Abstract This paper provides a novel incremental multigrid strategy for the equations of fluid dynamics. The (time dependent) governing equations are discretized in time by means of a two level implicit Euler scheme and linearized using Taylor series and the incremental (delta) form of Beam and Warming. The coefficients and the right hand side of the resulting linear systems are evaluated always at the finest grid level, whereas the (delta) unknowns are computed (approximately, by a single relaxation sweep) on a sequence of coarser meshes. At every grid level the computed deltas are interpolated up to the finest-grid level and used to update the solution, as well as the coefficients and the right hand side of the linear systems. The process is repeated, sweeping all grid levels successively, until a satisfactory convergence criterion is met. The validity of the proposed approach is demonstrated by solving a simple linear problem and the vorticity-stream function Navier-Stokes equations, using line relaxation methods as smoothers, and the lambda-formulation Euler equations, in conjunction with a simple explicit smoother. In all cases, the proposed multigrid strategy provides a considerable efficiency gain over the corresponding single-grid methods.

An Incremental Multigrid Strategy for the Fluid Dynamics Equations

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Introduction

In the last years, the author has been working on developing and improving implicit numerical methods for solving the Navier-Stokes equations and the lambda-formulation Euler Equations, mostly for the case of steady flows. The basic framework of all numerical schemes developed by the author and his coworkers (see, e. g., [1-7]) is the following: the time dependent governing equations are discretized in time by a two level implicit Euler time stepping and linearized using Taylor series and the incremental (delta) approach of Beam and Warming [8]; the resulting large block-sparse linear system obtained at every time step is reduced to a series of smaller block-tridiagonal systems by approximate factorization [1-5] or simple "mutilation" [6,7]; these systems are solved very efficiently by means of standard block-tridiagonal Gaussian elimination [9]; the solution is updated and the process is repeated until a satisfactory convergence criterion is met.

The rationale behind such a general approach (see also [8, 10-12]) is that a single relaxation-like procedure (requiring to solve only block-tridiagonal systems) is used to relax for both the nonlinearity of the governing equations and the extra-tridiagonal terms in the linear systems arising at every time level, leading to a globally efficient procedure. A major benefit of employing the incremental approach [8] is in that a deferred correction strategy [13,14] can be implemented very elegantly and straightforwardly (see, e.g., [4]): two-point first-order-accurate upwind differences are used for the advection terms in the LHS (left hand side) implicit operator, whereas second- or higher-order-accurate differences are used for the RHS (right hand side) steady-state equations, which are evaluated at the old time level explicitly. In this way, the linear systems to be solved at every time step are guaranteed to be diagonally dominant so that improved convergence rate and stability are obtained without lowering the accuracy of the final steady-state solution, in general, and classical point and line relaxation methods have regained a competitive edge with respect to more sophisticated approximate factorization methods [6,7,15], in particular. Is is noteworthy that from a fluid dynamics point of view (for the case of the Navier-Stokes equations), the use of such a deferred corrector approach is a means of stabilizing the pseudo-transient by adding artificial viscosity, which is proportional to a time derivative and thus diminishes (and eventually vanishes) when approaching (and finally reaching) the steady state.

The aforementioned "CFD philosophy" has been reasonably successful, especially from the point of view of its general applicability to the compressible Navier- Stokes [11,8,6] and Euler [1,2,7] equations as well as to the incompressible vorticity-stream function (and energy) equations [3-6] governing the motion of two-dimensional viscous flows (including buoyancy effects).

However, the convergence rate of all of these methods invariably deteriorates when the mesh is refined, so that for very complicated problems requiring a very fine computational grid, they tend to lose their "competitive edge".

The aim of this paper is to remove such a limitation by developing a simple incremental multigrid strategy especially "custom-tailored" for all of these methods, so as to improve their convergence rate especially at the later stages of the relaxation process.

There are two complementary and somewhat antithetic ways of looking at a multigrid strategy as a means for obtaining a desired numerical solution efficiently [16]: the first way is to consider every finer-grid computation as a means for improving the accuracy of a solution cheaply obtained on a coarser mesh; the second way is to regard every coarser-grid computation as a means for reducing a given frequency component in the residual as effectively as possible. Of course, every articulate multigrid strategy combines both the aforementioned aspects, insofar as it makes use of smoothing processes at all grid levels (except maybe at the coarsest one) as well as of prolongation and restriction operators among the various grid levels. However, by considering, for example, three very successful multigrid methods employed for solving flow problems [17-19], it is clear that the method of Ghia et al. [17] is more typical of the first point of view, insofar as a converged solution is obtained on successively finer grids as a starting point for smoothly and efficiently achieving the sought finest-grid solution; the methods of Ni [18] and Jameson [19] are instead more typical of the second point of view, insofar as they only obtain the finest-grid solution and use the coarser meshes just to annihilate the residual as fast as possible.

The present approach has been devised to improve the convergence rate of the various numerical methods developed and employed by the author to solve fluid dynamic problems [1-7]. From the very nature of these methods, it is clear that the proposed multigrid approach is more similar to those of Ni [18] and Jameson [19]. As a matter of fact, the present methodology goes a step further towards employing a single relaxation sweep on each grid to obtain the final steady state solution on the finest grid as fast as possible, insofar as the incremental variables computed from each coarse-grid relaxation sweep are immediately used to update the finest-grid solution: only the finest-grid solution is ever computed, whereas the deltas are evaluated sequentially on a series of coarser grids by means of a single relaxation sweep, immediately interpolated back, up to the finest-grid level and, finally, summed to the finest-grid solution, which is therefore updated after every relaxation sweep.

The proposed methodology is described in the particular case of the vorticity-stream function Navier-Stokes equations, in the next section, and is then applied to three different sets of equations, namely: the Laplace equation, the aforementioned Navier-Stokes equations and, finally, the lambda-formulation Euler equations.

Methodology

The proposed multigrid strategy is described here for convenience for the particular case of the vorticity stream function Navier-Stokes equations:

$$\omega_t + \psi_y \, \omega_x - \psi_x \omega_y - (\omega_{xx} + \omega_{yy})/Re = 0 \tag{1}$$

$$\psi_{xx} + \psi_{yy} + \omega - \psi_t = 0 \tag{2}$$

In eqns (1-2) Re is the Reynolds number, $-\omega$ is the vorticity, ψ is the stream function, x and y are the standard Cartesian coordinates, t is the time, subscripts indicate partial derivatives and a relaxation-like time derivative has been added to the stream function equation to parabolize it. Eqns (1, 2) are discretized and linearized in time using a two level implicit Euler scheme and the delta approach of Beam and Warming [8] to give:

$$\Delta\omega^H/\Delta t + \psi_y{}^n \, \Delta\omega_x{}^H + \omega_x{}^n \Delta\psi_y{}^H - \psi_x{}^n \Delta\omega_y{}^H - \omega_y{}^n \Delta\psi_x{}^H - (\Delta\omega_{xx}{}^H + \Delta\omega_{yy}{}^H)/Re =$$

$$-(\psi_y{}^n\,\omega^n)_x + (\psi_x{}^n\omega^n)_y + (\omega_{xx}{}^n + \omega_{yy}{}^n)/Re \;=\; RES(\omega)^n \qquad (3)$$

$$\Delta\psi^H/\Delta t - \Delta\psi_{xx}{}^H - \Delta\psi_{yy}{}^H - \Delta\omega^H \;=\; \psi_{xx}{}^n + \psi_{yy}{}^n + \omega^n \;=\; RES(\psi)^n \qquad (4)$$

In eqns (3, 4) Δt is the time step (which can be different in eqns (3) and (4) and can also vary at every iteration), $\Delta\omega = \omega^{n+1} - \omega^n$, the superscripts n+1 and n indicating the new and old time levels t^{n+1} and t^n, etc. and the superscript H indicates the mesh on which the superscripted variables are evaluated (notice that the superscript h, indicating the finest mesh employed in the computation, is always omitted, for convenience); finally, the conservative form of the convection terms is used in the RHS of eqn (3), since it has been shown to provide improved accuracy over the standard form, especially at high Re values (see, e. g., [17, 6]).

Eqns (3-4) are discretized in space using second-order-accurate central differences throughout (except for the incremental convective terms of the vorticity equation in the LHS of eqn (3), which are approximated using first-order-accurate upwind differences), so that a large linear system is obtained, characterized by a 2x2 block-pentadiagonal matrix. Such a linear system is then solved approximately (by a single-sweep relaxation method requiring to solve only block-tridiagonal systems). The process is started on the finest grid, with $\Delta\omega^H = \Delta\omega$, $\Delta\psi^H = \Delta\psi$, the solution (ψ^n, ω^n) is updated and Eqns (3, 4) are then solved on successively coarser grids (H = 2h, 4h,....), until the finest-grid residual is reduced to a suitable small value. In more detail, at every grid level H, the following steps are required by the proposed multigrid strategy: a) the coefficients of eqns (3, 4) are evaluated from the finest-grid (updated) solution ψ^n, ω^n, locally at the H-mesh gridpoints (which amounts to injecting the h-mesh coefficients into the H mesh); b) the RHS of eqns (3, 4) are replaced by the corresponding collected residuals, given as:

$$RES^H(\psi, \omega) = C_h{}^H\, RES(\psi, \omega)^n \qquad (5)$$

where $C_h{}^H$ indicates a suitable collection operator from the finest grid h to the current grid H; c) eqns (3, 4) are solved approximately, by a single sweep of the smoother, to provide $\Delta\psi^H$, $\Delta\omega^H$; d) $\Delta\psi$, $\Delta\omega$ are evaluated as

$$(\Delta\psi, \Delta\omega) = I_H{}^h\, (\Delta\psi^H, \Delta\omega^H) \qquad (6)$$

where $I_H{}^H$ is the standard bilinear interpolation operator from the current grid H to the finest one h; e) the (finest-grid) solution is finally updated as

$$(\psi, \omega)^n \leftarrow (\psi, \omega)^n + (\Delta\psi, \Delta\omega) \qquad (7)$$

It is worth noticing that the proposed approach, as described above, is applicable to any set of time-discretized systems of partial differential equations, is extremely simple and does not require any "logical choices" or "free parameters" to be tuned, insofar as a single relaxation sweep is performed at every grid level, nor any artificial dissipation in addition to that associated with the upwinding of the coefficients in the LHS of eqns (3, 4), which identically vanishes at convergence. Furthermore, by solving always for the deltas on every grid, simple homogeneous boundary conditions can be applied at every but the finest mesh, where, of course, the appropriate boundary conditions need to be imposed. This is a very desirable property of the method, especially for the present case of the vorticity stream function equations, for which the vorticity at the wall is obtained by a Neumann condition for the stream function, so that a complicated interpolation procedure would be required if the vorticity was to be computed also on the coarser meshes. Moreover, since the finest-grid solution is immediately updated after every relaxation sweep, the correct Neumann boundary condition can be applied not only at the finest-grid relaxation sweeps, implicitly, but also immediately after every updating of the solution, as an explixit correction. In this way the expected deterioration of the convergence rate of the method in the presence of Neumann boundary conditions (an unavoidable trade-off of the simplicity of the approach) can be reduced significantly. Of course, the extreme simplicity of the method has its drawbacks. In particular, the work per iteration of the proposed approach is slightly greater than that of other multigrid schemes and its convergence rate is rather slow at the beginning of the process, when the deltas computed on the coarser meshes contain high frequency errors, which are thus fed back into the sought finest-grid solution. However, the "pros" are believed to outperform the "cons", as it will be shown in the next section.

Results

The proposed methodology has been applied to solve three different problems.

The Laplace equation in a unit square was considered at first, to verify the validity of the proposed approach for the case of a simple elliptic linear equation with Dirichlet boundary conditions such that the exact solution of the differential problem was given as $\exp(\pi y/2)\sin(\pi x/2)$. A standard line Gauss-Seidel method was used as smoother for the present application, together with the collection operator (for the residual) called "9-point restriction" by Wesseling (see [17]). Convergence histories of the method are given in Figures 1, where the (finest-mesh) residual is plotted versus the number of iterations. In Figure 1a the effect of the number of grid levels, the finest grid being always a 65x65 uniform one, is considered: the convergence rate of the method appears to increase significantly with the number of grid levels used in the computation and is similar to that of well established multigrid methods. Figure 1b shows instead the influence of the size of the finest grid, when using always 5 grid-levels: the convergence rate is seen to be practically the same in all cases, as it should. The above results are considered a sufficient evidence of the potential validity of the proposed approach and of the correctness of its implementation.

The vorticity-stream function Navier-Stokes equations, applied to the classical driven cavity flow at Re = 1000, were then considered as a much more severe test for the present method. The numerical method of Ref. 6, namely a line Gauss-Seidel method sweeping in alternate directions, was used as the smoother in the present multigrid strategy, with a double sweep being performed at every grid level, at each iteration. The same collection operator already used in the previous problem and a nonoptimized unitary time step were used in this case. Furthermore, after updating the solution obtained from every coarse-grid calculation, the vorticity at the wall was corrected by imposing the Neumann boundary condition for the stream function at the wall on the finest grid. Only four grids were used, because of the nonlinearity of the equations and the presence of a Neumann boundary condition for the stream function. The convergence history of the method is given in Figure 2, for four different uniform (finest) grids. In all cases the residual is seen to decrease very slowly at first, but, after a reasonably small number of iterations, the method takes on a multigrid-type convergence rate. Such a behavior is not surprising and could have been easily anticipated. The equations are in fact nonlinear and a sequence of single relaxation cycles at every grid level cannot be expected to annihilate efficiently the various frequency errors of both the linear systems (which are solved incompletely) and of the nonlinear (Newton) iteration process, especially since the solution is obtained starting from rest, i. e., without any smoothing of the initial data. For completeness, the convergence history of the smoother (using a single 65x65 uniform grid) is also provided in Figure 2. It is noteworthy that the asymptotic convergence rate of the multigrid method is characterized by a reduction of the residual per iteration equal to 0.83, whereas the corresponding value of the single-grid method is equal to 0.984. The improvement is substantial even though the CPU cost per iteration of the multigrid method is more than twice that of the standard solver. The results for the maximum stream function in the cavity and the vorticity at the center of the moving wall, obtained using the finest 81x81 grid, are finally given in Table 1, where the very accurate results of Ref. 17 are also given for comparison.

The lambda-formulation Euler equations were finally considered in order to assess the capability of the proposed approach to deal also with hyperbolic systems. A model problem, namely the two-dimensional counterpart of the subsonic spherical source flow problem studied in Ref. 7 was considered as a test case. Solutions were obtained using the proposed multigrid strategy and the Block-Explicit method described in Ref. 7, as "smoother". A simple injection operator was used for obtaining the collected residuals on the mesh H from the h-mesh residuals, because, as already found in Ref. 20, for the case of the Euler equations, no advantage was obtained when using more complicated colletion operators. Also, the smoother being an explicit one step method, subject to a strict CFL condition for the time step, a different time step was used at every grid level, equal to the finest-grid time step multiplied times the ratio H/h. Results were obtained using from one to four grid levels, the finest grid having 25x25 and 49x49 gridpoints, respectively. The convergence histories are provided in Figures 3a and 3b. It appears that the same number of iterations (i.e.,$\sim$80) are required for convergence in both cases, when using four grid levels, a very typical property of "sound" multigrid schemes; incidentally, the single-grid scheme requires more than 300 and more than 600 iterations, respectively, to achieve the same convergence level. However, considering the reduced work, 3 grid levels appear to be the optimal number, especially for the first case; such a result is in agreement with the findings of Ref. 20, where a similar approach is used.

Conclusions and future work

A novel incremental multigrid strategy has been proposed to solve the equations of fluid dynamics, which provides considerable efficiency gains for the case of elliptic, mixed parabolic-elliptic and hyperbolic problems. With respect to current well established multigrid schemes, the proposed approach is conceptually simpler and does not require any parameters or additional artificial dissipation. However, it requires more work per iteration and, at present, it has been shown to be effective only for subsonic flow problems and uniform grids. Future work is required to overcome these two limitations and to demonstrate the applicability of the method to the compressible Navier-Stokes equations in conservation form.

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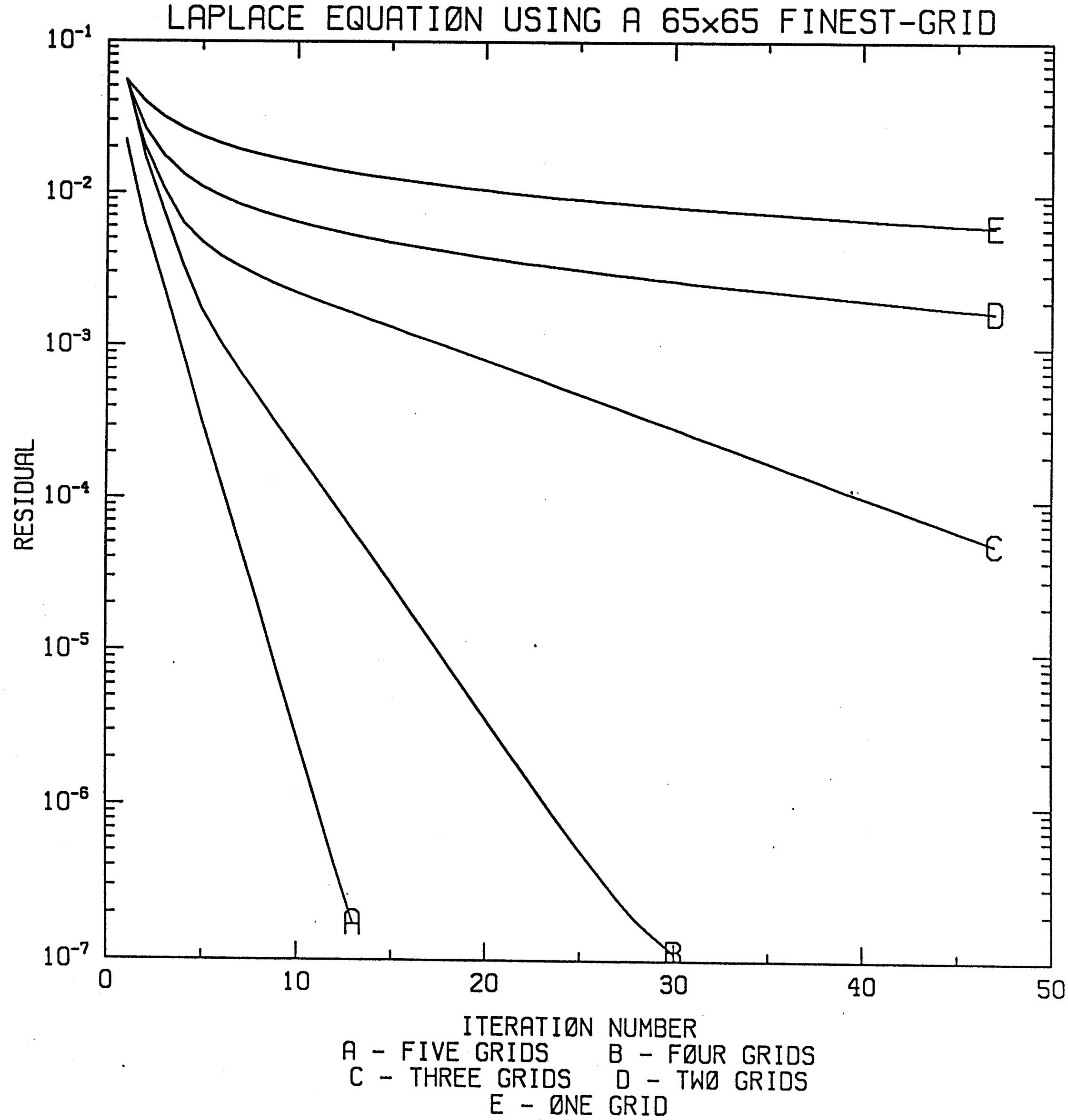

Figure 1a: Influence of grid levels on the convergence rate.

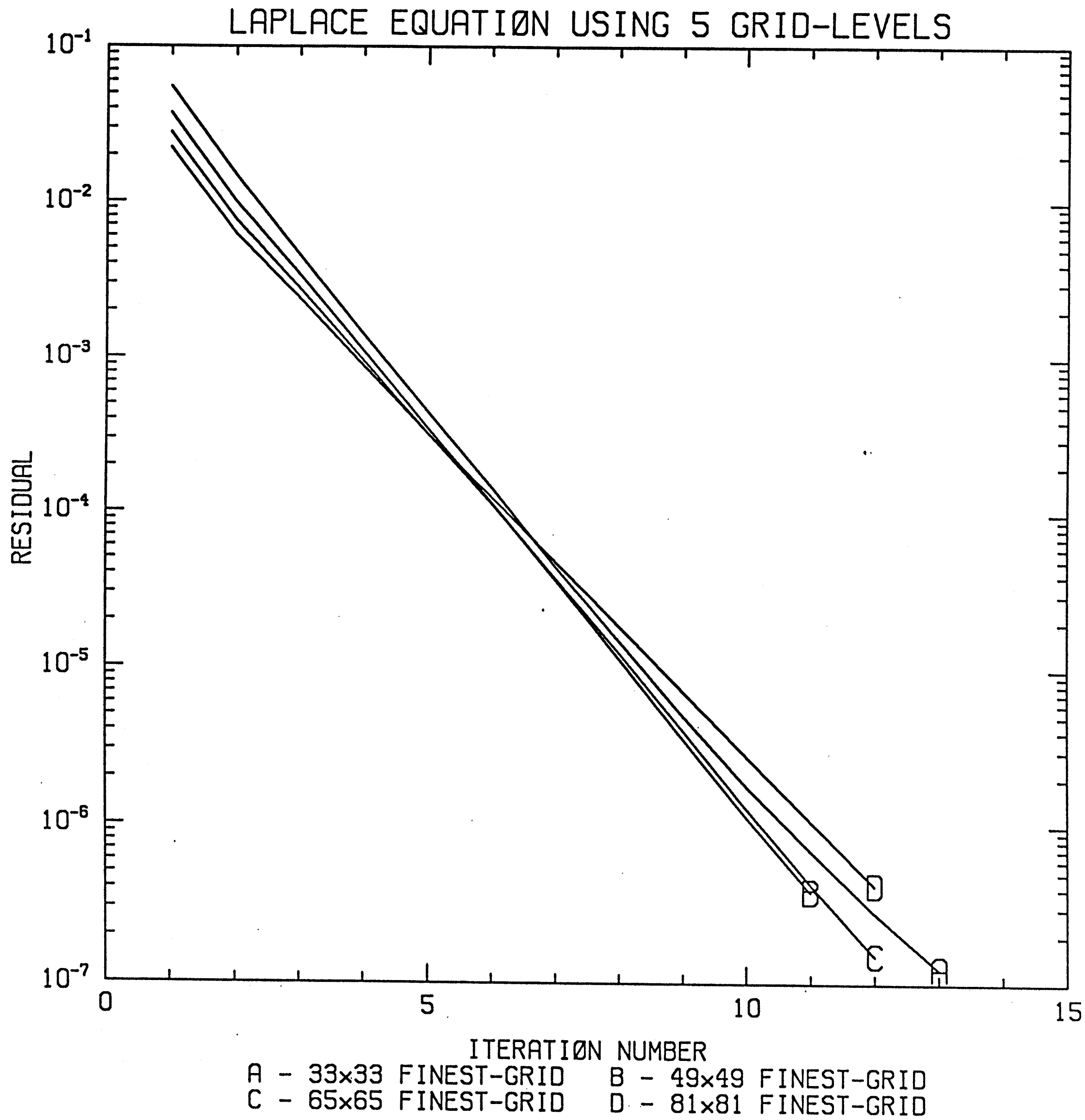

Figure 1b: Influence of finest-grid size on the convergence rate.

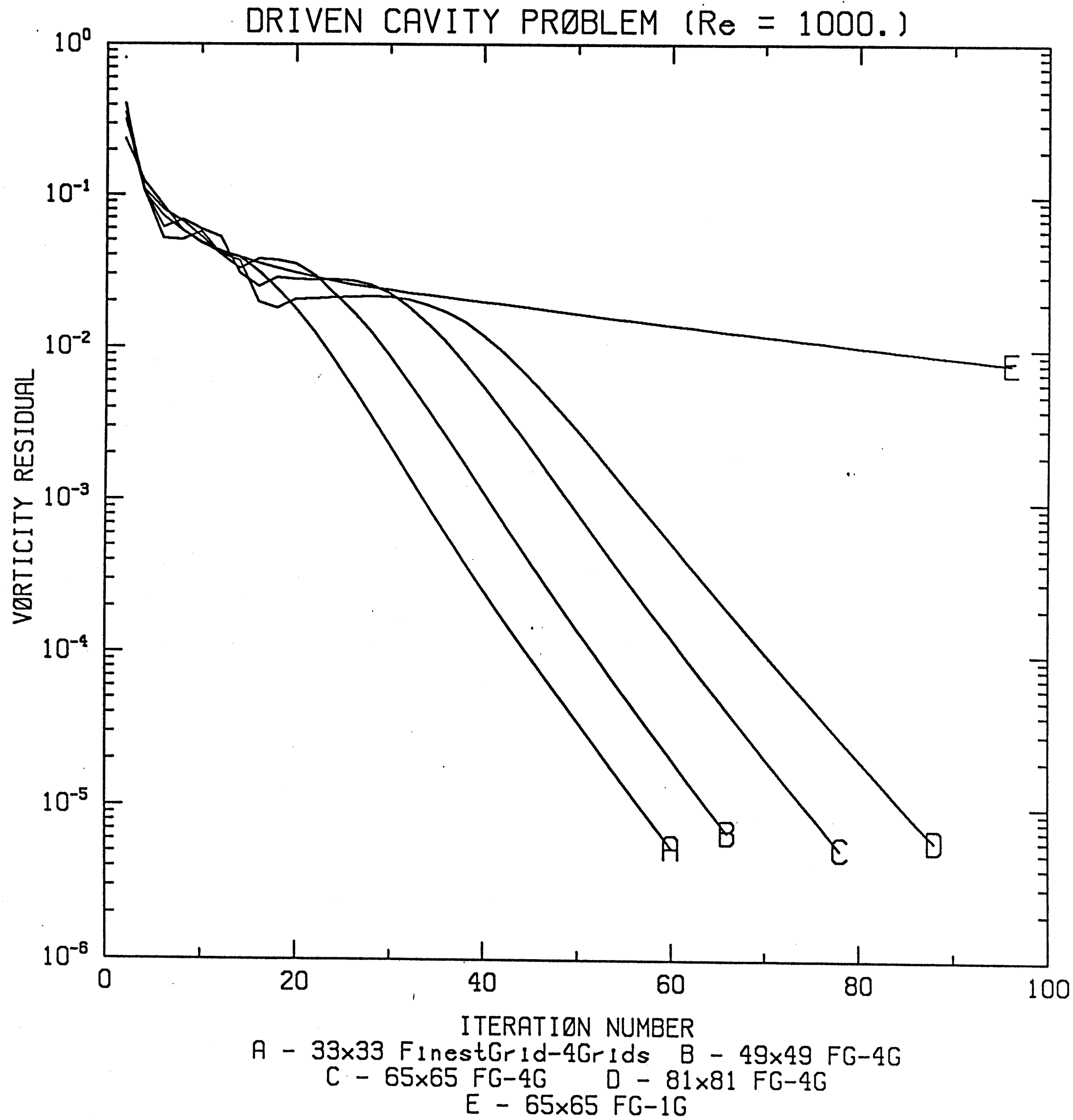

Figure 2: Convergence history for the Navier-Stokes equations

	Maximum Stream function	Midplate vorticity
Present results (81x81 grid)	.1167	15.0305
Reference results Ghia et al. /17/	.1179	14.8901

Table 1. Numerical results for Re = 1000. Driven cavity problem

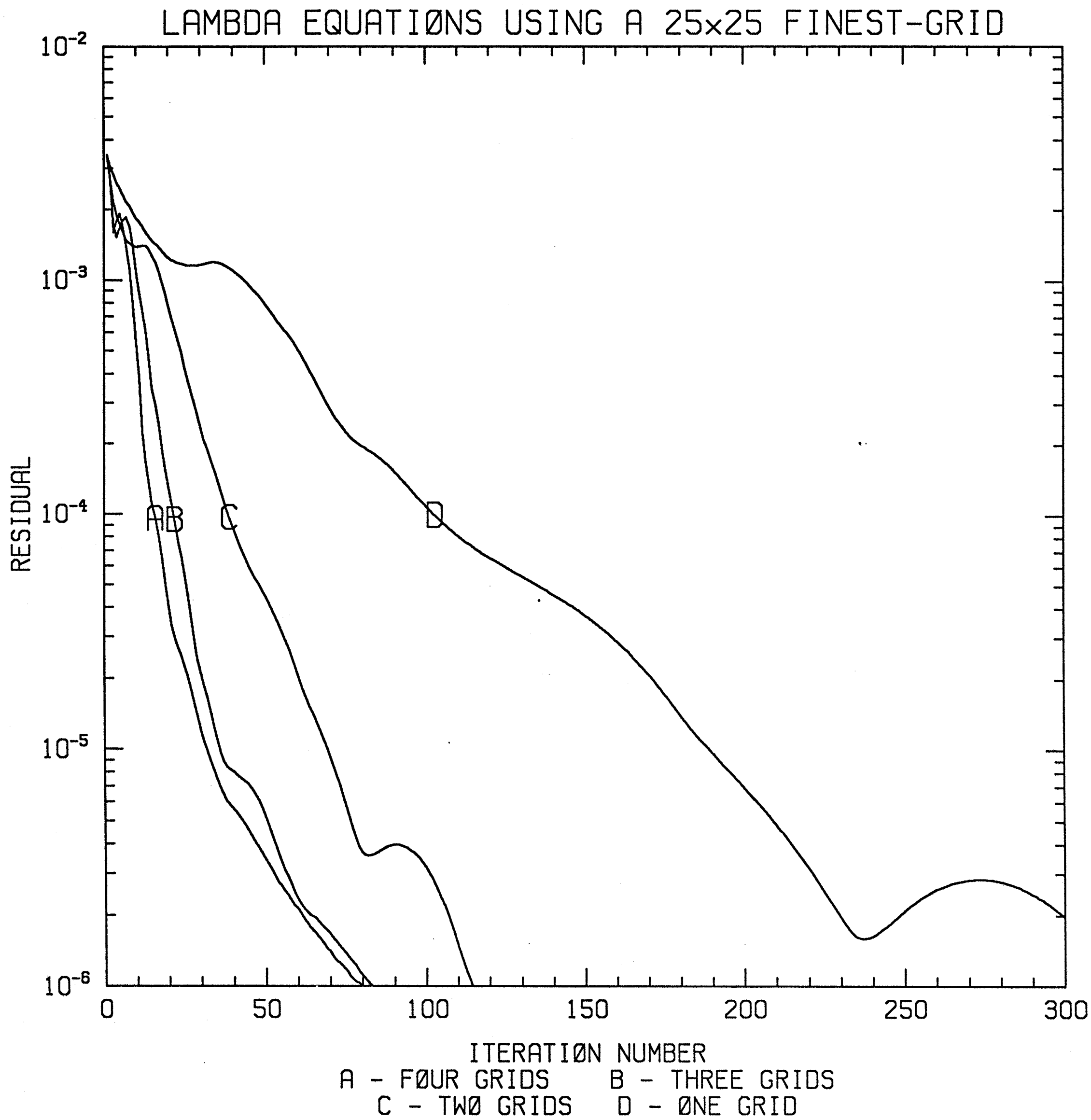

Figure 3a: Convergence history for the lambda equations

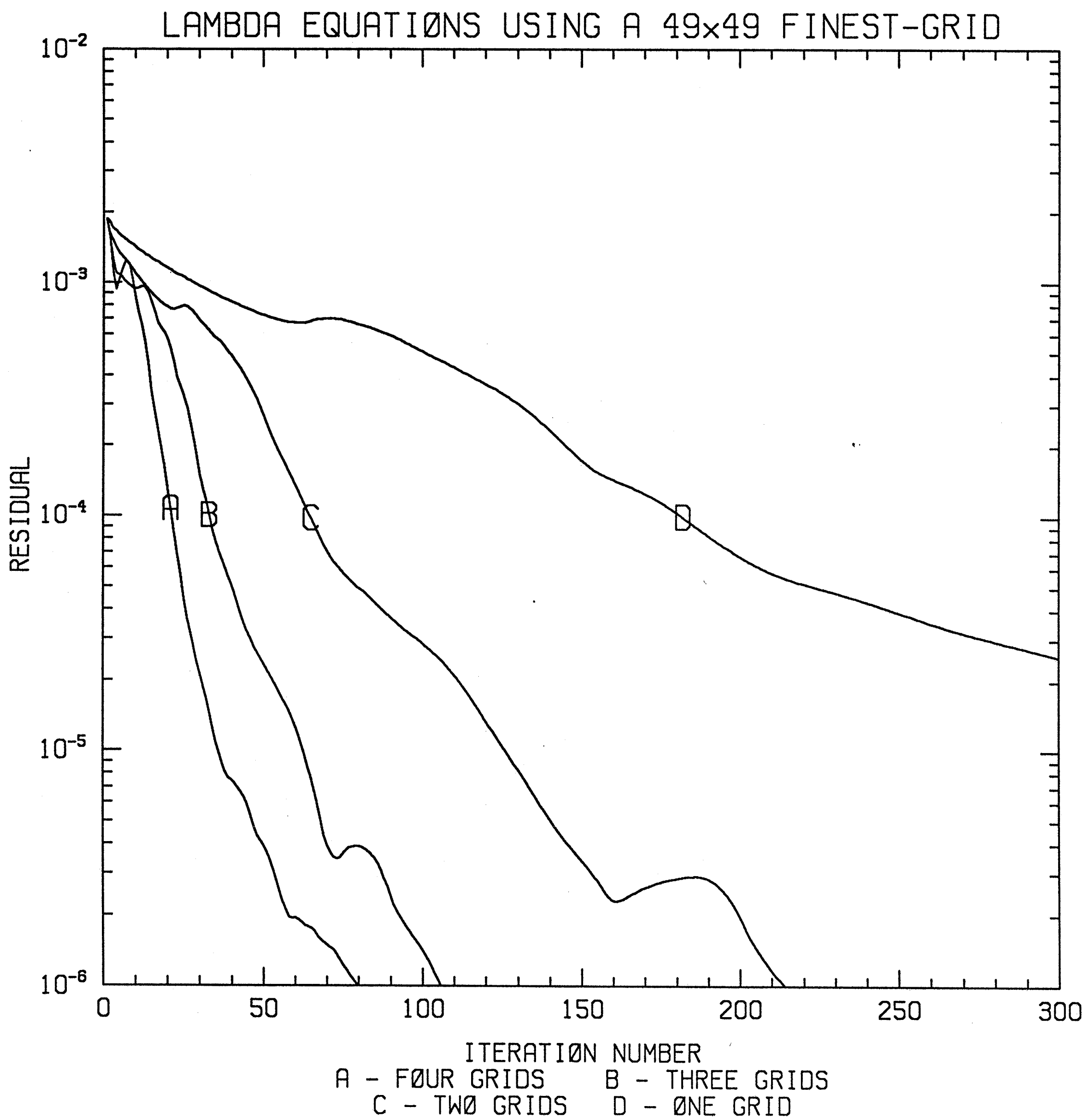

Figure 3b: Convergence history for the lambda equations