Prolate Spheroidal Wave Functions, Quadrature, Interpolation, and Asymptotic Formulae

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Abstract

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Whenever physical signals are measured or generated, the results tend to be band-limited (i.e. to have compactly supported Fourier transforms). Indeed, measurements of electromagnetic and acoustic data are band-limited due to the oscillatory character of the processes that have generated the quantities being measured. When the signals being measured come from heat propagation or diffusion processes, they are (practically speaking) band-limited, since the underlying physical processes operate as low-pass filters. The importance of band-limited functions has been recognized for hundreds of years; classical Fourier analysis can be viewed as an apparatus for dealing with such functions. When band-limited functions are defined on the whole line (or on the circle), classical tools are very satisfactory.

However, in many cases, we are confronted with band-limited functions defined on intervals (or, more generally, on compact regions in $\mathbb{R}^n$). In this environment, standard tools based on polynomials are often effective, but not optimal. In fact, the optimal approach was discovered more than 30 years ago by Slepian *et al*, who observed that for the analysis of band-limited functions on intervals, Prolate Spheroidal Wave Functions (PSWFs) are a natural tool. They built the requisite analytical apparatus in a sequence of famous papers,

and applied the resulting tools in many areas of signal processing, statistics, antenna theory, etc. However, their efforts have not lead to numerical techniques; the principal reason appears to be the lack at the time of effective numerical algorithms for the evaluation of PSWFs and related quantities.

In this dissertation, we start with noticing that in the modern numerical environment, evaluation of PSWFs presents no serious difficulties, and construct a straightforward procedure for the numerical evaluation of PSWFs. Then we use PSWFs to build analogues for band-limited functions of some of the classical numerical techniques: Gaussian quadratures and corresponding interpolation formulae (both exact on certain classes of band-limited functions). We also construct a new class of asymptotic formulae for PSWFs. Unlike the classical apparatus based on Legendre polynomials, our approach utilizes Hermite polynomials, and is valid when bandwidth is large. We illustrate our results with numerical examples.

To my parents.

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1

Introduction

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be band-limited if there exists a positive real c and a function $\sigma \in L^2[-1, 1]$ such that

$$f(x) = \int_{-1}^{1} e^{icxt} \, \sigma(t) \; dt. \tag{1.1}$$

Band-limited functions are ubiquitous: whenever physical signals are measured or generated, the results tend to be band-limited. Indeed, measurements of electromagnetic and acoustic data are band-limited due to the oscillatory character of the processes that have generated the quantities being measured. When the signals being measured come from heat propagation or diffusion processes, they are (practically speaking) band-limited, since the underlying physical processes operate as low-pass filters.

For band-limited functions that are well-behaved on the whole line (for example, in signal processing), numerical tools are well-studied and very satisfactory. Among them, classical Fourier analysis has been one of the most important for more than 150 years. However, in many cases, we are confronted with band-limited functions that are defined on intervals (or, more generally, on compact regions in $\mathbb{R}^n$). Wave phenomena are a rich source of such functions, both in the engineering and computational contexts; such functions are also encountered in fluid dynamics, signal processing, and many other areas. Often, band-limited functions on intervals can be effectively approximated by polynomials via standard tools of classical analysis. However, even when such approximations are feasible, they are usually not optimal. Smooth periodic functions are a good illustration of this observation: while they *can* be approximated by polynomials (for example, via Chebyshev or Legendre expansions), they are more efficiently approximated by Fourier expansions, both for analytical and numerical purposes. It would appear that an approach explicitly based on trigonometric polynomials could be more efficient in dealing with band-limited functions on intervals. In this dissertation, we present a set of numerical tools that are optimal when the underlying functions are limited both in the time domain and frequency domain. Our apparatus is based on the Prolate Spheroidal Wave Functions, a class of band-limited functions that were studied in some detail more than 30 years ago [31].

1.1 Brief History

Prolate Spheroidal Wave Functions (PSWFs) arose from the study of the wave equation in general ellipsoidal coordinates. When the wave equation is separated in prolate spheroidal coordinates, the resulting second-order ordinary differential equations are satisfied by PSWFs. During the last 30 years or so, PSWFs have been used in a broad range of physical and engineering environments such as acoustic radiation, scattering of electromagnetic waves, fluid dynamics, antenna design, communication theory, and many others.

Prolate Spheroidal Wave Functions have been studied extensively since the late nineteenth century, and a large number of analytical and numerical results have been obtained. In particular, numerous expansions of prolate spheroidal wave functions have been constructed, in associated Legendre polynomials, Gegenbauer functions, spherical Bessel functions, power series, etc. (see, for example, [27, 6, 36, 28]). These expansions usually have extremely general forms, with analogous formulae for other spheroidal wave functions. The expansion coefficients usually obey certain recursion formulae, and are often evaluated via continued-fractions (see, for example, [36]) or Bouwkamp's iterative scheme [2]. Detailed tabulations of the expansion coefficients, as well as tables of the functions, have been presented in several monographs published on this subject (see, for example, [36, 6]). Unfortunately, both continued-fractions and Bouwkamp's algorithm become unstable at high frequencies. Although a series of asymptotic formulae have been developed for sufficiently high frequencies [25], these formulae are incomplete (see [6]).

Among the early results on PSWFs, there were also a number of integral relations reported in fairly general terms. These results were used in various derivations of the analytical properties of PSWFs, but their significance was not realized until the 1960s, when Slepian *et al* observed that the PSWFs are eigenfunctions of the Fourier operator on intervals. The requisite analytical apparatus necessary for engineering applications was then built in a sequence of famous papers (see [17]–[19], [31]–[34]), and the resulting tools were utilized in many areas of signal processing, antenna theory, communication theory, etc. However, these results have not lead to numerical techniques, due to what appears to be the lack at the time of effective numerical algorithms for the evaluation of PSWFs and related quantities.

We observe that, in the modern numerical environment, the evaluation of PSWFs presents no serious difficulties, and construct a straightforward numerical evaluation procedure. We then construct a class of quadratures for band-limited functions that closely parallel the Gaussian quadratures for polynomials. The nodes are very close to being roots of appropriately chosen Prolate Spheroidal Wave Functions, the resulting quadratures are stable, and all weights are positive. Moreover, as is in the case of polynomials, there are interpolation, differentiation and indefinite integration schemes associated with the obtained quadratures, exact on certain classes of band-limited functions. We also construct a class of asymptotic formulae for PSWFs and related quantities, which are uniformly convergent on $\mathbb{R}^1$ and are valid for high frequencies.

1.2 Outline of the Dissertation

This dissertation is organized as follows. In Chapter 2, we summarize various elementary mathematical facts used in the remainder of the dissertation. In Chapter 3, we describe the algorithms for the evaluation of Prolate Spheroidal Wave Functions and associated eigenvalues. We describe two procedures for the construction of quadratures for band-limited functions in Sections 4.1 and 4.2, and show that roots of appropriately chosen Prolate Spheroidal Wave Functions can serve as quadrature nodes. We then analyze the use of PSWFs for interpolation in Section 4.3. The asymptotic formulae for Prolate Spheroidal Wave Functions and related quantities at high frequencies are given in Chapter 5. We present results of our numerical experiments with quadratures, interpolation, and asymptotic formulae in Chapter 6, and collect a number of miscellaneous properties of Prolate Spheroidal Wave Functions in Chapter 7. Finally, Chapter 8 contains generalizations and conclusions.

2

Mathematical Preliminaries

In this chapter, we summarize a number of well-known facts to be used in the remainder of this dissertation. These facts can be found or easily follow from facts that can be found in, for example, [1, 7].

As a matter of convention, unless stated otherwise, the norm of a function will refer to its L^2 norm:

$$\|f\| = \sqrt{\int |f(x)|^2 \, dx}. \tag{2.1}$$

We also frequently use $[a]$ (a is real) to denote the integer part of a.

2.1 Legendre Polynomials

In agreement with standard practice, we denote by P_n the classical Legendre polynomials, defined by the three-term recursion

$$P_{n+1}(x) = \frac{2n+1}{n+1} \, x \, P_n(x) - \frac{n}{n+1} \, P_{n-1}(x), \tag{2.2}$$

with the initial conditions

$$P_0(x) \;\; = \;\; 1\,, \tag{2.3}$$

$$P_1(x) \quad = \quad x\,.$$

As is well known,

$$P_n(1) = 1 \tag{2.4}$$

for all $n = 0, 1, 2, \ldots$, and each of the polynomials P_n satisfies the differential equation

$$(1-x^2)\,\frac{d^2P_n(x)}{dx^2} - 2x\,\frac{dP_n(x)}{dx} + n\cdot(n+1)\,P_n(x) = 0. \tag{2.5}$$

The polynomials defined by the formulae (2.2), (2.3) are orthogonal on the interval $[-1,1]$; however, they are not orthonormal, since for each $n \geq 0$,

$$\int_{-1}^{1} (P_n(x))^2\;dx = \frac{1}{n+1/2}. \tag{2.6}$$

We introduce a normalized version of the Legendre polynomials $\overline{P_n}$, so that

$$\overline{P_n}(x) = P_n(x)\cdot\sqrt{n+1/2}\,. \tag{2.7}$$

The following lemma follows immediately from the Cauchy-Schwartz inequality and from the orthogonality of the Legendre polynomials on the interval $[-1,1]$.

Lemma 2.1 *For all integer $k \geq n \geq 0$,*

$$\left|\int_{-1}^{1} x^k\,\overline{P_n}(x)\;dx\right| < \sqrt{\frac{2}{k+1}}\,; \tag{2.8}$$

for all integer $0 \leq k < n$,

$$\left|\int_{-1}^{1} x^k\,\overline{P_n}(x)\;dx\right| = 0\,. \tag{2.9}$$

2.2 Hermite Polynomials and Hermite Functions

As is well known, Hermite polynomials H_n satisfy the differential equation

$$H_n^{''}(x) - 2\cdot x\cdot H_n^{'}(x) + 2n\cdot H_n(x) = 0\,, \tag{2.10}$$

and are orthogonal on $\mathbb{R}$ with the weight function e^{-x^2}. That is, for all non-negative integers m and n,

$$\int_{-\infty}^{\infty} e^{-x^2} \cdot H_n(x) \cdot H_m(x)\, dx = \sqrt{\pi}\, 2^n\, n!\, \delta_{n,m}\,, \tag{2.11}$$

where $\delta_{n,m}$ is the Kronecker Delta function. Moreover, Hermite polynomials satisfy the three-term recursion

$$H_{n+1}(x) = 2 \cdot x \cdot H_n(x) - 2n \cdot H_{n-1}(x)\,, \tag{2.12}$$

with the initial conditions

$$H_0(x) = 1\,, \tag{2.13}$$

$$H_1(x) = 2\,x\,. \tag{2.14}$$

Furthermore, Hermite polynomials satisfy the inequality

$$|H_n(x)| \leq \frac{n!\, 2^{\frac{n}{2}-[\frac{n}{2}]}}{[\frac{n}{2}]!} \cdot e^{2\,x\sqrt{[\frac{n}{2}]}} \tag{2.15}$$

for all integer $n \geq 0$ and $x \in \mathbb{R}$, where $[\frac{n}{2}]$ denotes the integer part of $n/2$.

Throughout this dissertation, we use a scaled version of Hermite polynomials which we denote by H_n^a. Defining the polynomials $\{H_n^a\}$ to be orthonormal on $\mathbb{R}$ with the weight function $e^{-a^2 x^2}$ ($a \neq 0$), i.e.,

$$\int_{-\infty}^{\infty} e^{-a^2 x^2} \cdot H_n^a(x)\, H_m^a(x)\, dx = \delta_{n,m} \tag{2.16}$$

for all non-negative integers m and n, we see easily that

$$H_n^a(x) = \frac{\sqrt{a}}{\pi^{\frac{1}{4}} \cdot 2^{\frac{n}{2}} \cdot (n!)^{\frac{1}{2}}} \cdot H_n(a\,x)\,, \tag{2.17}$$

and that

$$\frac{1}{a^2} \cdot \frac{d^2\, H_n^a(x)}{d\,x^2} - 2\,x\,\frac{d\,H_n^a(x)}{d\,x} + 2n\,H_n^a(x) = 0\,. \tag{2.18}$$

2.2.1 Recursion Relations

As in the case of Hermite polynomials H_n, polynomials H_n^a have their own three-term recursion given by the formula

$$H_{n+1}^a(x) = a\,x\sqrt{\frac{2}{n+1}}\,H_n^a(x) - \sqrt{\frac{n}{n+1}}\,H_{n-1}^a(x)\,, \tag{2.19}$$

with the initial conditions

$$H_0^a(x) = \sqrt{a}\left(\frac{1}{\sqrt{\pi}}\right)^{\frac{1}{2}}\,, \tag{2.20}$$

$$H_1^a(x) = \sqrt{2\,a}\left(\frac{1}{\sqrt{\pi}}\right)^{\frac{1}{2}}\,a\,x\,. \tag{2.21}$$

Rearranging the terms in (2.19), we immediately obtain the following theorem.

Theorem 2.2 *For all real $a \neq 0$ and all integer $n \geq 1$,*

$$x\,H_n^a(x) = \frac{1}{a}\sqrt{\frac{n+1}{2}}\,H_{n+1}^a(x) + \frac{1}{a}\sqrt{\frac{n}{2}}\,H_{n-1}^a(x)\,. \tag{2.22}$$

Furthermore,

$$x\,H_0^a(x) = \frac{1}{\sqrt{2}\,a}\,H_1^a(x)\,. \tag{2.23}$$

Applying Theorem 2.2 twice, we obtain the following theorem.

Theorem 2.3 *For any real $a \neq 0$ and integer $n \geq 2$,*

$$\begin{aligned} x^2 \cdot H_n^a(x) \;=\; & \frac{1}{a^2}\cdot\sqrt{\frac{n+1}{2}\cdot\frac{n+2}{2}}\cdot H_{n+2}^a(x) \\ & +\frac{1}{a^2}\cdot\left(n+\frac{1}{2}\right)\cdot H_n^a(x) \\ & +\frac{1}{a^2}\cdot\sqrt{\frac{n}{2}\cdot\frac{n-1}{2}}\cdot H_{n-2}^a(x)\,. \end{aligned} \tag{2.24}$$

Furthermore,

$$x^2 \cdot H_1^a(x) = \frac{1}{a^2}\cdot\sqrt{\frac{3}{2}}\cdot H_3^a(x) + \frac{1}{a^2}\cdot\frac{3}{2}\cdot H_1^a(x)\,, \tag{2.25}$$

$$x^2 \cdot H_0^a(x) = \frac{1}{a^2}\cdot\frac{1}{\sqrt{2}}\cdot H_2^a(x) + \frac{1}{a^2}\cdot\frac{1}{2}\cdot H_0^a(x)\,. \tag{2.26}$$

Applying Theorem 2.3 twice, we obtain Theorem 2.4.

Theorem 2.4 *For any real a and integer $n \geq 4$,*

$$\begin{aligned} x^4 \cdot H_n^a(x) &= \frac{1}{a^4} \cdot \sqrt{(n+1)(n+2)(n+3)(n+4)} \cdot H_{n+4}^a(x) \\ &+ \frac{1}{a^4} \cdot \left(n + \frac{3}{2}\right) \cdot \sqrt{(n+1)(n+2)} \cdot H_{n+2}^a(x) \\ &+ \frac{3}{4} \cdot \frac{1}{a^4} \cdot (1 + 2n + 2n^2) \cdot H_n^a(x) \\ &+ \frac{1}{2a^4} \cdot \sqrt{(n-1)n} \cdot (2n-1) \cdot H_{n-2}^a(x) \\ &+ \frac{1}{4a^4} \cdot \sqrt{(n-3)(n-2)(n-1)n} \cdot H_{n-4}^a(x) . \end{aligned} \tag{2.27}$$

Moreover, for $n = 2$ and $n = 3$,

$$\begin{aligned} x^4 \cdot H_n^a(x) &= \frac{1}{a^4} \cdot \sqrt{(n+1)(n+2)(n+3)(n+4)} \cdot H_{n+4}^a(x) \\ &+ \frac{1}{a^4} \cdot \left(n + \frac{3}{2}\right) \cdot \sqrt{(n+1)(n+2)} \cdot H_{n+2}^a(x) \\ &+ \frac{3}{4} \cdot \frac{1}{a^4} \cdot (1 + 2n + 2n^2) \cdot H_n^a(x) \\ &+ \frac{1}{2a^4} \cdot \sqrt{(n-1)n} \cdot (2n-1) \cdot H_{n-2}^a(x) . \end{aligned} \tag{2.28}$$

Finally, for $n = 0$ and $n = 1$,

$$\begin{aligned} x^4 \cdot H_n^a(x) &= \frac{1}{a^4} \cdot \sqrt{(n+1)(n+2)(n+3)(n+4)} \cdot H_{n+4}^a(x) \\ &+ \frac{1}{a^4} \cdot \left(n + \frac{3}{2}\right) \cdot \sqrt{(n+1)(n+2)} \cdot H_{n+2}^a(x) \\ &+ \frac{3}{4} \cdot \frac{1}{a^4} \cdot (1 + 2n + 2n^2) \cdot H_n^a(x) . \end{aligned} \tag{2.29}$$

2.2.2 Hermite Functions

Given any non-zero real number a, we define the functions $\phi_0^a, \phi_1^a, \phi_2^a, \ldots : \mathbb{R} \to \mathbb{R}$ (frequently referred to as Hermite functions) via the formula

$$\phi_n^a(x) = e^{-a^2 x^2/2} \cdot H_n^a(x) . \tag{2.30}$$

The following theorem summarizes the well-known facts of Hermite functions.

Theorem 2.5 *Suppose that a is real and non-zero. Then, for all integers $m \geq 0, n \geq 0$,*

$$\int_{-\infty}^{\infty} \phi_n^a(x) \cdot \phi_m^a(x)\ dx = \delta_{m,n}\,. \tag{2.31}$$

Moreover, any function $f \in L^2[-\infty, \infty]$ can be expanded in a Hermite series so that

$$f(x) = \sum_{n=0}^{\infty} \alpha_n\, \phi_n^a(x)\,, \tag{2.32}$$

where α_n are constants defined by the formula

$$\alpha_n = \int_{-\infty}^{\infty} f(x)\, \phi_n^a(x)\, dx\,. \tag{2.33}$$

Furthermore, if f is even, $\alpha_{2n+1} = 0$ for all natural n; if f is odd, $\alpha_{2n} = 0$ for all natural n.

The following lemma is an immediate consequence of inequality (2.15). It shows that for all integer $n \geq 0$ and sufficiently large a, the values of the function ψ_n^a at points outside the interval $[-1, 1]$ decay exponentially with a.

Lemma 2.6 *Suppose that a is real and positive, and that m, n are non-negative integers. Suppose further that ϕ_n^a, ϕ_m^a are the n-th and m-th Hermite functions with the weight function $e^{-a^2x^2}$. Then,*

$$\int_{-\infty}^{-1} e^{-a^2x^2}\, \phi_n^a(x)\, \phi_m^a(x)\ dx + \int_{1}^{\infty} e^{-a^2x^2}\, \phi_n^a(x)\, \phi_m^a(x)\ dx \leq \gamma\, e^{-a} \tag{2.34}$$

for all a such that

$$a \geq 2\sqrt{\left[\frac{n}{2}\right]} + 2\sqrt{\left[\frac{m}{2}\right]} + 1\,, \tag{2.35}$$

where constant γ is is given by the formula

$$\gamma = \frac{2 \cdot \sqrt{m!\,n!}}{\sqrt{\pi} \cdot 2^{[\frac{m}{2}]+[\frac{n}{2}]} \cdot [\frac{n}{2}]!\ [\frac{m}{2}]!}\,. \tag{2.36}$$

The following technical lemma will be used in Section 5.4. It is an immediate consequence of Lemma 2.6.

Lemma 2.7 *Suppose that $p > 0$ is integer, and that c is a sufficiently large real number. Suppose further that function $\Psi^c : [-1,1] \to R$ is given by the Hermite expansion*

$$\Psi^c(x) = \sum_{i=0}^{p} \alpha_i \cdot \phi_i^{\sqrt{c}}(x) + O\left(c^{-p-1}\right), \tag{2.37}$$

where the expansion coefficients $\alpha_0, \alpha_1, \ldots, \alpha_p$ are p-th order polynomials in $1/c$. Suppose further that for all integer $k \in [0, p]$, $\alpha_k \neq 0$ when $c \to \infty$. Then, for all $x \in [-1,1]$,

$$\left| \frac{\Psi^c(x)}{\|\Psi^c\|_{[-1,1]}} - \frac{\Psi^c(x)}{l} \right| \leq M \cdot \left(\frac{1}{c}\right)^{p+1}, \tag{2.38}$$

where $M > 0$ is a constant, and

$$l = \sqrt{\sum_{i=0}^{p} \alpha_i^2}. \tag{2.39}$$

2.3 Prolate Spheroidal Wave Functions

In this section, we summarize a number of analytical properties of the Prolate Spheroidal Wave Functions. Unless stated otherwise, all of these facts can be found in [31, 17].

Given a real $c > 0$, we denote by F_c the operator $L^2[-1,1] \to L^2[-1,1]$ defined by the formula

$$F_c(\varphi)(x) = \int_{-1}^{1} e^{icxt}\, \varphi(t)\, dt. \tag{2.40}$$

Obviously, F_c is compact; we denote by $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ the eigenvalues of F_c ordered so that $|\lambda_{j-1}| \geq |\lambda_j|$ for all natural j. For each non-negative integer j, we denote by ψ_j the eigenfunctions corresponding to λ_j, so that

$$\lambda_j \psi_j(x) = \int_{-1}^{1} e^{icxt}\, \psi_j(t)\, dt, \tag{2.41}$$

for all $x \in [-1,1]$. We adopt the convention that the functions are normalized such that $\|\psi_j\|_{[-1,1]} = 1$ for all j.[1] The following theorem is a combination of several lemmas from

[1]This convention differs from that used in [31]; however, the present dissertation is concerned almost exclusively with approximation of functions on $[-1,1]$, and in that context, the convention that the functions $\{\psi_j\}$ have unit norm on that interval is by far the most convenient.

[31, 8, 14].

Theorem 2.8 *For any positive real c, the eigenfunctions $\psi_0, \psi_1, \ldots,$ of the operator F_c are purely real, are orthonormal, and are complete in $L^2[-1,1]$. The even-numbered eigenfunctions are even, and the odd-numbered ones are odd. All eigenvalues of F_c are non-zero and simple; the even-numbered eigenvalues are purely real, and the odd-numbered ones are purely imaginary; in particular, $\lambda_j = i^j|\lambda_j|$. The functions ψ_i constitute a Chebyshev system (see Section 2.4 below) on the interval $[-1,1]$; in particular, the function ψ_i has exactly i zeroes on that interval, for any $i = 0, 1, \ldots,$.*

We define the self-adjoint operator $Q_c : L^2[-1,1] \to L^2[-1,1]$ by the formula

$$Q_c(\varphi) = \frac{1}{\pi} \int_{-1}^{1} \frac{sin(c \cdot (x-t))}{x-t} \varphi(t)\ dt; \tag{2.42}$$

a simple calculation shows that

$$Q_c = \frac{c}{2\pi} \cdot F_c^* \cdot F_c\,, \tag{2.43}$$

that Q_c has the same eigenfunctions as F_c, and that the j-th (in descending order) eigenvalue μ_j of Q_c is connected with λ_j by the formula

$$\mu_j = \frac{c}{2\pi} \cdot |\lambda_j|^2. \tag{2.44}$$

The operator Q_c is obviously closely related to the operator $P_c : L^2[-\infty,\infty] \to L^2[-\infty,\infty]$ defined by the formula

$$P_c(\varphi) = \frac{1}{\pi} \cdot \int_{-\infty}^{\infty} \frac{sin(c \cdot (x-t))}{x-t} \cdot \varphi(t) dt, \tag{2.45}$$

which, as is well known, is the orthogonal projection operator onto the space of functions of band limit c on $(-\infty,\infty)$.

For large c, the spectrum of Q_c consists of three parts: about $2c/\pi$ eigenvalues that are very close to 1, followed by order $\log(c)$ eigenvalues which decay exponentially from 1 to nearly 0; the remaining eigenvalues are all very close to zero. The following theorem, proven in [19], describes the spectrum of Q_c more precisely.

Theorem 2.9 *For any positive real c and $0 < \alpha < 1$, the number N of eigenvalues of the operator Q_c that are greater than α satisfies the equation*

$$N = \frac{2c}{\pi} + \left(\frac{1}{\pi^2} \log \frac{1-\alpha}{\alpha}\right) \log(c) + o(\log(c)). \tag{2.46}$$

By a remarkable coincidence, the eigenfunctions $\psi_0, \psi_1, \cdots, \psi_n$ of the operator Q_c turn out to be the Prolate Spheroidal Wave Functions, well known from classical Mathematical Physics (see, for example, [26]). The following theorem formalizes this statement. It is proven in a considerably more general form in [32, 12].

Theorem 2.10 *For any $c > 0$, there exists a strictly increasing sequence of positive real numbers $\chi_0, \chi_1, \ldots$ such that for each $j \geq 0$, the differential equation*

$$(1 - x^2)\, \psi''(x) - 2\, x\, \psi'(x) + (\chi_j - c^2\, x^2)\, \psi(x) = 0 \tag{2.47}$$

has a solution that is continuous and bounded on the interval $[-1, 1]$. Moreover, for each $j \geq 0$, the function ψ_j (defined in Theorem 2.8) is the solution of (2.47). Furthermore, for $c \to \infty$,

$$\chi_j = (2j + 1)\, c + O(1)\,. \tag{2.48}$$

2.4 Generalized Gaussian Quadratures

A quadrature rule is an expression of the form

$$\sum_{j=1}^{n} w_j\, \phi(x_j) \tag{2.49}$$

viewed as an approximation to the integral of the form

$$\int_a^b \phi(x)\, \omega(x)\; dx\,. \tag{2.50}$$

The function ω is assumed to be integrable and non-negative, and is often referred to as the weight function; the points $x_j \in \mathbb{R}$ and coefficients $w_j \in \mathbb{R}$ are known as the nodes and weights of the quadrature rule, respectively.

Quadratures are typically constructed so that the formula (2.49) is exact for a preselected set of functions, commonly polynomials of some fixed order. Of these, the classical Gaussian quadratures are an important example: when the class of functions to be integrated are polynomials and the weight function ω is 1, an n-point Gaussian rule is exact for all polynomials of orders up to $2n - 1$, and no n-point rule is exact for all polynomials of order $2n$.

Gaussian quadratures admit fairly radical generalizations. Although the existence of generalized Gaussian quadratures was observed more than 100 years ago (see [23, 24], [8, 16], [13, 14]), the requisite numerical algorithms have been constructed only recently (see [21, 38, 3]); in the remainder of this subsection, we summarize several definitions and theorems regarding the generalized Gaussian quadratures.

Definition 2.1 *A quadrature formula will be referred to as* Gaussian *with respect to a set of* $2n$ *functions* $\phi_1, \ldots, \phi_{2n} : [a, b] \to \mathbb{R}$ *and a weight function* $\omega : [a, b] \to \mathbb{R}^+$*, if it consists of* n *weights and nodes, and integrates the functions* ϕ_i *exactly with the weight function* ω *for all* $i = 1, \ldots, 2n$*. The weights and nodes of a Gaussian quadrature will be referred to as Gaussian weights and nodes respectively.*

Definition 2.2 *A sequence of functions* $\phi_1, \ldots, \phi_n$ *will be referred to as a* Chebyshev system *on the interval* $[a, b]$ *if each of them is continuous and the determinant*

$$\begin{vmatrix} \phi_1(x_1) & \cdots & \phi_1(x_n) \\ \vdots & & \vdots \\ \phi_n(x_1) & \cdots & \phi_n(x_n) \end{vmatrix} \tag{2.51}$$

is nonzero for any sequence of points $x_1, \ldots, x_n$ *such that* $a \leq x_1 < x_2 \ldots < x_n \leq b$.

The following theorem appears to be due to Markov (see [23, 24]); proofs of it can also be found in [16] and [14] (in a somewhat different form).

Theorem 2.11 *Suppose that the functions* $\phi_1, \dots, \phi_{2n} : [a,b] \to \mathbb{R}$ *form a Chebyshev system on* $[a,b]$. *Suppose in addition that* $\omega : [a,b] \to \mathbb{R}$ *is a non-negative integrable function* $[a,b] \to \mathbb{R}$. *Then there exists a unique Gaussian quadrature for the functions* $\phi_1, \dots, \phi_{2n}$ *on* $[a,b]$ *with respect to the weight function* ω. *The weights of this quadrature are positive.*

Remark 2.12 *When a Generalized Gaussian quadrature is to be constructed, the determination of its nodes tends to be the critical step (though the procedure of [21, 38, 3] determines the nodes and weights simultaneously). Indeed, once the nodes* $x_1, x_2, \dots, x_n$ *have been found, the weights* $w_1, w_2, \dots, w_n$ *can be determined easily as the solution of the* $n \times n$ *system of linear equations*

$$\sum_{j=1}^{n} w_j \cdot \phi_i(x_j) = \int_a^b \omega(x)\ \phi_i(x)\ dx, \tag{2.52}$$

with $i = 1, 2, \dots, n$.

2.5 Convolutional Volterra Equations

A convolutional Volterra equation of the second kind is an expression of the form

$$\varphi(x) = \int_a^x K(x-t)\,\varphi(t)\ dt\ +\ \sigma(x) \tag{2.53}$$

where a, b are a pair of numbers such that $a < b$, the functions $\sigma, K : [a,b] \to \mathbb{C}$ are square-integrable, and $\varphi : [a,b] \to \mathbb{C}$ is the function to be determined. A proof of the following theorem can be found (for example) in [5].

Theorem 2.13 *The equation (2.53) always has a unique solution on the interval* $[a,b]$. *If both of the functions* K *and* σ *are* k *times continuously differentiable, the solution* φ *is also* k *times continuously differentiable.*

3

Evaluation of PSWFs and Related Quantities

The classical Bouwkamp's algorithm (see, for example, [2]) for the evaluation of the Prolate Spheroidal Wave Functions ψ_j, as well as the algorithm presented in this chapter for the same task, are based on the expansion of ψ_j in a Legendre series of the form

$$\psi_j(x) = \sum_{n=0}^{\infty} \alpha_n \, P_n(x) \,. \tag{3.1}$$

The coefficients α_k decay super-algebraically (see Theorems 3.1 and 3.2 below) once k is sufficiently large.

3.1 Decay of Legendre Coefficients of PSWFs

The following two theorems establish bounds for the rate of decay of the Legendre coefficients of PSWFs. Throughout this section, we use $[a]$ to denote the integer part of the real number a.

Theorem 3.1 *Suppose that $\overline{P_n}(x)$ is the n-th normalized Legendre polynomial (defined in (2.7)). Then, for all real positive a and non-negative n,*

$$\int_{-1}^{1} e^{i\,a\,x}\,\overline{P_n}(x)\,dx = \sum_{k=k_0}^{\infty} \alpha_k \int_{-1}^{1} x^{2k}\,\overline{P_n}(x)\,dx + i \sum_{k=k_0}^{\infty} \beta_k \int_{-1}^{1} x^{2k+1}\,\overline{P_n}(x)\,dx, \tag{3.2}$$

where

$$\alpha_k = (-1)^k\,\frac{a^{2k}}{(2k)!}, \tag{3.3}$$

$$\beta_k = (-1)^k\,\frac{a^{2k+1}}{(2k+1)!}, \tag{3.4}$$

$$k_0 = [n/2]\,. \tag{3.5}$$

Furthermore, for all integer $n \geq 0$ and all integer $m \geq [e \cdot a] + 1$,

$$\left| \int_{-1}^{1} e^{i\,a\,x}\,\overline{P_n}(x)\;dx - \sum_{k=k_0}^{m-1} \alpha_k \int_{-1}^{1} x^{2k}\,\overline{P_n}(x)\;dx - i \sum_{k=k_0}^{m-1} \beta_k \int_{-1}^{1} x^{2k+1}\,\overline{P_n}(x)\;dx \right| < \left(\frac{1}{2}\right)^{2m}. \tag{3.6}$$

In particular, if

$$n \geq 2\;([e \cdot a] + 1)\;, \tag{3.7}$$

then

$$\left| \int_{-1}^{1} e^{i\,a\,x}\,\overline{P_n}(x)\;dx \right| < \left(\frac{1}{2}\right)^{n-1}. \tag{3.8}$$

Proof. The formula (3.2) follows immediately from Lemma 2.1 and Taylor's expansion of $e^{i\,a\,x}$. In order to prove (3.6), we assume that m is an integer such that

$$m \geq [e \cdot a] + 1\,. \tag{3.9}$$

Introducing the notation

$$R_m = \sum_{k=m}^{\infty} \alpha_k \int_{-1}^{1} x^{2k}\,\overline{P_n}(x)\;dx + i \sum_{k=m}^{\infty} \beta_k \int_{-1}^{1} x^{2k+1}\,\overline{P_n}(x)\;dx, \tag{3.10}$$

we immediately observe that, due to Lemma 2.1 and the triangle inequality,

$$
\begin{aligned}
|R_m| &\leq \sum_{k=2m}^{\infty} \left(\frac{a^k}{k\,!} \cdot \sqrt{\frac{2}{k+1}} \right) \\
&< \sum_{k=2m}^{\infty} \frac{a^k}{k\,!}.
\end{aligned} \tag{3.11}
$$

Since (3.9) implies that

$$
\frac{a}{2m+k} < \frac{a}{2m} < \frac{1}{2e} < \frac{1}{2} \tag{3.12}
$$

for all integer $m, k > 0$, we rewrite (3.11) as

$$
\begin{aligned}
|R_m| &< \frac{a^{2m}}{(2m)\,!} \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \ldots \right) \\
&< \frac{2 \cdot a^{2m}}{(2m)\,!},
\end{aligned} \tag{3.13}
$$

and obtain (3.6) immediately using Stirling's formula. Finally, we obtain (3.8) by setting

$$
m = [e \cdot a] + 1\,. \tag{3.14}
$$

□

Theorem 3.2 *Suppose that $\overline{P_k}(x)$ is the k-th normalized Legendre polynomial (defined in (2.7)). Suppose further that ψ_m, λ_m are the m-th eigenfunction and corresponding eigenvalue of (2.40). Then for all integer $m \geq 0$ and all real positive c,*

$$
k \geq 2\ ([e \cdot c] + 1)\ , \tag{3.15}
$$

implies

$$
\left| \int_{-1}^{1} \psi_m(x)\, \overline{P_k}(x)\ dx \right| < \frac{1}{\lambda_m} \cdot \left(\frac{1}{2} \right)^{k-1/2}. \tag{3.16}
$$

Moreover, given any $\varepsilon > 0$,

$$
k \geq 2\,([e \cdot c] + 1) + \log_2\left(\frac{1}{\varepsilon}\right) + \log_2\left(\frac{1}{\lambda_m}\right), \tag{3.17}
$$

implies

$$\left| \int_{-1}^{1} \psi_m(x)\, \overline{P_k}(x)\ dx \right| < \varepsilon\,. \tag{3.18}$$

Proof. Obviously,

$$\begin{aligned} &\left| \int_{-1}^{1} \psi_m(x)\, \overline{P_k}(x)\ dx \right| \\ &= \frac{1}{|\lambda_m|} \cdot \left| \int_{-1}^{1} \psi_m(x) \left(\int_{-1}^{1} e^{icxt}\, \overline{P_k}(t)\ dt \right)\ dx \right| \\ &< \frac{1}{|\lambda_m|} \int_{-1}^{1} |\psi_m(x)| \cdot \left| \int_{-1}^{1} e^{icxt}\, \overline{P_k}(t)\ dt \right| dx\,. \end{aligned} \tag{3.19}$$

Observing that

$$\int_{-1}^{1} |\psi_m(x)|\, dx \le \sqrt{2} \tag{3.20}$$

(since $\|\psi_m\| = 1$), and introducing the notation

$$a = c\,x\,, \tag{3.21}$$

we conclude that the combination of (3.19), (3.21), (3.20), and Lemma 3.1 implies that

$$\begin{aligned} &\left| \int_{-1}^{1} \psi_m(x)\, \overline{P_k}(x)\ dx \right| \\ &< \frac{1}{|\lambda_m|} \cdot \left(\frac{1}{2}\right)^{k-1} \int_{-1}^{1} |\psi_m(x)|\, dx \\ &\le \frac{1}{|\lambda_m|} \left(\frac{1}{2}\right)^{k-1/2}\,. \end{aligned} \tag{3.22}$$

Now, substituting (3.17) into (3.16), we immediately obtain (3.18). □

3.2 Numerical Evaluation of PSWFs

The classical scheme for the numerical evaluation of prolate spheroidal wave functions is based on the connection between the PSWFs and the Legendre polynomials. Comparing the differential equations (2.47) and (2.5), we immediately observe that PSWFs converge to corresponding Legendre polynomials as the band limit c approaches zero. Substituting (3.1) into (2.47), and using (2.2) and (2.5), we immediately obtain the well-known three-term recursion

$$\begin{aligned}&\frac{(k+2)(k+1)}{(2k+3)(2k+5)} \cdot c^2 \cdot \alpha_{k+2} + \\ &\left(k(k+1) + \frac{2k(k+1)-1}{(2k+3)(2k-1)} \cdot c^2 - \chi_j\right) \cdot \alpha_k + \\ &\frac{k(k-1)}{(2k-3)(2k-1)} \cdot c^2 \cdot \alpha_{k-2} = 0.\end{aligned} \tag{3.23}$$

for the Legendre coefficients α_k in (3.1). Combining (3.23) with (2.7), we obtain the three-term recursion

$$\begin{aligned}&\frac{(k+2)(k+1)}{(2k+3)\sqrt{(2k+5)(2k+1)}} \cdot c^2 \cdot \beta^j_{k+2} + \\ &\left(k(k+1) + \frac{2k(k+1)-1}{(2k+3)(2k-1)} \cdot c^2 - \chi_j\right) \cdot \beta^j_k + \\ &\frac{k(k-1)}{(2k-1)\sqrt{(2k-3)(2k+1)}} \cdot c^2 \cdot \beta^j_{k-2} = 0\end{aligned} \tag{3.24}$$

for the coefficients $\beta^j_0, \beta^j_1, \ldots$ of the expansion

$$\psi_j(x) = \sum_{k=0}^{\infty} \beta^j_k \cdot \overline{P_k}(x). \tag{3.25}$$

In the remainder of this dissertation, we denote by β^j the vector in l^2 defined by the formula

$$\beta^j = (\beta^j_0, \beta^j_1, \beta^j_2, \ldots) \tag{3.26}$$

for each $j = 0, 1, 2, \ldots,$. The following theorem restates the recursion (3.24) in a slightly different form.

Theorem 3.3 *The coefficients χ_i are the eigenvalues and the vectors β^i are the corresponding eigenvectors of the operator $l^2 \to l^2$ represented by the symmetric matrix A given by the* *formulae*

$$A_{k,k} = k(k+1) + \frac{2k(k+1)-1}{(2k+3)(2k-1)} \cdot c^2, \tag{3.27}$$

$$A_{k,k+2} = \frac{(k+2)(k+1)}{(2k+3)\sqrt{(2k+1)(2k+5)}} \cdot c^2, \tag{3.28}$$

$$A_{k+2,k} = \frac{(k+2)(k+1)}{(2k+3)\sqrt{(2k+1)(2k+5)}} \cdot c^2, \tag{3.29}$$

for all $k = 0, 1, 2, \ldots$, *with the remainder of the entries being zero.*

In other words, the recursion (3.24) can be rewritten in the form

$$(A - \chi_j \cdot I)(\beta^j) = 0, \tag{3.30}$$

where A is separable into two symmetric tridiagonal matrices A_{even} and A_{odd}, the first consisting of the elements of A with even-numbered rows and columns and the second consisting of the elements of A with odd-numbered rows and columns. While these two matrices are infinite, and their entries do not decay much with increasing row or column number, the coordinates of the eigenvectors β^j of interest (those corresponding to the first m Prolate Spheroidal Wave Functions) decay rapidly (see Theorem 3.2). Thus, the evaluation of Prolate Spheroidal Wave Functions can be performed by the following procedure:

1. Generate the leading k rows and columns of A, where k is given by (3.17).

2. Separate the generated portion of A into A_{even} and A_{odd}, and use a solver for the symmetric tridiagonal eigenproblem (such as that in LAPACK) to compute their eigenvectors $\{\beta^j\}$ and eigenvalues $\{\chi_j\}$.

3. Use the obtained values of the coefficients $\beta_0^j, \beta_1^j, \beta_2^j, \ldots$ in the expansion (3.25) to evaluate the function ψ_j at arbitrary points on the interval $[-1, 1]$.

Obviously steps 1 and 2 can be precomputed for any given c. As a numerical diagonalization of a positive definite tridiagonal matrix with well-separated eigenvalues, this precomputation stage is numerically robust and efficient, requiring $O(cm)$ operations to construct the

Legendre expansions of the form (3.26) for the first m prolate spheroidal wave functions. Since m is in general proportional to c, the complexity of this scheme is roughly $O(c^2)$, with each subsequent evaluation of a prolate spheroidal wave function costing $O(c)$ operations.

3.3 Prolate Series

Since the functions $\psi_0, \psi_1, \ldots, \psi_n, \ldots$ constitute an orthonormal basis in $L^2[-1,1]$, any formula for the inner product of Prolate Spheroidal Wave Functions with another function f is also a formula for the coefficients of an expansion of f into Prolate Spheroidal Functions; we will refer to this expansion as the prolate expansion of f. The following theorem provides the coefficients of the prolate expansion of the derivative of a Prolate Spheroidal Wave Function, and the coefficients of the prolate expansion of a Prolate Spheroidal Wave Function multiplied by x. On the other hand, these coefficients are the entries of the matrix for differentiation of a prolate expansion (producing another prolate expansion), and the entries of the matrix for multiplication of a prolate expansion by x, respectively. The formulae in this section, however, are not suitable for the numerical evaluation of such matrices, since in many cases they exhibit catastrophic cancellation; one stable way to obtain such matrices is via formulae (3.1) or (3.25).

Theorem 3.4 *Suppose that* c *is real and positive, and that the integers* m *and* n *are non-negative. If* $m = n \pmod 2$*, then*

$$\int_{-1}^{1} \psi_n'(x)\, \psi_m(x)\; dx = \int_{-1}^{1} x\, \psi_n(x)\, \psi_m(x)\; dx = 0. \tag{3.31}$$

If $m \neq n \pmod 2$*, then*

$$\int_{-1}^{1} \psi_n'(x)\, \psi_m(x)\; dx = \frac{2\,\lambda_m^2}{\lambda_m^2 + \lambda_n^2}\, \psi_m(1)\, \psi_n(1)\,, \tag{3.32}$$

$$\int_{-1}^{1} x\, \psi_n(x)\, \psi_m(x)\; dx = \frac{2}{ic}\, \frac{\lambda_m \lambda_n}{\lambda_m^2 + \lambda_n^2}\, \psi_m(1)\, \psi_n(1)\,. \tag{3.33}$$

Proof. Since the functions ψ_j are alternately even and odd, (3.31) is obvious. In order to prove (3.32), we start with the identity

$$\lambda_n \psi_n(x) = \int_{-1}^{1} e^{icxt}\, \psi_n(t)\; dt \tag{3.34}$$

(see (2.41) in Subsection 2.3). Differentiating (3.34) with respect to x, we obtain

$$\lambda_n \psi_n'(x) = i\, c \int_{-1}^{1} t\, e^{icxt}\, \psi_n(t)\; dt. \tag{3.35}$$

Projecting both sides of (3.35) on ψ_m and using the identity (3.34) again (with n replaced with m), we have

$$\begin{aligned}
&\lambda_n \int_{-1}^{1} \psi_n'(x)\, \psi_m(x)\; dx \\
&= \; i\, c \int_{-1}^{1} \psi_m(x) \int_{-1}^{1} t\, e^{icxt}\, \psi_n(t)\; dt\, dx \\
&= \; i\, c \int_{-1}^{1} t\, \psi_n(t) \int_{-1}^{1} e^{icxt}\, \psi_m(x)\; dx\, dt \\
&= \; i\, c\, \lambda_m \int_{-1}^{1} t\, \psi_n(t)\, \psi_m(t)\; dt.
\end{aligned} \tag{3.36}$$

Obviously, the above calculation can be repeated with m and n exchanged, yielding the identity

$$\lambda_m \int_{-1}^{1} \psi_m'(x)\, \psi_n(x)\; dx = i\, c\, \lambda_n \int_{-1}^{1} t\, \psi_n(t)\, \psi_m(t)\; dt\,; \tag{3.37}$$

combining (3.36) with (3.37), we have

$$\int_{-1}^{1} \psi_m'(x)\, \psi_n(x)\; dx = \frac{\lambda_n^2}{\lambda_m^2} \int_{-1}^{1} \psi_m(x)\, \psi_n'(x)\; dx. \tag{3.38}$$

On the other hand, integrating the left side of (3.38) by parts, we have

$$\begin{aligned}
&\int_{-1}^{1} \psi_m'(x)\, \psi_n(x)\; dx \\
&= \; \psi_m(1)\, \psi_n(1) \; - \; \psi_m(-1)\, \psi_n(-1) \; - \; \int_{-1}^{1} \psi_n'(x)\, \psi_m(x)\; dx.
\end{aligned} \tag{3.39}$$

Since $m \neq n \pmod 2$, we rewrite (3.39) as

$$\int_{-1}^{1} \psi_m'(x)\,\psi_n(x)\;dx = 2\,\psi_m(1)\,\psi_n(1) - \int_{-1}^{1} \psi_n'(x)\,\psi_m(x)\;dx. \tag{3.40}$$

Now, combining (3.38) and (3.40) and rearranging terms, we get

$$\int_{-1}^{1} \psi_n'(x)\,\psi_m(x)\;dx = \frac{2\,\lambda_m^2}{\lambda_m^2+\lambda_n^2}\,\psi_m(1)\,\psi_n(1)\,. \tag{3.41}$$

Substituting (3.36) into (3.41), we get

$$\begin{aligned}\int_{-1}^{1} x\,\psi_n(x)\,\psi_m(x)\;dx &= \frac{1}{ic}\frac{\lambda_n}{\lambda_m}\int_{-1}^{1} \psi_n'(x)\,\psi_m(x)\;dx \\ &= \frac{1}{ic}\frac{\lambda_n}{\lambda_m}\frac{2\,\lambda_m^2}{\lambda_m^2+\lambda_n^2}\,\psi_m(1)\,\psi_n(1) \\ &= \frac{2}{ic}\frac{\lambda_m\lambda_n}{\lambda_m^2+\lambda_n^2}\,\psi_m(1)\,\psi_n(1)\,.\end{aligned} \tag{3.42}$$

□

The following corollary, which is an immediate consequence of (3.38), finds use in the numerical evaluation of the eigenvalues λ_j.

Corollary 3.5 *Suppose that c is real and positive, and that the integers m and n are non-negative. If $m \neq n \pmod 2$, then*

$$\frac{\lambda_m^2}{\lambda_n^2} = \frac{\int_{-1}^{1} \psi_n'(x)\,\psi_m(x)\;dx}{\int_{-1}^{1} \psi_m'(x)\,\psi_n(x)\;dx}\,. \tag{3.43}$$

3.4 Numerical Evaluation of Eigenvalues

Although the algorithm of Section 3.2 for the evaluation of Prolate Spheroidal Wave Functions also produces the eigenvalues χ_j of the differential operator (2.47), it does not produce the eigenvalues λ_j of the integral operator F_c (defined in (2.40)). Some of those eigenvalues can be computed using the formula

$$\lambda_j \psi_j(x) = \int_{-1}^{1} e^{icxt} \psi_j(t) \, dt, \tag{3.44}$$

by evaluating the integral on the right hand side numerically. Obviously, this scheme has a condition number of about $1/\lambda_j$, and is thus inappropriate for computing small λ_j. A well-conditioned procedure is as follows:

1. Use (3.44) to calculate λ_0 by evaluating the right hand side numerically at $x = 0$ (so that $\psi_0(x)$ is not small).

2. Use the obtained λ_0 and Corollary 3.5 to compute the absolute values $|\lambda_j|$, for $j = 1, 2, ..., m$. Compute each $|\lambda_j|$ from $|\lambda_{j-1}|$ (and again, evaluating the required integrals numerically).

3. Use the fact that $\lambda_j = i^j |\lambda_j|$ (see Theorem 2.8) to finish the computation.

4

Quadrature and Interpolation

In this chapter, we construct quadratures and interpolation schemes based on Prolate Spheroidal Wave Functions. As a matter of convention, all prolate functions ψ_j of this chapter correspond to the band limit c, which we omit from the notation.

4.1 Quadratures for Band-Limited Functions

Since the prolate spheroidal wave functions $\psi_0, \psi_1, \ldots, \psi_n, \ldots$ constitute an orthonormal basis in $L^2[-1,1]$,

$$e^{icxt} = \sum_{j=0}^{\infty} \left(\int_{-1}^{1} e^{icx\tau} \, \psi_j(\tau) \, d\tau \right) \psi_j(t) \tag{4.1}$$

for all $x, t \in [-1,1]$. Substituting (2.41) into (4.1), we have

$$e^{icxt} = \sum_{j=0}^{\infty} \lambda_j \, \psi_j(x) \, \psi_j(t) \, . \tag{4.2}$$

The following theorem provides a basis for the construction of quadratures for band-limited functions with Prolate Spheroidal Wave Functions.

Theorem 4.1 *Suppose that the n-point quadrature with nodes $x_1, x_2, \ldots, x_n \in (-1,1)$ and weights $w_1, w_2, \ldots, w_n$ integrate exactly each of the functions $\psi_0, \psi_1, \ldots, \psi_{m-1}$, so that*

$$\sum_{k=1}^{n} w_k \, \psi_j(x_k) = \int_{-1}^{1} \psi_j(x) \; dx \,. \tag{4.3}$$

Then, for all $a \in (-1,1)$ and real positive c,

$$\left| \sum_{k=1}^{n} w_k \, e^{i\,c\,a\,x_k} - \int_{-1}^{1} e^{i\,c\,a\,x} \; dx \right| \le M\,(n \cdot W \cdot M + \sqrt{2}) \sum_{j=m}^{\infty} |\lambda_j| \,, \tag{4.4}$$

with

$$W = \max_{1 \le j \le n} |w_j| \,, \tag{4.5}$$

$$M = \max_{j \ge m} \left(\max_{1 \le k \le n} |\psi_j(x_k)|, \psi_j(a) \right) . \tag{4.6}$$

Proof. Obviously,

$$\begin{aligned} &\sum_{k=1}^{n} w_k \, e^{i\,c\,a\,x_k} - \int_{-1}^{1} e^{i\,c\,a\,x} \; dx \\ &= \sum_{k=1}^{n} w_k \left(\sum_{j=0}^{\infty} \lambda_j \, \psi_j(a) \, \psi_j(x_k) \right) - \int_{-1}^{1} \left(\sum_{j=0}^{\infty} \lambda_j \, \psi_j(a) \, \psi_j(x) \right) \; dx \,, \end{aligned} \tag{4.7}$$

which is equivalent to

$$\begin{aligned} &\sum_{k=1}^{n} w_k \, e^{i\,c\,a\,x_k} - \int_{-1}^{1} e^{i\,c\,a\,x} \; dx \\ &= \sum_{j=m}^{\infty} \lambda_j \, \psi_j(a) \left(\sum_{k=1}^{n} w_k \, \psi_j(x_k) - \int_{-1}^{1} \psi_j(x) \; dx \right) , \end{aligned} \tag{4.8}$$

due to (4.3). Since for all $j \ge 0$, the functions ψ_j are analytic in $\mathbb{C}$ and have unit norm on $[-1,1]$, there exists a constant M defined by (4.6) (see, for example, [20], p.160). Now, the combination of (3.20) and the triangle inequality converts (4.7) into (4.4). □

Remark 4.2 *From Theorem 4.1, it is easily seen that the error of the integration (4.4) is proportional to $|\lambda_m|$, provided that m is in the range where the eigenvalues $\{\lambda_j\}$ decay* *exponentially (as is the case for quadratures of any useful accuracy; see Theorem 2.9). The constant factor of the error is determined by the combination of the magnitude of the weights* $\{w_k\}$*, the number of quadrature nodes, and the bound* M *of PSWFs.*

The existence of an $n/2$-point quadrature that is exact for the first n Prolate Spheroidal Wave Functions follows from the combination of Theorems 2.11 and 2.8. An algorithm for the numerical evaluation of nodes and weights of such quadratures can be found in [3]. An alternative procedure for the construction of quadrature formulae for band-limited functions (leading to slightly different nodes and weights) is described in the next section. In Chapter 6 below, we give a numerical comparison of the two algorithms.

Remark 4.3 *The above text considers only the error of integration of a single exponential. For a band-limited function* $g : [-1,1] \to \mathbb{C}$ *given by the formula*

$$g(x) = \int_{-1}^{1} G(t)\, e^{icxt}\, dt, \tag{4.9}$$

with function $G : [-1,1] \to \mathbb{C}$*, the error is bounded by the formula*

$$\left| \sum_{k=1}^{m} w_k g(x_k) - \int_{-1}^{1} g(x)\, dx \right| \le \varepsilon \cdot \|G\|, \tag{4.10}$$

where ε *is the maximum error of integration (4.4) of a single exponential, for any* $t \in [-1,1]$*. While* $\|G\|$ *might be much larger than* $\|g\|_{[-1,1]}$ *(when, for instance,* $g = \psi_{30 \cdot n}$*), if we extend* g *to the rest of the real line using equation (4.9), then by Parseval's formula,* $\|G\| = \|g\|_{(-\infty,\infty)}$*; that is to say, although the bound on the error of such a quadrature when applied to a band-limited function is not proportional to the norm of that function on the interval of integration, it is proportional to the norm of that function on the entire real line.*

4.2 Quadrature Nodes from Roots of the PSWFs

An alternative approach to that of the previous section is to use roots of appropriate Prolate Spheroidal Wave Functions as quadrature nodes, with the weights determined via the procedure described in Remark 2.12. The following theorems provide a basis for this construction. Numerically, the resulting quadrature nodes tend to be inferior to those produced by the optimization scheme of [21, 38, 3] (see Chapter 6); however, the former are useful as starting points for the latter, or as somewhat less efficient nodes which can be computed much faster.

4.2.1 Euclid Division Algorithm for Band-Limited Functions

The following two theorems constitute a straightforward extension to band-limited functions of Euclid's division algorithm for polynomials. Their proofs are simple, but are provided for completeness.

Theorem 4.4 *Suppose that $\sigma, \varphi : [0,1] \to \mathbb{C}$ are a pair of c^2–functions such that*

$$\varphi(1) \neq 0, \tag{4.11}$$

c is a positive real number, and the functions f, p are defined by the formulae

$$f(x) = \int_0^1 \sigma(t)\, e^{2icxt}\, dt, \tag{4.12}$$

$$p(x) = \int_0^1 \varphi(t)\, e^{icxt}\, dt. \tag{4.13}$$

Then there exist two c^1-functions $\eta, \xi : [0,1] \to \mathbb{C}$ such that

$$f(x) = p(x)\, q(x) + r(x) \tag{4.14}$$

for all $x \in \mathbb{R}$, with the functions $q, r : [0,1] \to \mathbb{R}$ defined by the formulae

$$q(x) = \int_0^1 \eta(t)\, e^{icxt}\, dt, \tag{4.15}$$

$$r(x) = \int_0^1 \xi(t)\, e^{icxt}\, dt. \tag{4.16}$$

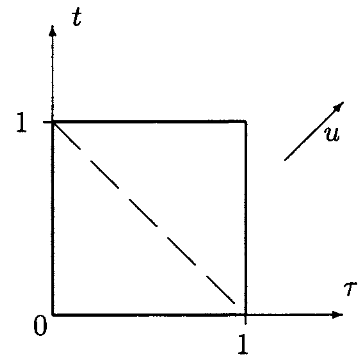


Figure 4.1: The split of integration range that yields (4.19)

Proof.

Obviously, for any functions p, q given by (4.13), (4.15),

$$\begin{aligned} p(x)\,q(x) &= \int_0^1 \varphi(t)\,e^{icxt}\,dt \cdot \int_0^1 \eta(\tau)\,e^{icx\tau}\,d\tau \\ &= \int_0^1 \int_0^1 \varphi(t)\,\eta(\tau)\,e^{icx(t+\tau)}\,d\tau\,dt. \end{aligned} \tag{4.17}$$

Defining the new independent variable u by the formula

$$u = t + \tau, \tag{4.18}$$

we rewrite (4.17) as

$$\begin{aligned} p(x)q(x) &= \int_0^1 e^{icux} \int_0^u \varphi(u-\tau)\,\eta(\tau)\,d\tau\,du \\ &+ \int_1^2 e^{icux} \int_{u-1}^1 \varphi(u-\tau)\,\eta(\tau)\,d\tau\,du \end{aligned} \tag{4.19}$$

(see Figure 4.1). Substituting (4.12), (4.16), and (4.19) into (4.14), we get

$$\begin{aligned} & \int_0^1 e^{icux} \int_0^u \varphi(u-\tau)\,\eta(\tau)\,d\tau\,du \\ + & \int_1^2 e^{icux} \int_{u-1}^1 \varphi(u-\tau)\,\eta(\tau)\,d\tau\,du + \int_0^1 \xi(t)\,e^{icxt}\,dt \\ = & \int_0^{1/2} \sigma(t)\,e^{2icxt}\,dt + \int_{1/2}^1 \sigma(t)\,e^{2icxt}\,dt. \end{aligned} \tag{4.20}$$

Due to the well-known uniqueness of the Fourier Transform, (4.20) is equivalent to two independent equations:

$$\int_0^1 e^{icux} \int_0^u \varphi(u-\tau)\,\eta(\tau)\;d\tau\,du + \int_0^1 \xi(t)\,e^{icxt}\;dt = \int_0^{1/2} \sigma(t)\,e^{2icxt}\;dt, \tag{4.21}$$

$$\int_1^2 e^{icux} \int_{u-1}^1 \varphi(u-\tau)\,\eta(\tau)\;d\tau\,du = \int_{1/2}^1 \sigma(t)\,e^{2icxt}\;dt. \tag{4.22}$$

Now, we observe that (4.22) does not contain ξ, and use it to obtain an expression of η as a function of φ and σ. After that, we will view (4.21) as an expression of ξ via φ, σ, η.

From (4.22) and the uniqueness of the Fourier Transform, we obtain

$$\int_{u-1}^1 \varphi(u-\tau)\,\eta(\tau)\;d\tau = \frac{1}{2}\,\sigma\left(\frac{u}{2}\right), \tag{4.23}$$

for all $u \in [1,2]$. Introducing the new variable v via the formula

$$v = u - 1, \tag{4.24}$$

we convert (4.23) into

$$\int_v^1 \varphi(v+1-\tau)\,\eta(\tau)\;d\tau = \frac{1}{2}\,\sigma\left(\frac{v+1}{2}\right), \tag{4.25}$$

which is a Volterra equation of the first kind with respect to η. Differentiating (4.25) with respect to v, we get

$$-\varphi(1)\,\eta(v) + \int_v^1 \varphi'(v+1-\tau)\,\eta(\tau)\;d\tau = \frac{1}{4}\,\sigma'\left(\frac{v+1}{2}\right), \tag{4.26}$$

which is a Volterra equation of the second kind. Now, the existence and uniqueness of the solution of (4.26) (and, therefore, of (4.23) and (4.22)) follows from Theorem 2.13 of Chapter 2.

With η defined as the solution of (4.23), we use (4.21) together with the uniqueness of the Fourier Transform to finally obtain

$$\xi(u) = \frac{1}{2}\,\sigma\left(\frac{u}{2}\right) - \int_0^u \varphi(u-\tau)\,\eta(\tau)\;d\tau, \tag{4.27}$$

for all $u \in [0,1]$.

□

The following theorem is a consequence of the preceding one.

Theorem 4.5 (Division Theorem) *Suppose that $\sigma, \varphi : [-1,1] \to \mathbb{C}$ are a pair of c^2−functions such that $\varphi(-1) \neq 0$, $\varphi(1) \neq 0$, c is a positive real number, and the functions f, p are defined by the formulae*

$$f(x) = \int_{-1}^{1} \sigma(t)\, e^{2icxt}\, dt, \tag{4.28}$$

$$p(x) = \int_{-1}^{1} \varphi(t)\, e^{icxt}\, dt. \tag{4.29}$$

Then there exist two c^1-functions $\eta, \xi : [-1,1] \to \mathbb{C}$ such that

$$f(x) = p(x)\, q(x) + r(x) \tag{4.30}$$

for all $x \in \mathbb{R}$, with the functions $q, r : [-1,1] \to \mathbb{R}$ defined by the formulae

$$q(x) = \int_{-1}^{1} \eta(t)\, e^{icxt}\, dt, \tag{4.31}$$

$$r(x) = \int_{-1}^{1} \xi(t)\, e^{icxt}\, dt. \tag{4.32}$$

Proof.

Defining the functions f_+, f_-,p_+, p_-, by the formulae

$$f_+(x) = \int_{0}^{1} \sigma(t)\, e^{2icxt}\, dt, \tag{4.33}$$

$$f_-(x) = \int_{-1}^{0} \sigma(t)\, e^{2icxt}\, dt, \tag{4.34}$$

$$p_+(x) = \int_{0}^{1} \varphi(t)\, e^{icxt}\, dt, \tag{4.35}$$

$$p_-(x) = \int_{-1}^{0} \varphi(t)\, e^{icxt}\, dt, \tag{4.36}$$

we observe that for all $x \in \mathbb{R}$,

$$f(x) = f_+(x) + f_-(x), \tag{4.37}$$

$$p(x) = p_+(x) + p_-(x). \tag{4.38}$$

Due to Theorem 4.4, there exist η_+, η_-, ξ_+, ξ_-, such that

$$f_+(x) = p_+(x)\, q_+(x) + r_+(x), \tag{4.39}$$

$$f_-(x) = p_-(x)\, q_-(x) + r_-(x), \tag{4.40}$$

with the functions q_+, q_-, r_+, r_- defined by the formulae

$$q_+(x) = \int_0^1 \eta_+(t)\, e^{icxt}\, dt, \tag{4.41}$$

$$q_-(x) = \int_{-1}^0 \eta_-(t)\, e^{icxt}\, dt, \tag{4.42}$$

$$r_+(x) = \int_0^1 \xi_+(t)\, e^{icxt}\, dt, \tag{4.43}$$

$$r_-(x) = \int_{-1}^0 \xi_-(t)\, e^{icxt}\, dt. \tag{4.44}$$

Now, if we define q by the formula

$$q(x) = q_-(x) + q_+(x) \tag{4.45}$$

for all $x \in [-1, 1]$, we have

$$\begin{aligned} p(x)\, q(x) &= (p_-(x) + p_+(x)) \cdot (q_-(x) + q_+(x)) \\ &= p_+(x)\, q_+(x) + p_-(x)\, q_-(x) + p_-(x)\, q_+(x) + p_+(x)\, q_-(x), \end{aligned} \tag{4.46}$$

and we define $r(x)$ by the formula

$$r(x) = r_-(x) + r_+(x) - (p_-(x)\, q_+(x) + p_+(x)\, q_-(x)). \tag{4.47}$$

The product $p_+(x)q_-(x)$ is given by the formula

$$p_+(x)q_-(x) = \int_0^1 \int_{-1}^0 \phi(t)\, \eta_+(t) e^{icx(t+\tau)}\, dt\, d\tau\, . \tag{4.48}$$

Since $-1 \le t + \tau \le 1$ in the integral in (4.48), the product $p_+(x)q_-(x)$ has the appropriate band limits; likewise for $p_-(x)q_+(x)$.

□

4.2.2 Quadrature Nodes from the Division Theorem

In much the same way that the division theorem for polynomials can be used to provide a constructive proof of the existence of Gaussian quadratures, Theorem 4.5 provides a method of constructing generalized Gaussian quadratures for band-limited functions.

To construct a quadrature for functions of bandwidth $2c$, we use the Prolate Spheroidal Wave Functions corresponding to bandwidth c. Thus the eigenvalues $\{\lambda_j\}$ and eigenfunctions $\{\psi_j\}$ of this section, as well as elsewhere in the dissertation, are those corresponding to bandwidth c. The following theorem provides a bound on the error of a quadrature whose nodes are the roots of the n-th Prolate Spheroidal Wave Function ψ_n, when applied to a function f that satisfies the conditions of the Division Theorem.

Theorem 4.6 *Suppose that $x_1, x_2, \ldots, x_n \in \mathbb{R}$ are the roots of ψ_n on the interval $[-1,1]$. Let the numbers $w_1, w_2, \ldots, w_n \in \mathbb{R}$ be such that*

$$\sum_{k=1}^{n} w_k \, \psi_j(x_k) = \int_{-1}^{1} \psi_j(x) \, dx, \tag{4.49}$$

for all $j = 0, 1, \ldots, n-1$. Then for any function $f : [-1,1] \to \mathbb{C}$ that satisfies the conditions of Theorem 4.5,

$$\begin{aligned} &\left| \sum_{k=1}^{n} w_k \, f(x_k) - \int_{-1}^{1} f(x) \, dx \right| \\ &\leq \; |\lambda_n| \cdot \|\eta\| \; + \; \|\xi\| \cdot \sum_{j=n}^{\infty} |\lambda_j| \cdot \|\psi_j\|_{\infty}^2 \cdot \left(2 + \sum_{k=1}^{m} |w_k| \right), \end{aligned} \tag{4.50}$$

where the functions $\eta, \xi : [-1,1] \to \mathbb{C}$ are as defined in Theorem 4.5.

Proof. Since f satisfies the conditions of Theorem 4.5, there exist functions $q, r : [-1,1] \to \mathbb{R}$ defined by (4.31),(4.32) such that

$$f(x) = \psi_n(x) \, q(x) + r(x) \, . \tag{4.51}$$

Then, defining the error of integration E_f for the function f by

$$E_f = \left| \sum_{k=1}^{n} w_k f(x_k) - \int_{-1}^{1} f(x) \, dx \right| , \tag{4.52}$$

we have

$$
\begin{aligned}
E_f &= \left| \sum_{k=1}^{n} w_k \left(\psi_n(x_k)\, q(x_k) + r(x_k)\right) - \int_{-1}^{1} \left(\psi_n(x)\, q(x) + r(x)\right) dx \right| \\
&\leq \left| \sum_{k=1}^{n} w_k\, \psi_n(x_k)\, q(x_k) - \int_{-1}^{1} \psi_n(x)\, q(x)\; dx \right| \\
&\quad + \left| \sum_{k=1}^{n} w_k r(x_k) - \int_{-1}^{1} r(x)\; dx \right| . \qquad (4.53)
\end{aligned}
$$

Since the nodes $\{x_k\}$ are the roots of ψ_n,

$$\sum_{k=1}^{n} w_k\, \psi_n(x_k)\, q(x_k) = 0. \qquad (4.54)$$

Thus

$$E_f \leq \left| \int_{-1}^{1} \psi_n(x)\, q(x)\; dx \right| + \left| \sum_{k=1}^{n} w_k r(x_k) - \int_{-1}^{1} r(x)\; dx \right| . \qquad (4.55)$$

On the other hand,

$$
\begin{aligned}
\int_{-1}^{1} \psi_n(x)\, q(x) dx &= \int_{-1}^{1} \psi_n(x) \int_{-1}^{1} \eta(t)\, e^{icxt}\; dt\, dx \\
&= \int_{-1}^{1} \eta(t) \int_{-1}^{1} \psi_n(x)\, e^{icxt}\; dx\, dt \\
&= \int_{-1}^{1} \eta(t)\, \lambda_n\, \psi_n(t)\; dt. \qquad (4.56)
\end{aligned}
$$

Using the Cauchy-Schwartz inequality, and the fact that ψ_n has unit norm, (4.56) implies that

$$\left| \int_{-1}^{1} \psi_n(x)\, q(x)\; dx \right| \;\leq\; |\lambda_n| \cdot \|\eta\|. \qquad (4.57)$$

We also have

$$
\begin{aligned}
& \sum_{k=1}^{n} w_k r(x_k) - \int_{-1}^{1} r(x)\; dx \\
&= \sum_{k=1}^{n} w_k \left(\int_{-1}^{1} \xi(t)\, e^{icx_k t}\; dt \right) - \int_{-1}^{1} \left(\int_{-1}^{1} \xi(t)\, e^{icxt}\; dt \right) dx \\
&= \int_{-1}^{1} \xi(t) \left(\sum_{k=1}^{n} w_k e^{icx_k t} - \int_{-1}^{1} e^{icxt}\; dx \right) dt. \qquad (4.58)
\end{aligned}
$$

Substituting (4.4) into (4.58), and using the Cauchy-Schwartz inequality, we get

$$
\begin{aligned}
&\sum_{k=1}^{n} w_k r(x_k) - \int_{-1}^{1} r(x)\, dx \\
&= \int_{-1}^{1} \xi(t) \left(\sum_{k=1}^{m} w_k \left(\sum_{j=n}^{\infty} \lambda_j\, \psi_j(t)\, \psi_j(x_k) \right) \right. \\
&\qquad \left. - \int_{-1}^{1} \left(\sum_{j=n}^{\infty} \lambda_j\, \psi_j(t)\, \psi_j(x) \right) dx \right) dt \\
&\leq \|\xi\| \cdot \sum_{j=n}^{\infty} |\lambda_j| \cdot \|\psi_j\|_{\infty}^2 \cdot \left(2 + \sum_{k=1}^{m} |w_k| \right).
\end{aligned}
\tag{4.59}
$$

Combining (4.55), (4.57), and (4.59), we get

$$
E_f \leq |\lambda_n| \cdot \|\eta\| + \|\xi\| \cdot \sum_{j=n}^{\infty} |\lambda_j| \cdot \|\psi_j\|_{\infty}^2 \cdot \left(2 + \sum_{k=1}^{m} |w_k| \right). \tag{4.60}
$$

□

Remark 4.7 *The use of Theorem 4.6 for the construction of quadrature rules for band-limited functions depends on the fact that the norms of the band-limited functions q and r in (4.51) are not large, compared with the norm of f (both sets of norms being on $[-\infty, \infty]$). Such estimates have been obtained for all $n > 2c/\pi + 10\log(c)$. In this dissertation, we demonstrate the performance of the obtained quadrature formulae numerically (see Chapter 6 below).*

Remark 4.8 *It is natural to view (4.51) as an analogue for band-limited functions of the Euclid division theorem for polynomials. However, there are certain differences. In particular, Theorem 4.4 admits extensions to band-limited functions of several variables, while the classical Euclid algorithm does not.*

4.3 Interpolation via Prolate Spheroidal Wave Functions

Given a fixed sequence of functions $\phi_1, \phi_2, \ldots, \phi_n : [a,b] \to \mathbb{C}$, an interpolation scheme of $f : [a,b] \to \mathbb{C}$ in functions $\{\phi_i\}$ is given by the formula

$$f(x) = c_1\,\phi_1(x) + c_2\,\phi_2(x) + \ldots + c_n\,\phi_n(x), \tag{4.61}$$

with the coefficients $c_1, c_2, \ldots, c_n$ usually determined by solving an $n \times n$ linear system from the values of f at the n interpolation nodes. The functions ϕ_i are referred to as the interpolation functions, and the formula (4.61) is then used to evaluate f wherever needed. As is well known, if f is well-approximated by a linear combination of the interpolation functions, and if the linear system to be solved is well-conditioned, this procedure is accurate.

As shown in Section 4.1 in the context of quadratures, a linear combination of the first n prolate spheroidal functions $\psi_0, \psi_1, \ldots, \psi_{n-1}$ for a band limit c can provide a good approximation to functions of the form e^{icxt}, with $t \in [-1,1]$ (see (4.2), (4.4)). In the regime where the accuracy is numerically useful, the error is of the same order of magnitude as $|\lambda_n|$. This, in turn, shows that such linear combinations provide a good approximation (in the same sense as in Remark 4.3) to any band-limited function of band limit c. Thus, if $\psi_0, \psi_1, \ldots, \psi_{n-1}$ are used as the interpolation functions, they can be expected to yield an accurate interpolation scheme for band-limited functions, provided that the matrix to be inverted is well-conditioned.

The following theorem shows that if the interpolation nodes are chosen to be quadrature nodes accurate up to twice the bandwidth of interpolation, with the quadrature formula being accurate to more than twice as many digits as the interpolation formula is to be accurate to, then the matrix inverted in the procedure is close to being a scaled version of an orthogonal matrix.

Theorem 4.9 *Given a real $c > 0$, suppose that the numbers $w_1, w_2, \ldots, w_n \in \mathbb{R}$ and*

$x_1, x_2, \ldots, x_n \in \mathbb{R}$ are such that

$$\left| \int_{-1}^{1} e^{2icax}\, dx - \sum_{j=1}^{n} w_j e^{2icax_j} \right| < \varepsilon \tag{4.62}$$

for all $a \in [-1, 1]$. Let the matrix A be given by the formula

$$A = \begin{pmatrix} \psi_0(x_1) & \psi_1(x_1) & \ldots & \psi_{n-1}(x_1) \\ \psi_0(x_2) & \psi_1(x_2) & \ldots & \psi_{n-1}(x_2) \\ \vdots & \vdots & & \vdots \\ \psi_0(x_n) & \psi_1(x_n) & \ldots & \psi_{n-1}(x_n) \end{pmatrix}, \tag{4.63}$$

let the matrix W be the diagonal matrix whose diagonal entries are $w_1, w_2, \ldots, w_n$, and let the matrix $E = (e_{jk})$ be given by the formula

$$E = I - A^* W A\,. \tag{4.64}$$

Then

$$|e_{jk}| < \left| \frac{2\,\varepsilon}{\lambda_{j-1}\lambda_{k-1}} \right|. \tag{4.65}$$

Proof. Clearly

$$e_{jk} = \delta_{j,k} - \sum_{l=1}^{n} w_l\, \psi_{j-1}(x_l)\, \psi_{k-1}(x_l), \tag{4.66}$$

where $\delta_{i,j}$ is the Kronecker Delta function. Using (2.41), (4.66) becomes

$$\begin{aligned} e_{jk} &= \delta_{jk} - \sum_{l=1}^{n} w_l \cdot \left(\frac{1}{\overline{\lambda}_{j-1}} \int_{-1}^{1} e^{-icx_l t}\, \psi_{j-1}(t)\, dt \right) \\ &\qquad \cdot \left(\frac{1}{\lambda_{k-1}} \int_{-1}^{1} e^{icx_l \tau}\, \psi_{k-1}(\tau)\, d\tau \right) \\ &= \delta_{jk} - \frac{1}{\overline{\lambda}_{j-1}\lambda_{k-1}} \int_{-1}^{1}\int_{-1}^{1} \psi_{j-1}(t)\, \psi_{k-1}(\tau) \sum_{l=1}^{n} w_l e^{-icx_l t}\, e^{icx_l \tau}\, dt\, d\tau\,. \end{aligned} \tag{4.67}$$

Using (4.62), (4.67) becomes

$$\begin{aligned} e_{jk} = \delta_{jk} - \frac{1}{\overline{\lambda}_{j-1}\lambda_{k-1}} \int_{-1}^{1}\int_{-1}^{1} & \psi_{j-1}(t)\, \psi_{k-1}(\tau) \\ & \cdot \left(\int_{-1}^{1} e^{-icst}\, e^{ics\tau}\, ds \;-\; f_\varepsilon(t+\tau) \right)\, dt\, d\tau, \end{aligned} \tag{4.68}$$

where $f_\varepsilon : [-2,2] \to \mathbb{C}$ is a function that satisfies the relation

$$|f_\varepsilon(x)| < \varepsilon, \tag{4.69}$$

for all $x \in [-2,2]$. Thus,

$$\begin{aligned} e_{jk} &= \delta_{jk} - \frac{1}{\overline{\lambda}_{j-1}\lambda_{k-1}} \int_{-1}^{1}\int_{-1}^{1} \psi_{j-1}(t)\,\psi_{k-1}(\tau) \int_{-1}^{1} e^{-icst}\, e^{ics\tau}\, ds\, dt\, d\tau \\ &+ \frac{1}{\overline{\lambda}_{j-1}\lambda_{k-1}} \int_{-1}^{1}\int_{-1}^{1} \psi_{j-1}(t)\,\psi_{k-1}(\tau)\, f_\varepsilon(t+\tau)\; dt\, d\tau\,. \end{aligned} \tag{4.70}$$

Using (2.41), (4.70) becomes

$$\begin{aligned} e_{jk} &= \delta_{jk} - \int_{-1}^{1} \psi_{j-1}(s)\,\psi_{k-1}(s)\; ds \\ &+ \frac{1}{\overline{\lambda}_{j-1}\lambda_{k-1}} \int_{-1}^{1} \psi_{k-1}(\tau) \int_{-1}^{1} \psi_{j-1}(t)\, f_\varepsilon(t+\tau)\; dt\, d\tau\,, \end{aligned} \tag{4.71}$$

which becomes

$$e_{jk} = \frac{1}{\overline{\lambda}_{j-1}\lambda_{k-1}} \int_{-1}^{1} \psi_{k-1}(\tau) \int_{-1}^{1} \psi_{j-1}(t)\, f_\varepsilon(t+\tau)\; dt\, d\tau\,, \tag{4.72}$$

due to the orthonormality of the functions $\{\psi_j\}$. Finally, using the Cauchy-Schwartz inequality, we have

$$\begin{aligned} |e_{jk}| &\le \left|\frac{1}{\lambda_{j-1}\lambda_{k-1}}\right| \|\psi_{k-1}\| \sqrt{\int_{-1}^{1} \left|\int_{-1}^{1} \psi_{j-1}(t)\, f_\varepsilon(t+\tau)\; dt\right|^2 d\tau} \\ &\le \left|\frac{1}{\lambda_{j-1}\lambda_{k-1}}\right| \sqrt{\int_{-1}^{1} \|\psi_{j-1}\|^2 \int_{-1}^{1} |f_\varepsilon(t+\tau)|^2\; dt\; d\tau} \\ &= \left|\frac{1}{\lambda_{j-1}\lambda_{k-1}}\right| \sqrt{\int_{-1}^{1}\int_{-1}^{1} |f_\varepsilon(t+\tau)|^2\; dt\; d\tau} \\ &< \left|\frac{2\varepsilon}{\lambda_{j-1}\lambda_{k-1}}\right|. \end{aligned} \tag{4.73}$$

□

It can be easily seen from Theorem 2.9 that the number N of eigenvalues needed for a bandwidth of $2c$ and an accuracy of ε^2 is roughly twice the number of eigenvalues needed for a bandwidth of c and an accuracy of ε. Thus a generalized Gaussian quadrature for a bandwidth $2c$ and an accuracy ε^2 has roughly the same number of nodes as those needed for interpolation of accuracy ε and bandwidth c.

In our numerical experiments, this correspondence was found to be much closer than the rough bounds in Theorem 2.9 indicate. In the results tabulated in Chapter 6, the number of nodes for an interpolation formula of a desired accuracy ε was always chosen to be the number of quadrature nodes for a desired accuracy ε^2 for twice the band limit. The correspondence between the desired accuracy and the experimentally measured maximum error can be seen in Tables 6.3 and 6.4.

The coefficients $c_1, c_2, \ldots, c_n$ produced by this interpolation procedure (see (4.61)) can, of course, be used for evaluating derivatives or indefinite integrals of the interpolated function, as they can for computing the function itself.

5

Asymptotic Expansions

In this chapter, we present asymptotic expressions for Prolate Spheroidal Wave Functions and corresponding eigenvalues at large bandwidths. These formulae are built upon a connection between PSWFs and Hermite polynomials, instead of the connection between PSWFs and Legendre polynomials. The latter is the basis of the classical evaluation scheme presented in Chapter 3. Nevertheless, the underlying procedure for constructing the asymptotic formulae of this chapter is a diagonalization process performed on symmetric tridiagonal matrices, as is the case in the classical scheme.

This chapter is organized as follows. In Section 5.1, we introduce the five-term recursion that connects a Prolate Spheroidal Wave Function to an arbitrary set of Hermite functions. We then show that a specific choice of Hermite functions reduces the five-term recursion to a three-term recursion. In Section 5.2, we analyze the resulting three-term recursion, and introduce several analytical facts which we will use in the construction of the formulae of this chapter. We present the obtained asymptotic formulae in Sections 5.3–5.5. Section 5.3 and Section 5.4 contain two sets of formulae for PSWFs, differ from each other by a scaling factor. Section 5.4 also include formulae of a slightly higher order for ψ_0 and ψ_1. We present the formulae for corresponding eigenvalues of equation (2.47) in Section 5.5.

eigenvalue eigenvalue

5.1 Recursion Relations

We start with revisiting the differential equation (2.47), and introduce the differential operator G_c as follows: given a real $c > 0$, G_c is defined by the formula

$$G_c(\psi)(x) = (1 - x^2)\,\psi''(x) - 2\,x\,\psi'(x) - c^2\,x^2\,\psi(x)\,. \tag{5.1}$$

As is easily seen, G_c is linear and self-adjoint. The following theorem is an immediate consequence of Theorems 2.2, 2.3, 2.4.

Theorem 5.1 *Suppose that $a \neq 0$ is real, and that $\{\phi_j^a\}$ are the sequence of Hermite functions for the weight function $e^{-a^2x^2}$ (defined in (2.30)). Then, for any non-negative real number c and any integer $n \geq 0$,*

$$G_c(\phi_n^a)\,(x) \;=\; \sum_{i=-2}^{2} d_{n,i}\cdot\phi_{n+2i}^a(x)\,, \tag{5.2}$$

where

$$d_{n,-2} \;=\; -\frac{1}{4}\sqrt{-3+n}\,\sqrt{-2+n}\,\sqrt{-n+n^2}\,, \tag{5.3}$$

$$d_{n,-1} \;=\; \frac{1}{2\,a^2}\left(a^2-c\right)\left(a^2+c\right)\sqrt{-n+n^2}\,, \tag{5.4}$$

$$d_{n,0} \;=\; -\frac{1}{4\,a^2}\left(-3\,a^2+2\,a^4+2\,c^2-2\,a^2\,n+4\,a^4\,n+4\,c^2\,n-2\,a^2\,n^2\right), \tag{5.5}$$

$$d_{n,1} \;=\; +\frac{1}{2\,a^2}\left(a^2-c\right)\left(a^2+c\right)\sqrt{2+3\,n+n^2}\,, \tag{5.6}$$

$$d_{n,2} \;=\; -\frac{1}{4}\sqrt{3+n}\,\sqrt{4+n}\,\sqrt{2+3\,n+n^2}\,, \tag{5.7}$$

for $n \geq 4$, and

$$d_{3,-2} \;=\; d_{2,-2} = 0\,, \tag{5.8}$$

$$d_{1,-2} \;=\; d_{0,-2} = 0\,, \tag{5.9}$$

$$d_{1,-1} \;=\; d_{0,-1} = 0\,. \tag{5.10}$$

Suppose now that ψ_m is an eigenfunction of G_c. Obviously, for all $a \neq 0$, ψ_m can be expanded in the Hermite series given by the formula

$$\psi_m(x) = \sum_{n=0}^{\infty} \alpha_n^m \, \phi_n^a(x) \tag{5.11}$$

(see Theorem 2.5). The following theorem gives the five-term recursion for the coefficients α_n^m, and is the principal purpose of this section.

Theorem 5.2 *Suppose that χ_m, ψ_m are the m-th eigenvalue and eigenfunction of the differential operator G_c. Suppose further that $\alpha_0^m, \alpha_1^m, \alpha_2^m, \ldots$ are the coefficients of the expansion (5.11). Then, for all $n \geq 4$, α_n^m satisfy the five-term recursion*

$$\begin{aligned}
&-\frac{1}{4}\sqrt{-3+n}\,\sqrt{-2+n}\,\sqrt{-n+n^2}\cdot\alpha_{n-4}^m \\
&+\frac{1}{2\,a^2}\left(a^2-c\right)\left(a^2+c\right)\sqrt{-n+n^2}\cdot\alpha_{n-2}^m \\
&+\left(\chi_m-\frac{1}{4\,a^2}\left(-3\,a^2+2\,a^4+2\,c^2-2\,a^2\,n+4\,a^4\,n+4\,c^2\,n-2\,a^2\,n^2\right)\right)\cdot\alpha_n^m \\
&+\frac{1}{2\,a^2}\left(a^2-c\right)\left(a^2+c\right)\sqrt{2+3\,n+n^2}\cdot\alpha_{n+2}^m \\
&-\frac{1}{4}\sqrt{3+n}\,\sqrt{4+n}\,\sqrt{2+3\,n+n^2}\cdot\alpha_{n+4}^m=0\,,
\end{aligned} \tag{5.12}$$

where c is the bandwidth parameter in (2.47). Furthermore,

$$\begin{aligned}
&\left(\chi_m-\frac{1}{4\,a^2}\left(-3\,a^2+2\,a^4+2\,c^2\right)\right)\cdot\alpha_0^m \\
&+\frac{1}{\sqrt{2}\,a^2}\left(a^2-c\right)\left(a^2+c\right)\cdot\alpha_2^m-\frac{1}{2}\sqrt{6}\cdot\alpha_4^m=0\,,
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
&\left(\chi_m-\frac{1}{4\,a^2}\left(-7\,a^2+6\,a^4+6\,c^2\right)\right)\cdot\alpha_1^m \\
&+\frac{1}{2\,a^2}\left(a^2-c\right)\left(a^2+c\right)\sqrt{6}\cdot\alpha_3^m-\frac{1}{2}\sqrt{30}\cdot\alpha_5^m=0\,,
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
&\frac{1}{\sqrt{2}\,a^2}\left(a^2-c\right)\left(a^2+c\right)\cdot\alpha_0^m \\
&+\left(\chi_m-\frac{1}{4\,a^2}\left(-15\,a^2+10\,a^4+10\,c^2\right)\right)\cdot\alpha_2^m \\
&+\frac{1}{a^2}\left(a^2-c\right)\left(a^2+c\right)\sqrt{3}\cdot\alpha_4^m-\frac{3}{2}\sqrt{20}\cdot\alpha_6^m=0\,,
\end{aligned} \tag{5.15}$$

$$\frac{1}{2a^2}\left(a^2-c\right)\left(a^2+c\right)\sqrt{6}\cdot\alpha_1^m$$
$$+\left(\chi_m-\frac{1}{4a^2}\left(-27a^2+14a^4+14c^2\right)\right)\cdot\alpha_3^m$$
$$+\frac{1}{2a^2}\left(a^2-c\right)\left(a^2+c\right)\sqrt{20}\cdot\alpha_5^m-\frac{1}{2}\sqrt{210}\cdot\alpha_7^m=0\,. \tag{5.16}$$

Proof. Applying G_c to both sides of (5.11), we obtain

$$G_c(\psi_m)(x)=G_c\left(\sum_{n=0}^{\infty}\alpha_n^m\,\phi_n^a(x)\right)\,. \tag{5.17}$$

Swapping the orders of the sum and G_c, we have

$$G_c(\psi_m)(x)=\sum_{n=0}^{\infty}\alpha_n^m\,G_c(\phi_n^a)(x)\,. \tag{5.18}$$

Due to Theorem 5.1, we have that for all $n\geq 0$,

$$G_c(\phi_n^a)(x)=\sum_{i=-2}^{2}d_{n,i}\cdot\phi_{n+2i}^a(x)\,, \tag{5.19}$$

where $d_{n,i}$ are given by (5.3)–(5.10). Substituting (5.19) into (5.18), we get

$$G_c(\psi_m)(x)=\sum_{n=0}^{\infty}\alpha_n^m\sum_{i=-2}^{2}d_{n,i}\cdot\phi_{n+2i}^a(x)\,, \tag{5.20}$$

which is equivalent to

$$G_c(\psi_m)(x)=\sum_{n=0}^{\infty}\left(\sum_{i=-2}^{2}\alpha_{n-2i}^m\,d_{n-2i,i}\right)\cdot\phi_n^a(x)\,, \tag{5.21}$$

with the additional definitions that $d_{-4,i}=d_{-3,i}=d_{-2,i}=d_{-1,i}=0$ for all $-2\leq i\leq 2$, and that $\alpha_{-4}^m=\alpha_{-3}^m=\alpha_{-2}^m=\alpha_{-1}^m=0$. On the other hand, we have

$$G_c(\psi_m)(x)=-\sum_{n=0}^{\infty}\alpha_n^m\,\chi_m\cdot\phi_n^a(x)\,. \tag{5.22}$$

Comparing the corresponding coefficients for $\phi_n^a(x)$ in (5.21) and (5.22), we obtain the recursion (5.12). Repeating the above exercise and setting $n=0,1,2,3$ respectively, we obtain formulae (5.13) through (5.16). □

Introducing the notation μ^m as the vector in l^2 defined by the formula

$$\mu^m = (\alpha_0^m, \alpha_1^m, \alpha_2^m, \ldots), \tag{5.23}$$

we restate Theorem 5.2 in a slightly different form below.

Theorem 5.3 *Suppose that B is a symmetric matrix defined by the formulae*

$$b_{n,n} = -\frac{1}{4\,a^2}\left(-3\,a^2 + 2\,a^4 + 2\,c^2 - 2\,a^2\,n + 4\,a^4\,n + 4\,c^2\,n - 2\,a^2\,n^2\right), \tag{5.24}$$

$$b_{n,n+2} = \frac{1}{2\,a^2}\left(a^2 - c\right)\left(a^2 + c\right)\sqrt{2 + 3\,n + n^2}, \tag{5.25}$$

$$b_{n+2,n} = \frac{1}{2\,a^2}\left(a^2 - c\right)\left(a^2 + c\right)\sqrt{2 + 3\,n + n^2}, \tag{5.26}$$

$$b_{n,n+4} = -\frac{1}{4}\sqrt{(n+1)(n+2)(n+3)(n+4)}, \tag{5.27}$$

$$b_{n+4,n} = -\frac{1}{4}\sqrt{(n+1)(n+2)(n+3)(n+4)}, \tag{5.28}$$

for all $n = 0, 1, 2, \ldots$, where all other entries are zero. Suppose further that χ_m is an eigenvalue of G_c, and that μ^m is the vector defined by (5.23) and (5.11). Then $-\chi_m$ and μ^m are eigenvalue and corresponding eigenvector of B, for all integers $m \geq 0$.

In other words, in the basis consisting of the functions $\phi_0^a, \phi_1^a, \phi_2^a, \ldots$, the differential equation (2.47) has the form

$$(B + \chi_m \cdot I) \cdot \mu^m = 0\,. \tag{5.29}$$

An inspection of the formulae (5.24)–(5.28) immediately yields the following observation.

Observation 5.4 *Suppose that*

$$a^2 = c\,. \tag{5.30}$$

Then the matrix B defined by (5.24)–(5.28) is tridiagonal.

Combining Observation 5.4 with Theorem 5.3, we introduce the tridiagonal matrix B^c below, which is the matrix representation of G_c in the basis consisting of the Hermite functions $\phi_0^{\sqrt{c}}, \phi_1^{\sqrt{c}}, \phi_2^{\sqrt{c}}, \ldots,$.

Definition 5.1 *We defined the tridiagonal matrix $B^c = (b^c_{i,j})$ by the formulae*

$$b^c_{n,n} = -(2n+1)\,c + \frac{1}{4}\left(3 + 2\,n + 2\,n^2\right), \tag{5.31}$$

$$b^c_{n,n+4} = -\frac{1}{4}\sqrt{(n+1)(n+2)(n+3)(n+4)}, \tag{5.32}$$

$$b^c_{n+4,n} = -\frac{1}{4}\sqrt{(n+1)(n+2)(n+3)(n+4)}, \tag{5.33}$$

where $n = 0, 1, 2, \ldots$, and the remainder of the entries being zero.

5.2 Construction of Asymptotic Expansions

We observe that all diagonal elements of B^c are proportional to c, while all subdiagonal elements are independent of c. Dividing all entries of B^c by c, we introduce the tridiagonal matrix T defined by the formula

$$T = \frac{1}{c} \cdot B^c. \tag{5.34}$$

Clearly, if (χ_m, μ^m) is a pair of eigenvalue and eigenvector of B, then $(\frac{\chi_m}{c}, \mu^m)$ is a pair of eigenvalue and eigenvector of T. In other words, in the basis consisting of the functions $\phi_0^{\sqrt{c}}, \phi_1^{\sqrt{c}}, \phi_2^{\sqrt{c}}, \ldots$, the differential equation (2.47) can be written in the form

$$\left(T + \frac{\chi_m}{c} \cdot I\right) \cdot \mu^m = 0. \tag{5.35}$$

Now, all entries of T (defined in (5.34)) depend on $\frac{1}{c}$. To emphasize this dependency, we introduce the notation

$$h = \frac{1}{c}, \tag{5.36}$$

and denote by $T(h)$ the matrix T in the remainder of this chapter.

Observation 5.5 *For $h = 0$ (that is, for $c = \infty$), T(0) is diagonal. Under this condition, the diagonal elements $t_{0,0}, t_{1,1}, \ldots, t_{n,n}, \ldots$, which assume the values of $-1, -3, -5, \ldots$, $-(2\,n+1), \ldots$, respectively, are the eigenvalues and the only eigenvalues of $T(0)$. In other* *words, the m-th Prolate Spheroidal Wave Function ψ_m projects only on the m-th Hermite Functions $\phi_m^{\sqrt{c}}$ for $c = \infty$; all projections of ψ_m on other Hermite Functions are zero. For all $0 < c < \infty$, $T(h)$ is tridiagonal.*

Obviously, $T(h)$ can be viewed as a perturbation from $T(0)$ when h is sufficiently small. The theorem below summarizes several well-known facts in the classical perturbation theory pertaining to the linear operator T. A proof (of a more general form) can be found in any classical treatise on perturbation theory for linear operators, for example [15].

We first introduce a notation for describing certain finite submatrices of $T(h)$. Supposing that integers μ, ν are such that $0 < \mu < \nu$, and denoting the (i,j)-th element of $T(h)$ by $t_{i,j}$, we define $T^{\mu,\nu}(h)$ as the $(\nu-\mu+1)$ by $(\nu-\mu+1)$ submatrix of $T(h)$ given by the formula

$$T^{\mu,\nu}(h) = \begin{bmatrix} t_{\mu,\mu} & t_{\mu,\mu+1} & & & \\ t_{\mu+1,\mu} & t_{\mu+1,\mu+1} & t_{\mu+1,\mu+2} & & \\ & t_{\mu+2,\mu+1} & \cdot & \cdot & \\ & & \cdot & t_{\nu-1,\nu-1} & t_{\nu-1,\nu} \\ & & & t_{\nu,\nu-1} & t_{\nu,\nu} \end{bmatrix}. \tag{5.37}$$

Theorem 5.6 *Suppose that $T(h)$ is the linear operator defined in (5.34) and (5.31)-(5.33). Then, for all integers $0 \le m_1 \le m_2$ and sufficiently small h, $T^{m_1,m_2}(h)$ has $(m_2 - m_1 + 1)$ isolated eigenvalues $\chi_0(h) < \chi_1(h) < \ldots < \chi_{m_2-m_1}$. Moreover, for all integers $0 \le i \le m_2 - m_1$, $\chi_i(h)$ is holomorphic in a neighborhood of $h = 0$. Furthermore, for all integers $0 \le i \le m_2 - m_1$, the corresponding eigenvector $\mu_i(h) \in l^{m_2-m_1+1}$ is holomorphic in a neighborhood of $h = 0$.*

Observation 5.7 *An inspection of the formulae (5.31)-(5.33) shows that for all integers $0 < \mu < \nu$ and sufficiently large c, the matrices $T^{\mu,\nu}(h)$ are diagonally dominant. The eigenproblem for such matrices (i.e. symmetric, tridiagonal and diagonally dominant) can be solved effectively using any classical techniques, such as the Inverse Power Method.*

The following two theorems summarize several technical facts regarding the asymptotic formulae we are to construct. We omit the proofs to these theorems, and show the convergence order numerically in Chapter 6.

Theorem 5.8 *Suppose that ψ_m is the m-th eigenfunction of G_c, and that α_n^m are the Hermite expansion coefficients (defined in (5.11)) in the basis $\{\phi_j^{\sqrt{c}}\}$. Then, for all $m, n \geq 0$, if $m \neq n \pmod 4$, then*

$$\alpha_n^m = 0\,; \tag{5.38}$$

if $m = n \pmod 4$, then α_n^m are non-vanishing polynomials of $1/c$. Furthermore, if ψ_m is scaled such that $\alpha_m^m = 1$, then $\alpha_{m+4j}^m = O(c^{-|j|})$, for all $j \geq -[m/4]$ (where $[m/4]$ denotes the integer part of $m/4$).

Theorem 5.9 *Suppose that integers $m, l \geq 0$, h is sufficiently small, and χ_m is the m-th eigenvalue of the differential operator G_c. Suppose that*

$$p = \min(m, l)\,, \tag{5.39}$$

and that $\mu^m = (\alpha_0^m, ..., \alpha_p^m, \ldots, \alpha_m^m, \ldots, \alpha_{m+l}^m, \ldots) \in l^2$ is the vector consisting of coefficients for the Hermite expansion of ψ_m in the basis $\{\phi_j^{\sqrt{c}}\}$ (see (5.23)). Suppose further that the eigenvalues of $T^{m-p,m+l}(h)$ are ordered such that $\tilde{\chi}_{-p}(h) < \tilde{\chi}_{-p+1}(h) < \ldots < \tilde{\chi}_l(h)$, and that the $(p+l+1)$-dimensional vector $(\nu_{-p}(h), \nu_{-p+1}(h), \ldots, \nu_0(h) = 1, \nu_1(h), \ldots, \nu_l(h))$ is an eigenvector corresponding to $\chi_0(h)$. Then,

$$\chi_m - \tilde{\chi}_0(h) = o(h^{2l})\,; \tag{5.40}$$

for all $-p \leq j \leq l$,

$$\alpha_{m+j}^m - \nu_j(h) = o(h^{|j|})\,. \tag{5.41}$$

Theorem 5.9 shows that for sufficiently large c, the appropriate eigenvector of the truncated submatrix $T^{m-p,m+l}(h)$ provides an asymptotic expansion of ψ_m in $\phi_j^{\sqrt{c}}$ to the order of l.

Before we proceed to describe our algorithm of the construction of asymptotic expansions for prolate functions, we introduce one last notation below. Clearly, each element of $T(h)$ only interacts with every fourth elements, therefore $T(h)$ separates into four submatrices, which we will denote by $T^0(h)$, $T^1(h)$, $T^2(h)$ and $T^3(h)$, respectively. These four matrices are defined by the formulae

$$t^0_{i,j} = t_{4i,4j}\,, \tag{5.42}$$

$$t^1_{i,j} = t_{4i+1,4j+1}\,, \tag{5.43}$$

$$t^2_{i,j} = t_{4i+2,4j+2}\,, \tag{5.44}$$

$$t^3_{i,j} = t_{4i+3,4j+3}\,, \tag{5.45}$$

where $i = 0, 1, 2, \ldots,$ and $j = 0, 1, 2, \ldots,$.

Example 5.1 *The 5 by 5 submatrix $T^{0,1,5}(h)$ of $T^0(h)$ is shown below.*

$$\begin{bmatrix} -1+\frac{3h}{4} & -\sqrt{\frac{3}{2}}\cdot h & 0 & 0 & 0 \\ -\sqrt{\frac{3}{2}}\cdot h & -9+\frac{43h}{4} & -\sqrt{105}\cdot h & 0 & 0 \\ 0 & -\sqrt{105}\cdot h & -17+\frac{147h}{4} & -3\sqrt{\frac{165}{2}}\cdot h & 0 \\ 0 & 0 & -3\sqrt{\frac{165}{2}}\cdot h & -25+\frac{315h}{4} & -\sqrt{2730}\cdot h \\ 0 & 0 & 0 & -\sqrt{2730}\cdot h & -33+\frac{547h}{4} \end{bmatrix}. \tag{5.46}$$

Thus, formulae for eigenvalues χ_m and the coefficients α^m_n of the Hermite expansion of ψ_m can be obtained by the following procedure.

1. Separate $T(h)$ into four matrices $T^0(h), T^1(h), T^2(h), T^3(h)$, and choose $T^k(h)$ where $k = m \bmod 4$.
2. Determine the size l of the truncated matrix (see Theorem 5.9) necessary for the required order.
3. Truncate $T^k(h)$ around the m-th diagonal element and create the matrix $T^{k,m-l,m+l}$; if $m < l$, create the matrix $T^{k,0,m+l}$.

4. Compute $\tilde{\chi}_0$, $\nu(h)$ (see Theorem 5.9) using the Inverse Power Method.

5.3 Formulae for Non-Normalized PSWFs

We have implemented the scheme described in the previous section in Mathematica and obtained several new asymptotic expansions. We present these expansions in the next three sections.

This current section and Section 5.4 both consist of formulae for PSWFs. These formulae differ only in the sense of scaling: the coefficients in this section are scaled such that the projection of the m-th prolate spheroidal wave function on the m-th Hermite function $\phi_m^{\sqrt{c}}$ is 1, while the coefficients of the next section are scaled so that the prolate functions have unit norm on the interval $[-1,1]$. We denote the m-th Prolate Spheroidal Wave Function scaled in this section by $\tilde{\psi}_m$, and assume it has an expansion of the form

$$\tilde{\psi}_m(x) = \sum_{i=0}^{n}\sum_{j=0}^{n} \tilde{\alpha}_{i,j} \cdot c^{-j} \cdot \phi_{m+4i}^{\sqrt{c}}(x) + \sum_{i=1}^{p}\sum_{j=0}^{n} \tilde{\beta}_{i,j} \cdot c^{-j} \cdot \phi_{m-4i}^{\sqrt{c}}(x) + O\left(c^{-n-1}\right). \tag{5.47}$$

The number n in (5.47) is often referred to as the order of the expansion; the coefficients $\tilde{\alpha}_{i,j}$ are real constants, and the integer p is the minimum of $[m/4]$ (i.e. the integer part of $m/4$) and n. As mentioned before, the functions $\phi_j^{\sqrt{c}}$ are the Hermite functions corresponding to the weight function e^{-cx^2}.

The following are coefficients of the expansion (5.47), accurate to the fifth order.

$$\tilde{\alpha}_{0,0} = 1 \tag{5.48}$$

$$\tilde{\alpha}_{0,1} = 0 \tag{5.49}$$

$$\tilde{\alpha}_{0,2} = 0 \tag{5.50}$$

$$\tilde{\alpha}_{0,3} = 0 \tag{5.51}$$

$$\tilde{\alpha}_{0,4} = 0 \tag{5.52}$$

$$\tilde{\alpha}_{0,5} = 0 \tag{5.53}$$

$$\tilde{\alpha}_{1,0} = 0 \tag{5.54}$$

$$\tilde{\alpha}_{1,1} = \frac{-1}{32}\sqrt{24+50\,m+35\,m^2+10\,m^3+m^4} \tag{5.55}$$

$$\tilde{\alpha}_{1,2} = \frac{-1}{128}\left((5+2\,m)\;\sqrt{24+50\,m+35\,m^2+10\,m^3+m^4}\right) \tag{5.56}$$

$$\tilde{\alpha}_{1,3} = \frac{-1}{65536}\left(\left(4832+3514\,m+715\,m^2-6\,m^3+m^4\right)\cdot \sqrt{24+50\,m+35\,m^2+10\,m^3+m^4}\right) \tag{5.57}$$

$$\tilde{\alpha}_{1,4} = \frac{-1}{262144}\left(\sqrt{24+50\,m+35\,m^2+10\,m^3+m^4}\;(47200+47662\,m +17277\,m^2+2352\,m^3-33\,m^4+6\,m^5)\right) \tag{5.58}$$

$$\tilde{\alpha}_{1,5} = \frac{-1}{100663296}\left(\sqrt{24+50\,m+35\,m^2+10\,m^3+m^4}\;(53561856 +67091736\,m+33702288\,m^2+8077280\,m^3+828745\,m^4 -14020\,m^5+2806\,m^6-4\,m^7+m^8)\right) \tag{5.59}$$

$$\tilde{\alpha}_{2,0} = 0 \tag{5.60}$$

$$\tilde{\alpha}_{2,1} = 0 \tag{5.61}$$

$$\tilde{\alpha}_{2,2} = \frac{1}{2048}\sqrt{(1+m)\;(2+m)\;(3+m)\;(4+m)}\cdot \sqrt{(5+m)\;(6+m)\;(7+m)\;(8+m)} \tag{5.62}$$

$$\tilde{\alpha}_{2,3} = \frac{1}{4096}\sqrt{(1+m)\;(2+m)\;(3+m)\;(4+m)}\cdot \sqrt{(5+m)\;(6+m)\;(7+m)\;(8+m)}\;(7+2\,m) \tag{5.63}$$

$$\tilde{\alpha}_{2,4} = \frac{1}{3145728}\sqrt{(1+m)\;(2+m)\;(3+m)\;(4+m)}\cdot \sqrt{(5+m)\;(6+m)\;(7+m)\;(8+m)}\cdot \left(18672+9882\,m+1451\,m^2-6\,m^3+m^4\right) \tag{5.64}$$

$$\tilde{\alpha}_{2,5} = \frac{1}{3145728}\sqrt{(1+m)\;(2+m)\;(3+m)\;(4+m)}\cdot \sqrt{(5+m)\;(6+m)\;(7+m)\;(8+m)}\cdot \left(70824+52470\,m+14109\,m^2+1396\,m^3-9\,m^4+2\,m^5\right) \tag{5.65}$$

$$\tilde{\alpha}_{3,0} = 0 \tag{5.66}$$

$$\tilde{\alpha}_{3,1} = 0 \tag{5.67}$$

$$\tilde{\alpha}_{3,2} = 0 \tag{5.68}$$

$$\tilde{\alpha}_{3,3} = \frac{-1}{196608}\left(\sqrt{(1+m)\,(2+m)\,(3+m)\,(4+m)}\cdot \sqrt{(5+m)\,(6+m)\,(7+m)\,(8+m)}\cdot \sqrt{(9+m)\,(10+m)\,(11+m)\,(12+m)}\right) \tag{5.69}$$

$$\tilde{\alpha}_{3,4} = \frac{-1}{262144}\left(\sqrt{(1+m)\,(2+m)\,(3+m)\,(4+m)}\cdot \sqrt{(5+m)\,(6+m)\,(7+m)\,(8+m)}\cdot \sqrt{(9+m)\,(10+m)\,(11+m)\,(12+m)}\,(9+2m)\right) \tag{5.70}$$

$$\tilde{\alpha}_{3,5} = \frac{-1}{268435456}\left(\sqrt{(1+m)\,(2+m)\,(3+m)\,(4+m)}\cdot \sqrt{(5+m)\,(6+m)\,(7+m)\,(8+m)}\cdot \sqrt{(9+m)\,(10+m)\,(11+m)\,(12+m)}\cdot \left(51392+21370\,m+2443\,m^2-6\,m^3+m^4\right)\right) \tag{5.71}$$

$$\tilde{\alpha}_{4,0} = 0 \tag{5.72}$$

$$\tilde{\alpha}_{4,1} = 0 \tag{5.73}$$

$$\tilde{\alpha}_{4,2} = 0 \tag{5.74}$$

$$\tilde{\alpha}_{4,3} = 0 \tag{5.75}$$

$$\tilde{\alpha}_{4,4} = \frac{1}{25165824}\sqrt{(1+m)\,(2+m)\,(3+m)\,(4+m)}\cdot \sqrt{(5+m)\,(6+m)\,(7+m)\,(8+m)}\cdot \sqrt{(9+m)\,(10+m)\,(11+m)\,(12+m)}\cdot \sqrt{(13+m)\,(14+m)\,(15+m)\,(16+m)} \tag{5.76}$$

$$\tilde{\alpha}_{4,5} = \frac{1}{25165824}\sqrt{(1+m)\,(2+m)\,(3+m)\,(4+m)}\cdot \sqrt{(5+m)\,(6+m)\,(7+m)\,(8+m)}\cdot \sqrt{(9+m)\,(10+m)\,(11+m)\,(12+m)}\cdot$$

$$\sqrt{(13+m)\ (14+m)\ (15+m)\ (16+m)}\ (11+2\,m) \tag{5.77}$$

$$\tilde{\alpha}_{5,0} = 0 \tag{5.78}$$

$$\tilde{\alpha}_{5,1} = 0 \tag{5.79}$$

$$\tilde{\alpha}_{5,2} = 0 \tag{5.80}$$

$$\tilde{\alpha}_{5,3} = 0 \tag{5.81}$$

$$\tilde{\alpha}_{5,4} = 0 \tag{5.82}$$

$$\begin{aligned}\tilde{\alpha}_{5,5} = {} & \frac{-1}{4026531840}\Big(\sqrt{(1+m)\ (2+m)\ (3+m)\ (4+m)}\cdot \\ & \sqrt{(5+m)\ (6+m)\ (7+m)\ (8+m)}\cdot \\ & \sqrt{(9+m)\ (10+m)\ (11+m)\ (12+m)}\cdot \\ & \sqrt{(13+m)\ (14+m)\ (15+m)\ (16+m)}\cdot \\ & \sqrt{(17+m)\ (18+m)\ (19+m)\ (20+m)}\Big)\end{aligned} \tag{5.83}$$

$$\tilde{\beta}_{1,0} = 0 \tag{5.84}$$

$$\tilde{\beta}_{1,1} = \frac{1}{32}\sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m} \tag{5.85}$$

$$\tilde{\beta}_{1,2} = \frac{1}{128}\sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m}\ (-3+2\,m) \tag{5.86}$$

$$\begin{aligned}\tilde{\beta}_{1,3} = {} & \frac{1}{65536}\sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m}\ (2040-2062\,m \\ & +739\,m^2+10\,m^3+m^4\big)\end{aligned} \tag{5.87}$$

$$\begin{aligned}\tilde{\beta}_{1,4} = {} & \frac{1}{262144}\sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m}\ (-14424+20326\,m \\ & -9963\,m^2+2544\,m^3+63\,m^4+6\,m^5\big)\end{aligned} \tag{5.88}$$

$$\begin{aligned}\tilde{\beta}_{1,5} = {} & \frac{1}{100663296}\sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m}\ (12940704 \\ & -20517048\,m+14625320\,m^2-4565784\,m^3+941145\,m^4 \\ & +30996\,m^5+2862\,m^6+12\,m^7+m^8\big)\end{aligned} \tag{5.89}$$

$$\tilde{\beta}_{2,0} = 0 \tag{5.90}$$

$$\tilde{\beta}_{2,1} = 0 \tag{5.91}$$

$$\tilde{\beta}_{2,2} = \frac{1}{2048}\sqrt{(-7+m)\,(-6+m)\,(-5+m)\,(-4+m)}\cdot \sqrt{(-3+m)\,(-2+m)\,(-1+m)\,m} \tag{5.92}$$

$$\tilde{\beta}_{2,3} = \frac{1}{4096}\sqrt{(-7+m)\,(-6+m)\,(-5+m)\,(-4+m)}\cdot \sqrt{(-3+m)\,(-2+m)\,(-1+m)\,m}\cdot (-5+2\,m) \tag{5.93}$$

$$\tilde{\beta}_{2,4} = \frac{1}{3145728}\sqrt{(-7+m)\,(-6+m)\,(-5+m)\,(-4+m)}\cdot \sqrt{(-3+m)\,(-2+m)\,(-1+m)\,m}\cdot \left(10248 - 6958\,m + 1475\,m^2 + 10\,m^3 + m^4\right) \tag{5.94}$$

$$\tilde{\beta}_{2,5} = \frac{1}{3145728}\sqrt{(-7+m)\,(-6+m)\,(-5+m)\,(-4+m)}\cdot \sqrt{(-3+m)\,(-2+m)\,(-1+m)\,m}\,(-31056 + 28486\,m -9847\,m^2 + 1452\,m^3 + 19\,m^4 + 2\,m^5\big) \tag{5.95}$$

$$\tilde{\beta}_{3,0} = 0 \tag{5.96}$$

$$\tilde{\beta}_{3,1} = 0 \tag{5.97}$$

$$\tilde{\beta}_{3,2} = 0 \tag{5.98}$$

$$\tilde{\beta}_{3,3} = \frac{1}{196608}\sqrt{(-11+m)\,(-10+m)\,(-9+m)\,(-8+m)}\cdot \sqrt{(-7+m)\,(-6+m)\,(-5+m)\,(-4+m)}\cdot \sqrt{(-3+m)\,(-2+m)\,(-1+m)\,m} \tag{5.99}$$

$$\tilde{\beta}_{3,4} = \frac{1}{262144}\sqrt{(-11+m)\,(-10+m)\,(-9+m)\,(-8+m)}\cdot \sqrt{(-7+m)\,(-6+m)\,(-5+m)\,(-4+m)}\cdot \sqrt{(-3+m)\,(-2+m)\,(-1+m)\,m}\,(-7+2\,m) \tag{5.100}$$

$$\tilde{\beta}_{3,5} = \frac{1}{268435456}\sqrt{(-11+m)\,(-10+m)\,(-9+m)\,(-8+m)}\cdot \sqrt{(-7+m)\,(-6+m)\,(-5+m)\,(-4+m)}\cdot \sqrt{(-3+m)\,(-2+m)\,(-1+m)\,m}\cdot$$

$$\left(32472 - 16462\, m + 2467\, m^2 + 10\, m^3 + m^4\right) \tag{5.101}$$

$$\tilde{\beta}_{4,0} = 0 \tag{5.102}$$

$$\tilde{\beta}_{4,1} = 0 \tag{5.103}$$

$$\tilde{\beta}_{4,2} = 0 \tag{5.104}$$

$$\tilde{\beta}_{4,3} = 0 \tag{5.105}$$

$$\begin{aligned}\tilde{\beta}_{4,4} = {} & \frac{1}{25165824} \sqrt{(-15+m)\,(-14+m)\,(-13+m)\,(-12+m)} \cdot \\ & \sqrt{(-11+m)\,(-10+m)\,(-9+m)\,(-8+m)} \cdot \\ & \sqrt{(-7+m)\,(-6+m)\,(-5+m)\,(-4+m)} \cdot \\ & \sqrt{(-3+m)\,(-2+m)\,(-1+m)\,m}\end{aligned} \tag{5.106}$$

$$\begin{aligned}\tilde{\beta}_{4,5} = {} & \frac{1}{25165824} \sqrt{(-15+m)\,(-14+m)\,(-13+m)\,(-12+m)} \cdot \\ & \sqrt{(-11+m)\,(-10+m)\,(-9+m)\,(-8+m)} \cdot \\ & \sqrt{(-7+m)\,(-6+m)\,(-5+m)\,(-4+m)} \cdot \\ & \sqrt{(-3+m)\,(-2+m)\,(-1+m)\,m}\,(-9+2\,m)\end{aligned} \tag{5.107}$$

$$\tilde{\beta}_{5,0} = 0 \tag{5.108}$$

$$\tilde{\beta}_{5,1} = 0 \tag{5.109}$$

$$\tilde{\beta}_{5,2} = 0 \tag{5.110}$$

$$\tilde{\beta}_{5,3} = 0 \tag{5.111}$$

$$\tilde{\beta}_{5,4} = 0 \tag{5.112}$$

$$\begin{aligned}\tilde{\beta}_{5,5} = {} & \frac{1}{4026531840} \sqrt{(-19+m)\,(-18+m)\,(-17+m)\,(-16+m)} \cdot \\ & \sqrt{(-15+m)\,(-14+m)\,(-13+m)\,(-12+m)} \cdot \\ & \sqrt{(-11+m)\,(-10+m)\,(-9+m)\,(-8+m)} \cdot \\ & \sqrt{(-7+m)\,(-6+m)\,(-5+m)\,(-4+m)} \cdot \\ & \sqrt{(-3+m)\,(-2+m)\,(-1+m)\,m}\end{aligned} \tag{5.113}$$

5.4 Formulae for Normalized PSWFs

In many situations (for example, in quadrature design), it is necessary to normalize the Prolate Spheroidal Wave Functions such that their L^2 norms on some specific interval are 1. In this section, we present formulae for PSWFs ψ_m after such a scaling. To be more specific, we assume that

$$\|\psi_m\|_{[-1,1]} = 1\,, \tag{5.114}$$

and that ψ_m is expanded in Hermite functions via the formula

$$\psi_m(x) = \sum_{i=0}^{n} \alpha_i \cdot \phi_{m+4i}^{\sqrt{c}}(x) + \sum_{i=1}^{p} \beta_i \cdot \phi_{m-4i}^{\sqrt{c}}(x) + O\left(c^{-n-1}\right). \tag{5.115}$$

The expansion coefficients α_i, β_i here are polynomials of $1/c$ up to the order n ; the summation limit p is defined by the formula

$$p = \min([m/4], n)\,, \tag{5.116}$$

where $[m/4]$ denotes the integer part of $m/4$. The following theorem establishes the basis for the computation of the expansion coefficients α_i, β_i of (5.115) via the coefficients of formula (5.47).

Theorem 5.10 *Suppose that $m, n \geq 0$ are integers, p is defined by (5.116) and that i, j are integers such that $0 \leq i \leq p$, $1 \leq j \leq n$. Suppose further that $\tilde{\alpha}_{i,k}, \tilde{\beta}_{j,k}$ are coefficients of the n-th order asymptotic expansion (5.47) for $\tilde{\psi}_m$, and that α_i, β_j are defined by the formulae*

$$\alpha_i = \frac{\tilde{\alpha}_i}{l}\,, \tag{5.117}$$

$$\beta_j = \frac{\tilde{\beta}_j}{l}\,, \tag{5.118}$$

where

$$l = \sqrt{\sum_{i=0}^{n} (\tilde{\alpha}_i)^2 + \sum_{i=1}^{p} \left(\tilde{\beta}_i\right)^2}\,, \tag{5.119}$$

$$\tilde{\alpha}_i = \sum_{j=0}^{n} \tilde{\alpha}_{i,j} \cdot c^{-j} , \tag{5.120}$$

$$\tilde{\beta}_j = \sum_{k=0}^{n} \tilde{\beta}_{j,k} \cdot c^{-k} . \tag{5.121}$$

Then, α_i, β_j are coefficients of the n-th order asymptotic expansion (5.115) for ψ_m.

Proof. Clearly,

$$\psi_m(x) = \frac{\tilde{\psi}_m(x)}{\|\tilde{\psi}_m(x)\|_{[-1,1]}} ; \tag{5.122}$$

formulae (5.117), (5.118) follow immediately due to Lemma 2.7.

□

The formulae for α_i, β_i are presented in the next three sections.

5.4.1 Normalized Formula for ψ_0

The Hermite expansion of the zero-th Prolate Spheroidal Wave Function ψ_0 contains no β_i coefficients, since $p = 0$ (see 5.116). The following are the coefficients α_i to the twelfth order. The symbol h denotes $1/c$ throughout this section.

$$\begin{aligned} \alpha_0 &= 1 - \frac{3}{256} h^2 \left(1 + \frac{5\,h}{2} + \frac{3417\,h^2}{512} + \frac{10345\,h^3}{512} + \frac{9070831\,h^4}{131072} + \frac{69656505\,h^5}{262144} \right. \\ &\quad + \frac{152146864319\,h^6}{134217728} + \frac{357807379735\,h^7}{67108864} + \frac{943629416863287\,h^8}{34359738368} \\ &\quad \left. + \frac{10582827757169135\,h^9}{68719476736} + \frac{16450747629554491685\,h^{10}}{17592186044416} \right) \end{aligned} \tag{5.123}$$

$$\begin{aligned} \alpha_1 &= \frac{-1}{8} \sqrt{\frac{3}{2}}\, h \left(1 + \frac{5\,h}{4} + \frac{601\,h^2}{256} + \frac{5855\,h^3}{1024} + \frac{2213069\,h^4}{131072} + \frac{30528325\,h^5}{524288} \right. \\ &\quad + \frac{7685368511\,h^6}{33554432} + \frac{135697600405\,h^7}{134217728} + \frac{170020042796819\,h^8}{34359738368} \\ &\quad + \frac{3655990749738775\,h^9}{137438953472} + \frac{1371061264355155055\,h^{10}}{8796093022208} \\ &\quad \left. + \frac{348027307708850847 85\,h^{11}}{35184372088832} \right) \end{aligned} \tag{5.124}$$

$$\alpha_2 = \frac{3}{128}\sqrt{\frac{35}{2}}\,h^2\left(1+\frac{7\,h}{2}+\frac{3109\,h^2}{256}+\frac{5893\,h^3}{128}+\frac{25298213\,h^4}{131072}+\frac{233987741\,h^5}{262144}\right.$$
$$+\frac{151942845467\,h^6}{33554432}+\frac{840588201457\,h^7}{33554432}+\frac{5163818737175267\,h^8}{34359738368}$$
$$\left.+\frac{66840158761504577\,h^9}{68719476736}+\frac{59457238265913265811\,h^{10}}{8796093022208}\right) \tag{5.125}$$

$$\alpha_3 = \frac{-15}{2048}\sqrt{231}\,h^3\left(1+\frac{27\,h}{4}+\frac{9633\,h^2}{256}+\frac{212349\,h^3}{1024}+\frac{155542269\,h^4}{131072}\right.$$
$$+\frac{3760444371\,h^5}{524288}+\frac{1545406615431\,h^6}{33554432}+\frac{42227250366039\,h^7}{134217728}$$
$$\left.+\frac{78501712973628723\,h^8}{34359738368}+\frac{2420049703810264281\,h^9}{137438953472}\right) \tag{5.126}$$

$$\alpha_4 = \frac{315}{32768}\sqrt{\frac{715}{2}}\,h^4\left(1+11\,h+\frac{23053\,h^2}{256}+\frac{349327\,h^3}{512}\right.$$
$$+\frac{672206933\,h^4}{131072}+\frac{1290151807\,h^5}{32768}+\frac{10500275949635\,h^6}{33554432}$$
$$\left.+\frac{173939724085367\,h^7}{67108864}+\frac{771001386332965379\,h^8}{34359738368}\right) \tag{5.127}$$

$$\alpha_5 = \frac{-945}{524288}\sqrt{46189}\,h^5\left(1+\frac{65\,h}{4}+\frac{47017\,h^2}{256}+\frac{1877195\,h^3}{1024}\right.$$
$$+\frac{2298838829\,h^4}{131072}+\frac{87502840945\,h^5}{524288}+\frac{54021922328207\,h^6}{33554432}$$
$$\left.+\frac{2135191974295465\,h^7}{134217728}\right) \tag{5.128}$$

$$\alpha_6 = \frac{10395}{8388608}\sqrt{676039}\,h^6\left(1+\frac{45\,h}{2}+\frac{85941\,h^2}{256}+\frac{1093425\,h^3}{256}\right.$$
$$\left.+\frac{6645913989\,h^4}{131072}+\frac{153626611995\,h^5}{262144}+\frac{226418271225771\,h^6}{33554432}\right) \tag{5.129}$$

$$\alpha_7 = \frac{-2027025}{67108864}\sqrt{\frac{22287}{2}}\,h^7\left(1+\frac{119\,h}{4}+\frac{145009\,h^2}{256}+\frac{9164489\,h^3}{1024}\right.$$

$$+\frac{16923941213\,h^4}{131072}+\frac{934327827487\,h^5}{524288}\Bigg) \tag{5.130}$$

$$\begin{aligned}\alpha_8 &= \frac{6081075}{2147483648}\sqrt{\frac{33393355}{2}}\,h^8\left(1+38\,h+\frac{230173\,h^2}{256}+\frac{8843917\,h^3}{512}\right.\\ &\left.+\frac{39019921589\,h^4}{131072}\right)\end{aligned} \tag{5.131}$$

$$\alpha_9 = \frac{-172297125}{34359738368}\sqrt{90751353}\,h^9\left(1+\frac{189\,h}{4}+\frac{348153\,h^2}{256}+\frac{31970679\,h^3}{1024}\right) \tag{5.132}$$

$$\alpha_{10} = \frac{1964187225}{549755813888}\sqrt{3829070245}\,h^{10}\left(1+\frac{115\,h}{2}+\frac{506437\,h^2}{256}\right) \tag{5.133}$$

$$\alpha_{11} = \frac{-13749310575}{4398046511104}\sqrt{\frac{263012370465}{2}}\,h^{11}\left(1+\frac{275\,h}{4}\right) \tag{5.134}$$

$$\alpha_{12} = \frac{4743512148375}{140737488355328}\sqrt{35830670759}\,h^{12} \tag{5.135}$$

5.4.2 Normalized Formula for ψ_1

As in the previous section, no β_i coefficients are present in the Hermite expansion of ψ_1. The following are coefficients of the expansion to the twelfth order, with h again denoting $1/c$.

$$\begin{aligned}\alpha_0 &= 1-\frac{15\,h^2}{256}\left(1+\frac{7\,h}{2}+\frac{6429\,h^2}{512}+\frac{25319\,h^3}{512}+\frac{28065463\,h^4}{131072}+\frac{265524987\,h^5}{262144}\right.\\ &+\frac{698069990147\,h^6}{134217728}+\frac{1934996005313\,h^7}{67108864}+\frac{5904057036771207\,h^8}{34359738368}\\ &\left.+\frac{75369254618179333\,h^9}{68719476736}+\frac{131493205293826048457\,h^{10}}{17592186044416}\right)\end{aligned} \tag{5.136}$$

$$\begin{aligned}\alpha_1 \;=\;& \frac{-1}{8}\sqrt{\frac{15}{2}}\,h\left(1+\frac{7\,h}{4}+\frac{1117\,h^2}{256}+\frac{13993\,h^3}{1024}+\frac{6624677\,h^4}{131072}+\frac{111799247\,h^5}{524288}\right.\\ &+\frac{33704624963\,h^6}{33554432}+\frac{699057017771\,h^7}{134217728}+\frac{1011316141666979\,h^8}{34359738368}\\ &+\frac{24732042380045021\,h^9}{137438953472}+\frac{10410060698312917787\,h^{10}}{8796093022208}\\ &\left.+\frac{293232914038243737047\,h^{11}}{35184372088832}\right)\end{aligned} \tag{5.137}$$

$$\begin{aligned}\alpha_2 \;=\;& \frac{9}{128}\sqrt{\frac{35}{2}}\,h^2\left(1+\frac{9\,h}{2}+\frac{4985\,h^2}{256}+\frac{5753\,h^3}{64}+\frac{58923133\,h^4}{131072}+\frac{638386595\,h^5}{262144}\right.\\ &+\frac{477800166223\,h^6}{33554432}+\frac{3003575504521\,h^7}{33554432}+\frac{20706560133141235\,h^8}{34359738368}\\ &\left.+\frac{297552339017127023\,h^9}{68719476736}+\frac{291117166121432916463\,h^{10}}{8796093022208}\right)\end{aligned} \tag{5.138}$$

$$\begin{aligned}\alpha_3 \;=\;& \frac{-15}{2048}\sqrt{3003}\,h^3\left(1+\frac{33\,h}{4}+\frac{14085\,h^2}{256}+\frac{364803\,h^3}{1024}+\frac{309051093\,h^4}{131072}\right.\\ &+\frac{8522527905\,h^5}{524288}+\frac{3946521040683\,h^6}{33554432}+\frac{120213842316609\,h^7}{134217728}\\ &\left.+\frac{246811766645833155\,h^8}{34359738368}+\frac{8334751311293039691\,h^9}{137438953472}\right)\end{aligned} \tag{5.139}$$

$$\begin{aligned}\alpha_4 \;=\;& \frac{315}{32768}\sqrt{\frac{12155}{2}}\,h^4\left(1+13\,h+\frac{31681\,h^2}{256}+\frac{550481\,h^3}{512}+\frac{1199798189\,h^4}{131072}\right.\\ &+\frac{645014009\,h^5}{8192}+\frac{23303419676567\,h^6}{33554432}+\frac{424809328266721\,h^7}{67108864}\\ &\left.+\frac{2056902054145938707\,h^8}{34359738368}\right)\end{aligned} \tag{5.140}$$

$$\begin{aligned}\alpha_5 \;=\;& \frac{-945}{524288}\sqrt{969969}\,h^5\left(1+\frac{75\,h}{4}+\frac{61805\,h^2}{256}+\frac{2780045\,h^3}{1024}\right.\\ &+\frac{3797895685\,h^4}{131072}+\frac{159867137155\,h^5}{524288}+\frac{108306606244435\,h^6}{33554432}\\ &\left.+\frac{4665516063967975\,h^7}{134217728}\right)\end{aligned} \tag{5.141}$$

$$\alpha_6 = \frac{51975}{8388608}\sqrt{676039}\,h^6\left(1+\frac{51\,h}{2}+\frac{109257\,h^2}{256}+\frac{1545177\,h^3}{256}\right.$$
$$\left.+\frac{10355608797\,h^4}{131072}+\frac{262058558181\,h^5}{262144}+\frac{420122902825503\,h^6}{33554432}\right) \quad (5.142)$$

$$\alpha_7 = \frac{-2027025}{67108864}\sqrt{\frac{646323}{2}}\,h^7\left(1+\frac{133\,h}{4}+\frac{179605\,h^2}{256}+\frac{12484423\,h^3}{1024}\right.$$
$$\left.+\frac{25186533173\,h^4}{131072}+\frac{1509916948085\,h^5}{524288}\right) \quad (5.143)$$

$$\alpha_8 = \frac{6081075}{2147483648}\sqrt{\frac{1101980715}{2}}\,h^8\left(1+42\,h+\frac{279185\,h^2}{256}+\frac{11697259\,h^3}{512}\right.$$
$$\left.+\frac{55955374093\,h^4}{131072}\right) \quad (5.144)$$

$$\alpha_9 = \frac{-172297125}{34359738368}\sqrt{3357800061}\,h^9\left(1+\frac{207\,h}{4}+\frac{415101\,h^2}{256}\right.$$
$$\left.+\frac{41270577\,h^3}{1024}\right) \quad (5.145)$$

$$\alpha_{10} = \frac{1964187225}{549755813888}\sqrt{156991880045}\,h^{10}\left(1+\frac{125\,h}{2}+\frac{595225\,h^2}{256}\right) \quad (5.146)$$

$$\alpha_{11} = \frac{-206239658625}{4398046511104}\sqrt{\frac{52602474093}{2}}\,h^{11}\left(1+\frac{297\,h}{4}\right) \quad (5.147)$$

$$\alpha_{12} = \frac{33204585038625}{140737488355328}\sqrt{35830670759}\,h^{12} \quad (5.148)$$

5.4.3 Normalized Formula for ψ_m

The general formula for the Hermite expansion of ψ_m at large band-limit c is given in this section. The expansion is assumed to have the form

$$\psi_m(x) = \sum_{i=0}^{n}\sum_{j=0}^{n} \alpha_{i,j} \cdot \phi_{m+4i}^{\sqrt{c}}(x) \cdot c^{-j} + \sum_{i=1}^{p}\sum_{j=1}^{n} \beta_{i,j} \cdot \phi_{m-4i}^{\sqrt{c}}(x) \cdot c^{-j}, \tag{5.149}$$

where

$$p = \min([m/4], n), \tag{5.150}$$

and $[m/4]$ denotes the integer part of $m/4$. The coefficients $\alpha_{i,j}$, $\beta_{i,j}$ are given to the fifth order below.

$$\alpha_{0,0} = 1 \tag{5.151}$$

$$\alpha_{0,1} = 0 \tag{5.152}$$

$$\alpha_{0,2} = \frac{1}{1024}(-12 - 22\,m - 23\,m^2 - 2\,m^3 - m^4) \tag{5.153}$$

$$\alpha_{0,3} = \frac{1}{2048}(-60 - 158\,m - 115\,m^2 - 80\,m^3 - 5\,m^4 - 2\,m^5) \tag{5.154}$$

$$\begin{aligned}\alpha_{0,4} = {} & \frac{1}{4194304}(-328032 - 891024\,m - 1127140\,m^2 \\ & -476156\,m^3 - 247887\,m^4 - 11768\,m^5 - 3918\,m^6 \\ & +4\,m^7 + m^8)\end{aligned} \tag{5.155}$$

$$\begin{aligned}\alpha_{0,5} = {} & \frac{1}{4194304}(-993120 - 3161552\,m - 3698884\,m^2 \\ & -3044356\,m^3 - 874439\,m^4 - 363350\,m^5 - 13566\,m^6 \\ & -3864\,m^7 + 9\,m^8 + 2\,m^9)\end{aligned} \tag{5.156}$$

$$\alpha_{1,0} = 0 \tag{5.157}$$

$$\alpha_{1,1} = \frac{-\sqrt{24 + 50\,m + 35\,m^2 + 10\,m^3 + m^4}}{32} \tag{5.158}$$

$$\alpha_{1,2} = \frac{-(5 + 2\,m)\,\sqrt{24 + 50\,m + 35\,m^2 + 10\,m^3 + m^4}}{128} \tag{5.159}$$

$$\begin{aligned}\alpha_{1,3} = {} & \frac{1}{65536}\sqrt{(1+m)\,(2+m)\,(3+m)\,(4+m)}\,(-4808 \\ & -3470\,m - 669\,m^2 + 10\,m^3 + m^4\big)\end{aligned} \tag{5.160}$$

$$\alpha_{1,4} = \frac{1}{262144}\sqrt{(1+m)\,(2+m)\,(3+m)\,(4+m)}\,\Big(-46840 \\ -46762\,m - 16499\,m^2 - 1920\,m^3 + 71\,m^4 + 6\,m^5\Big) \tag{5.161}$$

$$\alpha_{1,5} = \frac{-1}{402653184}\Big(\sqrt{(1+m)\,(2+m)\,(3+m)\,(4+m)}\cdot \\ \Big(212454624 + 263405280\,m + 128877012\,m^2 \\ +29276108\,m^3 + 2118049\,m^4 - 151072\,m^5 \\ -11030\,m^6 + 20\,m^7 + m^8\Big)\Big) \tag{5.162}$$

$$\alpha_{2,0} = 0 \tag{5.163}$$

$$\alpha_{2,1} = 0 \tag{5.164}$$

$$\alpha_{2,2} = \frac{1}{2048}\sqrt{(1+m)\,(2+m)\,(3+m)\,(4+m)}\cdot \\ \sqrt{(5+m)\,(6+m)\,(7+m)\,(8+m)} \tag{5.165}$$

$$\alpha_{2,3} = \frac{1}{4096}\sqrt{(1+m)\,(2+m)\,(3+m)\,(4+m)}\cdot \\ \sqrt{(5+m)\,(6+m)\,(7+m)\,(8+m)}\,(7+2\,m) \tag{5.166}$$

$$\alpha_{2,4} = \frac{-1}{6291456}\Big(\sqrt{(1+m)\,(2+m)\,(3+m)\,(4+m)}\cdot \\ \sqrt{(5+m)\,(6+m)\,(7+m)\,(8+m)}\,(-37308 \\ -19698\,m - 2833\,m^2 + 18\,m^3 + m^4)\Big) \tag{5.167}$$

$$\alpha_{2,5} = \frac{1}{3145728}\sqrt{(1+m)\,(2+m)\,(3+m)\,(4+m)}\cdot \\ \sqrt{(5+m)\,(6+m)\,(7+m)\,(8+m)}\,\Big(70716 \\ +52218\,m + 13869\,m^2 + 1291\,m^3 - 21\,m^4 - m^5\Big) \tag{5.168}$$

$$\alpha_{3,0} = 0 \tag{5.169}$$

$$\alpha_{3,1} = 0 \tag{5.170}$$

$$\alpha_{3,2} = 0 \tag{5.171}$$

$$\alpha_{3,3} = \frac{-1}{196608}\Big(\sqrt{(1+m)\,(2+m)\,(3+m)\,(4+m)}\cdot \\ \sqrt{(5+m)\,(6+m)\,(7+m)\,(8+m)}\cdot$$

$$\sqrt{(9+m)\ (10+m)\ (11+m)\ (12+m)}\Big) \tag{5.172}$$

$$\begin{aligned}\alpha_{3,4} &= \frac{-1}{262144}\left(\sqrt{(1+m)\ (2+m)\ (3+m)\ (4+m)}\cdot\right.\\ &\sqrt{(5+m)\ (6+m)\ (7+m)\ (8+m)}\cdot\\ &\left.\sqrt{(9+m)\ (10+m)\ (11+m)\ (12+m)}\ (9+2\,m)\right)\end{aligned} \tag{5.173}$$

$$\begin{aligned}\alpha_{3,5} &= \frac{1}{805306368}\sqrt{(1+m)\ (2+m)\ (3+m)\ (4+m)}\cdot\\ &\sqrt{(5+m)\ (6+m)\ (7+m)\ (8+m)}\cdot\\ &\sqrt{(9+m)\ (10+m)\ (11+m)\ (12+m)}\ (-154128\\ &-64022\,m-7237\,m^2+26\,m^3+m^4\big)\end{aligned} \tag{5.174}$$

$$\alpha_{4,0} = 0 \tag{5.175}$$

$$\alpha_{4,1} = 0 \tag{5.176}$$

$$\alpha_{4,2} = 0 \tag{5.177}$$

$$\alpha_{4,3} = 0 \tag{5.178}$$

$$\begin{aligned}\alpha_{4,4} &= \frac{1}{25165824}\sqrt{(1+m)\ (2+m)\ (3+m)\ (4+m)}\cdot\\ &\sqrt{(5+m)\ (6+m)\ (7+m)\ (8+m)}\cdot\\ &\sqrt{(9+m)\ (10+m)\ (11+m)\ (12+m)}\cdot\\ &\sqrt{(13+m)\ (14+m)\ (15+m)\ (16+m)}\end{aligned} \tag{5.179}$$

$$\begin{aligned}\alpha_{4,5} &= \frac{1}{25165824}\sqrt{(1+m)\ (2+m)\ (3+m)\ (4+m)}\cdot\\ &\sqrt{(5+m)\ (6+m)\ (7+m)\ (8+m)}\cdot\\ &\sqrt{(9+m)\ (10+m)\ (11+m)\ (12+m)}\cdot\\ &\sqrt{(13+m)\ (14+m)\ (15+m)\ (16+m)}\ (11+2\,m)\end{aligned} \tag{5.180}$$

$$\alpha_{5,0} = 0 \tag{5.181}$$

$$\alpha_{5,1} = 0 \tag{5.182}$$

$$\alpha_{5,2} = 0 \tag{5.183}$$

$$\alpha_{5,3} = 0 \quad (5.184)$$

$$\alpha_{5,4} = 0 \quad (5.185)$$

$$\alpha_{5,5} = \frac{-1}{4026531840} \Big(\sqrt{(1+m)\ (2+m)\ (3+m)\ (4+m)}\cdot \sqrt{(5+m)\ (6+m)\ (7+m)\ (8+m)}\cdot \sqrt{(9+m)\ (10+m)\ (11+m)\ (12+m)}\cdot \sqrt{(13+m)\ (14+m)\ (15+m)\ (16+m)}\cdot \sqrt{(17+m)\ (18+m)\ (19+m)\ (20+m)}\Big) \quad (5.186)$$

$$\beta_{1,1} = \frac{\sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m}}{32} \quad (5.187)$$

$$\beta_{1,2} = \frac{\sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m}\ (-3+2\,m)}{128} \quad (5.188)$$

$$\beta_{1,3} = \frac{-1}{65536} \Big(\sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m}\ (-2016 +2106\,m - 693\,m^2 - 6\,m^3 + m^4)\Big) \quad (5.189)$$

$$\beta_{1,4} = \frac{1}{262144} \sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m}\ (-14592 +19778\,m - 10373\,m^2 + 2144\,m^3 + 41\,m^4 - 6\,m^5) \quad (5.190)$$

$$\beta_{1,5} = \frac{1}{402653184} \sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m}\cdot \Big(50908320 - 84318336\,m + 55101860\,m^2 -19514436\,m^3 + 2707329\,m^4 + 84528\,m^5 -11142\,m^6 - 12\,m^7 + m^8\Big) \quad (5.191)$$

$$\beta_{2,1} = 0 \quad (5.192)$$

$$\beta_{2,2} = \frac{1}{2048} \sqrt{(-7+m)\ (-6+m)\ (-5+m)\ (-4+m)}\cdot \sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m} \quad (5.193)$$

$$\beta_{2,3} = \frac{1}{4096} \sqrt{(-7+m)\ (-6+m)\ (-5+m)\ (-4+m)}\cdot \sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m}\ (-5+2\,m) \quad (5.194)$$

$$\beta_{2,4} = \frac{-1}{6291456} \Big(\sqrt{(-7+m)\ (-6+m)\ (-5+m)\ (-4+m)}\cdot$$

$$\sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m}\ (-20460+13982\,m$$
$$-2881\,m^2-14\,m^3+m^4\Big)\Big) \tag{5.195}$$

$$\beta_{2,5} = \frac{-1}{3145728}\left(\sqrt{(-7+m)\ (-6+m)\ (-5+m)\ (-4+m)}\cdot\right.$$
$$\sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m}\ (31056-28432\,m$$
$$+9880\,m^2-1365\,m^3-16\,m^4+m^5\Big)\Big) \tag{5.196}$$

$$\beta_{3,1} = 0 \tag{5.197}$$

$$\beta_{3,2} = 0 \tag{5.198}$$

$$\beta_{3,3} = \frac{1}{196608}\sqrt{(-11+m)\ (-10+m)\ (-9+m)\ (-8+m)}\cdot$$
$$\sqrt{(-7+m)\ (-6+m)\ (-5+m)\ (-4+m)}\cdot$$
$$\sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m} \tag{5.199}$$

$$\beta_{3,4} = \frac{1}{262144}\sqrt{(-11+m)\ (-10+m)\ (-9+m)\ (-8+m)}\cdot$$
$$\sqrt{(-7+m)\ (-6+m)\ (-5+m)\ (-4+m)}\cdot$$
$$\sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m}\ (-7+2\,m) \tag{5.200}$$

$$\beta_{3,5} = \frac{-1}{805306368}\sqrt{(-11+m)\ (-10+m)\ (-9+m)\ (-8+m)}\cdot$$
$$\sqrt{(-7+m)\ (-6+m)\ (-5+m)\ (-4+m)}\cdot$$
$$\sqrt{(-3+m)\ (-2+m)\ (-1+m)\ m}\ (-97368$$
$$+49474\,m-7309\,m^2-22\,m^3+m^4\Big) \tag{5.201}$$

$$\beta_{4,1} = 0 \tag{5.202}$$

$$\beta_{4,2} = 0 \tag{5.203}$$

$$\beta_{4,3} = 0 \tag{5.204}$$

$$\beta_{4,4} = \frac{1}{25165824}\sqrt{(-15+m)\ (-14+m)\ (-13+m)\ (-12+m)}\cdot$$
$$\sqrt{(-11+m)\ (-10+m)\ (-9+m)\ (-8+m)}\cdot$$
$$\sqrt{(-7+m)\ (-6+m)\ (-5+m)\ (-4+m)}\cdot$$

$$\sqrt{(-3+m)\,(-2+m)\,(-1+m)\,m} \tag{5.205}$$

$$\begin{aligned}\beta_{4,5} &= \frac{1}{25165824}\sqrt{(-15+m)\,(-14+m)\,(-13+m)\,(-12+m)}\,\cdot \\ &\sqrt{(-11+m)\,(-10+m)\,(-9+m)\,(-8+m)}\,\cdot \\ &\sqrt{(-7+m)\,(-6+m)\,(-5+m)\,(-4+m)}\,\cdot \\ &\sqrt{(-3+m)\,(-2+m)\,(-1+m)\,m}\,(-9+2\,m)\end{aligned} \tag{5.206}$$

$$\beta_{5,1} = 0 \tag{5.207}$$

$$\beta_{5,2} = 0 \tag{5.208}$$

$$\beta_{5,3} = 0 \tag{5.209}$$

$$\beta_{5,4} = 0 \tag{5.210}$$

$$\begin{aligned}\beta_{5,5} &= \frac{1}{4026531840}\sqrt{(-19+m)\,(-18+m)\,(-17+m)\,(-16+m)}\,\cdot \\ &\sqrt{(-15+m)\,(-14+m)\,(-13+m)\,(-12+m)}\,\cdot \\ &\sqrt{(-11+m)\,(-10+m)\,(-9+m)\,(-8+m)}\,\cdot \\ &\sqrt{(-7+m)\,(-6+m)\,(-5+m)\,(-4+m)}\,\cdot \\ &\sqrt{(-3+m)\,(-2+m)\,(-1+m)\,m}\end{aligned} \tag{5.211}$$

5.5 Asymptotic Expansions for Eigenvalues χ_m

In this section, we present formulae for the eigenvalues χ_m in equation (2.47) (also referred to as the separation parameters) for $c \to \infty$.

5.5.1 Formula for χ_0

The asymptotic formula for χ_0 to the 14-th order is given below.

$$\begin{aligned}\chi_0 &= c - \frac{3}{4} - \frac{3}{16\,c} - \frac{15}{64\,c^2} - \frac{453}{1024\,c^3} - \frac{4425}{4096\,c^4} - \frac{104613}{32768\,c^5} \\ &- \frac{1442595}{131072\,c^6} - \frac{181431165}{4194304\,c^7} - \frac{3200304885}{16777216\,c^8}\end{aligned}$$

$$-\frac{125185972551}{134217728\,c^9}-\frac{2689647087045}{536870912\,c^{10}}$$
$$-\frac{251987915369193}{8589934592\,c^{11}}-\frac{6392700476893245}{34359738368\,c^{12}}$$
$$-\frac{349366400286979629}{274877906944\,c^{13}}-\frac{4095046504712829315}{4398046511104\,c^{14}}$$
$$+O\left(c^{-15}\right). \tag{5.212}$$

5.5.2 Formula for χ_1

The following is the asymptotic expansion for χ_1 to the 14-th order.

$$\chi_1 = 3\,c-\frac{7}{4}-\frac{15}{16\,c}-\frac{105}{64\,c^2}-\frac{4245}{1024\,c^3}-\frac{53655}{4096\,c^4}-\frac{1594245}{32768\,c^5}$$
$$-\frac{26929245}{131072\,c^6}-\frac{4055050365}{4194304\,c^7}-\frac{83932150155}{16777216\,c^8}$$
$$-\frac{3785103105735}{134217728\,c^9}-\frac{92331184512315}{536870912\,c^{10}}$$
$$-\frac{9692588898867945}{8589934592\,c^{11}}-\frac{272433662232864195}{34359738368\,c^{12}}$$
$$-\frac{16338528501230653485}{274877906944\,c^{13}}-\frac{520891433271407619645}{1099511627776\,c^{14}}$$
$$+O\left(c^{-15}\right) \tag{5.213}$$

5.5.3 Formula for χ_m

The general formula for χ_m (to the sixth order) as $c \to \infty$ is given below; m is any non-negative integer.

$$\chi_m = c\,(1+2\,m)-\left(\frac{3}{4}+\frac{m}{2}+\frac{m^2}{2}\right)-\frac{1}{c}\left(\frac{3}{16}+\frac{7\,m}{16}+\frac{3\,m^2}{16}+\frac{m^3}{8}\right)$$
$$-\frac{1}{c^2}\left(\frac{15}{64}+\frac{35\,m}{64}+\frac{5\,m^2}{8}+\frac{5\,m^3}{32}+\frac{5\,m^4}{64}\right)$$
$$-\frac{1}{c^3}\left(\frac{453}{1024}+\frac{1321\,m}{1024}+\frac{639\,m^2}{512}+\frac{481\,m^3}{512}+\frac{165\,m^4}{1024}+\frac{33\,m^5}{512}\right)$$
$$-\frac{1}{c^4}\left(\frac{4425}{4096}+\frac{13349\,m}{4096}+\frac{9239\,m^2}{2048}+\frac{5255\,m^3}{2048}+\frac{5885\,m^4}{4096}\right.$$
$$\left.+\frac{189\,m^5}{1024}+\frac{63\,m^6}{1024}\right)$$

$$
\begin{aligned}
&-\frac{1}{c^5}\left(\frac{104613}{32768}+\frac{355301\,m}{32768}+\frac{117445\,m^2}{8192}+\frac{13107\,m^3}{1024}\right.\\
&\left.+\frac{163045\,m^4}{32768}+\frac{18149\,m^5}{8192}+\frac{3689\,m^6}{16384}+\frac{527\,m^7}{8192}\right)\\
&-\frac{1}{c^6}\left(\frac{1442595}{131072}+\frac{5046979\,m}{131072}+\frac{1008819\,m^2}{16384}+\frac{201271\,m^3}{4096}\right.\\
&\left.+\frac{2106769\,m^4}{65536}+\frac{609063\,m^5}{65536}+\frac{56231\,m^6}{16384}+\frac{9387\,m^7}{32768}+\frac{9387\,m^8}{131072}\right)\\
&+O\left(c^{-7}\right) \qquad (5.214)
\end{aligned}
$$

6

Numerical Results

This chapter contains numerical results primarily of two categories. In Section 6.1, we present the performance results of the quadratures and interpolation schemes described in Chapter 4. We also show some elementary numerical properties of PSWFs and related eigenvalues. In Section 6.2, we present the performance of the asymptotic expansions presented in Chapter 5. The algorithms of Sections 4.1–4.3 have been implemented in double precision (64-bit floating point) arithmetic in Fortran, and the experiments on the asymptotic formulae in extended precision (128-bit floating point) arithmetic in Fortran.

6.1 Quadratures and Interpolation

The results of the algorithms of Sections 4.1–4.3 are summarized in Tables 6.1–6.1. Tables 6.1 and 6.2 show the performance of quadrature nodes produced by the schemes of Sections 4.1 and 4.2, when used as quadrature nodes. Tables 6.3 and 6.4 show their performance when used as interpolation nodes. The quadrature nodes and the interpolation nodes are actually not the same sets of nodes: even with the bandwidth c for interpolation being half of the bandwidth for quadrature (as it is in the tables), the required accuracy of the nodes are higher (ε^2 as opposed to ε) when used for interpolation than for integration in achieving the same accuracy of ε. This discrepancy can be seen by comparing the number of nodes (printed in the column labeled n in each table).

Table 6.5 and Table 6.6 tabulate the numbers of nodes required for integrating or interpolating band-limited functions to double precision accuracy. For each frequency, the average number of nodes per wavelength is listed in the column labeled "nodes/λ".

The error figures in these tables are approximations of the maximum error of the quadrature or of interpolation, when applied to functions of the form $\cos(ax)$ and $\sin(ax)$, with $0 \leq a \leq c$. The errors were computed by measuring the error at a large number of points in a (for interpolation, in both a and x).

In Tables 6.1–6.4, the columns labeled "Roots" contain the errors for the nodes produced by the scheme of Section 4.2; the columns labeled "Refined" contain the errors after the scheme of Section 4.1 was performed using the former results as a starting point. The variable ε that appears in the tables is the requested accuracy, used to determine the number of nodes in the ways described in Sections 4.1 and 4.3. Also tabulated are the numbers of Legendre nodes required to achieve the same accuracy ε using polynomial interpolation or quadrature schemes. Since Chebyshev nodes are generally known to be superior for interpolation, the numbers of Chebyshev nodes required to achieve the same accuracy are also tabulated.

Finally, Tables 6.7 and 6.1 contain samples of quadrature weights and nodes.

Table 6.1: Quadrature performance for varying band limits, for $\varepsilon = 10^{-7}$

c	n	Maximum Errors		N_{pol}
		Roots	Refined	
10.0	9	0.96E-05	0.51E-07	13
20.0	13	0.17E-04	0.94E-07	19
30.0	17	0.12E-04	0.50E-07	25
40.0	20	0.70E-05	0.30E-06	31
50.0	24	0.35E-05	0.83E-07	37
60.0	27	0.25E-04	0.27E-06	43
70.0	31	0.11E-04	0.66E-07	48
80.0	34	0.48E-05	0.17E-06	54
90.0	38	0.21E-05	0.40E-07	59
100.0	41	0.12E-04	0.91E-07	65
200.0	74	0.24E-05	0.86E-07	118
300.0	106	0.32E-05	0.21E-06	171
400.0	139	0.52E-05	0.62E-07	223
500.0	171	0.56E-05	0.88E-07	275
1000.0	331	0.50E-05	0.14E-06	530
2000.0	651	0.23E-05	0.64E-07	1038
4000.0	1288	0.37E-05	0.17E-06	2047

Table 6.2: Quadrature performance for varying precisions, for $c = 50$

ε	n	Maximum Errors		N_{pol}
		Roots	Refined	
0.10E-01	19	0.45E-01	0.10E-01	30
0.10E-02	20	0.70E-02	0.13E-02	32
0.10E-03	21	0.91E-03	0.14E-03	33
0.10E-04	22	0.82E-04	0.13E-04	34
0.10E-05	23	0.54E-04	0.11E-05	36
0.10E-06	24	0.35E-05	0.83E-07	37
0.10E-07	25	0.33E-05	0.57E-08	38
0.10E-08	26	0.18E-06	0.36E-09	39
0.10E-09	26	0.18E-06	0.36E-09	40
0.10E-10	27	0.17E-06	0.21E-10	42
0.10E-11	28	0.79E-08	0.11E-11	43
0.10E-12	29	0.78E-08	0.56E-13	45
0.10E-13	30	0.31E-09	0.27E-14	55

Table 6.3: Interpolation performance for varying band limits, for $\varepsilon = 10^{-7}$

c	n	Maximum Errors		N_{pol}	
		Roots	Refined	Cheb.	Leg.
5.0	13	0.12E-06	0.12E-06	17	17
10.0	18	0.12E-06	0.13E-06	24	25
15.0	22	0.24E-06	0.25E-06	31	32
20.0	26	0.26E-06	0.28E-06	37	39
25.0	30	0.22E-06	0.23E-06	43	45
30.0	33	0.67E-06	0.73E-06	49	51
35.0	37	0.42E-06	0.46E-06	55	57
40.0	41	0.25E-06	0.27E-06	61	63
45.0	44	0.54E-06	0.60E-06	67	69
50.0	48	0.29E-06	0.33E-06	73	75
100.0	82	0.39E-06	0.46E-06	128	131
200.0	147	0.12E-05	0.15E-05	235	239
300.0	212	0.13E-05	0.17E-05	340	345
400.0	277	0.10E-05	0.14E-05	443	450
500.0	341	0.16E-05	0.22E-05	547	554
1000.0	662	0.16E-05	0.24E-05	1058	1068
1500.0	982	0.15E-05	0.25E-05	1566	1578
2000.0	1301	0.20E-05	0.35E-05	2072	2086

Table 6.4: Interpolation performance for varying precisions, for $c = 25$

ε	n	Maximum Errors		N_{pol}	
		Roots	Refined	Cheb.	Leg.
0.10E-01	21	0.38E-01	0.43E-01	31	34
0.10E-02	23	0.37E-02	0.41E-02	34	36
0.10E-03	25	0.29E-03	0.31E-03	37	39
0.10E-04	26	0.74E-04	0.81E-04	39	41
0.10E-05	28	0.44E-05	0.47E-05	41	43
0.10E-06	30	0.22E-06	0.23E-06	43	45
0.10E-07	31	0.46E-07	0.49E-07	45	47
0.10E-08	32	0.95E-08	0.10E-07	47	49
0.10E-09	34	0.36E-09	0.38E-09	49	51
0.10E-10	35	0.67E-10	0.70E-10	51	52
0.10E-11	37	0.21E-11	0.22E-11	53	54
0.10E-12	38	0.36E-12	0.37E-12	54	56
0.10E-13	39	0.59E-13	0.63E-13	98	61

Table 6.5: Quadrature performance and number of nodes per wavelength for varying band limits, for $\varepsilon = 10^{-14}$

c	n	Error	nodes/λ
10.0	13	0.17764E-14	4.08
20.0	18	0.15543E-14	2.83
30.0	22	0.44270E-14	2.30
40.0	26	0.46491E-14	2.04
50.0	30	0.20366E-14	1.88
60.0	33	0.21550E-13	1.73
70.0	37	0.95202E-14	1.66
80.0	41	0.38337E-14	1.61
90.0	44	0.12216E-13	1.54
100.0	48	0.29126E-14	1.51
200.0	82	0.37951E-14	1.29
300.0	115	0.65867E-14	1.20
400.0	147	0.24807E-13	1.15
500.0	180	0.12677E-13	1.13
1000.0	341	0.23376E-13	1.07
2000.0	662	0.15834E-13	1.04
4000.0	1302	0.19924E-13	1.02

Table 6.6: Interpolation performance and number of nodes per wavelength for varying band limits, for $\varepsilon = 10^{-14}$

c	n	Error	nodes/λ
10.0	25	0.44E-13	7.85
20.0	35	0.32E-13	5.50
30.0	43	0.98E-13	4.50
40.0	51	0.81E-13	4.01
50.0	59	0.57E-13	3.71
60.0	66	0.98E-13	3.46
70.0	74	0.48E-13	3.27
80.0	81	0.47E-13	3.18
90.0	88	0.71E-13	3.07
100.0	95	0.62E-13	2.98
200.0	163	0.11E-12	2.56
300.0	229	0.20E-12	2.40
400.0	295	0.28E-12	2.32
500.0	360	0.41E-12	2.26
1000.0	682	0.31E-12	2.14
2000.0	1324	0.18E-11	2.08
4000.0	2601	0.90E-11	2.04

Table 6.7: Quadrature nodes for band-limited functions, with $c = 50$ and $\varepsilon = 10^{-7}$

This table contains only half of the nodes and weights, in particular those for which the node is less than or equal to zero. Reflecting these nodes around zero yields the remaining nodes; the weight for the node at $-x$ is the same as the weight for the node at x.

Node	Weight
−.9904522459960804E+00	0.2413064234922188E−01
−.9525601106643832E+00	0.5024347217095568E−01
−.8927960861459153E+00	0.6801787677830858E−01
−.8186117530609125E+00	0.7952155999100788E−01
−.7350624131965875E+00	0.8706680708376023E−01
−.6452878027260844E+00	0.9216240765763570E−01
−.5512554698695428E+00	0.9569254015486106E−01
−.4542505281525226E+00	0.9817257766311556E−01
−.3551568458127944E+00	0.9990914516102242E−01
−.2546173463813596E+00	0.1010880172648715E+00
−.1531287781860989E+00	0.1018214308931439E+00
−.5110121484050418E−01	0.1021735189986602E+00

Table 6.8: Quadrature nodes for band-limited functions, with $c = 150$ and $\varepsilon = 10^{-14}$

This table contains only half of the nodes and weights, in particular those for which the node is less than or equal to zero. Reflecting these nodes around zero yields the remaining nodes; the weight for the node at $-x$ is the same as the weight for the node at x.

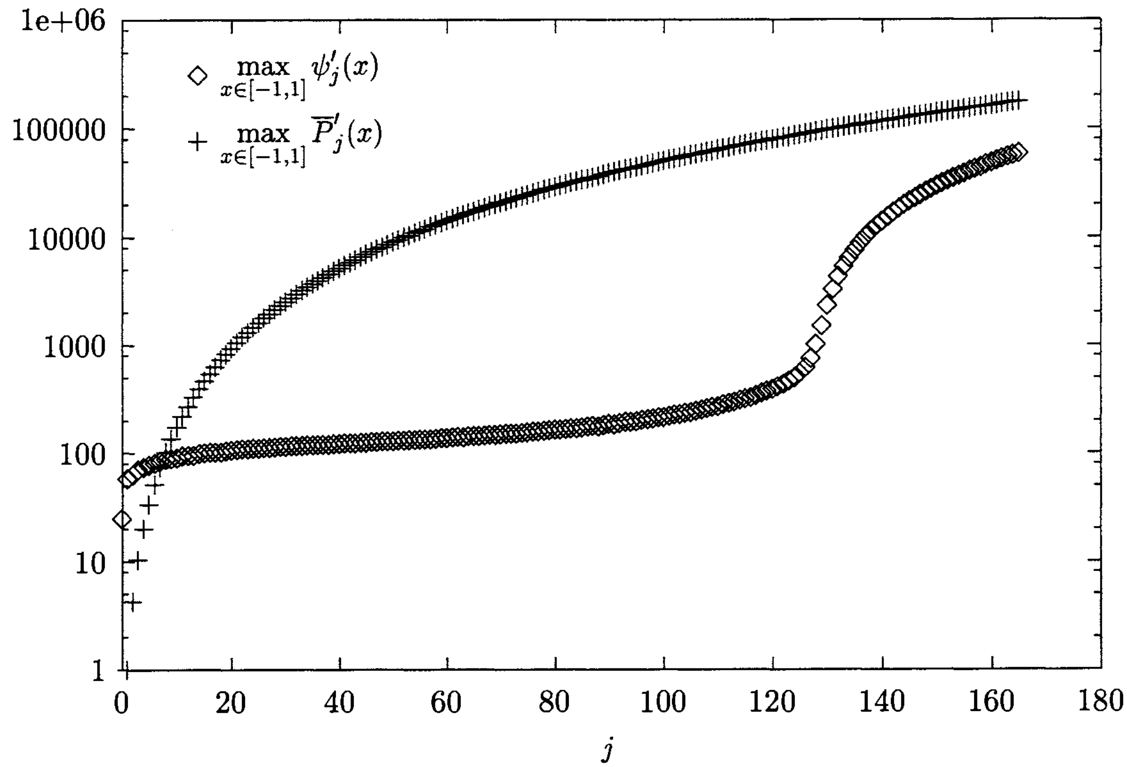
Node	Weight
−.9982883010959975E+00	0.4374483371752129E−02
−.9911354691596528E+00	0.9842619236149078E−02
−.9788315280982487E+00	0.1463518300250369E−01
−.9621348937901911E+00	0.1862396111287527E−01
−.9418386698454396E+00	0.2184988739217138E−01
−.9186509576802944E+00	0.2442858670932862E−01
−.8931541850293142E+00	0.2648864579258096E−01
−.8658083894041821E+00	0.2814375940413615E−01
−.8369709588254746E+00	0.2948528624795690E−01
−.8069187108185302E+00	0.3058356160435090E−01
−.7758670331396409E+00	0.3149181066633766E−01
−.7439849501152674E+00	0.3225015506203403E−01
−.7114064976175457E+00	0.3288893713079314E−01
−.6782391686910609E+00	0.3343126421620424E−01
−.6445701594098660E+00	0.3389488931551181E−01
−.6104710013384929E+00	0.3429358206877410E−01
−.5760010202980960E+00	0.3463812513892117E−01

(continued)	
−.5412099413257457E+00	0.3493704033879884E−01
−.5061398697742787E+00	0.3519712095895683E−01
−.4708268134473433E+00	0.3542382499917732E−01
−.4353018643598344E+00	0.3562156808557525E−01
−.3995921259242572E+00	0.3579394352776868E−01
−.3637214481257228E+00	0.3594388900778062E−01
−.3277110167114320E+00	0.3607381381247460E−01
−.2915798305819667E+00	0.3618569660385742E−01
−.2553450930388687E+00	0.3628116095737887E−01
−.2190225363501577E+00	0.3636153393399723E−01
−.1826266945721476E+00	0.3642789154364812E−01
−.1461711362450572E+00	0.3648109393796617E−01
−.1096686661347072E+00	0.3652181242257066E−01
−.7313150339365902E−01	0.3655054982303338E−01
−.3657144220122915E−01	0.3656765531685031E−01
0	0.3657333451556860E−01

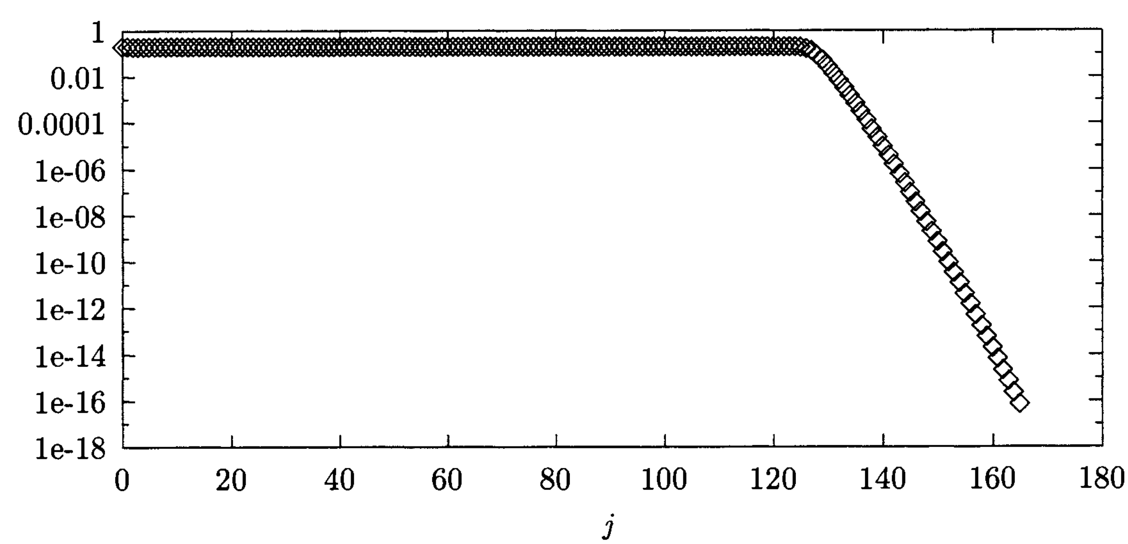
6.1.1 Miscellaneous Numerical Properties

Figure 6.1 contains the maximum norm of the derivative of each prolate spheroidal wave function ψ_j as a function of j, for $c = 200$ and $x \in [-1, 1]$. Also graphed, for comparison, is the maximum norm of the derivative of each normalized Legendre polynomial $\overline{P}_j(x)$ over the same range. The absolute values of the eigenvalues λ_j are graphed below, on the same horizontal scale. The graph clearly shows that, for this value of c, computing the derivatives of a function given by a prolate series is a better-conditioned operation than computing the derivatives of a function given by a Legendre series of the same number of terms. Obviously, if the number of terms can also be reduced, as in the situations of Tables 6.1–6.4, there is a further improvement in the condition number. The same general pattern of behavior is exhibited for other values of c. As c approaches zero, the value of j at which the maximum norm of the derivative rises sharply also approaches zero. This phenomenon is to be expected, since for $c = 0$ the prolate functions reduce to Legendre polynomials.

Figure 6.1: Maximum norms of derivatives of prolate spheroidal wave functions for $c = 200$, and of normalized Legendre polynomials



Norms of eigenvalues λ_j for $c = 200$:



The following observations can be made from the examples presented in this section, and from the more extensive tests performed by the author.

1. When the nodes obtained via the algorithm of [3] are used for the integration of band-limited functions, the resulting quadrature rules are significantly more accurate than the quadratures obtained from the nodes of appropriately chosen prolate wave functions; however, the *difference* between the numbers of nodes required by the two approaches to obtain a *prescribed* precision is not large. When the nodes obtained via the two approaches are used for the interpolation (as opposed to the integration) of band-limited functions, the performances of the two are virtually identical.

2. For large c, the number of nodes required by a quadrature rule for the integration of band-limited functions with the band-limit c is close to c/π; the dependence on the required precision of integration is weak (as one would expect from Theorem 2.9 and subsequent developments).

3. The number of nodes required by our quadratures rules to integrate band-limited functions is roughly $\pi/2$ times less than the number of Gaussian nodes; the number of nodes required by our interpolation formulae in order to interpolate band-limited functions is roughly $\pi/2$ times less than the number of Chebyshev (or Gaussian) nodes. Again, the dependence of the required number of nodes on the accuracy requirements is weak.

4. The norm of the differentiation operator based on our nodes is of the order $c^{3/2}$, as compared to the norm of the spectral differentiation operators obtained from classical polynomial expansions. This might be useful in the design of spectral (or pseudo-spectral) techniques.

6.2 Performance of Asymptotic Expansions

6.2.1 Approximations of ψ_m

We compared the results from the asymptotic formulae of ψ_m and those from the evaluation procedure described in Chapter 3. The experiments are summarized below. The errors reported in Tables 6.9–6.11 are L^2 errors on the interval $[-1,1]$, which were approximated by measuring the error at a large number of equispaced points on $[-1,1]$.

Table 6.9 and Table 6.10 show at various values of c the accuracy of the twelfth-order approximations of ψ_0, ψ_1, and that of the fifth-order approximations of several arbitrarily chosen Prolate Spheroidal Wave Functions. In Table 6.11, we show the accuracy of the fifth-order approximations of Prolate Spheroidal Wave Functions ψ_0 through ψ_{10} at three arbitrarily chosen frequencies: $c = 15\pi$, $c = 20\pi$ and $c = 25\pi$. Table 6.11 shows that for moderate values of c (such as $c = 20\pi$), the fifth-order approximation for each of the first eleven prolate functions has roughly single precision accuracy.

Table 6.9: Errors of the twelfth-order approximations of ψ_0, ψ_1 for varying band limits

c	Error of ψ_0	Error of ψ_1
10	$0.131732210659618E-05$	$0.117720227326312E-04$
20	$0.130769291525087E-08$	$0.114923274403983E-07$
40	$0.228781718888588E-12$	$0.230610279001231E-11$
80	$0.228105840274304E-16$	$0.229431254878355E-15$
160	$0.253798504532040E-20$	$0.254176969938198E-19$
320	$0.296804337674577E-24$	$0.296697705527968E-23$
640	$0.521517427857163E-28$	$0.360557579827469E-27$
1280	$0.647635038949054E-28$	$0.608485856537912E-28$

Table 6.10: Errors of the fifth-order approximations of ψ_7, ψ_{10}, ψ_{20} for varying band limits

c	Error of ψ_7	Error of ψ_{10}	Error of ψ_{20}
10	$0.23242E+00$	$0.10145E+01$	$0.12817E+02$
20	$0.31189E-02$	$0.24351E-01$	$0.14292E+01$
40	$0.31435E-04$	$0.23830E-03$	$0.28909E-01$
80	$0.40183E-06$	$0.28233E-05$	$0.24885E-03$
160	$0.57643E-08$	$0.39464E-07$	$0.31490E-05$
320	$0.86551E-10$	$0.58604E-09$	$0.45010E-07$
640	$0.13266E-11$	$0.89358E-11$	$0.67473E-09$
1280	$0.20532E-13$	$0.13795E-12$	$0.10333E-10$

Table 6.11: Errors of the fifth-order approximations of ψ_j for $0 \leq j \leq 10$

j	Errors		
	$c = 15\pi$	$c = 20\pi$	$c = 25\pi$
0	$0.26462E-08$	$0.45417E-09$	$0.11659E-09$
1	$0.17320E-07$	$0.29598E-08$	$0.75792E-09$
2	$0.77196E-07$	$0.33501E-08$	$0.43215E-09$
3	$0.27195E-06$	$0.11688E-07$	$0.15022E-08$
4	$0.80974E-06$	$0.34422E-07$	$0.44059E-08$
5	$0.21224E-05$	$0.89146E-07$	$0.11360E-07$
6	$0.50328E-05$	$0.20865E-06$	$0.26464E-07$
7	$0.11014E-04$	$0.45029E-06$	$0.56833E-07$
8	$0.22570E-04$	$0.90913E-06$	$0.11416E-06$
9	$0.43804E-04$	$0.17369E-05$	$0.21695E-06$
10	$0.81208E-04$	$0.31673E-05$	$0.39345E-06$

Remark 6.1 *We observe that the above accuracies are strictly in L^2 sense. At the ends of the interval $[-1, 1]$, the asymptotic expansion formulae of ψ_m are only accurate in the sense that both the actual values and the approximated ones are small, for sufficiently large c. This observation can be easily seen from Table 6.12 and Table 6.13.*

Table 6.12: $\psi_0(1)$ and the 6^{th}-order and 12^{th}-order approximations for varying band limits

c	Exact	6^{th}-order Appr.	12^{th}-order Appr.
10	$0.65478E-03$	$0.67200E-03$	$0.66236E-03$
20	$0.50983E-07$	$0.36693E-06$	$0.56739E-07$
40	$0.17835E-15$	$0.10497E-09$	$-0.27294E-12$
80	$0.69770E-27$	$0.46657E-15$	$-0.11568E-18$
160	$-0.22067E-27$	$0.11222E-29$	$0.11986E-29$
320	$-0.10514E-26$	$0.31654E-62$	$0.34152E-59$
640	$-0.54266E-27$	$0.10814E-129$	$0.19188E-124$
1280	$-0.92841E-27$	$0.10170E-266$	$0.17473E-259$
2560	$0.96445E-27$	$0.93887E-543$	$0.12551E-533$

Table 6.13: $\psi_4(1)$ and the 12^{th}-order approximations for varying band limits

c	Exact $\psi_4(1)$	12^{th} Order Appr.
10	$0.41938E+00$	$0.38453E+00$
20	$0.20273E-03$	$-0.17051E-03$
40	$0.32847E-11$	$-0.19154E-06$
80	$0.61521E-27$	$-0.16503E-11$
160	$0.42283E-27$	$-0.74239E-26$
320	$-0.63142E-27$	$-0.40735E-58$
640	$-0.97616E-27$	$-0.27470E-125$
1280	$0.95964E-27$	$-0.51343E-262$
2560	$-0.10846E-26$	$-0.94494E-538$

6.2.2 Approximations of χ_m

We rewrite (5.214) in the form

$$\chi_m = p_{-1}(m) \cdot c + \sum_{i=0}^{n} p_i(m) \cdot \left(\frac{1}{c}\right)^i + O\left(c^{-n-1}\right), \tag{6.1}$$

where $p_{-1}(m), p_0(m), \ldots, p_n(m)$ are polynomials in m of n-th order, as given in (5.214). In the rest of this section, we refer to the number n as the order of the expansion (6.1).

Table 6.14 and Table 6.15 consist of results of the approximation of χ_0 and χ_1 using formulae (5.212) and (5.213), respectively; these results were obtained using all 16 terms of each expansion. In Tables 6.16 and 6.17, we list results of the approximation of χ_2 and χ_3 with formula (5.214) to the sixth order.

Tables 6.18–6.20 contain the minimum order n of the expansion (5.214) that is required for achieving a specific accuracy ε for each of the first several eigenvalues. The columns labeled "j" give the indices of the eigenvalues χ_j.

Finally, we show in the Tables 6.21 and 6.22 the accuracy of (5.214) for the first eleven eigenvalues of the differential equation (2.47) at frequencies $c = 20\pi$ and $c = 200$.

Table 6.14: Errors of the fourteenth-order approximation of χ_0 for varying band limits

c	Exact χ_0	Absolute error	Relative error
10	$0.92283E+01$	$0.23974E-06$	$0.25978E-07$
20	$0.19240E+02$	$0.39399E-11$	$0.20477E-12$
40	$0.39245E+02$	$0.85702E-16$	$0.21838E-17$
80	$0.79248E+02$	$0.22987E-20$	$0.29007E-22$
160	$0.15925E+03$	$0.65932E-25$	$0.41402E-27$
320	$0.31925E+03$	$0.25638E-29$	$0.79926E-32$
640	$0.63925E+03$	$0.23054E-27$	$0.36063E-30$
1280	$0.12792E+04$	$0.12129E-27$	$0.94852E-31$

Table 6.15: Errors of the fourteenth-order approximation of χ_1 for varying band limits

c	Exact χ_1	Absolute error	Relative error
10	$0.28133E+02$	$0.10593E-04$	$0.37654E-06$
20	$0.58198E+02$	$0.23096E-09$	$0.39684E-11$
40	$0.11823E+03$	$0.48209E-14$	$0.40777E-16$
80	$0.23824E+03$	$0.12815E-18$	$0.53791E-21$
160	$0.47824E+03$	$0.36772E-23$	$0.76889E-26$
320	$0.95825E+03$	$0.15688E-27$	$0.16370E-30$
640	$0.19182E+04$	$0.16921E-27$	$0.88208E-31$
1280	$0.38382E+04$	$0.66225E-27$	$0.17256E-30$

Table 6.16: Errors of the sixth-order approximation of χ_2 for varying band limits

c	Exact χ_2	Absolute error	Relative error
10	$0.45869E+02$	$0.30534E-02$	$0.66567E-04$
20	$0.96090E+02$	$0.11477E-04$	$0.11944E-06$
40	$0.19618E+03$	$0.72761E-07$	$0.37090E-09$
80	$0.39621E+03$	$0.52104E-09$	$0.13150E-11$
160	$0.79623E+03$	$0.39094E-11$	$0.49099E-14$
320	$0.15962E+04$	$0.29953E-13$	$0.18764E-16$
640	$0.31962E+04$	$0.23177E-15$	$0.72514E-19$
1280	$0.63962E+04$	$0.18021E-17$	$0.28175E-21$

Table 6.17: Errors of the sixth-order approximation of χ_3 for varying band limits

c	Exact χ_3	Absolute error	Relative error
10	$0.62258E+02$	$0.25241E-01$	$0.40543E-03$
20	$0.13287E+03$	$0.83058E-04$	$0.62513E-06$
40	$0.27307E+03$	$0.50295E-06$	$0.18418E-08$
80	$0.55317E+03$	$0.35461E-08$	$0.64106E-11$
160	$0.11132E+04$	$0.26431E-10$	$0.23743E-13$
320	$0.22332E+04$	$0.20188E-12$	$0.90398E-16$
640	$0.44732E+04$	$0.15598E-14$	$0.34869E-18$
1280	$0.89532E+04$	$0.12119E-16$	$0.13536E-20$

Table 6.18: Minimum order n required for $\varepsilon = 10^{-3}$, for $c = 20$

j	Min n	Relative error
0	1	$0.337422864120107E-04$
1	1	$0.811202216861174E-04$
2	1	$0.197595874877010E-03$
3	1	$0.426434771008809E-03$
4	1	$0.831173278939073E-03$
5	2	$0.424991987300546E-03$
6	2	$0.833156789212658E-03$
7	3	$0.678544959628742E-03$
8	4	$0.754566227646631E-03$

Table 6.19: Minimum order n required for $\varepsilon = 10^{-6}$, for $c = 50$

j	Min n	Relative error
0	2	$0.755976896517343E-07$
1	2	$0.239020755340339E-06$
2	2	$0.711980808974688E-06$
3	3	$0.171031787786026E-06$
4	3	$0.442130435042915E-06$

Table 6.20: Minimum order n required for $\varepsilon = 10^{-9}$, for $c = 150$

j	Min n	Relative error
0	2	$0.892827592625061E-09$
1	3	$0.591968282468442E-10$
2	3	$0.215659115085051E-09$
3	3	$0.641923516105837E-09$
4	4	$0.681751546447221E-10$
5	4	$0.175118080642807E-09$

Table 6.21: Errors of (5.214) for $\chi_0, \chi_1, \ldots, \chi_{10}$, for $c \approx 62.832$

j	χ_j	Relative error
0	$0.620788076925242E+02$	$0.19397E-12$
1	$0.186730205258151E+03$	$0.14605E-11$
2	$0.310362813313921E+03$	$0.93179E-11$
3	$0.432963651106072E+03$	$0.45638E-10$
4	$0.554519171938024E+03$	$0.18073E-09$
5	$0.675015221252847E+03$	$0.13948E-08$
6	$0.794436990272064E+03$	$0.15186E-07$
7	$0.912768964569442E+03$	$0.29778E-07$
8	$0.102999486684922E+04$	$0.60436E-07$
9	$0.114609759306556E+04$	$0.59101E-06$
10	$0.126105914085799E+04$	$0.87981E-06$

Table 6.22: Errors of (5.214) for $\chi_0, \chi_1, \ldots, \chi_{10}$, for $c = 200$

j	χ_j	Relative error
0	$0.199249056584642E+03$	$0.17344E-16$
1	$0.598245270957844E+03$	$0.12962E-15$
2	$0.996235776724989E+03$	$0.81653E-15$
3	$0.139321672741520E+04$	$0.39425E-14$
4	$0.178918422715135E+04$	$0.15374E-13$
5	$0.218413432959437E+04$	$0.28512E-12$
6	$0.257806303685598E+04$	$0.41183E-11$
7	$0.297096629837867E+04$	$0.77894E-11$
8	$0.336284000978153E+04$	$0.15362E-10$
9	$0.375368001167075E+04$	$0.46737E-09$
10	$0.414348208841325E+04$	$0.65268E-09$

The following observations can be made from the examples presented in this section, and from the more extensive tests performed by the author.

1. For the approximation of the zeroth and the first Prolate Spheroidal Wave Functions at moderately large c's (such as $c = 40$), the asymptotic expansions of Sections 5.4.1 and 5.4.2 give double precision accuracy (in the L^2 sense, see Remark 6.1). For the approximation of the other Prolate Spheroidal Wave Functions, the general formula of Section 5.4.3 has single precision accuracy only. Nevertheless, such approximations are still useful when used as starting points in other more elaborate schemes. Furthermore, expansions of higher orders can always be constructed individually for any prolate function, although the accuracy of these formulae tend to degrade slowly as the orders of the PSWFs increase.

2. The number of terms needed in the general asymptotic expansion of χ_m for achieving single precision accuracy is fairly small: for $c = 50$, the approximation of the first six eigenvalues require roughly three to six terms. Of course, the formulae given in Section 5.5.1 and Section 5.5.2 have much higher accuracies; other higher-order formulae can be constructed for each individual case.

7

Miscellaneous Properties

Prolate Spheroidal Wave Functions possess a rich set of properties, resembling those of Bessel functions. This chapter establishes several such properties, some of which can be found in [6, 31, 17] or easily derivable from the facts in [6, 31, 17]. Through out this chapter, unless otherwise stated, we will use ψ_m to denote the m-th Prolate Spheroidal Wave Function corresponding to band limit c, and use λ_m, μ_m, χ_m to denote corresponding eigenvalues of the operators F_c, Q_c, G_c, respectively (see (2.40), (2.42), (5.1)). The band limit c associated with these notations is omitted whenever the context is clear.

The following five lemmas are immediate consequences of the identity

$$e^{icxt} = \sum_{m=0}^{\infty} \lambda_m \, \psi_m(x) \, \psi_m(t) \tag{7.1}$$

(see Section 4.1). Although these properties are fairly obvious, we record them here since similar properties of other special functions have often been found useful.

Lemma 7.1 *For all $x, t \in [-1, 1]$, $c \in (0, \infty)$, and all integers $j, k \geq 0$,*

$$x^j t^k e^{icxt} = \left(\frac{1}{ic}\right)^{(j+k)} \sum_{m=0}^{\infty} \lambda_m \, \psi_m^{(j)}(x) \, \psi_m^{(k)}(t) \,. \tag{7.2}$$

Proof. Differentiating (7.1) j times with respect to x and k times with respect to t yields the above formula. □

Lemma 7.2 *For all* $x, u \in [-1, 1]$ *and* $c \in (0, \infty)$,

$$\frac{sin(c \cdot (x-u))}{x-u} = \frac{c}{2} \sum_{m=0}^{\infty} \lambda_m^2 \, \psi_m(x) \, \psi_m(u), \tag{7.3}$$

Proof. Multiplying (7.1) by e^{-icut}, and integrating with respect to t, we immediately obtain (7.3). □

Lemma 7.3 *The eigenvalues* λ_m *satisfy*

$$\sum_{m=0}^{\infty} |\lambda_m|^2 = 4\,. \tag{7.4}$$

Proof. Taking the squared norm of (7.1), and integrating with respect to x and t, we immediately obtain the above formula. □

Lemma 7.4 *The eigenvalues* μ_m *satisfy*

$$\sum_{m=0}^{\infty} \mu_m = \frac{2c}{\pi}. \tag{7.5}$$

Proof. Combining (7.4) with (2.44) yields identity (7.5). □

Lemma 7.5 *For all real c,*

$$e^{ic} = \sum_{m=0}^{\infty} \lambda_m \, \psi_m^2(1). \tag{7.6}$$

Proof. Setting $x = t = 1$ converts (7.1) into (7.6). □

The identity

$$\lambda_m \psi_m(x) = \int_{-1}^{1} e^{icxt}\, \psi_m(t)\, dt \tag{7.7}$$

(see Section 2.3) also has a number of simple but potentially useful consequences.

Lemma 7.6 *For all $x \in [-1, 1]$, $c \in (0, \infty)$, and all integers $m, k \geq 0$,*

$$\lambda_m \psi_m^{(k)}(x) = (ic)^k \int_{-1}^{1} e^{icxt}\, t^k\, \psi_m(t)\, dt. \tag{7.8}$$

Proof. Differentiating (7.7) k times with respect to x yields the above identity. □

The following lemma provides a recursion connecting the values of the k-th derivative of the function ψ_m with its derivatives of orders $k-1$, $k-2$, $k-3$, $k-4$.

Lemma 7.7 *For all $c \in (0, \infty)$, integer $m \geq 0$, and $x \in (-\infty, +\infty)$,*

$$\begin{aligned} &(1 - x^2)\, \psi_m^{(k+2)}(x) - 2\,(k+1)\, x\, \psi_m^{(k+1)}(x) \\ &+ (\chi_m - k\,(k+1) - c^2 x^2)\, \psi_m^{(k)}(x) \\ &- 2\, c^2\, k\, x\, \psi_m^{(k-1)}(x) - c^2\, k\,(k-1)\, \psi_m^{(k-2)}(x) = 0 \end{aligned} \tag{7.9}$$

for all $k \geq 2$, with the initial condition

$$\begin{aligned} &(1 - x^2)\, \psi_m'''(x) - 4\, x\, \psi_m''(x) + (\chi_m - 2 - c^2 x^2)\, \psi_m'(x) \\ &- 2\, c^2\, x\, \psi_m(x) = 0\,. \end{aligned} \tag{7.10}$$

Moreover,

$$\begin{aligned} &- 2\,(k+1)\, \psi_m^{(k+1)}(1) + (\chi_m - k\,(k+1) - c^2)\, \psi_m^{(k)}(1) \\ &- 2\, c^2\, k\, \psi_m^{(k-1)}(1) - c^2\, k\,(k-1)\, \psi_m^{(k-2)}(1) = 0 \end{aligned} \tag{7.11}$$

for all $k \geq 2$, *and*

$$-2\,\psi'_m(1) + (\chi_m - c^2)\,\psi_m(1) = 0\,, \tag{7.12}$$

$$-4\,\psi''_m(1) + (\chi_m - 2 - c^2)\,\psi'_m(1) - 2\,c^2\,\psi_m(1) = 0\,. \tag{7.13}$$

Furthermore, for all integer $m \geq 0$ *and* $k \geq 2$,

$$\psi_m^{(k+2)}(0) + (\chi_m - k\,(k+1))\,\psi_m^{(k)}(0) - c^2\,k\,(k-1)\,\psi_m^{(k-2)}(0) = 0\,. \tag{7.14}$$

For all odd m,

$$\psi'''_m(0) + (\chi_m - 2)\,\psi'_m(0) = 0\,, \tag{7.15}$$

and for all even m,

$$\psi''_m(0) + \chi_m\,\psi_m(0) = 0\,. \tag{7.16}$$

Finally, for all integer $m \geq 0$, $k \geq 0$,

$$\psi_m(1) \neq 0\,, \tag{7.17}$$

$$\psi_{2m+1}^{(2k)}(0) = 0\,, \tag{7.18}$$

$$\psi_{2m}^{(2k+1)}(0) = 0\,. \tag{7.19}$$

Proof. Identities (7.9) – (7.16), (7.18), (7.19) are immediately obtained by repeated differentiation of (2.47). In order to prove (7.17), we assume the contrary that

$$\psi_m(1) = 0 \tag{7.20}$$

is valid for some integer $m \geq 0$, and observe that the combination of (7.20) with (7.11), (7.12), (7.13) implies that

$$\psi_m^{(k)}(1) = 0 \tag{7.21}$$

for all $k = 0, 1, 2, \ldots$. Due to the analyticity of $\psi_m(x)$ in the complex plane, this would imply that

$$\psi_m(x) = 0 \tag{7.22}$$

for all $x \in \mathbb{R}$.

□

The following is an immediate consequence of the identity (7.12) of Lemma 7.7.

Corollary 7.8 *For all integer $m, n \geq 0$,*

$$\psi_m'(1) \cdot \psi_n(1) - \psi_n'(1) \cdot \psi_m(1) = (\chi_n - \chi_m) \cdot \psi_n(1) \cdot \psi_m(1)\,. \tag{7.23}$$

Theorem 3.4 in Section 3.3 gives a set of formulae for the entries of matrices for differentiation of prolate series and for multiplication of prolate series by x. Matrices for any combination of differentiation and of multiplication by a polynomial can obviously be constructed from these two matrices. For instance, calling the differentiation matrix D, and the multiplication-by-x matrix X, the matrix for taking the second derivative of a prolate series, then multiplying it by $5 - x^2$, is equal to $(5\,I - X^2)\,D^2$.

In many cases, however, there are simpler formulae for the entries of such matrices, that is, for inner products of ψ_m with its derivatives and with polynomials. The following lemmas establish several such formulae, as well as a few formulae for inner products which do not involve ψ_m itself but only its derivatives. We start with Theorem 3.4, restated here for consistency.

Lemma 7.9 *Suppose that c is real and positive, and that the integers m and n are non-negative. If $m = n \pmod 2$, then*

$$\int_{-1}^{1} \psi_n'(x)\,\psi_m(x)\;dx = \int_{-1}^{1} x\,\psi_n(x)\,\psi_m(x)\;dx = 0. \tag{7.24}$$

If $m \neq n \pmod 2$, then

$$\int_{-1}^{1} \psi_n'(x)\,\psi_m(x)\;dx \;=\; \frac{2\,\lambda_m^2}{\lambda_m^2 + \lambda_n^2}\,\psi_m(1)\,\psi_n(1)\,, \tag{7.25}$$

$$\int_{-1}^{1} x\,\psi_n(x)\,\psi_m(x)\;dx \;=\; \frac{2}{ic}\,\frac{\lambda_m\lambda_n}{\lambda_m^2 + \lambda_n^2}\,\psi_m(1)\,\psi_n(1)\,. \tag{7.26}$$

Lemma 7.10 *For all integers $m, n \geq 0$, if $m \neq n \pmod 2$, then*

$$\int_{-1}^{1} x\,\psi_n'(x)\,\psi_m(x)\;dx = 0\,; \tag{7.27}$$

if $m = n \pmod 2$, then

$$\int_{-1}^{1} x\,\psi_n'(x)\,\psi_m(x)\;dx = \frac{\lambda_m}{\lambda_m + \lambda_n}\,\left(2\,\psi_m(1)\,\psi_n(1) - \delta_{m,n}\right)\,, \tag{7.28}$$

where $\delta_{m,n}$ is the Kronecker Delta function.

Proof. Identity (7.27) is obvious since the functions ψ_m are alternately even and odd (see Lemma 2.8). In order to prove (7.28), we consider the integral

$$\begin{aligned}
&\int_{-1}^{1} x\,\psi_n'(x)\,\psi_m(x)\;dx \\
&= \frac{1}{\lambda_n}\int_{-1}^{1} x\left(\int_{-1}^{1} e^{icxt}\,\psi_n(t)\;dt\right)_x'\,\psi_m(x)\;dx \\
&= \frac{ic}{\lambda_n}\int_{-1}^{1} x\,\psi_m(x)\left(\int_{-1}^{1} t\,\psi_n(t)\,e^{icxt}\;dt\right)\;dx \\
&= \frac{ic}{\lambda_n}\int_{-1}^{1} t\left(\int_{-1}^{1} x\,\psi_m(x)\,e^{icxt}\;dx\right)\psi_n(t)\;dt \\
&= \frac{\lambda_m}{\lambda_n}\int_{-1}^{1} t\,\psi_m'(t)\,\psi_n(t)\;dt\,.
\end{aligned}$$

In other words,

$$\int_{-1}^{1} x\,\psi_n'(x)\,\psi_m(x)\;dx = \frac{\lambda_m}{\lambda_n}\int_{-1}^{1} x\,\psi_m'(x)\,\psi_n(x)\;dx\,. \tag{7.29}$$

On the other hand, integrating the left side of (7.29) by parts, we obtain

$$\begin{aligned}
&\int_{-1}^{1} x\,\psi_n'(x)\,\psi_m(x)\;dx \\
&= 2\,\psi_m(1)\,\psi_n(1) - \int_{-1}^{1}\left(\psi_n(x)\,\psi_m'(x)\,x + \psi_n(x)\,\psi_m(x)\right)\;dx \\
&= 2\,\psi_m(1)\,\psi_n(1) - \int_{-1}^{1} x\,\psi_n(x)\,\psi_m'(x)\;dx - \delta_{mn}\,.
\end{aligned}$$

Combining (7.29) and (7.30), we have

$$\begin{aligned}
&\frac{\lambda_m}{\lambda_n}\int_{-1}^{1} x\,\psi_m'(x)\,\psi_n(x)\;dx \\
&= 2\psi_m(1)\,\psi_n(1) - \int_{-1}^{1} x\,\psi_m'(x)\,\psi_n(x)\;dx - \delta_{mn}\,,
\end{aligned}$$

from which (7.28) follows directly. □

Lemma 7.11 *For all integers $m, n \geq 0$, if $m \neq n \pmod 2$, then*

$$\int_{-1}^{1} x^2\, \psi_n''(x)\, \psi_m(x)\; dx = 0\,; \tag{7.30}$$

if $m = n \pmod 2$ and $m \neq n$, then

$$\begin{aligned} &\int_{-1}^{1} x^2\, \psi_m''(x)\, \psi_n(x)\; dx \\ &= \frac{2\lambda_n}{\lambda_m - \lambda_n} \left(\psi_n'(1)\, \psi_m(1) - \psi_m'(1)\, \psi_n(1)\right) - \frac{4\lambda_n}{\lambda_n + \lambda_m}\, \psi_n(1)\, \psi_m(1) \qquad (7.31) \\ &= \frac{\lambda_n}{\lambda_m - \lambda_n} (\chi_n - \chi_m)\, \psi_n(1)\, \psi_m(1) - \frac{4\lambda_n}{\lambda_n + \lambda_m}\, \psi_n(1)\, \psi_m(1)\,. \qquad (7.32) \end{aligned}$$

Proof. Clearly, (7.30) is true, since the functions ψ_j are alternately even and odd. In order to prove (7.31) and (7.32), supposing that $m = n \pmod 2$ and $m \neq n$, we consider the integral

$$\begin{aligned} &\int_{-1}^{1} x^2\, \psi_n''(x)\, \psi_m(x)\; dx \\ &= \frac{1}{\lambda_n} \int_{-1}^{1} x^2 \cdot \left(\int_{-1}^{1} e^{icxt}\, \psi_n(t)\; dt\right)_x''\, \psi_m(x)\; dx \\ &= -\frac{c^2}{\lambda_n} \int_{-1}^{1} \psi_m(x)\, x^2 \cdot \left(\int_{-1}^{1} t^2\, \psi_n(t)\, e^{icxt}\; dt\right)\; dx \\ &= -\frac{c^2}{\lambda_n} \int_{-1}^{1} \left(\int_{-1}^{1} \psi_m(x)\, x^2\, e^{icxt}\, dx\right)\, \psi_n(t)\, t^2\; dt \\ &= \frac{\lambda_m}{\lambda_n} \int_{-1}^{1} t^2\, \psi_n(t)\, \psi_m''(t)\; dt\,, \end{aligned}$$

which is summarized as

$$\int_{-1}^{1} x^2\, \psi_n''(x)\, \psi_m(x)\; dx = \frac{\lambda_m}{\lambda_n} \int_{-1}^{1} x^2\, \psi_m''(x)\, \psi_n(x)\; dx\,. \tag{7.33}$$

On the other hand, integrating the left side of (7.33) by parts, we have

$$\int_{-1}^{1} x^2\, \psi_n''(x)\, \psi_m(x)\; dx$$

$$= 2\,\psi_n'(1)\,\psi_m(1) - \int_{-1}^{1} \psi_n'(x)\,\left(\psi_m'(x)\,x^2 + 2\,x\,\psi_m(x)\right)\,dx$$
$$= 2\,\psi_n'(1)\,\psi_m(1) - 2\int_{-1}^{1} \psi_n'(x)\,\psi_m(x)\,x\,dx$$
$$- \int_{-1}^{1} \psi_n'(x)\,\psi_m'(x)\,x^2\,dx\,. \tag{7.34}$$

Due to Lemma 7.10 and the fact that $m \neq n$, we immediately rewrite (7.34) as

$$\int_{-1}^{1} x^2\,\psi_n''(x)\,\psi_m(x)\,dx$$
$$= 2\psi_n'(1)\,\psi_m(1) - \frac{2\lambda_m}{\lambda_m+\lambda_n}\,2\,\psi_n(1)\,\psi_m(1) - \int_{-1}^{1} x^2\,\psi_n'(x)\,\psi_m'(x)\,dx\,, \tag{7.35}$$

which we rewrite as

$$\int_{-1}^{1} x^2\,\psi_n'(x)\,\psi_m'(x)\,dx$$
$$= 2\,\psi_n'(1)\,\psi_m(1) - \frac{4\,\lambda_m}{\lambda_m+\lambda_n}\,\psi_n(1)\,\psi_m(1) - \int_{-1}^{1} x^2\,\psi_n''(x)\,\psi_m(x)\,dx\,. \tag{7.36}$$

Swapping m with n, we convert (7.36) into

$$\int_{-1}^{1} x^2\,\psi_n'(x)\,\psi_m'(x)\,dx$$
$$= 2\,\psi_m'(1)\,\psi_n(1) - \frac{4\,\lambda_n}{\lambda_m+\lambda_n}\,\psi_n(1)\,\psi_m(1) - \int_{-1}^{1} x^2\,\psi_m''(x)\,\psi_n(x)\,dx. \tag{7.37}$$

Combining (7.36) and (7.37), we obtain

$$\int_{-1}^{1} x^2\,\psi_n''(x)\,\psi_m(x)\,dx - 2\,\psi_n'(1)\,\psi_m(1) + \frac{4\,\lambda_m}{\lambda_m+\lambda_n}\,\psi_n(1)\,\psi_m(1)$$
$$= \int_{-1}^{1} x^2\,\psi_m''(x)\,\psi_n(x)\,dx - 2\,\psi_m'(1)\,\psi_n(1) + \frac{4\,\lambda_n}{\lambda_m+\lambda_n}\,\psi_n(1)\,\psi_m(1)\,, \tag{7.38}$$

which is obviously equivalent to

$$\int_{-1}^{1} x^2\,\psi_n''(x)\,\psi_m(x)\,dx$$
$$= \int_{-1}^{1} x^2\,\psi_m''(x)\,\psi_n(x)\,dx + 2\,\left(\psi_n'(1)\,\psi_m(1) - \psi_m'(1)\,\psi_n(1)\right)$$
$$+ 4\,\frac{\lambda_n-\lambda_m}{\lambda_n+\lambda_m}\,\psi_n(1)\,\psi_m(1)\,. \tag{7.39}$$

Finally, combining (7.33) with (7.39), we have

$$\begin{aligned}\frac{\lambda_m}{\lambda_n}\int_{-1}^{1} & x^2\,\psi_m''(x)\,\psi_n(x)\;dx \\ = & \int_{-1}^{1} x^2\,\psi_m''(x)\,\psi_n(x)\;dx + 2\,\left(\psi_n'(1)\,\psi_m(1) - \psi_m'(1)\,\psi_n(1)\right) \\ & + 4\,\frac{\lambda_n-\lambda_m}{\lambda_n+\lambda_m}\,\psi_n(1)\,\psi_m(1)\,, \qquad (7.40)\end{aligned}$$

which is easily rewritten as

$$\begin{aligned}\left(\frac{\lambda_m}{\lambda_n}-1\right) & \int_{-1}^{1} x^2\,\psi_m''(x)\,\psi_n(x)\;dx \\ = & \; 2\,\left(\psi_n'(1)\,\psi_m(1) - \psi_m'(1)\,\psi_n(1)\right) + 4\,\frac{\lambda_n-\lambda_m}{\lambda_n+\lambda_m}\,\psi_n(1)\,\psi_m(1)\,,\end{aligned}$$

or

$$\begin{aligned}\int_{-1}^{1} & x^2\,\psi_m''(x)\,\psi_n(x)\;dx \\ = & \; \frac{2\,\lambda_n}{\lambda_m-\lambda_n}\,\left(\psi_n'(1)\,\psi_m(1) - \psi_m'(1)\,\psi_n(1)\right) - \frac{4\,\lambda_n}{\lambda_n+\lambda_m}\,\psi_n(1)\,\psi_m(1)\,. \qquad (7.41)\end{aligned}$$

We finally rewrite (7.41) as (7.32) using Corollary 7.8. □

The following lemma is an immediate consequence of the combination of the preceding lemma and equation (7.36).

Lemma 7.12 *Suppose that c is real and positive, and that the integers m and n are non-negative. If $m \neq n \pmod 2$, then*

$$\int_{-1}^{1} x^2\,\psi_n'(x)\,\psi_m'(x)\;dx = 0\,. \qquad (7.42)$$

If $m = n \pmod 2$ and $m \neq n$,

$$\begin{aligned}\int_{-1}^{1} & x^2\,\psi_m'(x)\,\psi_n'(x)\;dx \\ = & \; 2\,\psi_m'(1)\,\psi_n(1) + \frac{2\lambda_n}{\lambda_m-\lambda_n}\,\left(\psi_m'(1)\,\psi_n(1) - \psi_n'(1)\,\psi_m(1)\right) \qquad (7.43) \\ = & \; 2\,\psi_n'(1)\,\psi_m(1) + \frac{2\lambda_m}{\lambda_n-\lambda_m}\,\left(\psi_n'(1)\,\psi_m(1) - \psi_m'(1)\,\psi_n(1)\right) \qquad (7.44) \\ = & \; \psi_m(1)\,\psi_n(1)\left(\frac{\lambda_m\chi_m-\lambda_n\chi_n}{\lambda_m-\lambda_n} - c^2\right). \qquad (7.45)\end{aligned}$$

Lemma 7.13 *Suppose that c is real and positive, and that the integers m and n are non-negative. If $m \neq n \pmod 2$, then*

$$\int_{-1}^{1} \psi_n(x)\, \psi_m''(x)\ dx = \int_{-1}^{1} x^2\, \psi_n(x)\, \psi_m(x)\ dx = 0 \tag{7.46}$$

If $m = n \pmod 2$ and $m \neq n$, then

$$\int_{-1}^{1} \psi_n(x)\, \psi_m''(x)\ dx$$

$$= \frac{2\,\lambda_n^2}{\lambda_m^2 - \lambda_n^2}\ (\psi_n'(1)\, \psi_m(1) - \psi_n(1)\, \psi_m'(1)) \tag{7.47}$$

$$= \frac{\lambda_n^2}{\lambda_m^2 - \lambda_n^2}\, (\chi_n - \chi_m)\, \psi_m(1)\, \psi_n(1), \tag{7.48}$$

$$\int_{-1}^{1} x^2\, \psi_n(x)\, \psi_m(x)\ dx$$

$$= -\frac{2}{c^2}\, \frac{\lambda_m \lambda_n}{\lambda_m^2 - \lambda_n^2}\ (\psi_n'(1)\, \psi_m(1) - \psi_n(1)\, \psi_m'(1)) \tag{7.49}$$

$$= -\frac{1}{c^2}\, \frac{\lambda_m \lambda_n}{\lambda_m^2 - \lambda_n^2}\, (\chi_n - \chi_m)\, \psi_m(1)\, \psi_n(1)\,. \tag{7.50}$$

Proof. Identity (7.46) is obvious, since the functions ψ_m are alternately even and odd. In order to prove (7.47)–(7.50), we start with the expression

$$\lambda_n \psi_n''(x) = -c^2 \int_{-1}^{1} t^2\, e^{icxt}\, \psi_n(t)\ dt\,. \tag{7.51}$$

Taking the inner product of (7.51) with $\psi_m(x)$, we have

$$\lambda_n \int_{-1}^{1} \psi_n''(x)\, \psi_m(x)\ dx$$

$$= -c^2 \int_{-1}^{1} \left(\int_{-1}^{1} t^2\, \psi_n(t)\, e^{icxt}\ dt \right) \psi_m(x)\ dx$$

$$= -c^2 \int_{-1}^{1} t^2\, \psi_n(t) \left(\int_{-1}^{1} \psi_m(x)\, e^{icxt}\ dx \right)\ dt$$

$$= -c^2 \lambda_m \int_{-1}^{1} t^2\, \psi_n(t)\, \psi_m(t)\ dt\,,$$

which we summarize as

$$\int_{-1}^{1} x^2\, \psi_n(x)\, \psi_m(x)\ dx = -\frac{1}{c^2}\, \frac{\lambda_n}{\lambda_m} \int_{-1}^{1} \psi_n''(x)\, \psi_m(x)\ dx\,. \tag{7.52}$$

Swapping n, m, we rewrite (7.52) in the form of

$$\int_{-1}^{1} x^2 \psi_n(x)\, \psi_m(x) dx = -\frac{1}{c^2} \frac{\lambda_m}{\lambda_n} \int_{-1}^{1} \psi_m''(x)\, \psi_n(x)\ dx\,. \tag{7.53}$$

Combining (7.52) and (7.53), we get

$$\int_{-1}^{1} \psi_n''(x)\, \psi_m(x)\ dx = \frac{\lambda_m^2}{\lambda_n^2} \int_{-1}^{1} \psi_m''(x)\, \psi_n(x)\ dx\,. \tag{7.54}$$

On the other hand, integrating the left side of (7.54) by parts, we have

$$\begin{aligned} &\int_{-1}^{1} \psi_n''(x)\, \psi_m(x)\ dx \\ &= \psi_n'(1)\, \psi_m(1) - \psi_n'(-1)\, \psi_m(-1) - \int_{-1}^{1} \psi_n'(x)\, \psi_m'(x)\ dx \\ &= 2\, \psi_n'(1)\, \psi_m(1) - (\psi_n(1)\, \psi_m'(1) - \psi_n(-1)\, \psi_m'(-1)) \\ &\quad + \int_{-1}^{1} \psi_n(x)\, \psi_m''(x)\ dx. \end{aligned} \tag{7.55}$$

We rewrite (7.55) in the form of

$$\begin{aligned} &\int_{-1}^{1} \psi_n''(x)\, \psi_m(x)\ dx \\ &= 2\, (\psi_n'(1)\, \psi_m(1) - \psi_n(1)\, \psi_m'(1)) + \int_{-1}^{1} \psi_n(x)\, \psi_m''(x)\ dx\,. \end{aligned}$$

We combine (7.54) and (7.56) and get

$$\left(\frac{\lambda_m^2}{\lambda_n^2} - 1 \right) \int_{-1}^{1} \psi_n(x)\, \psi_m''(x)\ dx = 2\ (\psi_n'(1)\, \psi_m(1) - \psi_n(1)\, \psi_m'(1))\,. \tag{7.56}$$

Since $m \neq n$, we easily rewrite (7.56) as (7.47). We obtain expression (7.49) by combining (7.53) and (7.47). The identities (7.48), (7.50) follow from (7.47), (7.49) immediately due to Corollary 7.8. □

Lemma 7.14 *Suppose that c is real and positive, and that the integers m and n are non-negative. Let the function Ψ_n be defined by the formula*

$$\Psi_n(y) = \int_{0}^{y} \psi_n(x)\ dx\,. \tag{7.57}$$

Then, $m \neq n \pmod 2$ *implies that*

$$\int_{-1}^{1} \frac{1}{t}\,\psi_n(t)\,\psi_m(t)\;dt \tag{7.58}$$

$$= \; i\,c\,\frac{2\,\lambda_m\,\lambda_n}{\lambda_n^2+\lambda_m^2}\,\Psi_n(1)\,\Psi_m(1) + 2\,\frac{\lambda_m}{\lambda_n^2+\lambda_m^2}\,\Psi_m(1)\,\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\;dt\,; \tag{7.59}$$

$m = n \pmod 2$ *implies that*

$$\int_{-1}^{1} \frac{1}{t}\,\psi_n(t)\,\psi_m(t)\;dt = 0\,. \tag{7.60}$$

Proof. We start with the identity

$$\lambda_n\,\psi_n(x) = \int_{-1}^{1} e^{icxt}\,\psi_n(t)\;dt\,. \tag{7.61}$$

Integrating (7.61) with respect to x, we have

$$\lambda_n \int_0^y \psi_n(x)\,dx$$

$$= \int_0^y \left(\int_{-1}^{1} e^{icxt}\,\psi_n(t)\;dt\right)\;dx \tag{7.62}$$

$$= \int_{-1}^{1} \psi_n(t)\int_0^y e^{ixct}\;dx\,dt \tag{7.63}$$

$$= \frac{1}{i\,c}\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\,e^{icyt}\;dt - \frac{1}{i\,c}\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\;dt\,, \tag{7.64}$$

which we summarize as

$$\lambda_n\,\Psi_n(y) = \frac{1}{i\,c}\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\,e^{icyt}\;dt - \frac{1}{i\,c}\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\;dt\,. \tag{7.65}$$

Taking the inner product of (7.65) and $\psi_m(y)$, we obtain

$$\lambda_n\int_{-1}^{1}\Psi_n(y)\,\psi_m(y)\;dy$$

$$= \frac{1}{i\,c}\int_{-1}^{1}\psi_m(y)\cdot\left(\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\,e^{icyt}\;dt\right)\;dy$$

$$-\frac{1}{i\,c}\int_{-1}^{1}\psi_m(y)\cdot\left(\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\;dt\right)\;dy \tag{7.66}$$

$$= \frac{1}{i\,c}\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\cdot\left(\int_{-1}^{1}e^{icyt}\,\psi_m(y)\;dy\right)\;dt$$

$$-\frac{1}{i\,c}\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\;dt\cdot\int_{-1}^{1}\psi_m(y)\;dy \tag{7.67}$$

$$\begin{aligned} =\;& \frac{\lambda_m}{i\,c}\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\,\psi_m(t)\;dt \\ & -\frac{1}{i\,c}\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\;dt\cdot\int_{-1}^{1}\psi_m(y)\;dy\,, \end{aligned} \tag{7.68}$$

which we summarize as

$$\begin{aligned} &\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\,\psi_m(t)\;dt \\ &= i\,c\,\frac{\lambda_n}{\lambda_m}\int_{-1}^{1}\Psi_n(t)\,\psi_m(t)\;dt+\frac{1}{\lambda_m}\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\;dt\cdot\int_{-1}^{1}\psi_m(y)\;dy\,. \end{aligned} \tag{7.69}$$

Exchanging m with n, we convert (7.69) into

$$\begin{aligned} &\int_{-1}^{1}\frac{1}{t}\,\psi_m(t)\,\psi_n(t)\;dt \\ &= i\,c\,\frac{\lambda_m}{\lambda_n}\int_{-1}^{1}\Psi_m(t)\,\psi_n(t)\;dt+\frac{1}{\lambda_n}\int_{-1}^{1}\frac{1}{t}\,\psi_m(t)\;dt\cdot\int_{-1}^{1}\psi_n(y)\;dy\,. \end{aligned} \tag{7.70}$$

Combining (7.69) and (7.70), we get

$$\begin{aligned} &\frac{\lambda_n}{\lambda_m}\,i\,c\int_{-1}^{1}\Psi_n(t)\,\psi_m(t)\;dt-\frac{\lambda_m}{\lambda_n}\,i\,c\int_{-1}^{1}\Psi_m(t)\,\psi_n(t)\;dt \\ &= \frac{1}{\lambda_n}\int_{-1}^{1}\frac{1}{t}\,\psi_m(t)\;dt\cdot\int_{-1}^{1}\psi_n(t)\;dt-\frac{1}{\lambda_m}\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\;dt\cdot\int_{-1}^{1}\psi_m(t)\;dt\,. \end{aligned} \tag{7.71}$$

Now suppose that m is even and n is odd. Then the first product in the right hand side of (7.71) is zero, so that

$$\begin{aligned} &\frac{\lambda_n}{\lambda_m}\,i\,c\int_{-1}^{1}\Psi_n(t)\,\psi_m(t)\;dt-\frac{\lambda_m}{\lambda_n}\,i\,c\int_{-1}^{1}\Psi_m(t)\,\psi_n(t)\;dt \\ &= -\frac{1}{\lambda_m}\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\;dt\cdot\int_{-1}^{1}\psi_m(t)\;dt\,, \end{aligned} \tag{7.72}$$

which is equivalent to

$$\begin{aligned} &\int_{-1}^{1}\Psi_n(t)\,\psi_m(t)\;dt \\ &= \frac{\lambda_m^2}{\lambda_n^2}\int_{-1}^{1}\Psi_m(t)\,\psi_n(t)\;dt-\frac{1}{\lambda_n}\,\frac{1}{i\,c}\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\;dt\cdot\int_{-1}^{1}\psi_m(t)\;dt\,, \end{aligned} \tag{7.73}$$

or

$$\int_{-1}^{1} \Psi_m(t)\,\psi_n(t)\;dt$$
$$= \frac{\lambda_n^2}{\lambda_m^2}\int_{-1}^{1} \Psi_n(t)\,\psi_m(t)\;dt + \frac{\lambda_n}{\lambda_m^2}\,\frac{1}{i\,c}\int_{-1}^{1} \frac{1}{t}\,\psi_n(t)\;dt \cdot \int_{-1}^{1} \psi_m(t)\;dt\,. \qquad (7.74)$$

On the other hand, integrating the left side of (7.74) by parts, we obtain

$$\int_{-1}^{1} \Psi_m(t)\,\psi_n(t)\;dt$$
$$= \Psi_n(1)\,\Psi_m(1) - \Psi_n(-1)\,\Psi_m(-1) - \int_{-1}^{1} \Psi_n(t)\,\psi_m(t)\;dt\,. \qquad (7.75)$$

Since the product $\Psi_m(x)\,\Psi_n(x)$ is an odd function when $m \neq n \pmod 2$, we rewrite (7.75) as

$$\int_{-1}^{1} \Psi_m(t)\,\psi_n(t)\;dt = 2\,\Psi_n(1)\,\Psi_m(1) - \int_{-1}^{1} \Psi_n(t)\,\psi_m(t)\;dt\,. \qquad (7.76)$$

The combination of (7.74) and (7.76) implies that

$$\int_{-1}^{1} \Psi_n(t)\,\psi_m(t)\;dt + \frac{\lambda_n^2}{\lambda_m^2}\int_{-1}^{1} \Psi_n(t)\,\psi_m(t)\;dt$$
$$= 2\,\Psi_n(1)\,\Psi_m(1) - \frac{\lambda_n}{\lambda_m^2}\,\frac{1}{i\,c}\int_{-1}^{1} \frac{1}{t}\,\psi_n(t)\;dt \cdot \int_{-1}^{1} \psi_m(t)\;dt\,, \qquad (7.77)$$

or

$$\frac{\lambda_m^2 + \lambda_n^2}{\lambda_m^2}\int_{-1}^{1} \Psi_n(t)\,\psi_m(t)\;dt$$
$$= 2\,\Psi_n(1)\,\Psi_m(1) - \frac{\lambda_n}{\lambda_m^2}\,\frac{1}{i\,c}\int_{-1}^{1} \frac{1}{t}\,\psi_n(t)\;dt \cdot \int_{-1}^{1} \psi_m(t)\;dt\,, \qquad (7.78)$$

which is equivalent to

$$\int_{-1}^{1} \Psi_n(t)\,\psi_m(t)\;dt$$
$$= \frac{2\,\lambda_m^2}{\lambda_n^2 + \lambda_m^2}\,\Psi_n(1)\,\Psi_m(1) - \frac{\lambda_n}{\lambda_n^2 + \lambda_m^2}\,\frac{1}{i\,c}\int_{-1}^{1} \frac{1}{t}\,\psi_n(t)\;dt \cdot \int_{-1}^{1} \psi_m(t)\;dt\,. \qquad (7.79)$$

Finally, combining (7.69) and (7.79), we have

$$\int_{-1}^{1} \frac{1}{t}\,\psi_n(t)\,\psi_m(t)\,dt$$
$$= \; i\,c\,\frac{2\,\lambda_m\,\lambda_n}{\lambda_n^2+\lambda_m^2}\,\Psi_n(1)\,\Psi_m(1) + \frac{\lambda_m}{\lambda_n^2+\lambda_m^2}\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\,dt\cdot\int_{-1}^{1}\psi_m(t)\,dt\,. \tag{7.80}$$

Equation (7.60) is easily proven since the product $\frac{1}{t}\,\psi_m(x)\,\psi_n(x)$ is an odd function whenever $m = n \pmod 2$. □

The above lemmas do not use much of the detailed structure of the integral operators of which the functions $\{\psi_j\}$ are eigenfunctions. Thus many of them generalize easily to the case of an operator $L : L^2[0,1] \to L^2[0,1]$ defined via the formula

$$L(\psi)(x) = \int_0^1 K(xt)\,\psi(t)\,dt, \tag{7.81}$$

for some function $K : [0,1] \to \mathbb{C}$. The following lemma is an example of this.

Lemma 7.15 *Let λ_1, λ_2 be two eigenvalues of the operator L defined by (7.81), that is,*

$$\int_0^1 K(xt)\,\psi_1(t)\,dt \;=\; \lambda_1\psi_1(x)\,, \tag{7.82}$$

$$\int_0^1 K(xt)\,\psi_2(t)\,dt \;=\; \lambda_2\psi_2(x)\,. \tag{7.83}$$

Then

$$\frac{\lambda_2}{\lambda_1} = \frac{\displaystyle\int_0^1 x\,\psi_1'(x)\,\psi_2(x)\,dx}{\displaystyle\int_0^1 x\,\psi_2'(x)\,\psi_1(x)\,dx}\,, \tag{7.84}$$

provided that neither λ_1 nor the denominator of the right hand side of (7.84) is zero.

Proof. Differentiating (7.82), (7.83) with respect to x, we get

$$\int_0^1 t\,K'(xt)\,\psi_1(t)\,dt \;=\; \lambda_1\psi_1'(x)\,, \tag{7.85}$$

$$\int_0^1 t\,K'(xt)\,\psi_2(t)\,dt \;=\; \lambda_2\psi_2'(x)\,. \tag{7.86}$$

Multiplying (7.85) by $x\,\psi_2(x)$, we have

$$\lambda_1\, x\, \psi_1'(x)\, \psi_2(x) = x\, \psi_2(x) \int_0^1 t\, K'(xt)\, \psi_1(t)\; dt\,. \tag{7.87}$$

Integrating on the interval $[0,1]$, we obtain

$$\begin{aligned} \lambda_1 \int_0^1 x\, \psi_1'(x)\, \psi_2(x)\; dx &= \int_0^1 x\, \psi_2(x) \int_0^1 t\, K'(xt)\, \psi_1(t)\; dt\, dx \\ &= \int_0^1 t\, \psi_1(t) \int_0^1 x\, K'(xt)\, \psi_2(x)\; dx\, dt\,. \end{aligned} \tag{7.88}$$

Renaming the variables of integration on the right hand side from x to t and vice versa, we get

$$\lambda_1 \int_0^1 x\, \psi_1'(x)\, \psi_2(x)\; dx \;=\; \int_0^1 x\, \psi_1(x) \int_0^1 t\, K'(xt)\, \psi_2(t)\; dt\, dx. \tag{7.89}$$

Substituting (7.86) into (7.89), we obtain

$$\lambda_1 \int_0^1 x\, \psi_1'(x)\, \psi_2(x)\; dx \;=\; \lambda_2 \int_0^1 x\, \psi_1(x)\, \psi_2'(x)\, dx, \tag{7.90}$$

from which (7.84) follows immediately. □

The following lemma establishes the relation between the norm of each function ψ_m on $[-1,1]$ (which in this dissertation is taken to be one), and its norm on $(-\infty,\infty)$.

Lemma 7.16 *For all integer $m \geq 0$,*

$$\int_{-\infty}^{\infty} \psi_m^2(x)\; dx = \frac{1}{\mu_m}\,. \tag{7.91}$$

Proof. Since ψ_m is an eigenfunction of the operator Q_c, we have

$$\int_{-\infty}^{\infty} \psi_m^2(x)\; dx = \int_{-\infty}^{\infty} \left(\frac{1}{\pi \mu_m} \int_{-1}^{1} \psi_m(t)\, \frac{\sin(c\cdot(x-t))}{x-t}\; dt \right) \psi_m(x)\; dx\,. \tag{7.92}$$

Swapping integrals, we have

$$\int_{-\infty}^{\infty} \psi_m^2(x)\; dx = \frac{1}{\mu_m} \int_{-1}^{1} \psi_m(t) \cdot \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(c\cdot(x-t))}{x-t}\, \psi_m(x)\; dx \right) dt\,, \tag{7.93}$$

which is equivalent to

$$\int_{-\infty}^{\infty} \psi_m^2(x)\ dx = \frac{1}{\mu_m} \int_{-1}^{1} \psi_m^2(t)\ dt\,. \tag{7.94}$$

Identity (7.91) immediately follows from (7.94) due to the unit norm of ψ_m. □

The following lemma extends Lemma 7.16 to any band-limited function with band limit c.

Lemma 7.17 *Suppose that c is real and positive, and that $f : \mathbb{R} \to \mathbb{C}$ is a band-limited function with band limit c. Then for all integer $m \geq 0$,*

$$\int_{-\infty}^{\infty} \psi_m(x)\, f(x)\ dx = \frac{1}{\mu_m} \int_{-1}^{1} \psi_m(x)\, f(x)\ dx. \tag{7.95}$$

Proof. Since ψ_m is an eigenfunction of the operator Q_c, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \psi_m(x)\, f(x)\ dx \\ &= \int_{-\infty}^{\infty} \left(\frac{1}{\pi \mu_m} \int_{-1}^{1} \frac{\sin(c \cdot (x-t)\,)}{x-t}\, \psi_m(t)\ dt \right) f(x)\ dx\,. \end{aligned} \tag{7.96}$$

Exchanging integrals, we convert (7.96) into

$$\begin{aligned} &\int_{-\infty}^{\infty} \psi_m(x)\, f(x)\ dx \\ &= \frac{1}{\mu_m} \int_{-1}^{1} \psi_m(t) \cdot \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(c \cdot (x-t))}{x-t}\, f(x)\ dx \right) dt\,. \end{aligned} \tag{7.97}$$

Obviously, the second integral on the right-hand side of (7.97) equals to $f(t)$, since f belongs to the range of the projection operator P_c (see Section 2.3). We thus have

$$\int_{-\infty}^{\infty} \psi_m(x)\, f(x)\ dx = \frac{1}{\mu_m} \int_{-1}^{1} \psi_m(t)\, f(t)\ dt\,. \tag{7.98}$$

□

Lemma 7.18 *Suppose that c is real and positive, and that the integer n is non-negative. Then,*

$$\int_{-\infty}^{\infty} e^{icxt}\,\psi_m(t)dt = \begin{cases} \dfrac{\lambda_m}{\mu_m}\,\psi_m(x), & \textit{if } -1 < x < 1, \\ 0, & \textit{if } x > 1 \ \textit{ or } \ x < -1. \end{cases} \tag{7.99}$$

Proof. Remembering the identity

$$\mu_m \psi_m(t) = \frac{1}{\pi}\int_{-1}^{1} \frac{sin(c\cdot(x-u))}{x-u}\,\psi_m(u)\,du\,, \tag{7.100}$$

we have

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{icxt}\,\psi_m(t)\,dt \\ &= \frac{1}{\mu_m}\int_{-\infty}^{\infty} e^{icxt}\left(\frac{1}{\pi}\int_{-1}^{1}\frac{sin(c\cdot(x-u))}{x-u}\,\psi_m(u)\,du\right)\,dt \qquad (7.101) \\ &= \frac{1}{\mu_m}\int_{-1}^{1}\psi_m(u)\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{sin(c\cdot(x-u))}{x-u}\,e^{icxt}\,dt\right)\,du\,. \qquad (7.102) \end{aligned}$$

Since the innermost integral is the orthogonal projection operator P_c onto the space of functions of band limit c on $(-\infty,\infty)$, it follows that:

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{icxt}\,\psi_m(t)\,dt \\ &= \frac{1}{\mu_m}\int_{-1}^{1}\psi_m(u)\left(\begin{cases} e^{icxu}, & \text{if } -1 < x < 1, \\ 0, & \text{if } x > 1 \text{ or } x < -1 \end{cases}\right)\,du \qquad (7.103) \\ &= \begin{cases} \dfrac{1}{\mu_m}\displaystyle\int_{-1}^{1}\psi_m(u)\,e^{icxu}\,du, & \text{if } -1 < x < 1, \\ 0, & \text{if } x > 1 \text{ or } x < -1, \end{cases} \qquad (7.104) \end{aligned}$$

from which (7.99) immediately follows. □

The following five lemmas provide some simple formulae for the calculation of derivatives of ψ_m, λ_m, μ_m with respect to c.

Lemma 7.19 *For all positive real c and non-negative integer m,*

$$\frac{\partial \lambda_m}{\partial c} = \lambda_m \frac{2\,\psi_m^2(1) - 1}{2\,c}. \tag{7.105}$$

Proof. We start with

$$\lambda_m \psi_m(x) = \int_{-1}^{1} e^{icxt}\,\psi_m(t)\;dt. \tag{7.106}$$

Differentiating (7.106) with respect to c, we obtain

$$\begin{aligned} &\frac{\partial \lambda_m}{\partial c}\,\psi_m(x) + \lambda_m \frac{\partial \psi_m(x)}{\partial c} \\ &= \int_{-1}^{1} i\,x\,t\,e^{icxt}\,\psi_m(t)\;dt + \int_{-1}^{1} e^{icxt}\,\frac{\partial \psi_m(t)}{\partial c}\;dt. \end{aligned} \tag{7.107}$$

Multiplying by $\psi_m(x)$ on both sides of (7.107), and integrating on the interval $[-1,1]$, we have

$$\begin{aligned} &\int_{-1}^{1} \psi_m(x)\left(\frac{\partial \lambda_m}{\partial c}\,\psi_m(x) + \lambda_m \frac{\partial \psi_m(x)}{\partial c}\right)\,dx \qquad (7.108) \\ &= \int_{-1}^{1} \psi_m(x) \int_{-1}^{1} i\,x\,t\,e^{icxt}\,\psi_m(t)\;dt\;dx \\ &\quad + \int_{-1}^{1} \psi_m(x) \int_{-1}^{1} e^{icxt}\,\frac{\partial \psi_m(t)}{\partial c}\;dt\;dx, \end{aligned}$$

which we rewrite as

$$\begin{aligned} &\frac{\partial \lambda_m}{\partial c} + \lambda_m \int_{-1}^{1} \frac{\partial \psi_m(x)}{\partial c}\,\psi_m(x)\;dx \\ &= \int_{-1}^{1} i\,t\,\psi_m(t) \int_{-1}^{1} e^{icxt}\,x\,\psi_m(x)\;dx\;dt \\ &\quad + \int_{-1}^{1} \frac{\partial \psi_m(t)}{\partial c} \int_{-1}^{1} e^{icxt}\,\psi_m(x)\;dx\;dt \qquad (7.109) \\ &= \lambda_m \int_{-1}^{1} i\,t\,\psi_m(t)\,\frac{1}{i\,c}\,\frac{\partial \psi_m(t)}{\partial t}\;dt \\ &\quad + \lambda_m \int_{-1}^{1} \frac{\partial \psi_m(t)}{\partial c}\,\psi_m(t)\;dt. \qquad (7.110) \end{aligned}$$

We summarize the preceding derivations as

$$\frac{\partial \lambda_m}{\partial c} = \frac{\lambda_m}{c} \int_{-1}^{1} t\,\psi_m(t)\,\frac{\partial \psi_m(t)}{\partial t}\;dt. \tag{7.111}$$

On the other hand, integrating the right-hand side of (7.111) by parts, we have

$$\begin{aligned}\int_{-1}^{1} t\,\psi_m(t)\,\frac{\partial\psi_m(t)}{\partial t}\,dt \qquad\qquad (7.112)\\ = \psi_m^2(1)+\psi_m^2(-1)-1-\int_{-1}^{1}\psi_m(t)\,t\,\frac{\partial\psi_m(t)}{\partial t}\,dt\,,\end{aligned}$$

which we rewrite as

$$\int_{-1}^{1} t\,\psi_m(t)\,\frac{\partial\psi_m(t)}{\partial t}\,dt=\psi_m^2(1)-\frac{1}{2}\,. \qquad (7.113)$$

Finally, substituting (7.113) into (7.111), we immediately have

$$\frac{\partial\lambda_m}{\partial c}=\lambda_m\,\frac{2\,\psi_m^2(1)-1}{2\,c}\,. \qquad (7.114)$$

□

Lemma 7.20 *For any positive real c and non-negative integer m,*

$$\frac{\partial\mu_m}{\partial c}=\frac{2}{c}\,\mu_m\,\psi_m^2(1)\,. \qquad (7.115)$$

Proof. We start with the identity

$$\mu_m=\frac{2c}{\pi}\,\overline{\lambda}_m\,\lambda_m. \qquad (7.116)$$

Differentiating (7.116) with respect to c, we get

$$\frac{\partial\mu_m}{\partial c}=\frac{2c}{\pi}\left(\overline{\lambda}_m\,\frac{\partial\lambda_m}{\partial c}+\lambda_m\,\frac{\partial\overline{\lambda}_m}{\partial c}\right)+\frac{2}{\pi}\,\overline{\lambda}_m\,\lambda_m\,. \qquad (7.117)$$

Substituting Lemma 7.19 into (7.117), we get

$$\begin{aligned}\frac{\partial\mu_m}{\partial c} &= \frac{2c}{\pi}\cdot 2\,\overline{\lambda}_m\,\lambda_m\,\frac{2\,\psi_m^2(1)-1}{2\,c}+\frac{2}{\pi}\,\overline{\lambda}_m\,\lambda_m \qquad (7.118)\\ &= 2\,\mu_m\,\frac{2\,\psi_m^2(1)-1}{2\,c}+\frac{1}{c}\,\mu_m\\ &= \frac{2}{c}\,\mu_m\,\psi_m^2(1)-\frac{1}{c}\,\mu_m+\frac{1}{c}\,\mu_m\\ &= \frac{2}{c}\,\mu_m\,\psi_m^2(1)\,. \qquad (7.119)\end{aligned}$$

□

The following lemma immediately follows from Lemmas 7.19 and 7.20.

Lemma 7.21 *For all positive real c and non-negative integer m, n,*

$$\left(\frac{\lambda_m}{\lambda_n}\right)' = \frac{\lambda_m}{\lambda_n}\,\frac{1}{c}\,\left(\psi_m^2(1) - \psi_n^2(1)\right), \tag{7.120}$$

$$\left(\frac{\mu_m}{\mu_n}\right)' = \frac{\mu_m}{\mu_n}\,\frac{2}{c}\,\left(\psi_m^2(1) - \psi_n^2(1)\right). \tag{7.121}$$

Lemma 7.22 *Suppose that c is real and positive, and the integers m, n are non-negative. Then, for all $m \neq n$,*

$$\int_{-1}^{1} \psi_m(t)\,\frac{\partial \psi_n}{\partial c}(t)\;dt = -\frac{2}{c}\,\frac{\lambda_n\,\lambda_m}{\lambda_m^2 - \lambda_n^2}\,\psi_m(1)\,\psi_n(1)\,; \tag{7.122}$$

for all $m = n$,

$$\int_{-1}^{1} \psi_m(t)\,\frac{\partial \psi_n}{\partial c}(t)\;dt = 0. \tag{7.123}$$

Proof. Since the norm of ψ_n on $[-1,1]$ remains constant as c varies, ψ_n must be orthogonal on $[-1,1]$ to its own derivative with respect to c, which immediately yields (7.123). To establish (7.122), we start with the identity

$$\lambda_n \psi_n(x) = \int_{-1}^{1} e^{icxt}\,\psi_n(t)\;dt\,. \tag{7.124}$$

Differentiating (7.124) with respect to c, we get

$$\begin{aligned}&\frac{\partial \lambda_n}{\partial c}\,\psi_n(x) + \lambda_n\,\frac{\partial \psi_n}{\partial c}\\ &= \int_{-1}^{1} \left(i\,x\,t\,e^{icxt}\,\psi_n(t) + e^{icxt}\,\frac{\partial \psi_n(t)}{\partial c}\right)\;dt\,.\end{aligned} \tag{7.125}$$

Multiplying both sides of (7.125) by $\psi_m(x)$ and integrating with respect to x, we have

$$\begin{aligned}&\lambda_n \int_{-1}^{1} \psi_m(x)\,\frac{\partial \psi_n(x)}{\partial c}\;dx\\ &= \frac{\lambda_n}{c}\int_{-1}^{1} x\,\psi_n'(x)\,\psi_m(x)\;dx + \lambda_m \int_{-1}^{1} \psi_m(t)\,\frac{\partial \psi_n(t)}{\partial c}\;dt\,,\end{aligned} \tag{7.126}$$

which, using (7.28), we rewrite as

$$(\lambda_n - \lambda_m) \int_{-1}^{1} \psi_m(t) \, \frac{\partial \psi_n(t)}{\partial c} \, dt$$
$$= \frac{\lambda_n}{c} \frac{\lambda_m}{\lambda_m + \lambda_n} (2\,\psi_m(1)\,\psi_n(1) - \delta_{mn}) \,. \tag{7.127}$$

Assuming that $m \neq n$, and dividing by $\lambda_n - \lambda_m$, we obtain (7.122). □

Lemma 7.23 *Suppose that c is real and positive, and integer m is non-negative. Then*

$$\frac{\partial \chi_m}{\partial c} = 2c \int_{-1}^{1} x^2 \psi_m^2(x). \tag{7.128}$$

Proof. Due to Lemma 2.10,

$$(1 - x^2)\psi_m''(x) - 2\,x\,\psi_m'(x) + (\chi_m - c^2x^2)\,\psi_m(x) = 0. \tag{7.129}$$

Making the infinitesimal changes $c = c + h$, $\chi_m = \chi_m + \varepsilon$, and $\psi_m(x) = \psi_m(x) + \delta(x)$, this becomes

$$(1 - x^2) \cdot (\psi_m''(x) + \delta''(x)) - 2x \cdot (\psi_m'(x) + \delta'(x))$$
$$+ (\chi_m + \varepsilon - (c + h)^2 x^2) \cdot (\psi_m(x) + \delta(x)) = 0. \tag{7.130}$$

Expanding each term, discarding infinitesimals of the second order and greater (i.e. products of two or more of the quantities h, ε, and $\delta(x)$), and subtracting (7.129), we have

$$(1 - x^2)\,\delta''(x) - 2x\delta'(x) + (\chi_m - c^2x^2)\,\delta(x) + (\varepsilon - 2chx^2)\psi_m(x) = 0. \tag{7.131}$$

Defining the self-adjoint differential operator L by the formula

$$L(f)(x) = (1 - x^2)f''(x) - 2xf'(x) + (\chi_m - c^2x^2)f(x)\,, \tag{7.132}$$

multiplying (7.131) by $\psi_m(x)/h$ and integrating on $[-1, 1]$, we get

$$\int_{-1}^{1} L\left(\frac{\partial \psi_m}{\partial c}\right)(x)\,\psi_m(x)\;dx + \frac{\varepsilon}{h} - \int_{-1}^{1} 2cx^2\psi_m^2(x) = 0. \tag{7.133}$$

Now $\frac{\varepsilon}{h} = \frac{\partial \chi_m}{\partial c}$. In addition, since L is self-adjoint,

$$\int_{-1}^{1} L\left(\frac{\partial \psi_m}{\partial c}\right)(x)\,\psi_m(x)\;dx = \int_{-1}^{1} \frac{\partial \psi_m}{\partial c}(x)\,L(\psi_m)(x)\;dx\,. \tag{7.134}$$

But due to (7.129), $L(\psi_m)(x) = 0$ for all $x \in [-1,1]$, so the integrals on both sides of (7.134) are zero. Thus, (7.133) becomes

$$\frac{\partial \chi_m}{\partial c} = 2c\int_{-1}^{1} x^2\psi_m^2(x). \tag{7.135}$$

□

Finally, we consider the integral

$$f(x) = f(a,x) = \int_{-1}^{1} \frac{e^{icxt}}{t-a}\psi_m(t)\;dt. \tag{7.136}$$

Differentiating (7.136) with respect to x, we have

$$\frac{d}{dx}f(a,x) = ic\int_{-1}^{1} \frac{te^{icxt}}{t-a}\psi_m(t)\;dt. \tag{7.137}$$

Multiplying (7.136) by ica, and subtracting it from (7.137), we obtain

$$\begin{aligned} \frac{d}{dx}f(a,x) - icaf(a,x) &= ic\int_{-1}^{1} e^{icxt}\,\psi_m(t)\;dt \\ &= ic\lambda_m\psi_m(x). \end{aligned} \tag{7.138}$$

In other words, f satisfies the differential equation

$$f'(x) - icaf(x) = ic\lambda_m\psi_m(x). \tag{7.139}$$

The standard "variation of parameter" calculation provides the solution to (7.139):

$$f(x) = ic\lambda_m\int_{0}^{x} e^{-ica(x-t)}\psi_m(t)\;dt + f(0)\,e^{icax}. \tag{7.140}$$

Introducing the notation

$$\mathcal{D} = \frac{1}{ic}\circ\frac{d}{dx} \tag{7.141}$$

(i.e. $\mathcal{D}$ is the product of multiplication by $1/ic$ and differentiation), we rewrite (7.8) as

$$\mathcal{D}^k(\psi_m)(x) = \frac{1}{\lambda_m}\int_{-1}^{1} t^k\, e^{icxt}\,\psi_m(t)dt; \tag{7.142}$$

for an arbitrary polynomial P (with real or complex coefficients),

$$P(\mathcal{D})(\psi_m)(x) = \frac{1}{\lambda_m}\int_{-1}^{1} P(t)\, e^{icxt}\,\psi_m(t)dt. \tag{7.143}$$

By the same token, the function ϕ defined by the formula

$$\phi(x) = \int_{-1}^{1} \frac{e^{icxt}}{P(t)}\psi_m(t)dt \tag{7.144}$$

satisfies the differential equation

$$P(\mathcal{D})(\phi)(x) = \lambda_m\psi_m(x). \tag{7.145}$$

8

Generalizations and Conclusions

In this dissertation, we have designed quadrature rules for band-limited functions, based on the properties of Prolate Spheroidal Wave Functions (PSWFs), and the connections of the latter with certain fundamental integral operators (see (2.40), (2.42) in Section 2.3). The quadratures are a surprisingly close analogue for band-limited functions of Gaussian quadratures for polynomials, in that they have positive weights, are optimal in the appropriately defined sense, and their nodes, when used for approximation (as opposed to integration), result in extremely efficient interpolation formulae. Thus, Sections 4.1-4.3 of this dissertation can be viewed as reproducing for band-limited functions much of the standard polynomial-based approximation theory (see, for example, [35]). Generally, there is a striking analogy between the band-limited functions and polynomials.

Obviously, there are certain differences between the resulting apparatus and the standard numerical analysis. To start with, where the classical techniques are optimal for polynomials, the approach of this dissertation is optimal for band-limited functions. Whenever the functions to be dealt with are naturally represented by trigonometric expansions on finite intervals, our quadrature and interpolation formulae tend to be more efficient than those based on polynomials. When the functions to be dealt with are naturally represented by polynomials, the classical approach is more efficient; however, many physical phenomena involve band-limited functions, while few involve polynomials.

Qualitatively, the quadrature (and interpolation) nodes obtained in this dissertation behave like a compromise between the Gaussian nodes and the equispaced nodes: near the middle of the interval, they are very nearly equispaced, and near the ends, they concentrate somewhat, but much less than the Gaussian (or Chebyshev) nodes do. For large c, the distance between nodes near the ends of the interval is of the order $(1/c)^{3/2}$, with the total number of nodes close to c/π. In contrast, the distance between the Gaussian nodes near the ends of the interval is of the order $1/n^2$, with n being the total number of nodes. A closely related phenomenon is the reduced norm of the differentiation operator based on the Prolate expansions: for an $n-$point differentiation formula, the norm is of the order $n^{3/2}$, as opposed to n^2 for polynomial-based spectral differentiation. Thus, PSWFs are likely to be a better tool for the design of spectral and pseudo-spectral techniques, rather than orthogonal polynomials and related functions.

The asymptotic expressions of Prolate Spheroidal Wave Functions and eigenvalues of differential equation (2.47) are a small addition to the literature of prolate functions. The formulae for ψ_m, which converge uniformly to ψ_m on $\mathbb{R}$, show a striking connection between Hermite functions and PSWFs. In the numerical aspect, these formulae are also an effective tool for the evaluation of PSWFs, eigenvalues of related operators, and, to some degree, derivatives of PSWFs. Since the number of operations involved in the evaluation of a prolate wave function of band-limit c using the classical scheme is $O(c^2)$, the traditional process becomes cumbersome when c becomes large. In such cases, the asymptotic expansions provide a convenient alternative; the accuracies of these formulae are satisfactory, especially when the desired prolate functions have relatively low orders.

Much of the analytical apparatus we use was developed more than 30 years ago (see [31]-[32], [17], [18]); the fundamental importance of these results in certain areas of electrical engineering and physics has also been understood for a long time. However, there appears to have been no prior attempt made to view band-limited functions as a source of *numerical* algorithms. Generally, there is a fairly limited amount of information in the literature about PSWFs, especially when compared to the wealth of facts on many other special functions. Chapter 7 of this dissertation is an attempt to remedy this situation to a small degree.

The apparatus built in this dissertation is strictly one-dimensional. Obviously, one can construct discretizations of rectangles, cubes, etc. by using direct products of one-dimensional grids. The resulting numerical algorithms are satisfactory but not optimal. Furthermore, representation of band-limited functions on regions in higher dimensions is of both theoretical and engineering interest. Obvious applications include seismic data collection and processing, antenna theory, NMR imaging, and many others. When the region of interest is a disc, most of the necessary analytical apparatus can be found in [32]. Applications on more general regions pose much more difficult questions.

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