of all stable marriages. The relation of weak dominance defines a partial order P on $S(I)$. In fact [K], P is a distributive lattice. Knuth [K] asked whether the converse is true, i.e. is every finite distributive lattice the weak dominance relation for some instance of the stable marriage problem? This question was recently answered in the affirmative by C. Blair [BL] who gave a constructive method to generate an instance $B(L)$ of the stable marriage problem given any finite distributive lattice L , such that L is the weak dominance relation of $S(B(L))$. However, as pointed out in [BL], there may be other problem instances whose weak dominance relation is also L , and which have fewer people than $B(L)$. Further, the only known bound (proved in [BL]) on the number of people in $B(L)$ is 2^{k+1} , where k is the number of elements in L .

In this note we describe a method which, given a distributive lattice L with k elements, will construct a small stable marriage instance $I(L)$ such that L is the weak dominance relation of the set of stable marriages $S(I(L))$. The following theorems will be noted:

Theorem 1 *If L has k elements then $I(L)$ will have at most $k^2 - k + 4$ people.*

Theorem 2 *If h is the height of the distributive lattice L , then $I(L)$ will have at most $h^2 - h + 4$ people.*

Theorem 3 *If L is the weak dominance relation for some problem instance with $2n$ people, then $I(L)$ will have at most $n^4/4$ people.*

While the upper bound of $n^4/4$ in Theorem 3 seems large in comparison to n , it compares favorably to the doubly exponential bound (for a $2n$ person instance, there may be an exponential number of stable marriages) which is the best bound implied by the results in [BL]. The results in this note follow immediately from several facts, some of which are classic, and the others established in recent work by the first three authors. We will state these facts, give references to the proofs, and describe the construction, but will provide no proofs in this note.

2 Facts and Construction

Definition 1 *For a partial order P , a subset $R \subseteq P$ is called closed if $x \in R$ and $y \leq x$ implies $y \in R$, for any $y \in P$.*

Definition 2 *For a partial order P , let $L(P)$ denote the distributive lattice whose elements are the closed subsets of P , under the relation of set containment, i.e. for closed subsets C and C' in P , $C \leq C'$ in $L(P)$ if and only if $C \subseteq C'$ in P .*

Fact 1 [B,G] Any finite distributive lattice L is isomorphic to $L(P)$ for some partial order P , where P has fewer elements than $L(P)$.

Fact 2 [IL] If P is a partial order with k elements, of which u are maximal and d are minimal, and if P has m edges in its Hasse diagram, then there is a stable marriage instance $I(P)$ with $2(m+u+d)$ people, such that there is a one-one correspondence between stable marriages in $S(I(P))$ and closed subsets in P . Further, if stable marriages M and M' in $S(I(P))$ correspond respectively to closed subsets C and C' in P , then M (weakly) dominates M' if and only if $C \subseteq C'$.

For a partial order with k elements, of which u are maximal and d are minimal, and b are both,

$$m \leq \binom{k}{2} - \binom{d}{2} - \binom{u}{2} + \binom{b}{2} - b(k - u - d + b) \leq \binom{k}{2} - \binom{d}{2} - \binom{u}{2}.$$

So $2(m + u + d) \leq k^2 - k + 4$. Hence Theorem 1 follows from facts 1 and 2.

2.1 Theorem 2

The statement of Fact 1 above was sufficient to obtain Theorem 1, but in order to sharpen that result and make the construction explicit, we give Fact 1 in a fuller form.

Definition 3 In a distributive lattice L , let $M(L)$ be the subset of elements of L , such that $x \in M(L)$ if and only if x is the meet of two distinct elements, $y \neq z$, only when $y = x$ or $z = x$. Graphically, $x \in M(L)$ if and only if x has exactly one predecessor in the Hasse diagram of L .

Fact 3 [B,G] Every maximal chain in a distributive lattice L has exactly $h = |M(L)|$ edges.

Definition 4 Let $P(L)$ denote the partial order on the set of elements $M(L)$, where the relation between the elements is as given in L . As in Definition 2, let $L(P(L))$ be the distributive lattice defined on the closed subsets of $P(L)$.

Fact 1' (B, G): L is isomorphic to $L(P(L))$.

Facts 1', 2 and 3 imply Theorem 2.

2.2 Construction

We now describe the construction of $I(L)$ from L . The first step is clearly to construct $P(L)$ from L . The second step is the construction of $[IL]$ mentioned in Fact 2, which constructs instance $I(L)$ from partial order $P(L)$. Full details and proof of correctness of this step are found in $[IL]$.

1. Given partial order P with k elements, number the elements from 1 to k so that each element has a larger number than any of its predecessors. Append a source node 0 and connect it to each of the minimal elements in P ; append a sink node $k+1$ and connect it to each of the maximal elements in P . The resulting partial order P' has unique minimal element 0 and unique maximal element $k+1$.
2. Let $H(P')$ be the Hasse diagram of P' with m' edges. Label each edge in $H(P')$ with a distinct integer between 1 and m' . The instance $I(P)$ will have m' men and m' women.
3. For each i from 1 to m' place woman i on man i 's list, place man i on woman i 's list.
4. For i from 1 to k , iterate the following:

Let $E(i) = \{m(1), \dots, m(r)\}$ be an arbitrary ordering of the set of numbers on the edges incident with node i . Let $W(i) = \{w(1), \dots, w(r)\}$ be the ordered set of women such that for each j

from 1 to r , $w(j)$ is the last choice on man $m(j)$'s list constructed to this point. Then for $j = 1$ to r , place $w(j+1)$ on the end of man $m(j)$'s list, where $j+1$ is taken mod r . Similarly, for $j = 1$ to r , place man $m(j)$ at the head of woman $w(j+1)$'s list.

5. To complete the lists, add any missing entries at the end of the appropriate list, in any order.

Note that if P has m edges, then the construction has $2(m+u+d)$ people as claimed in Fact 2.

As an example, consider the 6 element lattice L discussed in [BL], shown in figure 1a. Figure 1b, shows $P(M(L))$, and figure 1c shows P' with nodes and edges labelled. The resulting stable marriage instance is shown in figure 2 and, interestingly, it has 10 people, the same number as in the example in [BL] used to show that $B(L)$, with 16 people, is not the smallest possible.

2.3 Theorem 3

The following fact follows easily from results in [IL], and is made explicit in [GU].

Fact 4 [IL, GU] If I is a stable marriage instance with $2n$ people, and L is the weak dominance relation on $S(I)$, then L has height $(|M(L)|)$ of at most $n(n-1)/2$.

Theorem 3 follows from fact 4 and Theorem 2.

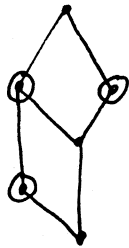
2.4 Comment

Fact 1' provides a compact way to understand and prove many of the results in [IL], where the structure of the set of all stable marriages $S(I)$ is derived in the context of the stable marriage problem, without reference to lattice theory. However, the approach in [IL] has great algorithmic import, as it allows the efficient construction of $P(L)$ from I , i.e. without first finding L . This is refined and exploited in [ILG] and [GU], and generalized to the stable roommate problem in [GU2].

3 References

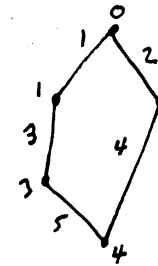
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L

1b. $P(M(L))$



1c. P'

Figure 1a.

elements of $M(L)$ are circled.

Fig. 2

Mens Lists

1	2	3	4	5
1	2	3	4	5
3	4	1	2	1
		5		

Womens Lists

1	2	3	4	5
5	4	1	2	1
3	2	3	4	5
1				