This paper generalizes Malvar-Coifman-Meyer (MCM) wavelets by extending the choice of bell functions. We dispense with the orthonormality of MCM wavelets to produce a family of smooth local trigonometric bases that efficiently compress trigonometric functions. Any such basis is, in general, not orthogonal, but any element of the dual basis differs from the corresponding element of the original basis only by the shape of the bell. Furthermore, in our scheme the bell functions are bounded by 1 and the dual bell functions are bounded by $(2^{1/2}+1)/2 \approx 1.2$. These bounds ensure the numerical stability of the forward and the inverse transformations in these bases. Numerical examples demonstrate that in many cases the proposed bases provide substantially better (up to a factor of two) compression than the standard MCM wavelets.

Optimized Local Trigonometric Bases

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1. Introduction

The problem of compression of digital data by means of transform coding has been attracting an increasing amount of interest during the last decade both in signal processing (see, for example, [12], [17]) and numerical analysis (see, for example, [2], [4]). This type of data compression usually involves approximating (with a specified error) a function defined in its domain at n points by a linear combination of $m \ll n$ elements of an appropriately chosen basis. In particular, efficient compression of trigonometric functions of the form

$$f(x) = \cos(\omega x + \alpha) \tag{1}$$

where ω and α are arbitrary constants, is a problem of significant importance. The interest in this problem is stimulated by the fact that in many areas of signal processing and numerical analysis one often encounters functions $c \in L^2(\mathbb{R})$ whose domain can be divided into a relatively small number of segments I_j in such a way that

$$c(x) = \sum_{n=0}^{M_j} A_n^j \cos(\omega_n^j x + \alpha_n^j) + \epsilon_j(x) \quad \text{for all } x \in I_j, \tag{2}$$

where A_n^j, ω_n^j, and α_n^j are real and independent of x, $M_j \approx 1$, and $\max\limits_{x \in I_j} |\epsilon_j(x)| / \max\limits_{\{n\}} \left|A_n^j\right| \ll 1$.

A powerful tool for the compression of functions (1) are orthonormal local trigonometric bases discovered by Malvar [14] and Coifman and Meyer [8]. These bases are a generalization of the well known trigonometric bases on $[0, 1]$ (see, for example, Chap. 2 of Tolstov [19]), in the sense that any element of these bases is a product of a function b, that has a compact support, and an appropriate trigonometric function (see Subsection 2.1 for more details) . We will refer to such bases as *Malvar-Coifman-Meyer (MCM) bases* or *MCM wavelets.*

The principal goal of this paper is to introduce a family of non-orthonormal bases that provide an efficient compression of functions (1), and bases dual to them. These objects are closely related to MCM wavelets: any element of a basis is a product of a compactly supported function b and a trigonometric function, while the corresponding element of the dual basis is a product of a compactly supported function $\tilde{b}$ (uniquely determined by the function b) and the same trigonometric function. Moreover, the functions b and $\tilde{b}$ can be chosen in such a manner that for all $x \in \mathbb{R}$,

$$0 \le b(x) \le 1, \tag{3}$$

$$0 \le \tilde{b}(x) \le (2^{1/2}+1)/2 \approx 1.2, \tag{4}$$

which ensures the numerical stability of the transformations to and from the resulting bases. Note that any such basis is uniquely determined by the function b.

The plan of the paper is as follows. In the remainder of this section we review some of the existing techniques for the compression of functions. In Section 2 we construct bi-orthonormal bases that are a generalization of MCM wavelets. In Section 3 we formulate the variational problem of the computation of functions b leading to efficient compression of trigonometric functions (1) while maintaining the bounds (3) and (4). This problem is solved exactly in Section 4. In Section 5 we compare the performance of our scheme with that of other algorithms. Finally, in Appendix we discuss an alternative construction leading to a somewhat different type of bases.

1.1. MCM Bases

The point of departure for the construction of MCM wavelets is the following well known theorem, that in a slightly different form can be found, for example, in Chap. 2 of Tolstov [19].

Theorem 1.1. *The sequence of functions*

$$\{2^{1/2}\sin(n+1/2)\pi x\}, \tag{5}$$

where $n = 0, 1, 2, \ldots$, *is an orthonormal basis on* $[0,1]$. •

The simplest version of MCM wavelets based on the system (5) is given by the theorem below (see, for example, Chap. 6 of Meyer [16])

Theorem 1.2. *Let* $b : \mathbb{R} \to \mathbb{R}$ *be an arbitrary function such that*

$$b^2(x) + b^2(-x) = 1 \quad \text{for all } -1/2 \le x \le 1/2, \tag{6}$$

$$b(x) = b(1-x) \quad \text{for all } 1/2 < x \le 3/2, \tag{7}$$

$$b(x) = 0 \quad \text{otherwise.} \tag{8}$$

Then the sequence of functions

$$u_n^k(x) = u_n^0(x-k), \tag{9}$$

where $n = 0, 1, 2, ...$, $k = 0, \pm 1, \pm 2, ...$, *and*

$$u_n^0(x) = 2^{1/2} b(x) \sin(n + 1/2)\pi x, \tag{10}$$

is an orthonormal basis on the real line. •

The function b in (10) is usually referred to as *the bell function* or *the bell.* Interesting and useful extensions of Theorem 1.2 can be found, for example, in [3] and [16].

Remark 1.1. Let the sequence of coefficients $\{f_n^k\}$ be defined via the formula

$$f_n^k = \int_{-\infty}^{\infty} f(x) u_n^k(x) dx. \tag{11}$$

Then Theorem 1.2 implies that

$$f(x) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} f_n^k u_n^k(x) \tag{12}$$

almost everywhere on the real line. •

Recent applications of MCM bases to compression of functions can be found in [1], [5], and [6].

1.2. Bi-Orthonormal Bases

An elegant and important generalization of the concept of orthonormal wavelet bases are the so called *bi-orthonormal* or *Riesz bases* (see, for example, Chap. 8 of Daubechies [9]). The principal analytical tool in this case is the following well known theorem, that in a slightly different form can be found, for example, in Chap. 4 of Meyer [16].

Theorem 1.3. *Let* $\{\phi_n\}$ *and* $\{\tilde{\phi}_n\}$ *be two bases in* $L^2(\mathbb{R})$ *such that*

$$\int_{-\infty}^{\infty} \phi_n(x) \tilde{\phi}_m(x) dx = \delta_{nm}, \tag{13}$$

where δ_{nm} *is Kronecker's delta. Then for any* $f \in L^2(\mathbb{R})$,

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n \tilde{\phi}_n(x), \tag{14}$$

where

$$\alpha_n = \int_{-\infty}^{\infty} f(x)\phi_n(x)dx. \tag{15}$$

Similarly,

$$f(x) = \sum_{n=-\infty}^{\infty} \tilde{\alpha}_n \phi_n(x). \tag{16}$$

where

$$\tilde{\alpha}_n = \int_{-\infty}^{\infty} f(x)\tilde{\phi}_n(x)dx. \bullet \tag{17}$$

Usually the system $\{\tilde{\phi}_n\}$ is referred to as the basis *dual* to $\{\phi_n\}$.

The following theorem and its corollary are immediate consequences of Theorem 1.3.

Theorem 1.4. *Let $\{\phi_n\}$ and $\{\tilde{\phi}_n\}$ be bi-orthonormal bases in $L^2(\mathbb{R})$. Then for any $f \in L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R})$,*

$$\int_{-\infty}^{\infty} f(x)g(x)dx = \sum_{n=-\infty}^{\infty} f_n \tilde{g}_n = \sum_{n=-\infty}^{\infty} \tilde{f}_n g_n \tag{18}$$

where

$$f_n = \int_{-\infty}^{\infty} f(x)\phi_n(x)dx, \tag{19}$$

$$g_n = \int_{-\infty}^{\infty} g(x)\phi_n(x)dx, \tag{20}$$

and

$$\tilde{f}_n = \int_{-\infty}^{\infty} f(x)\tilde{\phi}_n(x)dx, \tag{21}$$

$$\tilde{g}_n = \int_{-\infty}^{\infty} g(x)\tilde{\phi}_n(x)dx. \bullet \tag{22}$$

Corollary 1.1. *For any $f \in L^2(\mathbb{R})$,*

$$\int_{-\infty}^{\infty} f^2(x)dx = \sum_{n=-\infty}^{\infty} f_n \tilde{f}_n, \tag{23}$$

where the coefficients f_n and $\tilde{f}_n$ are defined in (19) and (21). $\bullet$

A recent example of data compression in bi-orthonormal bases can be found in [13].

2. Construction of Local Trigonometric Bases

In this section we construct local trigonometric bases (and their dual bases) with bell functions that do not necessarily satisfy the condition (6). These bi-orthonormal bases are a generalization of MCM wavelets.

2.1. Notation and Definitions

This subsection contains basic notation and definitions to be used in the remainder of the paper.

Let a continuous function $w : [-1/2, 1/2] \rightarrow \mathbb{R}$ satisfy the condition

$$w^2(x) + w^2(-x) \neq 0. \tag{24}$$

The function $\tilde{w} : [-1/2, 1/2] \rightarrow \mathbb{R}$ is defined by the formula

$$\tilde{w}(x) \stackrel{\text{def}}{=} \frac{w(x)}{w^2(x) + w^2(-x)}. \tag{25}$$

The bell function (or *the bell*) $b : \mathbb{R} \rightarrow \mathbb{R}$ will be defined by the formula

$$b(x) \stackrel{\text{def}}{=} \begin{cases} w(x) & \text{for all} \quad -1/2 \leq x \leq 1/2, \\ w(1-x) & \text{for all} \quad 1/2 < x \leq 3/2, \\ 0 & \text{otherwise,} \end{cases} \tag{26}$$

where w is an arbitrary function satisfying (24). The function $\tilde{b} : \mathbb{R} \rightarrow \mathbb{R}$ defined by the formula

$$\tilde{b}(x) \stackrel{\text{def}}{=} \begin{cases} \tilde{w}(x) & \text{for all} \quad -1/2 \leq x \leq 1/2, \\ \tilde{w}(1-x) & \text{for all} \quad 1/2 < x \leq 3/2, \\ 0 & \text{otherwise,} \end{cases} \tag{27}$$

with $\tilde{w}$ defined in (25) will be called *the dual bell function* (or *the dual bell*) to b.

Let $\theta : \mathbb{R} \rightarrow \mathbb{R}$ be defined by the formula

$$\theta(x) \stackrel{\text{def}}{=} \left(\sum_{k=-\infty}^{\infty} b^2(x-k) \right)^{-1/2}. \tag{28}$$

The following properties of θ are immediate consequences of (26) and (28):

$$\theta(x-m) = \theta(x) \quad \text{for all } m = 0, \pm 1, \pm 2, ..., \tag{29}$$

$$\theta^2(-x) = \theta^2(x) = \frac{1}{b^2(x)+b^2(-x)} \equiv \frac{1}{w^2(x)+w^2(-x)} \quad \text{for all } -1/2 \le x \le 1/2, \tag{30}$$

$$\theta(1-x) = \theta(x) \quad \text{for all } 1/2 < x \le 3/2. \tag{31}$$

The combination of (25) – (27), (30), and (31) yields

$$\tilde{b}(x) = \theta^2(x) \cdot b(x). \tag{32}$$

Next, for $k = 0, \pm1, \pm2...$ and $n = 0, 1, 2, ...$ we define (locally supported) functions $u_n^k : \mathbb{R} \to \mathbb{R}$ and $\tilde{u}_n^k : \mathbb{R} \to \mathbb{R}$ by the formulae

$$u_n^k(x) \stackrel{\text{def}}{=} u_n^0(x-k), \tag{33}$$

$$\tilde{u}_n^k(x) \stackrel{\text{def}}{=} \tilde{u}_n^0(x-k), \tag{34}$$

where

$$u_n^0(x) \stackrel{\text{def}}{=} 2^{1/2} b(x) \sin(n+1/2)\pi x, \tag{35}$$

$$\tilde{u}_n^0(x) \stackrel{\text{def}}{=} 2^{1/2} \tilde{b}(x) \sin(n+1/2)\pi x. \tag{36}$$

The functions b and $\tilde{b}$ in (35) and (36) are defined by (26) and (27), respectively. Since functions u_n^k and $\tilde{u}_n^k$ are locally supported, they will be referred to as functions belonging to a k-th interval. In Theorem 2.1 below we show that the collections of functions $\{u_n^k\}$ and $\{\tilde{u}_n^k\}$ are bi-orthonormal bases.

Let α be a real number and suppose that $f \in L^2([\alpha-1/2, \alpha+1/2])$ and $w \in L^2([-1/2, 1/2])$. Suppose further that w satisfies the condition (24). Then *the folding operator* $F_w^\alpha : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and *the unfolding operator* $U_w^\alpha : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are defined by the formulae

$$F_w^\alpha(f)(x) \stackrel{\text{def}}{=} \begin{cases} f(x)\cdot w(x-\alpha) - f(2\alpha - x)\cdot w(\alpha - x) & \text{for all} \quad \alpha \le x \le \alpha + 1/2, \\ f(x)\cdot w(\alpha - x) + f(2\alpha - x)\cdot w(x-\alpha) & \text{for all} \quad \alpha - 1/2 \le x < \alpha, \\ 0 & \text{otherwise,} \end{cases} \tag{37}$$

and

$$U_w^\alpha(f)(x) \stackrel{\text{def}}{=} \begin{cases} f(x)\cdot w(x-\alpha) + f(2\alpha - x)\cdot w(\alpha - x) & \text{for all} \quad \alpha \le x \le \alpha + 1/2, \\ f(x)\cdot w(\alpha - x) - f(2\alpha - x)\cdot w(x-\alpha) & \text{for all} \quad \alpha - 1/2 \le x < \alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

The combination of (25), (29) – (31), and (37) yields

$$F_{\tilde{w}}^{\alpha}(f)(x) = \theta^2(x) \cdot F_{w}^{\alpha}(f)(x). \tag{39}$$

Next, for $k = 0, \pm 1, \pm 2...$ and $n = 0, 1, 2, ...$ we define auxiliary (locally supported) functions $v_n^k : \mathbb{R} \to \mathbb{R}$ via the formula

$$v_n^k(x) \stackrel{\text{def}}{=} u_n^k(x) \cdot \theta(x), \tag{40}$$

where the functions u_n^k are defined by (33) and (35). Note that due to (32) the relation (40) can be rewritten in an equivalent form

$$v_n^k(x) = \tilde{u}_n^k(x)/\theta(x), \tag{41}$$

where the functions $\tilde{u}_n^k$ are defined by (34) and (36).

Finally, for any basis $\{u_n^k\}$ we define its *condition number* r_{cond} by the formula

$$r_{cond} \stackrel{\text{def}}{=} \max_{x \in [-1/2, 3/2]} \tilde{b}(x). \tag{42}$$

Remark 2.1. It is easy to see that as long as (3) holds, $\tilde{b} \geq 0$, and r_{cond} is reasonably small ($r_{cond} \approx 1$), both functions u_n^k and $\tilde{u}_n^k$ are bounded by a number of order 1, and therefore forward and inverse transforms in bases $\{u_n^k\}$ are numerically stable. In this respect the definition (42) serves the same purpose as the definitions of the condition number for other linear transformations (see, for example, the corresponding definition for matrices in Chap. 4 of Stoer and Bulirsch [18]).•

2.2. Bases $\{u_n^k\}$ and $\{\tilde{u}_n^k\}$

In this subsection we establish analogues of Theorems 1.3 and 1.4 for the collections of functions $\{u_n^k\}$ and $\{\tilde{u}_n^k\}$ defined in (33) – (36). We start with the following lemma, which will be used to reduce the proofs of bi-orthonormality and completeness of $\{u_n^k\}$ and $\{\tilde{u}_n^k\}$ to simple manipulations with MCM bases.

Lemma 2.1. *Suppose that an arbitrary function $w \in L^2([-1/2, 1/2])$ satisfies (24) and the bell function b is defined in (26). Then any set of functions $\{v_n^k\}$ defined in (40) is an MCM basis.*

Proof. For any $k = 0, \pm 1, \pm 2, \ldots$ and $n = 0, 1, 2, \ldots$ the combination of (29), (33), (35), and (40) yields

$$v_n^k(x) = v_n^0(x - k), \tag{43}$$

with

$$v_n^0(x) = 2^{1/2} \cdot b(x) \cdot \theta(x) \sin(n + 1/2)\pi x, \tag{44}$$

while from (26), (30), and (31) we have

$$b^2(x) \cdot \theta^2(x) + b^2(-x) \cdot \theta^2(-x) = 1 \quad \text{for all } -1/2 \leq x \leq 1/2, \tag{45}$$

and

$$b(x) \cdot \theta(x) = b(1 - x) \cdot \theta(1 - x) \quad \text{for all } 1/2 < x \leq 3/2. \tag{46}$$

Next, due to (26),

$$b(x) \cdot \theta(x) = 0 \quad \text{for all } |x - 1/2| > 1. \tag{47}$$

Now we see from (43) – (47) that the collection of functions $\{v_n^k\}$ satisfies all the conditions of Theorem 1.2 (with the product $\theta \cdot b$ playing the role of the bell function b), and therefore it is an MCM basis. •

The following theorem proves the bi-orthonormality and completeness of the collections of functions $\{u_n^k\}$ and $\{\tilde{u}_n^k\}$.

Theorem 2.1. *Suppose that the collections of functions $\{u_n^k\}$ and $\{\tilde{u}_n^k\}$ are defined in (33) – (36) with the bell functions b (26) defined in Lemma 2.1. Let $n, m, k,$ and l be integers such that $n \geq 0$, $m \geq 0$. Then*

$$\int_{-\infty}^{\infty} u_n^k(x)\tilde{u}_m^l(x)dx = \delta_{kl}\delta_{nm}. \tag{48}$$

Moreover, let for an arbitrary function $f \in L^2(\mathbb{R})$ the coefficients f_n^k be defined via the formula

$$f_n^k = \int_{-\infty}^{\infty} f(x)u_n^k(x)dx. \tag{49}$$

Then

$$f(x) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} f_n^k \tilde{u}_n^k(x), \tag{50}$$

almost everywhere on the real line.

Similarly, if

$$\tilde{f}_n^k = \int_{-\infty}^{\infty} f(x)\tilde{u}_n^k(x)dx, \tag{51}$$

then

$$f(x) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} \tilde{f}_n^k u_n^k(x), \tag{52}$$

almost everywhere on the real line.

Proof. The formula (48) is an immediate consequence of the orthonormality of the system $\{v_n^k\}$ and the relation

$$u_n^k(x) \cdot \tilde{u}_m^l(x) = v_n^k(x) \cdot v_m^l(x), \tag{53}$$

which in turn immediately follows from (40) and (41).

The equality (50) can be easily proven by expanding an auxiliary function $p \in L^2(\mathbb{R})$, defined by the formula

$$p(x) = f(x)/\theta(x), \tag{54}$$

in the basis $\{v_n^k\}$. In fact, it follows from (11) and (12) that if

$$p_n^k = \int_{-\infty}^{\infty} p(x)v_n^k(x)dx, \tag{55}$$

then

$$p(x) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} p_n^k v_n^k(x) \tag{56}$$

almost everywhere on the real line. Now (50) is a consequence of (54) – (56), (40), and (41).

The proof of (52) can be obtained by expanding an auxiliary function $q \in L^2(\mathbb{R})$ defined by the formula

$$q(x) = f(x) \cdot \theta(x) \tag{57}$$

in the basis $\{v_n^k\}$, and repeating the proof of (50) almost verbatim. •

Remark 2.2. Since the functions u_n^k and $\tilde{u}_n^k$ ($k = 0, \pm 1, \pm 2, ..., n = 0, 1, 2, ...,$) are periodic with period 1 (see the formulae (33) and (34)) while the functions u_n^0 and $\tilde{u}_n^0$ are locally supported on $[-1/2, 3/2]$, the definitions (49) and (51) can be rewritten in the form

$$f_n^k = \int_{k-1/2}^{k+3/2} f(x)u_n^k(x)dx, \tag{58}$$

and

$$\tilde{f}_n^k = \int_{k-1/2}^{k+3/2} f(x)\tilde{u}_n^k(x)dx. \bullet \tag{59}$$

Finally, Theorem 2.2 and Corollary 2.1 below are particular cases of Theorem 1.4 and Corollary 1.1, and they immediately follow from Theorem 2.1.

Theorem 2.2. *For any $f \in L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R})$,*

$$\int_{-\infty}^{\infty} f(x)g(x)dx = \sum_{k=-\infty}^{\infty}\sum_{n=0}^{\infty} f_n^k \tilde{g}_n^k = \sum_{k=-\infty}^{\infty}\sum_{n=0}^{\infty} \tilde{f}_n^k g_n^k, \tag{60}$$

where f_n^k, g_n^k and $\tilde{f}_n^k, \tilde{g}_n^k$ are the expansion coefficients of the functions f and g in the bases $\{u_n^k\}$ and $\{\tilde{u}_n^k\}$, respectively. $\bullet$

Corollary 2.1. *For any $f \in L^2(\mathbb{R})$,*

$$\int_{-\infty}^{\infty} f^2(x)dx = \sum_{k=-\infty}^{\infty} \|f\|_{(k)}^2, \tag{61}$$

where

$$\|f\|_{(k)} \stackrel{\text{def}}{=} \left(\sum_{n=0}^{\infty} f_n^k \tilde{f}_n^k\right)^{1/2}. \quad \bullet \tag{62}$$

In Remark 2.3 below we show that $\|f\|_{(k)}$ (62) satisfies all the conditions of a norm and we will call this parameter *the norm* of f on the k-th interval.

2.3. Folding of a Function.

The implementation of forward and backward transforms in bases $\{u_n^k\}$ as well as certain proofs can be simplified by means of *foldings* [3], [1]. We begin with a theorem, that establishes a connection between folding F (37) and unfolding U (38) operators; its proof is an immediate consequence of the definitions (25), (37), and (38)

Theorem 2.3. *Let $w : [-1/2, 1/2] \to \mathbb{R}$ satisfy the condition (24) and $\tilde{w} : [-1/2, 1/2] \to \mathbb{R}$ be defined via (25). Then*

$$U_{\tilde{w}}^{\alpha} \cdot F_w^{\alpha} = 1, \tag{63}$$

and

$$U_w^{\alpha} \cdot F_{\tilde{w}}^{\alpha} = 1, \tag{64}$$

where the operators F and U are defined by the formulae (37) and (38), respectively. •

Theorem 2.4 below expresses coefficients f_n^k and $\tilde{f}_n^k$ of an arbitrary function $f \in L^2(\mathbb{R})$ as standard Fourier coefficients of certain auxiliary functions.

Theorem 2.4. *Suppose that the function $\hat{f}_k : [k, k+1] \to \mathbb{R}$ is defined via the formula*

$$\hat{f}_k(x) \stackrel{\text{def}}{=} F_w^k(f)(x) + F_w^{k+1}(f)(x), \tag{65}$$

where $f \in L^2(\mathbb{R})$ is an arbitrary function and the operator F is defined by (37). Then

$$f_n^k = 2^{1/2} \int_k^{k+1} \hat{f}_k(x) \cdot \sin(n + 1/2)\pi(x - k) dx, \tag{66}$$

and

$$\hat{f}_k(x) = 2^{1/2} \sum_{n=0}^{\infty} f_n^k \sin(n + 1/2)\pi(x - k) \tag{67}$$

almost everywhere on $[k, k+1]$.

Similarly,

$$\tilde{f}_n^k = 2^{1/2} \int_k^{k+1} \theta^2(x) \cdot \hat{f}_k(x) \cdot \sin(n + 1/2)\pi(x - k) dx, \tag{68}$$

and

$$\theta^2(x) \cdot \hat{f}_k(x) = 2^{1/2} \sum_{n=0}^{\infty} \tilde{f}_n^k \sin(n + 1/2)\pi(x - k), \tag{69}$$

almost everywhere on $[k, k+1]$.

Proof. Observing that $\sin(n + 1/2)\pi(x - k)$ is an odd function of $x - k$ and an even function of $x - k - 1$, and combining (58) and (37) we immediately obtain (66). Analogously, (68) immediately follows from (59), (39) and (65). The formulae (67) and (69) are immediate consequences of (66) and (68), respectively, and Theorem 1.1. •

Now the expansion coefficients f_n^k can be obtained by first folding the function f at every integer point $\alpha = k$ by means of (37) (which produces $\hat{f}_k$ on every segment $x \in [k, k+1]$), and after that evaluate standard sine coefficients (66) on all the segments. Conversely, to apply the inversion formula (50) one can first compute the functions (65) for every integer k by summing the sine series (67), and then unfold the obtained function for every k via (38) and (63).

Remark 2.3 It is easy to show that the parameter $||f||_{(k)}$ defined in (62) is a norm. Indeed, combining (67) and (69) we have

$$\int_k^{k+1} \theta^2(x) \cdot \hat{f}_k^2(x) dx = \sum_{n=0}^{\infty} f_n^k \tilde{f}_n^k \equiv ||f||_{(k)}^2, \tag{70}$$

i.e. $||f||_{(k)}$ is the L^2 norm of $\hat{f}_k$ with the weight θ^2 •

Remark 2.4 It immediately follows from (67) that

$$||\hat{f}_k||_2 = \left(\sum_{n=0}^{\infty} \left(f_n^k \right)^2 \right)^{1/2} . \bullet \tag{71}$$

3. Optimization of Bell Functions: Statement of the Problem

In this section we formulate a problem of the construction of bases $\{u_n^k\}$ that efficiently compress trigonometric functions (1) and whose condition number $r_{cond} \approx 1$. Since any basis $\{u_n^k\}$ is uniquely determined by the bell function b this problem can be formulated as the problem of an appropriate selection of b. The principal result of this section consists in reducing the problem of an appropriate selection of the bell b to a minimization problem for a quadratic functional (see the formula (125) below).

3.1. Compression in Bases $\{u_n^k\}$

Since bases $\{u_n^k\}$ are, in general, not orthogonal the computation of relative errors of compression of functions in these bases can become very difficult. Consequently, any optimization of these bases (i.e. selection of such bases that with a given error approximate functions of a specified class with the least number of coefficients) becomes a complicated problem. In this subsection we observe, however, that the optimization of a wide class of bases $\{u_n^k\}$ is almost equivalent to a simpler problem. We begin with a lemma that expresses (62) in a form containing only coefficients f_n^k; its proof is an immediate consequence of (67) and (70).

Lemma 3.1. *For any integer k,*

$$||f||_{(k)}^2 = 2 \int_k^{k+1} \theta^2(x) \cdot \left(\sum_{n=0}^{\infty} f_n^k \sin(n + 1/2)\pi(x-k) \right)^2 dx, \tag{72}$$

where $\|f\|_{(k)}$ *is defined by (62).* •

The formulae (73) and (74) below introduce two parameters that play an important role in the formulation of the optimization problem.

For a function $f \in L^2$ let $\{f_n^k\}$ denote its set of coefficients with respect to a basis $\{u_n^k\}$. For every k, we denote by $S_\epsilon^k(f,b)$ any subset of $\{f_n^k\}$ that consists of the least number of coefficients for which the following inequality is satisfied:

$$\delta(S_\epsilon^k(f,b)) \stackrel{\text{def}}{=} \left(\frac{\|f\|_{(k)}^2 - 2\int_k^{k+1} \theta^2(x) \cdot \left(\sum_{n \in S_\epsilon^k(f,b)} f_n^k \sin(n+1/2)\pi(x-k) \right)^2 dx}{\|f\|_{(k)}^2} \right)^{1/2} =$$

$$\left(\frac{\int_k^{k+1} \theta^2(x) \cdot \left(\sum_{n \notin S_\epsilon^k(f,b)} f_n^k \sin(n+1/2)\pi(x-k) \right)^2 dx}{\int_k^{k+1} \theta^2(x) \cdot \left(\sum_{n=0}^{\infty} f_n^k \sin(n+1/2)\pi(x-k) \right)^2 dx} \right)^{1/2} \leq \epsilon. \tag{73}$$

The number of coefficients belonging to $S_\epsilon^k(f,b)$ will be denoted by $N_\epsilon^k(f,b)$). In (73), $\epsilon \in (0,1)$ is a specified (small) number and b is the bell of the basis $\{u_n^k\}$. The parameter $\delta(S_\epsilon^k(f,b))$ is the relative error in the norm (62) of compressing f on a k-th interval, i.e. setting to zero all the coefficients f_n^k that do not belong to the subset $S_\epsilon^k(f,b)$. Note that $S_\epsilon^k(f,b)$ depends on f, b, and ϵ.

Similarly to (73), for every k we choose a finite subset $\hat{S}_\epsilon^k(f,b)$ consisting of the least number of coefficients $\hat{N}_\epsilon^k(f,b)$ for which the following inequality holds:

$$\hat{\delta}(\hat{S}_\epsilon^k(f,b)) \stackrel{\text{def}}{=} \left(\frac{\int_k^{k+1} \left(\sum_{n \notin \hat{S}_\epsilon^k(f,b)} f_n^k \sin(n+1/2)\pi(x-k) \right)^2 dx}{\int_k^{k+1} \hat{f}_k^2(x) dx} \right)^{1/2} = \left(\frac{\sum_{n \notin \hat{S}_\epsilon^k(f,b)} (f_n^k)^2}{\sum_{n=0}^{\infty} (f_n^k)^2} \right)^{1/2} \leq \epsilon. \tag{74}$$

In (74) the equality of the first and second fractions is a consequence of the orthogonality of functions $\sin(n+1/2)\pi(x-k)$ $(n=0,1,...)$ on $[k,k+1]$ and the formula (67). Analogously to (73) the parameter $\hat{\delta}(\hat{S}_\epsilon^k(f,b))$ is the relative L^2 error of compressing the function $\hat{f}_k$, which immediately follows from (71).

Theorem 3.1 below establishes a connection between relative errors δ (73) and $\hat{\delta}$ (74).

Theorem 3.1. *For any continuous θ defined in (28) and $\hat{f}_k$ defined in (67) there exists $\kappa \in \mathbb{R}_+$ such that*

$$\delta(\hat{S}_\epsilon^k(f,b)) \le \kappa \cdot \epsilon, \tag{75}$$

and

$$\hat{\delta}(S_\epsilon^k(f,b)) \le \kappa \cdot \epsilon. \tag{76}$$

Furthermore, κ satisfies the inequality

$$1/\Theta \le \kappa \le \Theta, \tag{77}$$

where Θ is defined by the formula

$$\Theta \stackrel{\text{def}}{=} \frac{\max\limits_{x\in[0,1/2]} \theta(x)}{\min\limits_{x\in[0,1/2]} \theta(x)}. \tag{78}$$

Proof. We start with the proof of (75). This inequality can be written as

$$\left(\frac{\int_k^{k+1} \theta^2(x) \cdot \left(\sum\limits_{n \notin \hat{S}_\epsilon^k(f,b)} f_n^k \sin(n+1/2)\pi(x-k) \right)^2 dx}{\int_k^{k+1} \theta^2(x) \cdot \left(\sum\limits_{n=0}^{\infty} f_n^k \sin(n+1/2)\pi(x-k) \right)^2 dx} \right)^{1/2} \le \kappa \cdot \epsilon, \tag{79}$$

which is an immediate consequence of (73).

Combining the mean value theorem with (29) and (31) we obtain

$$\begin{aligned} &\int_k^{k+1} \theta^2(x) \cdot \left(\sum_{n \notin \hat{S}_\epsilon^k(s,b)} f_n^k \sin(n+1/2)\pi(x-k) \right)^2 dx = \\ &\theta^2(\xi_1) \cdot \int_k^{k+1} \left(\sum_{n \notin \hat{S}_\epsilon^k(f,b)} f_n^k \sin(n+1/2)\pi(x-k) \right)^2 dx, \end{aligned} \tag{80}$$

and

$$\int_k^{k+1} \theta^2(x) \cdot \left(\sum_{n=0}^{\infty} f_n^k \sin(n+1/2)\pi(x-k) \right)^2 dx =$$

$$\theta^2(\xi_2) \cdot \int_k^{k+1} \left(\sum_{n=0}^{\infty} f_n^k \sin(n+1/2)\pi(x-k) \right)^2 dx, \tag{81}$$

where $0 \le \xi_1, \xi_2 \le 1/2$. Now the inequality (75) is a consequence of (78), (80), and (81). The proof of (76) is analogous to that of (75) and we omit it. •

Remark 3.1. Obviously, the natural norm of computing errors of functions' compression is the norm (62) that is closely related to the L^2 norm. Then the optimization of bases for a particular function f and fixed ϵ consists in finding such bell function b for which the subset $S_\epsilon^k(f,b)$ consists of the least number of coefficients $N_\epsilon^k(f,b)$. However optimization in the norm (62) is a complicated problem because this norm involves both coefficients f_n^k and $\tilde{f}_n^k$. In addition, subsets $S_\epsilon^k(f,b)$ do not necessarily contain $N_\epsilon^k(f,b)$ largest coefficients f_n^k. In this remark we observe that for a wide selection of functions f the optimization of bells b in the norm (62) is almost equivalent to their optimization in the norm (71). Note that the norm (71) involves only coefficients f_n^k, while all the subsets $\hat{S}_\epsilon^k(f,b)$ consist of $\hat{N}_\epsilon^k(f,b)$ largest coefficients f_n^k.

In fact, the formulae (73), (74), and (75) show that if (for any given b and f, and fixed ϵ) we set to zero all the coefficients f_n^k that do not belong to a subset $\hat{S}_\epsilon^k$ then the upper bound of the relative error of such an approximation in the norm (62) does not exceed $\kappa \cdot \epsilon$. Therefore for $\kappa \approx 1$ (i.e. when $\Theta \approx 1$) and sufficiently smooth b and f we expect that the subsets S_ϵ^k and $\hat{S}_\epsilon^k$ almost coincide. •

Now we turn to discussing the compression of the functions (1) in bases $\{u_n^k\}$. Due to periodicity of both cosine and functions u_n^k without loss of generality we will consider expansions of these functions only on the interval $[-1/2, 3/2]$ (i.e. for $k=0$). We will write the coefficients of functions (1) on this interval as f_n (instead of f_n^0). Finally, for these functions numbers $\hat{N}_\epsilon^0(f,b)$ (see their definition in Subsection 3.1) will be denoted by $\hat{N}_\epsilon(\alpha,\omega,b)$.

Lemma 3.2. *For any ω, α, and $n \ge 0$ the coefficients f_n of the function f defined in (1) are given by the formula*

$$f_n = 2^{-1/2} \left(\sin(\alpha + A_+^n/2) \cdot B(A_+^n) + \sin(-\alpha + A_-^n/2) \cdot B(A_-^n) \right), \tag{82}$$

where

$$B(q) = \int_{-\infty}^{\infty} b(x+1/2) \cdot \exp(iqx)dx, \tag{83}$$

and

$$A_+^n = \omega + (n+1/2)\pi, \quad A_-^n = -\omega + (n+1/2)\pi. \tag{84}$$

Proof. First we observe that $b(x+1/2)$ is an even function of x, which immediately follows from the definition (26), and thus its Fourier transform B (83) is a real even function. Therefore combining (1), (35), and (49), we have

$$\begin{aligned} f_n = \; & 2^{1/2} \int_{-\infty}^{\infty} b(x) \cdot \cos(\omega x + \alpha) \cdot \sin(n+1/2)\pi x)dx = \\ & 2^{-1/2}\Im\left(\exp(i\alpha)\int_{-\infty}^{\infty} b(x)\exp(i \cdot A_+^n \cdot x)dx\right) + \\ & 2^{-1/2}\Im\left(\exp(-i\alpha)\int_{-\infty}^{\infty} b(x)\exp(i \cdot A_-^n \cdot x)dx\right) = \\ & 2^{-1/2}\Im\left(\exp i(\alpha + A_+^n/2) \cdot \int_{-\infty}^{\infty} b(x+1/2)\exp(iA_+^n)dx\right) + \\ & 2^{-1/2}\Im\left(\exp i(-\alpha + A_-^n/2) \cdot \int_{-\infty}^{\infty} b(x+1/2)\exp(iA_-^n)dx\right), \end{aligned} \tag{85}$$

and now the formula (82) follows from (83) and (85). •

Remark 3.2. In this remark we observe that for $\omega \gg 1$, $\alpha \neq \pi l/2$ ($l = \pm1, \pm3, \pm5...,$) and sufficiently smooth bells b,

$$\hat{N}_\epsilon(\alpha, \omega, b) \approx 2\hat{N}_\epsilon(0,0,b). \tag{86}$$

Indeed, since the function b in (26) is integrable, we have (see, for example, Chap. 7 of Tolstov [19])

$$\lim_{q \to \pm\infty} B(q) = 0, \tag{87}$$

and therefore for any $n \geq 0$ and $\omega \gg 1$,

$$f_n \approx 2^{-1/2}\sin(\alpha + A_-^n/2) \cdot B(A_-^n) = 2^{-1/2}\sin(\alpha + A_-^n/2) \cdot B(-\omega + (n+1/2)\pi), \tag{88}$$

which is a consequence of (82), (84), and (87). On the other hand, for $\omega = 0$ the combination of (82) and (84) yields

$$f_n = 2^{1/2} \cos(\alpha) \cdot \sin(\pi n/2 + \pi/4) \cdot B((n + 1/2)\pi). \tag{89}$$

Combining (88) and (89) we see that if B decays sufficiently fast (i.e. when b is sufficiently smooth), $\alpha \neq \pi l/2$ ($l = \pm 1, \pm 3, \pm 5...$), and $\omega \gg 1$,

$$\hat{N}_\epsilon(\alpha, \omega, b) \approx 2\hat{N}_\epsilon(\alpha, 0, b). \tag{90}$$

Next, for arbitrary b, ϵ, and $\alpha \neq \pi l/2$ ($l = \pm 1, \pm 3, \pm 5...$) the numbers $\hat{N}_\epsilon(\alpha, 0, b)$ do not depend on α and, due to (74), coincide with numbers $\hat{N}^0_\epsilon(f, b)$ for the function

$$f(x) = 1. \tag{91}$$

Now combining this observation with (90) we immediately obtain (86). •

3.2. Optimized Bases for Trigonometric Functions

In this subsection we give an informal description of the construction of the bases $\{u^k_n\}$, hereafter referred to as *optimized bases*, that efficiently compress trigonometric functions (1). This scheme involves two observations that significantly simplify this problem.

1) The selection of bases $\{u^k_n\}$ with $\Theta \approx 1$ and $r_{cond} \approx 1$ ensures the numerical stability of forward and inverse transforms in these bases (see Remark 2.1), and reduces their optimization in the norm (62) to an almost equivalent but a much simpler problem of optimizing them in the norm (71) (see Remark 3.1).

2) Furthermore, for sufficiently smooth bells the bases optimized for the function (91) in the norm (71) are almost optimal (in the same norm) for the functions (1) (see the relation (86) in Remark 3.2).

3.3. Construction of Optimized Bells as a Variational Problem

In this subsection we formalize constructing of optimized bases discussed in Subsection 3.2. We start with the explicit formula for the expansion coefficients of the function (91).

Combining (26), (35), and (58) with (91) we have

$$f_n = 2^{1/2}\int_{-1/2}^{3/2} b(x)\sin(n+1/2)\pi x dx = 2\int_{-1/2}^{1/2} w(x)\phi_n(x)dx, \tag{92}$$

where

$$\begin{aligned}\phi_n(x) = \quad & 2^{-1/2}(\sin(n+1/2)\pi x + (-1)^n\cos(n+1/2)\pi x) = \\ & (-1)^m \sin(n+1/2)\pi(x+1/2).\end{aligned} \tag{93}$$

In (93) $m = [(n+1)/2]$ with the symbol $[x]$ denoting the integer part of x.

It is natural to suppose that for sufficiently smooth bells b the first several coefficients f_n (92) decrease as a function of n. Therefore the bell optimal for the function (91) in the norm (71) can be obtained from the condition

$$\min_{\{w\}} \sum_{n=N}^{\infty} f_n^2. \tag{94}$$

We expect, that as N (the number of the optimized coefficients) in (94) increases, the value of this sum decreases thus providing a smaller value of the relative error $\hat{\delta}$ (74).

Remark 3.3. It is tempting to solve (94) under the MCM condition (6) thus obtaining an optimized orthonormal basis with $r_{cond} = \Theta = 1$. However, the solution of the variational problem (94) for MCM bells (6) does not yield a bell suitable for the compression of functions (1).

In fact, the function

$$b(x) = \begin{cases} \phi_0(x) = \sin(\pi(x+1/2)/2) & \text{for all} \quad -1/2 \le x \le 3/2, \\ 0 & \text{otherwise} \end{cases} \tag{95}$$

satisfies the conditions (6) – (8) (i.e. the basis $\{u_n^k\}$ with this bell is an MCM basis) as well as the condition (3). Now the combination of (92) and (95) yields

$$\begin{aligned} f_n = 2^{1/2}\int_{-1/2}^{3/2} & \sin(\pi(x+1/2)/2)\sin(n+1/2)\pi x dx = \\ (-1)^m 2^{3/2}\int_{-1/2}^{1/2} & \sin(\pi(x+1/2)/2)\cdot\sin(n+1/2)\pi(x+1/2)dx = 2^{1/2}\delta_{0n}, \end{aligned} \tag{96}$$

where $m = [(n+1)/2]$, and thus the function (95) provides the solution for (94). However, the Fourier transform of b (95) decays very slowly and therefore the basis with such a bell cannot efficiently compress functions (1) with $\omega \neq 0$. •

Remark 3.4. The Gaussian bell function

$$b(x) = \begin{cases} \exp\left(-a(x-1/2)^2\right) & \text{for all} \quad -1/2 \le x \le 3/2, \\ 0 & \text{otherwise}, \end{cases} \tag{97}$$

where $a = 1/\ln(1/\epsilon)$ provides a value of the sum in (94) that is close to its absolute minimum. This property of the Gaussian (97) (with an appropriately chosen $\epsilon = \epsilon(N)$) is a consequence of Heisenberg's inequality (see, for example, Chap. 2 of Dym and McKean [10]). However, it is easy to see that for bases with the bell (97) we have

$$r_{cond} \approx \epsilon^{-1/4}/2, \tag{98}$$

and

$$\Theta \approx \epsilon^{-1/4}/2^{1/2}, \tag{99}$$

and thus for small ϵ the inverse transformation (50) in such bases becomes numerically unstable. An example of this instability is discussed in Subsection 5.5 below. •

In Lemma 3.3. below we establish a class of bells for which $r_{cond} \approx 1$ and $\Theta \approx 1$.

Lemma 3.3. *Let a real-valued function $w \in C^1[-1/2, 1/2]$ satisfy (24) and suppose that*

$$w(x) + w(-x) = 1, \tag{100}$$

and

$$\frac{dw(x)}{dx} > 0 \quad \text{for all } 0 \le x < 1/2. \tag{101}$$

Furthermore, suppose that for some $x_0 \in [0, 1/2)$,

$$w(x_0) = 2^{-1/2}. \tag{102}$$

Then on $(-1/2, 1/2)$ the function $\tilde{w}$ defined in (25) has the unique maximum at $x = x_0$ and

$$\tilde{w}(x_0) = \left(2^{1/2} + 1\right)/2, \tag{103}$$

while on the same interval the function θ defined in (47) has the unique maximum at $x = 0$, and

$$\theta(0) = 2^{1/2}. \tag{104}$$

Proof. We start with the analysis of the function $\tilde{w}$ at the point $x = x_0$. We first observe that

$$\frac{dw(x)}{dx} > 0 \quad \text{for all } -1/2 \leq x < 1/2, \tag{105}$$

which is an immediate consequence of (100) and (101). Substituting (100) into (25) we have

$$\tilde{w}(x) = \frac{w(x)}{2w^2(x) - 2w(x) + 1}, \tag{106}$$

and thus

$$\frac{d\tilde{w}(x)}{dx} = \frac{dw(x)}{dx} \cdot \frac{1 - 2w^2(x)}{(2w^2(x) - 2w(x) + 1)^2}, \tag{107}$$

which in combination with (102) yields

$$\frac{d\tilde{w}(x_0)}{dx} = 0. \tag{108}$$

Next, combining (102), (105), and (107) we have

$$\frac{d\tilde{w}(x)}{dx} > 0 \quad \text{for all } -1/2 < x < x_0, \tag{109}$$

and

$$\frac{d\tilde{w}(x)}{dx} < 0 \quad \text{for all } x_0 < x < 1/2. \tag{110}$$

It immediately follows from (108) – (110) that $x = x_0$ is the unique maximum of the function $\tilde{w}$ on $(-1/2, 1/2)$. Finally, the relation (103) is a consequence of (102) and (106).

We now turn to the proof of (104). First we observe that

$$w(0) = 1/2, \tag{111}$$

which immediately follows from (100). Substituting (100) into (30) we have

$$\theta(x) = \frac{1}{(2w^2(x) - 2w(x) + 1)^{1/2}}, \tag{112}$$

and therefore

$$\frac{d\theta(x)}{dx} = \frac{dw(x)}{dx} \cdot \frac{1-2w(x)}{(2w^2(x)-2w(x)+1)^{3/2}}, \tag{113}$$

which in combination with (111) yields

$$\frac{d\theta(0)}{dx} = 0. \tag{114}$$

Combining (105), (111), and (113) we have

$$\frac{d\theta(x)}{dx} > 0 \quad \text{for all } -1/2 < x < 0, \tag{115}$$

and

$$\frac{d\theta(x)}{dx} < 0 \quad \text{for all } 0 < x < 1/2. \tag{116}$$

Formulae (114) – (116) show that $x = 0$ is the unique maximum of θ on $(-1/2, 1/2)$. Finally (104) is obtained by substituting (111) into (112). •

Remark 3.5. In this remark we compute the condition number r_{cond} (42) and the parameter Θ (78) for bells that satisfy the conditions (3) and (100) – (102). Note that (3) is equivalent to

$$0 \le w(x) \le 1, \tag{117}$$

which follows from (26).

Combining (103) and (42) we obtain

$$r_{cond} = \left(2^{1/2}+1\right)/2. \tag{118}$$

Next, combining (112) and (117) we have

$$\min_{x\in[0,1/2]} \theta(x) \ge 1, \tag{119}$$

which in combination with (104) yields

$$1 \le \Theta \le 2^{1/2}. \bullet \tag{120}$$

Remark 3.6. In this remark we establish an interesting relation between an arbitrary bell b (26) and the corresponding dual bell $\tilde{b}$ (27) when (3), (100), (101) and (117) hold.

Combining (106) and (117) we have

$$w(x) \leq \tilde{w}(x) \leq 2w(x), \tag{121}$$

which is equivalent to

$$b(x) \leq \tilde{b}(x) \leq 2b(x). \tag{122}$$

Obviously, the equality $b(x) = \tilde{b}(x)$ can hold only for x such that $b(x) = 1$, whereas the equality $b(x) = 2\tilde{b}(x)$ is satisfied only for x such that $b(x) = 1/2$, i.e. only at the points $x = 0$ or $x = 1$. Finally we observe that the last two relations in combination with (111) imply $\tilde{b}(0) = \tilde{b}(1) = 1$. •

We solve the minimization problem (94) (for arbitrary N) under the constraint (100) only. It turns out, however (see Subsection 4.3), that for any N the resulting bells also satisfy (3), (101), and (102), and therefore for the basis $\{u_n^k\}$ with such a bell r_{cond} and Θ are given by (118) and (120), respectively (see Remark 3.5). In Appendix we discuss the solution of (94) without any constraints.

Since w satisfies (100), there exists an odd function g such that

$$w(x) = (1 + g(x))/2. \tag{123}$$

Combining (92), (93), and (123) we have

$$f_n = 2^{1/2}\left(\frac{(-1)^n}{\pi(n+1/2)}\sin\left((n/2+1/4)\pi\right) + \int_0^{1/2}\sin(n+1/2)\pi x \cdot g(x)dx\right). \tag{124}$$

The relations (94) and (124) define the variational problem for the computation of the optimized bell: substituting (124) into (94) we see that formally it consists in finding the absolute minimum (with respect to g) of the functional

$$\begin{aligned} I\{g\} \stackrel{\text{def}}{=} \sum_{n=N}^{\infty} f_n^2 = \frac{4}{\pi}\sum_{n=N}^{\infty}\frac{(-1)^n}{n+1/2}\sin(n/2+1/4)\pi \cdot \int_0^{1/2} g(x)\sin(n+1/2)\pi x dx + \\ 2\sum_{n=N}^{\infty}\int_0^{1/2}\int_0^{1/2} g(x)g(y)\cdot\sin(n+1/2)\pi x\cdot\sin(n+1/2)\pi y dx dy + C, \end{aligned} \tag{125}$$

where C is a functional independent of g.

4. The Solution of the Variational Problem

In this section we find the absolute minimum of the functional I defined in (125).

4.1. Mathematical Background

Here we present a number of relevant mathematical facts to be used in the next subsection. The main result of this subsection, an establishment of a different representation of the functional (125), is Lemma 4.3 below. We start with the well known trigonometric expansion that (in a slightly different form) can be found, for example, in Chap. 5 of Tolstov [19].

Lemma 4.1. *For any* $|x| < \pi$,

$$\sum_{n=0}^{\infty}(-1)^n \frac{\cos(2n+1)x}{2n+1} = -\frac{\pi}{4}(2\vartheta(|x|-\pi/2)-1), \tag{126}$$

where ϑ *is the Heaviside unit step function.* •

The following lemma is an immediate consequence of the formula (126).

Lemma 4.2. *For any* $|x| < 1/2$,

$$\sum_{n=0}^{\infty}(-1)^n \left(\frac{\cos(n+1/2)\pi(x+1/2)}{n+1/2} - \frac{\cos(n+1/2)\pi(x-1/2)}{n+1/2}\right) = 0. \quad \bullet \tag{127}$$

Lemma 4.3. *For any* $g \in L^2([0,1/2])$ *and* $0 < N < \infty$,

$$I\{g\} = \int_0^{1/2} g^2(x)dx - \frac{4}{\pi}\sum_{n=0}^{N-1}\frac{(-1)^n}{n+1/2}\sin(n/2+1/4)\pi \cdot \int_0^{1/2} g(x)\sin(n+1/2)\pi x dx -$$

$$2\sum_{n=0}^{N-1}\int_0^{1/2}\int_0^{1/2} g(x)g(y)\cdot\sin(n+1/2)\pi x\cdot\sin(n+1/2)\pi y dx dy + C, \tag{128}$$

where the functional I *is defined by (125) and* C *is a functional independent of* g.

Proof. Throughout the proof of the lemma any functional independent of g will be denoted by C. We begin with rewriting (125) in the form

$$I\{g\} = I_1\{g\} - I_2\{g\}, \tag{129}$$

where

$$I_1\{g\} = \sum_{n=0}^{\infty} f_n^2, \tag{130}$$

and

$$I_2\{g\} = \sum_{n=0}^{N-1} f_n^2 = \frac{4}{\pi}\sum_{n=0}^{N-1}\frac{(-1)^n}{n+1/2}\sin(n/2+1/4)\pi\cdot\int_0^{1/2} g(x)\sin(n+1/2)\pi x dx +$$
$$2\sum_{n=0}^{N-1}\int_0^{1/2}\int_0^{1/2} g(x)g(y)\cdot\sin(n+1/2)\pi x\cdot\sin(n+1/2)\pi y dx dy + C. \tag{131}$$

Substituting (124) into (130) we obtain

$$I_1\{g\} = \sum_{n=0}^{\infty}\overline{f}_n^2 +$$
$$\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1/2}\int_0^{1/2} g(x)(\cos(n+1/2)\pi(x-1/2)-\cos(n+1/2)\pi(x+1/2))dx + C, \tag{132}$$

where

$$\overline{f}_n = 2^{1/2}\int_0^{1/2} g(x)\sin(n+1/2)\pi x dx. \tag{133}$$

Combining (133) with Parseval's theorem we have

$$\sum_{n=0}^{\infty}\overline{f}_n^2 = \int_0^{1/2} g^2(x)dx. \tag{134}$$

Next, changing the order of integration and summation in the second sum in (132) and using the formula (127) we see that this sum vanishes, which in combination with (134) yields

$$I_1\{g\} = 2\int_0^{1/2} g^2(x)dx + C. \tag{135}$$

Now (128) immediately follows from (129), (131), and (135). •

4.2. The Absolute Minimum of the Functional I

In this subsection we show that the function $G:\mathbb{R}\to\mathbb{R}$ defined by the formula

$$G(x) = \sum_{n=0}^{N-1} g_n\sin(n+1/2)\pi x, \tag{136}$$

is the unique minimum of the functional I. In (136) the coefficients g_n are the solution of the $N \times N$ linear system

$$\sum_{n=0}^{N-1} g_n d_{nm} = b_m, \quad m = 0, 1, ..., N-1, \tag{137}$$

where

$$b_n = \frac{(-1)^{n+1}}{\pi(n/2 + 1/4)} \sin(n/2 + 1/4)\pi, \tag{138}$$

and

$$d_{nm} = \begin{cases} -\frac{1}{2} - \frac{(-1)^n}{\pi(2n+1)} & \text{for all } m = n, \\ \frac{\sin(n-m)\pi/2}{\pi(n-m)} - \frac{\sin(n+m+1)\pi/2}{\pi(n+m+1)} & \text{for all } m \neq n. \end{cases} \tag{139}$$

This result is based on the standard analysis of the first and second variations of the functional I (see, for example, Chaps. 1 and 2 of Fox [11]).

Theorem 4.1. *The function G defined by the formulae (136) – (139) is an extremal of the functional I, i. e. $\delta I\{G\} = 0$*

Proof. We assume that the functions g in (128) (and their variations δg) belong to $L^2([0, 1/2])$.

Obviously, the first variation of the functional (128) has the form

$$\begin{aligned} \delta I\{g\} = \; & 2\int_0^{1/2} \delta g(x) \Bigg(g(x) - \frac{2}{\pi} \sum_{n=0}^{N-1} \frac{(-1)^n}{n+1/2} \sin(n/2 + 1/4)\pi \cdot \sin(n+1/2)\pi x - \\ & 2\sum_{n=0}^{N-1} \sin(n+1/2)\pi x \int_0^{1/2} g(y) \cdot \sin(n+1/2)\pi y dy \Bigg) dx. \end{aligned} \tag{140}$$

The combination of (140) with the condition $\delta I\{G\} = 0$ yields

$$\begin{aligned} & G(x) - \frac{2}{\pi} \sum_{n=0}^{N-1} \frac{(-1)^n}{n+1/2} \sin(n/2 + 1/4)\pi \cdot \sin(n+1/2)\pi x - \\ & 2\int_0^{1/2} G(y) \left(\sum_{n=0}^{N-1} \sin(n+1/2)\pi y \cdot \sin(n+1/2)\pi x \right) dy = 0, \end{aligned} \tag{141}$$

i.e. any extremal of I satisfies Fredholm's integral equation of the second kind with the so called Pincherle-Goursat kernel (see, for example, Chap. 2 of Tricomi [20]). Note, that now the representation (136) is an immediate consequence of (141).

The standard method for the solution of the integral equations with Pincherle-Goursat kernels consists in replacing them with suitable linear systems (see, for example, Chap. 2 of Tricomi [20]). Substituting (136) into (141) we obtain

$$\sum_{n=0}^{N-1} g_n \sin(n+1/2)\pi x - \frac{2}{\pi}\sum_{n=0}^{N-1}\frac{(-1)^n}{n+1/2}\sin(n/2+1/4)\pi\cdot\sin(n+1/2)\pi x -$$
$$2\sum_{n=0}^{N-1} g_n \sin(n+1/2)\pi x \sum_{m=0}^{N-1}\int_0^{1/2}\sin(n+1/2)\pi y\cdot\sin(m+1/2)\pi y dy = 0. \tag{142}$$

Now using linear independence of the functions $\sin(n+1/2)\pi x$ on $[0,1/2]$ and evaluating the integrals in (142) we immediately obtain (137) – (139). •

Theorem 4.2. *The function G defined by (136) – (139) is the unique minimum of the functional I .*

Proof. Obviously, in order to prove this theorem it is sufficient to show that for any nontrivial function δg (i.e such that $\delta g \not\equiv 0$ on $[0,1/2]$) the second variation of I is strictly positive.

From (125) we immediately obtain

$$\delta^2 I\{g\} = 2\sum_{n=N}^{\infty}\left(\int_0^{1/2}\delta g(x)\sin(n+1/2)\pi x dx\right)^2, \tag{143}$$

and thus

$$\delta^2 I\{g\} \geq 0. \tag{144}$$

Now we will prove that for any nontrivial δg the relation (144) can hold only as a strict inequality. Suppose that there exists a nontrivial δg such that

$$\delta^2 I\{g\} = 0. \tag{145}$$

Then combining (143) and (145) we have

$$\int_0^1 \overline{\delta g}(x)\sin(n+1/2)\pi x dx = 0 \quad \text{for all } n = N, N+1, ..., \tag{146}$$

where the function $\overline{\delta g} : [0,1] \to \mathbb{R}$ is defined by the formula

$$\overline{\delta g}(x) = \begin{cases} \delta g(x) & \text{for all } 0 \leq x \leq 1/2, \\ 0 & \text{otherwise.} \end{cases} \tag{147}$$

Since the system $\{\sin(n+1/2)\pi x\}$ is complete and orthogonal on $[0,1]$, the conditions (146) imply that

$$\overline{\delta g}(x) = \sum_{n=0}^{N-1} c_n \sin(n+1/2)\pi x, \tag{148}$$

where the coefficients c_n are independent of x. However, for any finite N (147) and (148) cannot simultaneously hold unless $\overline{\delta g}(x) = \delta g(x) \equiv 0$ for all $x \in [0,1/2]$. Therefore, for any nontrivial δg,

$$\delta^2 I\{g\} > 0, \tag{149}$$

which concludes the proof of the theorem. •

Remark 4.1. It immediately follows from (136) – (139), (123), and (26) that the functions w and b are uniquely determined by the number of optimized coefficients N. In order to emphasize this fact we will write w_N and b_N instead of w and b. •

4.3. Certain Properties of the Optimized Bells

The explicit formulae for the bells found in the preceding subsection are obtained by first substituting (136) into (123), which produces

$$w_N(x) = \frac{1}{2}\left(1 + \sum_{n=0}^{N-1} g_n \sin(n+1/2)\pi x\right), \tag{150}$$

followed by combining (26) and (150), which yields

$$b_N(x) = \begin{cases} \frac{1}{2}\left(1 + \sum_{n=0}^{N-1} g_n \sin(n+1/2)\pi x\right) & \text{for all} \quad -1/2 \le x \le 1/2, \\ \frac{1}{2}\left(1 + \sum_{n=0}^{N-1} (-1)^n g_n \cos(n+1/2)\pi x\right) & \text{for all} \quad 1/2 < x \le 3/2, \\ 0 & \text{otherwise.} \end{cases} \tag{151}$$

In (150) and (151) the coefficients g_n are the solution of the linear system (137) – (139). The numerical values of these coefficients for $N \le 20$ are listed in Table 1. [1]

[1] It turns out, that the condition number of the matrix (139) increases rapidly with N, and in order to avoid the lost of accuracy this sequence of linear systems was (numerically) solved in extended precision arithmetics.

Numerical computations of functions b_N for $N \leq 20$ via (150) with g_n's from Table 1 show that for these functions the conditions (3), (101), and (102) are satisfied. Therefore for bases $\{u_n^k\}$ with such bells the condition number r_{cond} (42) and the parameter Θ (78) are given by (118) and (120), respectively (see Remark 3.5). Note that independently of N, r_{cond} and Θ are sufficiently close to 1 to ensure numerical stability of the transformations in bases with the optimized bells.

Examples of optimized bells (151) and corresponding dual bells are shown in Fig. 1.

5. Numerical Results.

In this section we compare compression provided by the optimized bases with that obtained by two of nonoptimized ones whose bells are described in Subsection 5.2 below. We also present the corresponding results for the bases with the Gaussian bells (97).

5.1. Implementation of the Algorithm

The implementation of the algorithm is based on the results of Subsection 2.3. We assume that the expanded function f is defined at the nodes of the equally spaced mesh

$$x_i = i \cdot h, \quad i = 0, \pm 1, \pm 2, ..., \quad h = 1/(l-1), \quad l = 2p+1, \tag{152}$$

where $p > 0$ is an integer.

Remark 5.1. In most applications one usually uses bells b such that

$$b(-1/2) = b(3/2) = \tau, \tag{153}$$

where $0 \leq \tau \ll 1$. In this paper we will assume that τ in (153) is sufficiently small so that one can neglect the contribution of the points $x = k - 1/2$ and $x = k + 3/2$ while (numerically) computing the integral in (58) on the mesh (152). •

We start with the explicit formulae for the implementation of the folding (37) and unfolding (38) operators. Under the conditions described in Remark 5.1 the folding operator on the interval $x \in (k-1/2, k+1/2]$ has the following discrete form:

$$F_w^k(f)(k+h(i-1)) \equiv \hat{f}_k(k+h(i-1)) =$$
$$f(k+h(i-1)) \cdot w(h(i-1)) - f(k-h(i-1)) \cdot w(-h(i-1)) \quad \text{for all } 1 \leq i \leq p, \tag{154}$$

$$F_w^k(f)(k-h(i-1)) \equiv \hat{f}_{k-1}(k-h(i-1)) =$$
$$f(k+h(i-1)) \cdot w(-h(i-1)) + f(k-h(i-1)) \cdot w(h(i-1)) \quad \text{for all } 1 \leq i \leq p, \tag{155}$$

$$F_w^k(f)(k+hp) \equiv \hat{f}_k(k+hp) = f(k+hp) \cdot w(hp) \quad \text{for } i = p+1. \tag{156}$$

The discretized version of the unfolding operator (38) enables us to obtain the function f for $x \in (k-1/2, k+1/2]$ at the points of the mesh (152) as linear combinations of functions $\hat{f}_k$ and $\hat{f}_{k-1}$. In fact, combining (63) with (154) – (156) we have

$$f(k+h(i-1)) =$$
$$\hat{f}_k(k+h(i-1)) \cdot \tilde{w}(h(i-1)) + \hat{f}_{k-1}(k-h(i-1)) \cdot \tilde{w}(-h(i-1)) \quad \text{for all } 1 \leq i \leq p, \tag{157}$$

$$f(k-h(i-1)) =$$
$$\hat{f}_{k-1}(k-h(i-1)) \cdot \tilde{w}(h(i-1)) - \hat{f}_k(k+h(i-1)) \cdot \tilde{w}(-h(i-1)) \quad \text{for all } 1 \leq i \leq p, \tag{158}$$

$$f(k+hp) = \hat{f}_k(k+hp) \cdot \tilde{w}(hp) \quad \text{for } i = p+1. \tag{159}$$

Next, one can easily see from (154) – (156) that on the interval $[k, k+1]$ the values of the function $\hat{f}_k$ at the nodes of the mesh (152) are given by

$$\hat{f}_k(k+h(i-1)) = \begin{cases} f(k+h(i-1)) \cdot w(h(i-1)) - \\ f(k-h(i-1)) \cdot w(-h(i-1)) & \text{for all} \quad 1 \leq i \leq p, \\ \\ f(k+1+h(i-p-1)) \cdot w(h(p+1-i)) + \\ f(k+h(i-1)) \cdot w(h(l-i)) & \text{for all} \quad p+2 \leq x \leq l \\ \\ f(k+ph) \cdot w(hp) & \text{for} \quad i = p+1. \end{cases} \tag{160}$$

Observing that for any k

$$\hat{f}_k(k) = 0, \tag{161}$$

we obtain the trapezoidal approximation for the formula (58) in the form

$$f_n^k = 2^{1/2} \cdot h \cdot \sum_{i=2}^{l} \hat{f}_k(k + h(i-1)) \cdot \sin(\pi(n+1/2) \cdot h \cdot (i-1)), \quad n = 0, 1, ..., l-2, \tag{162}$$

where the numbers $\hat{f}_k(k + h(i-1))$ are computed in (160).

Conversely, in order to compute a function f on all the intervals $x \in (k - 1/2, k + 1/2]$ ($k = 0, \pm 1, \pm 2 ...,$) at the nodes of the mesh (152) from its coefficients f_n^{k-1} and f_n^k one first computes the numbers $\hat{f}_{k-1}(k - h(i-1))$ with $1 \le i \le p$ and $\hat{f}_k(k + h(i-1))$ with $1 \le i \le p+1$ via the discretization of (67) which has the form

$$\hat{f}_k(k + h(i-1)) = 2^{1/2} \sum_{n=0}^{l-1} f_n^k \sin(\pi(n+1/2) \cdot h \cdot (i-1)),$$

$$\hat{f}_{k-1}(k - h(i-1)) = 2^{1/2} \sum_{n=0}^{l-1} f_n^{k-1} \sin(\pi(n+1/2) \cdot h \cdot (i-1)), \tag{163}$$

and after that uses the formulae (157) – (159).

The cost of the algorithm's implementation for sufficiently large l is dominated by the Fast Fourier Transform (FFT) in (162) for forward transform and the FFT in (163) for inverse transform, and therefore has the complexity estimate $O(l \log l)$.

Now we will discuss the compression procedure used in this paper. In most applications the compression of a function f in a certain basis is usually achieved by neglecting all the coefficients whose absolute values are smaller than a given number. In this paper on every k-th interval we set to zero all the coefficients that do not satisfy the inequality

$$|f_n^k| \ge \epsilon \left(\sum_{n=0}^{l-2} \left(f_n^k\right)^2 \right)^{1/2}, \tag{164}$$

where ϵ is a given number. The subset of coefficients f_n^k that satisfy (164) will be denoted by D_ϵ^k. Now instead of the exact relations (163) we have the approximate formulae

$$\hat{f}_k(k + h(i-1)) \approx 2^{1/2} \sum_{n \in D_\epsilon^k} f_n^k \sin(\pi(n+1/2) \cdot h \cdot (i-1)),$$

$$\hat{f}_{k-1}(k - h(i-1)) \approx 2^{1/2} \sum_{n \in D_\epsilon^{k-1}}^{l-1} f_n^{k-1} \sin(\pi(n+1/2) \cdot h \cdot (i-1)), \tag{165}$$

which in combination with (157) – (159) produce approximate (i.e. based on compressed expansions (165)) values of f at the nodes of the mesh (152). The arguments, presented in Subsection 3.1 (especially in Theorem 3.1 and Remark 3.1) show, that ϵ in (164) is close to the relative error of this compression in the norm (62) provided that $r_{cond} \approx \Theta \approx 1$.

Finally, we compute the relative error of the approximation (165) in the norm (62). Namely, replacing the subset S_ϵ^k by the subset D_ϵ^k in (73) and observing that the total number of coefficients f_n^k on the k-th interval is equal to $l-1$ (see (162)), we have this error δ in the form

$$\delta(D_\epsilon^k) = \left(\frac{\displaystyle\int_k^{k+1} \theta^2(x) \cdot \left(\sum_{n \notin D_\epsilon^k} f_n^k \sin(n+1/2)\pi(x-k) \right)^2 dx}{\displaystyle\int_k^{k+1} \theta^2(x) \cdot \left(\sum_{n=0}^{l-1} f_n^k \sin(n+1/2)\pi(x-k) \right)^2 dx} \right)^{1/2}. \tag{166}$$

5.2. Two Nonoptimized Bell Functions

In this subsection we discuss certain basic properties of collections of bells proposed in [1] and [7]. These bells are computed via the formula (26) in which w is replaced by the functions w_N defined in (167) and (169) below.

The following bells were described by Coifman [7] and are based on the sequence of functions

$$w_N(x) = \sin\left(\frac{\pi}{2t_N} \int_{-1/2}^{x} (1/4 - t^2)^N dx \right), \tag{167}$$

where

$$t_N = \int_{-1/2}^{1/2} (1/4 - t^2)^N dx = \frac{(2N)!!}{2^{2N}(2N+1)!!}. \tag{168}$$

Clearly, the functions (167) satisfy the MCM condition (6) for any N. The parameter N controls the smoothness of the bell. In fact, combining (26) and (167) it is easy to see that for $N > 0$ the obtained bell is analytic on the real line except at the points $x = -1/2$ and $x = 3/2$, where it has N continuous derivatives, and at the point $x = 1/2$, where it has $2N+1$ continuous derivatives. For $N = 0$ the Coifman bell coincides with the function $\phi_0(x)$ from (92).

The bells proposed by Aharoni *et. al.* [1] (the AACI bells) are based on functions $w_N(x)$ that are recursively defined as

$$w_N(x) = (1 + y_N)/2; \quad y_0 = \sin \pi x, \quad y_{n+1} = \sin(\pi y_n/2), \quad n = 0, 1, ..., N-1, \tag{169}$$

and therefore for any N they satisfy the condition (100). Similarly to (167), the parameter N controls the bells' smoothness: one can verify that the combination of (26) and (169) produces a function that is analytic on the real line except at the points $x = -1/2$ and $x = 3/2$, where it has $2^N - 1$ continuous derivatives.

5.3. Compression of Trigonometric Functions

In this subsection we introduce a number of parameters estimating the compression of trigonometric functions (1) in bases $\{u_n^k\}$.

First, we consider a sequence of functions

$$f_j(x) = \cos(\omega_j x + \alpha_j), \tag{170}$$

where $j = 1, 2, ..., M$, ω_j and α_j are random variables uniformly distributed on $[\omega_1, \omega_2]$ and $[0, 2\pi]$, respectively (see the particular choice of M, ω_1 and ω_2 in (175) below). We begin with evaluating the expansion coefficients $f_n(j)$ of these functions via (162) (we again drop the superscript for brevity) on the interval $[-1/2, 3/2]$ (i.e. for $k = 0$). Next, we compute the integers m_j, equal to the number of coefficients that satisfy the inequality (164) for each of the functions $f_j(x)$. In addition, for every function $f_j(x)$ we evaluate the relative error of compression (hereafter denoted by δ_j) via (166). Finally, in accordance with the common practice (see, for example, Chap. 3 of Mandel [15]), we compute the sample estimates of the means $\bar{m}$ and $\bar{\delta}$, and their standard deviations σ and σ_δ via

$$\bar{m} = \sum_{j=1}^{M} m_j/M, \tag{171}$$

$$\bar{\delta} = \sum_{j=1}^{M} \delta_j/M, \tag{172}$$

and

$$\sigma = \left(\sum_{j=1}^{M} (m_j - \bar{m})^2 / (M-1) \right)^{1/2}, \tag{173}$$

$$\sigma_\delta = \left(\sum_{j=1}^{M} (\delta_j - \bar{\delta})^2 / (M-1) \right)^{1/2}. \tag{174}$$

We will also denote by $\bar{m}$ the number of coefficients satisfying the inequality (166) for the function (91) (formally this number can be obtained from (171) if $\omega_j = 0$ for all j). Naturally, for this function $\bar{\delta} = \sigma = \sigma_\delta = 0$.

Below we give the numerical values of the parameters M, ω_1, and ω_2 that were used in our experiments:

$$M = 100, \quad \omega_1 = 50, \quad \omega_2 = 1000. \tag{175}$$

Our experiments show that the parameters (171) – (174) only weakly depend on M, ω_1 and ω_2 provided that $M \gtrsim 30$, $\omega_1 \gtrsim 50$, and $\omega_2 \gg \omega_1$.

5.4. The Choice of the Optimal N

The bells, defined by functions (167) and (169), as well as the optimized bells (151), depend on a parameter N. In this subsection we discuss the procedure for choosing such N that for the given type of the bell and specified ϵ provides the best compression for functions (1) (i.e. the smallest value of $\bar{m}$ (171)).

In Table 2 we show the parameters $\bar{m}$ (171) and $\bar{\delta}$ (172) obtained for bases with AACI bells for certain values of N. For comparison this table also contains the corresponding results for the function (91) in the same bases. As we see from Table 2, for any ϵ there exists the optimal choice of N which provides the best compression for both functions (170) and (91). Our numerical experiments show, that for all other types of bells considered in this paper there also exists the optimal bell (i.e the optimal choice of N that depends on ϵ) with the best compression properties; these values are presented in Table 3.

5.5. Comparison of Compression in Optimized and Nonoptimized Bases

The main part of this subsection is devoted to the results of compression on the interval $[-1/2, 3/2]$ of functions (1) and (91) in bases $\{u_n^k\}$ with the Coifman (167), AACI (169), Gaussian (97), and optimized (151) bell functions for certain values of ϵ. The parameters ω_j, α_j, m_j, σ_j, $\bar{m}$, $\bar{\delta}$, σ, and σ_δ are defined in Subsection 5.3. All the computations are performed with the optimal (for the given ϵ and type of the bell function) values of N (see Table 3). In addition, we investigate certain properties of expansions of functions other than (1) and (91) in bases $\{u_n^k\}$.

Table 4 shows that for all the considered examples the approximate relation (86) is satisfied. Note, that we always have $\sigma/\bar{m} \lesssim 0.1$ which indicates that the numbers m_j only weakly depend on α_j and ω_j (for sufficiently large ω_j). Next, the compression of the test functions in bases with the Gaussian and optimized bells is essentially the same, and it is substantially better (especially for higher accuracies) than that in bases with the AACI and Coifman bells.

The data in Table 5 are closely related to that in Table 4 and contain values of parameters $\bar{\delta}$ (172) and σ_δ (174). These data show that for all the bells excluding the Gaussian one,

$$\bar{\delta}/\epsilon \approx 1 \tag{176}$$

independently of ϵ, i.e. in such cases the parameter ϵ in (164) is close to the relative error (in the norm (62)) of the approximation (165). However, in the case of the Gaussian bell the ratio $\bar{\delta}/\epsilon$ increases as ϵ decreases, which is a consequence of the growing r_{cond} and Θ (see the formulae (98) and (99)).

Fig. 2 shows the expansion coefficients f_n of the function (91) in the four bases, with the parameters N chosen to be optimal for $\epsilon = 10^{-7}$ (see Table 3). Naturally, these dependencies make sense only for the integer values of the argument, but for clarity they are drawn as continuous functions. As we see from Fig. 2 the large coefficients in the four bases are close to each other, and the advantage of the bases with the Gaussian and the optimized bells consists in generating fewer small coefficients. Similar behavior of the expansion coefficients of the function (91) is observed for other values of ϵ.

We now turn to the investigation of the expansions of functions that are almost singular at one point. Note that one cannot improve the compression of a function on a certain interval by the optimization of the bell if the function has an integrable discontinuity on this interval.

Table 6 contains the numbers of expansion coefficients f_n that satisfy the inequality (166) for functions

$$f_L(x) = A_L(x)\cos(200x), \tag{177}$$

where

$$A_L(x) = \sum_{n=0}^{L-1} \frac{\sin(2n+1)x}{2n+1}. \tag{178}$$

Note, that for $L = \infty$ the amplitude A_L coincides with the discontinuous function

$$A_\infty(x) \equiv \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1} = \frac{\pi}{4}(2\vartheta(t) - 1), \tag{179}$$

which immediately follows from (126) after the change of variable $x \to x + \pi/2$. The graphs of $A_L(x)$ for several values of L are presented in Fig. 3.

The following two observations can be made from Table 6, that are typical for the compression of functions which are more complex than trigonometric ones.

1. The compression in bases with the optimized bells is essentially the same as in bases with the Gaussian bells.

2. For higher accuracies the bases with the optimized bells usually provide a much better compression than the bases with nonoptimized bells. The advantage of the optimized bases becomes more prominent for smaller L, i.e. when the amplitudes A_L (178) are less steep in the vicinity of the point $x = 0$.

Finally, we discuss the numerical stability of the algorithm. Although Tables 4 and 6 show that the bases with the Gaussian and optimized bells provide similar compression, the formulae (98) and (118) indicate, that for sufficiently small ϵ the optimized bases have much smaller condition number. Therefore, in such cases the C^0 error of the approximation (165) will be larger for bases with the Gaussian bell than for optimized bases.

To illustrate this phenomenon, we approximate the function

$$f(x) = 1 + \exp\left(-10x^2\right)\cos(100x) \tag{180}$$

on $[-1/2, 1/2]$ via the combination of (165), where we keep only coefficients satisfying (164), with (157) – (159). Fig. 4 shows the difference Δ between f (180) and its approximation for $\epsilon = 10^{-7}$ in the bases with the Gaussian bell and the optimized bell with $N = 9$. One can see from that figure that while for the optimized basis the maximum of $|\Delta(x)|$ is close to ϵ for all $x \in [-1/2, 1/2]$, in case of the basis with the Gaussian bell we have $|\Delta(x)| \gg \epsilon$ for certain x from the vicinity of the folding point $x = 0$, i.e. where the dual bell is sharply peaked.

6. Conclusions and Generalizations

In this paper we present a family of non-orthogonal bases that efficiently compress trigonometric functions, as well as some non-trigonometric ones. The dual bases of these optimized bases are easy to construct: any element of a dual basis differs from the corresponding element of the original basis only by the shape of the bell. Moreover, the value of the condition number of these bases $r_{cond} = (2^{1/2} + 1)/2 \approx 1.2$ ensures numerical stability of forward and inverse transformations in these bases. The CPU time required for forward and backward transforms in the optimized bases is the same as in the standard MCM scheme, i.e. it is dominated by that of the FFT. The methods of this paper can be easily applied to the construction of bases optimized for other classes of functions. This work is now in progress and its results will be reported elsewhere.

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Appendix: The Unconditional Solution of (94)

Here we obtain the solution of the minimization problem (94) without the constraint (100). Our starting point is the formula (92).

Let the functions ϕ_n be given by (93) and suppose that the function $w_K : [-1/2, 1/2] \to \mathbb{R}$ is defined via the formula

$$w_K(x) = \sum_{n=0}^{K-1} c_n \phi_n(x), \tag{181}$$

where c_n $(n = 0, 1, ..., K-1)$ are arbitrary constants. Observing that the collection of functions ϕ_n $(n = 0, 1, 2, ...)$ is an orthogonal basis on $[-1/2, 1/2]$, and substituting (181) into (92) we have

$$f_n = 2 \int_{-1/2}^{1/2} w_K(x) \phi_n(x) dx = \begin{cases} c_n & \text{for all } n \leq K-1, \\ 0 & \text{otherwise.} \end{cases} \tag{182}$$

Combining (94) and (182) we see that

$$\sum_{n=N}^{\infty} f_n^2 = 0 \quad \text{for all } K \leq N, \tag{183}$$

i.e. any linear combination (181) with $K \leq N$ provides the absolute minimum for the functional (94).

In an attempt to construct a bell function from (26) and (181) that satisfies (3) and on the real line has as many continuous derivatives as possible, we require that the K arbitrary parameters c_n satisfy the conditions

$$w_K(x)|_{x=1/2} = 1, \tag{184}$$

$$\left. \frac{d^{2l+1} w_K(x)}{dx^{2l+1}} \right|_{x=-1/2} = 0 \quad \text{for all } l = 0, 1, ..., K-2. \tag{185}$$

Note, that for any nonnegative integer K and n,

$$\left. \frac{d^{2l} \phi_n(x)}{dx^{2l}} \right|_{x=-1/2} = \left. \frac{d^{2l} w_K(x)}{dx^{2l}} \right|_{x=-1/2} = 0 \quad \text{for all } l = 0, 1, \ldots . \tag{186}$$

It immediately follows from (185) and (186) that the bell function, defined via (26), (181), (184), and (185), has at least $2K-2$ continuous derivatives on the real line. Furthermore, it follows from (182) that in the basis with this bell a constant is reproduced from its first K coefficients per interval *exactly.*

Remark A. It is worth noting that the conditions (181), (184) and (185) in combination with (26) do produce bell-shaped functions at least for (numerically tested values) $K \leq 10$. Graphs of these bells for certain values of K can be found in Fig. 5. •

For example, the explicit formula for the bell with $K = 2$ is

$$b_2(x) = \begin{cases} \frac{1}{4}(3\sin(\pi(x+1/2)/2) - \sin(3\pi(x+1/2)/2)) & \text{for all } -1/2 \leq x \leq 3/2, \\ 0 & \text{otherwise.} \end{cases} \tag{187}$$

The bell (187) has two continuous derivatives on the real line, the basis with such a bell reproduces a constant from its first two coefficients per interval exactly, and its condition number $r_{cond} \approx 1.67$ is relatively small. These properties suggest that the basis with the bell (187) can be useful in signal processing where the required accuracy usually is not very high (normally in (164) $\epsilon \gtrsim 10^{-5}$), and where the low frequency components of the signals often play an important role.

However, due to the rapid growth of r_{cond} with K the application of the unconditionally optimized bells in case of $K > 2$ is limited.

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Table 1: Coefficients g_n for the computation of the optimized bells (151).

n	N=1	$N = 2$	$N = 3$
1	1.1002143947640111085 d0	1.1723768006269012949 d0	1.2031447668472587192 d0
2		0.1855148479250006034 d0	0.2487749850071917170 d0
3			0.0475141123801596348 d0

n	N=4	$N = 5$	$N = 6$
1	1.2196727213232474166 d0	1.2299263341780548351 d0	1.2368960390161255733 d0
2	0.2853161868567129887 d0	0.3091936162560012950 d0	0.3260454684070418003 d0
3	0.0789155484618257136 d0	0.1022547499107077172 d0	0.1202144284772571658 d0
4	0.0135550276613148530 d0	0.0270080848154667718 d0	0.0392824672307590613 d0
5		0.0040643028367457815 d0	0.0094646390135188831 d0
6			0.0012540398189263445 d0

n	N=7	$N = 8$	$N = 9$
1	1.2419380540495429195 d0	1.2457536263723204704 d0	1.2487411472878298253 d0
2	0.3385842628287665721 d0	0.3482817014907560773 d0	0.3560069611425606312 d0
3	0.1344330305316184567 d0	0.1459560151659402844 d0	0.1554771240803420597 d0
4	0.0501467090476806012 d0	0.0596831230559218041 d0	0.0680536775001451859 d0
5	0.0153116614831942916 d0	0.0211609941444967639 d0	0.0267991398186155608 d0
6	0.0033448841387432324 d0	0.0059658781972807312 d0	0.0088894805175620360 d0
7	0.0003942039965801714 d0	0.0011854380315612793 d0	0.0023126782210685839 d0
8		0.0001255469017738763 d0	0.0004203150366958263 d0
9			0.0000403732905776057 d0

n	N=10	$N = 11$	$N = 12$
1	1.2511435101979626018 d0	1.2531171572760240244 d0	1.2547673684254528441 d0
2	0.3623069592485838316 d0	0.3675432476836647534 d0	0.3719644997534442686 d0
3	0.1634729471600794903 d0	0.1702809341651916143 d0	0.1761463604899294465 d0
4	0.0754255125823668245 d0	0.0819484046917260704 d0	0.0877499693755897336 d0
5	0.0321305559573675648 d0	0.0371207535134114532 d0	0.0417667654440872536 d0
6	0.0119592651970663211 d0	0.0150712511279723392 d0	0.0181579310814718085 d0
7	0.0036995960643158676 d0	0.0052752483692935777 d0	0.0069802273692443469 d0
8	0.0008907386396059369 d0	0.0015235383929696997 d0	0.0022969056584170223 d0
9	0.0001489420206098608 d0	0.0003408100998412218 d0	0.0006209177060710873 d0
10	0.0000130803773243655 d0	0.0000527237866578197 d0	0.0001295783756168840 d0
11		0.426301975610614 d-5	0.0000186407482263636 d0
12			0.139606350016921 d-5

n	N=13	$N = 14$	$N = 15$
1	1.2561675816035252352 d0	1.2573705781037171767 d0	1.2584152648035577478 d0
2	0.3757474138575439717 d0	0.3790210170105753012 d0	0.3818817363437593838 d0
3	0.1812516131847340424 d0	0.1857350426141542667 d0	0.1897034477160271649 d0
4	0.0929368147130266026 d0	0.0975974055372798749 d0	0.1018051162625708735 d0
5	0.0460814756936615327 d0	0.0500851855065012145 d0	0.0538010556771918339 d0
6	0.0211765890308308775 d0	0.0241011932143827745 d0	0.0269168795850161325 d0
7	0.0087672206344614442 d0	0.0105996145856369849 d0	0.0124496130486740615 d0
8	0.0031866825949287158 d0	0.0041695173336949991 d0	0.0052243034619146703 d0
9	0.0009873442127753202 d0	0.0014340335627767149 d0	0.0019527481969350935 d0
10	0.0002505909631277034 d0	0.0004193026849064740 d0	0.0006365376324034804 d0
11	0.0000489789669579668 d0	0.0001002259697370695 d0	0.0001760791467702309 d0
12	0.658213072318501 d-5	0.0000184143983824276 d0	0.0000397571893420880 d0
13	0.459014688967825 d-6	0.232126216952405 d-5	0.688941970620248 d-5
14		0.151428882710068 d-6	0.817633371026949 d-6
15			0.501000810832705 d-7

n	N=16	$N = 17$	$N = 18$
1	1.2593309500457438031 d0	1.2601401306174552497 d0	1.2608603612954268205 d0
2	0.3844030891706786254 d0	0.3866421087683765354 d0	0.3886437212387816550 d0
3	0.1932405574881542732 d0	0.1964129247635806935 d0	0.1992741058079349344 d0
4	0.1056209666415645249 d0	0.1090959178203754145 d0	0.1122727446284484620 d0
5	0.0572526660933526002 d0	0.0604627480475656409 d0	0.0634525707675491656 d0
6	0.0296162216589041811 d0	0.0321967100439478970 d0	0.0346590468664805595 d0
7	0.0142964336695234923 d0	0.0161247652829157535 d0	0.0179235156823607947 d0
8	0.0063326892901278450 d0	0.0074791070877270307 d0	0.0086505731134735900 d0
9	0.0025343672760701497 d0	0.0031697006503196943 d0	0.0038499687030354075 d0
10	0.0009011351818553216 d0	0.0012105885859207094 d0	0.0015615520391816049 d0
11	0.0002789521059156221 d0	0.0004101047807666002 d0	0.0005698390644760264 d0
12	0.0000731823780789826 d0	0.0001208077758892354 d0	0.0001842373381822920 d0
13	0.0000156526972929684 d0	0.0000301301370711848 d0	0.0000517518322815702 d0
14	0.256611058982070 d-5	0.612055142365566 d-5	0.0000122981314895988 d0
15	0.287672400786250 d-6	0.951935962296931 d-6	0.237837400561446 d-5
16	0.166167484200405 d-7	0.101105205879303 d-6	0.351830571000310 d-6
17		0.552324489986036 d-8	0.354989120817137 d-7
18			0.183937626456071 d-8

n	N=19	$N=20$
1	1.2615055408626026438 d0	1.2620868165186817100 d0
2	0.3904437975327983132 d0	0.3920713262058702161 d0
3	0.2018676789041392205 d0	0.2042294606490294914 d0
4	0.1151875431220117202 d0	0.1178709389362636122 d0
5	0.0662416923898752204 d0	0.0688479112856821096 d0
6	0.0370059895821450853 d0	0.0392415664712484188 d0
7	0.0196848223338375282 d0	0.0214032823468272275 d0
8	0.0098363945921014495 d0	0.0110278550409693667 d0
9	0.0045670612090700171 d0	0.0053136548560657154 d0
10	0.0019502209236860229 d0	0.0023726050667205839 d0
11	0.0007577056719391676 d0	0.0009726951425515420 d0
12	0.0002645671093113601 d0	0.0003624288874381589 d0
13	0.0000817808996953258 d0	0.0001212704239660943 d0
14	0.0000219482849485672 d0	0.0000359012169826721 d0
15	0.498006589408313 d-5	0.922285969549705 d-5
16	0.918946754735698 d-6	0.200203409841185 d-5
17	0.129595480966266 d-6	0.353205918170597 d-6
18	0.124523708171851 d-7	0.475881033651720 d-7
19	0.613594334772978 d-9	0.436427442377405 d-8
20		0.204997258761082 d-9

Table 2: Compression of functions (91) (I) and (1) (II) in bases $\{u_n^k\}$ with AACI bells for certain values of N.

ϵ	N	$\bar{m}$		σ
		I	II	
10^{-3}	1	7	10.2	2.2
	2	5	9.7	1.0
	3	9	15.4	1.5
10^{-7}	2	30	48.6	7.2
	3	17	33.3	1.2
	4	28	53.3	2.2
10^{-13}	3	63	108.2	12.5
	4	48	92.2	6.1
	5	77	144.3	14.4

Table 3: The optimal parameters N for certain values of ϵ

Type of bell	AACI	Coifman	Optimized
$\epsilon = 10^{-3}$	2	2	4
$\epsilon = 10^{-7}$	3	7	9
$\epsilon = 10^{-13}$	4	13	16

Table 4: Comparison of compression of functions (91) (I) and (1) (II). For each value of ϵ the choice of N is the same as in Table 3.

Type of bell	AACI			Coifman			Gaussian			Optimized		
ϵ	$\bar{m}$		σ	$\bar{m}$		σ	$\bar{m}$		σ	$\bar{m}$		σ
	I	II		I	II		I	II		I	II	
10^{-3}	5	9.7	1.0	6	9.7	1.0	4	8.3	0.7	4	7.6	0.7
10^{-7}	17	33.3	1.2	16	33.8	1.2	10	19.7	1.2	9	19.1	1.3
10^{-13}	48	92.2	6.1	41	81.3	4.7	19	36.7	3.0	16	37.8	3.2

Table 5: The sample mean error $\bar{\delta}$ (172) and standard deviation σ_δ (174) (see text for details). For each value of ϵ the choice of N is the same as in Table 3.

Type of bell	AACI		Coifman		Gaussian		Optimized	
ϵ	$\bar{\delta}/\epsilon$	σ_δ/ϵ	$\bar{\delta}/\epsilon$	σ_δ/ϵ	$\bar{\delta}/\epsilon$	σ_δ/ϵ	$\bar{\delta}/\epsilon$	σ_δ/ϵ
10^{-3}	0.5	0.3	1.1	0.4	0.8	0.5	0.5	0.3
10^{-7}	0.9	0.3	1.1	0.3	3.5	2.0	0.8	0.3
10^{-13}	1.3	0.3	1.0	0.4	60	41	1.2	0.3

Table 6: Compression the function f_L (177) for $L = 2$ (I), $L = 4$ (II), $L = 8$ (III), and $L = 16$ (IV). For each value of ϵ the choice of N is the same as in Table 3.

Type of bell	AACI				Coifman				Gaussian				Optimized			
ϵ	I	II	III	IV	I	II	III	IV	I	II	III	IV	I	II	III	IV
10^{-3}	9	11	13	23	12	12	16	25	10	12	15	25	8	10	13	23
10^{-7}	33	30	36	45	35	36	41	49	22	22	27	37	20	22	27	38
10^{-13}	95	93	95	102	85	84	88	96	38	41	45	55	39	40	44	54

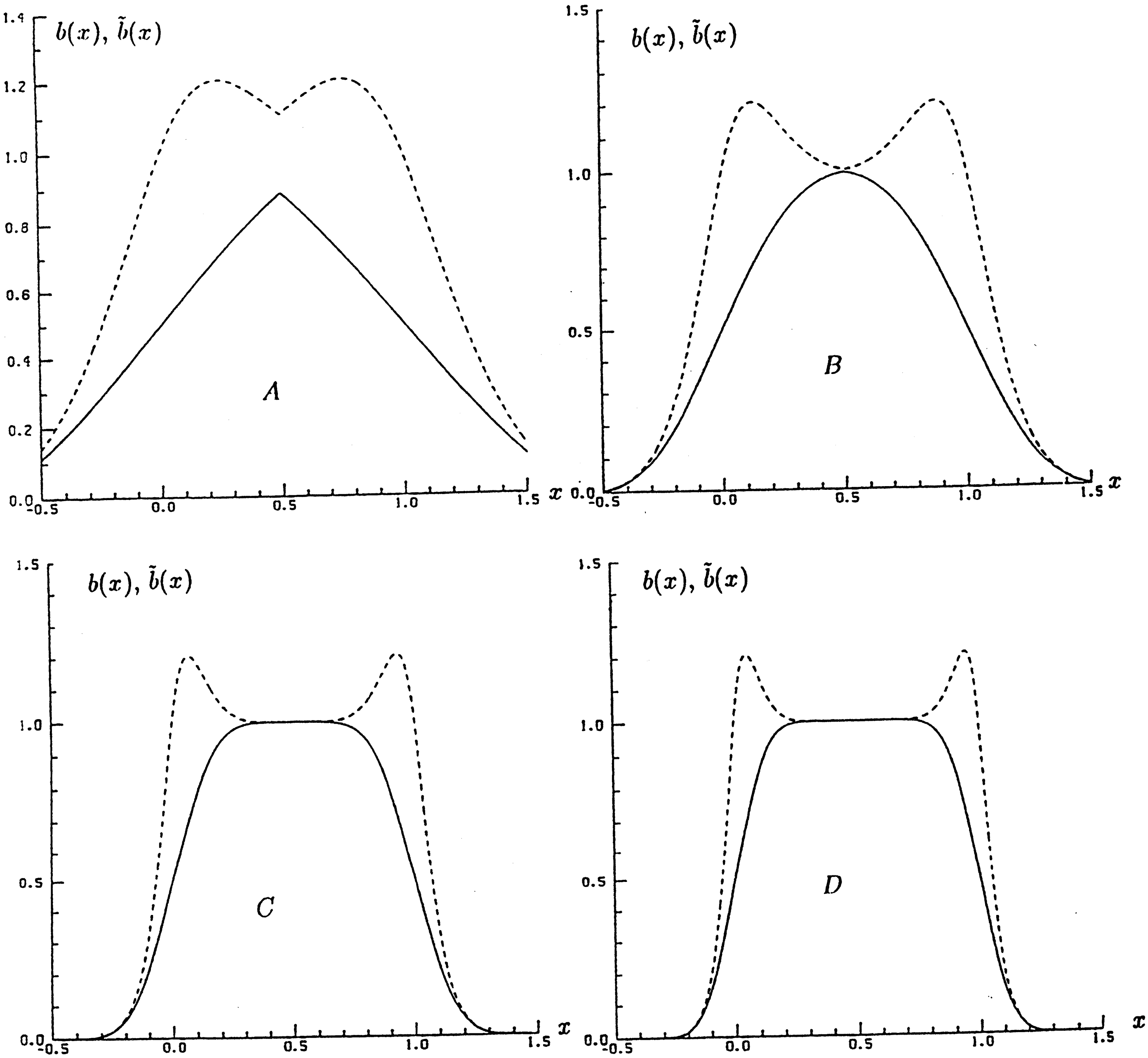

FIG. 1. Optimized bells (solid line) and corresponding dual bells (dashed line) for $N = 1$ (A), $N = 3$ (B), $N = 9$ (C), and $N = 16$ (D).

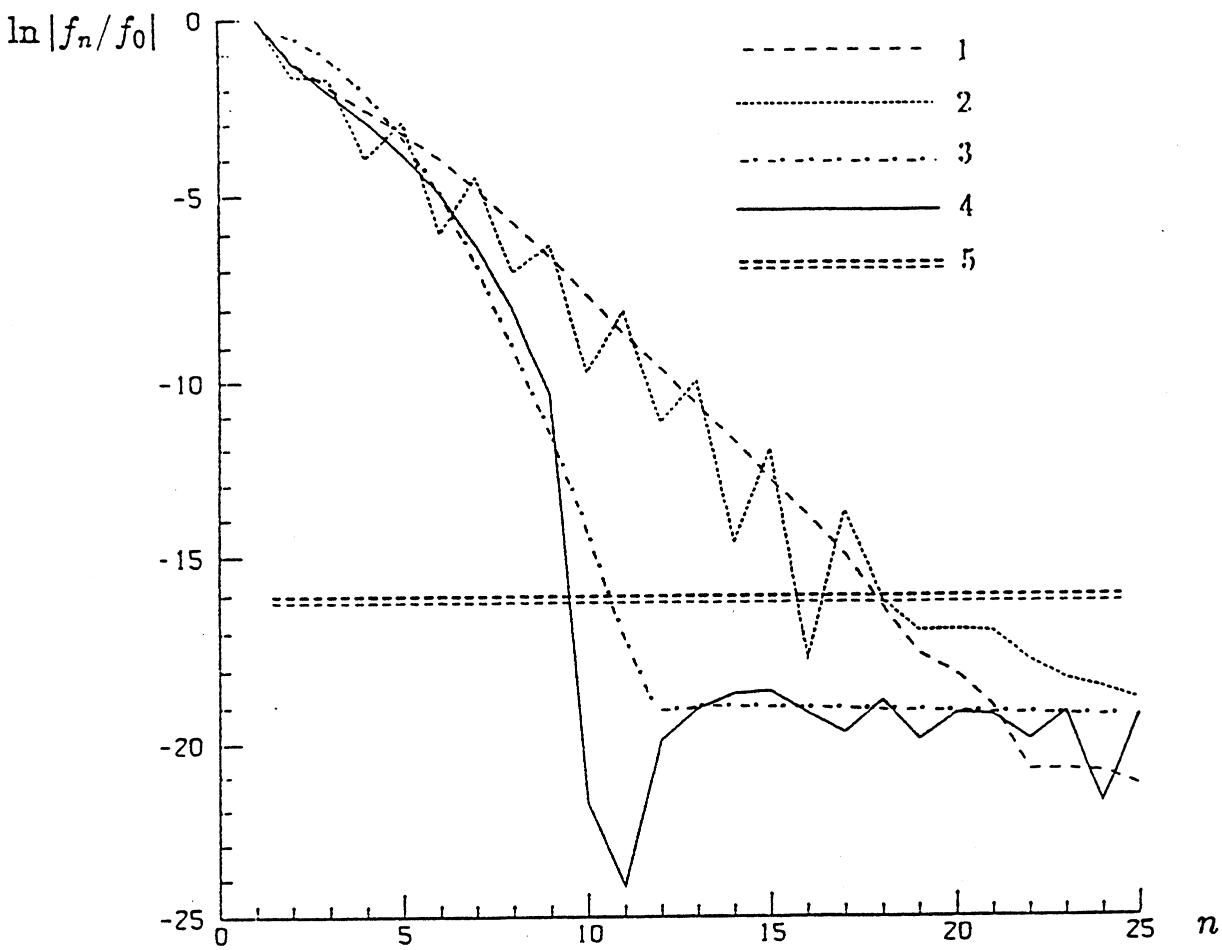

FIG. 2. Expansion coefficients f_n of the function (91) in bases with the AACI (1), the Coifman (2), the Gaussian (3), and the optimized (4) bells for $\epsilon = 10^{-7}$. The straight line (5) corresponds to $\ln \epsilon$.

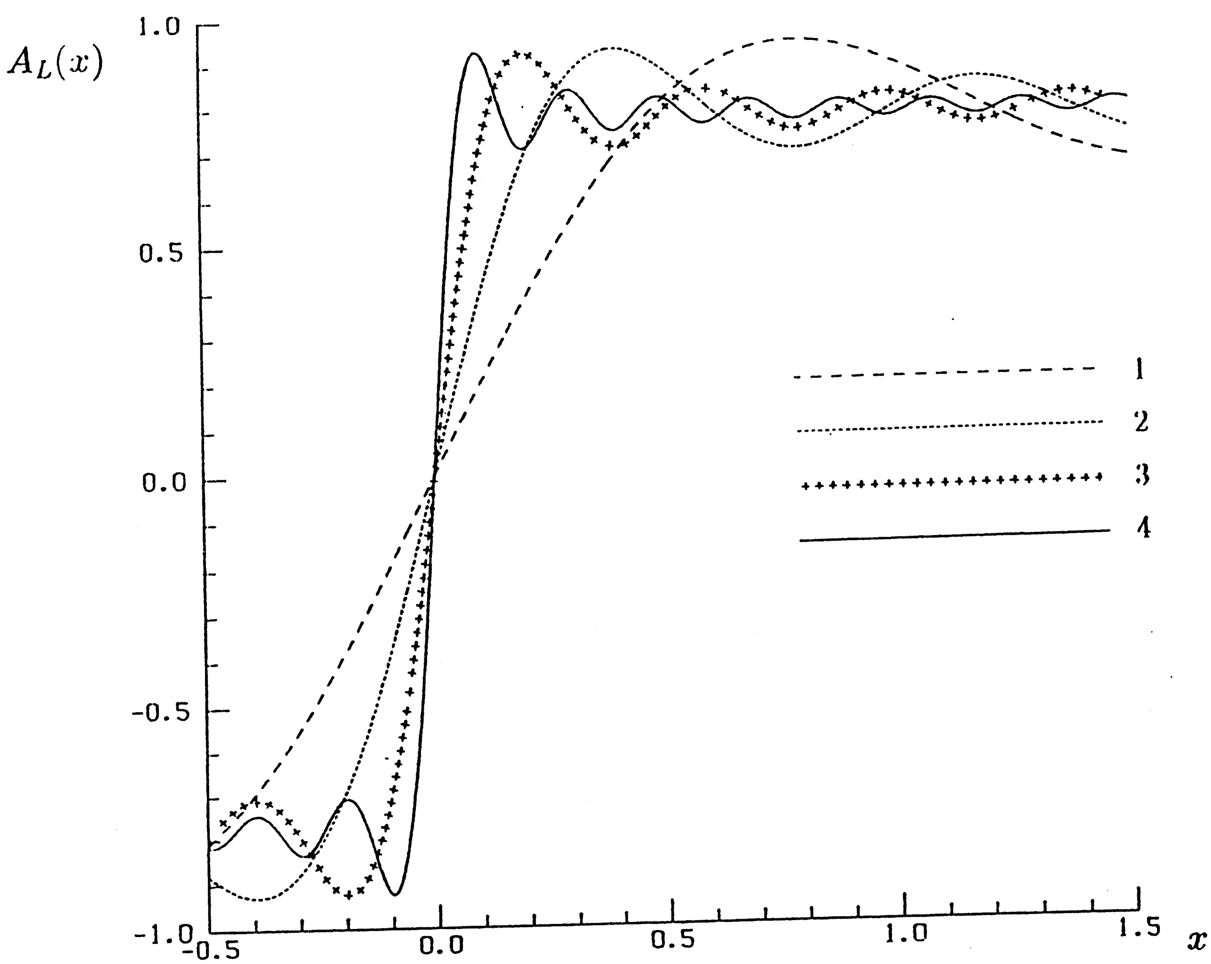

FIG. 3. The function A_L (177) for $L = 2$ (1), $L = 4$ (2), $L = 8$ (3), and $L = 16$ (4).

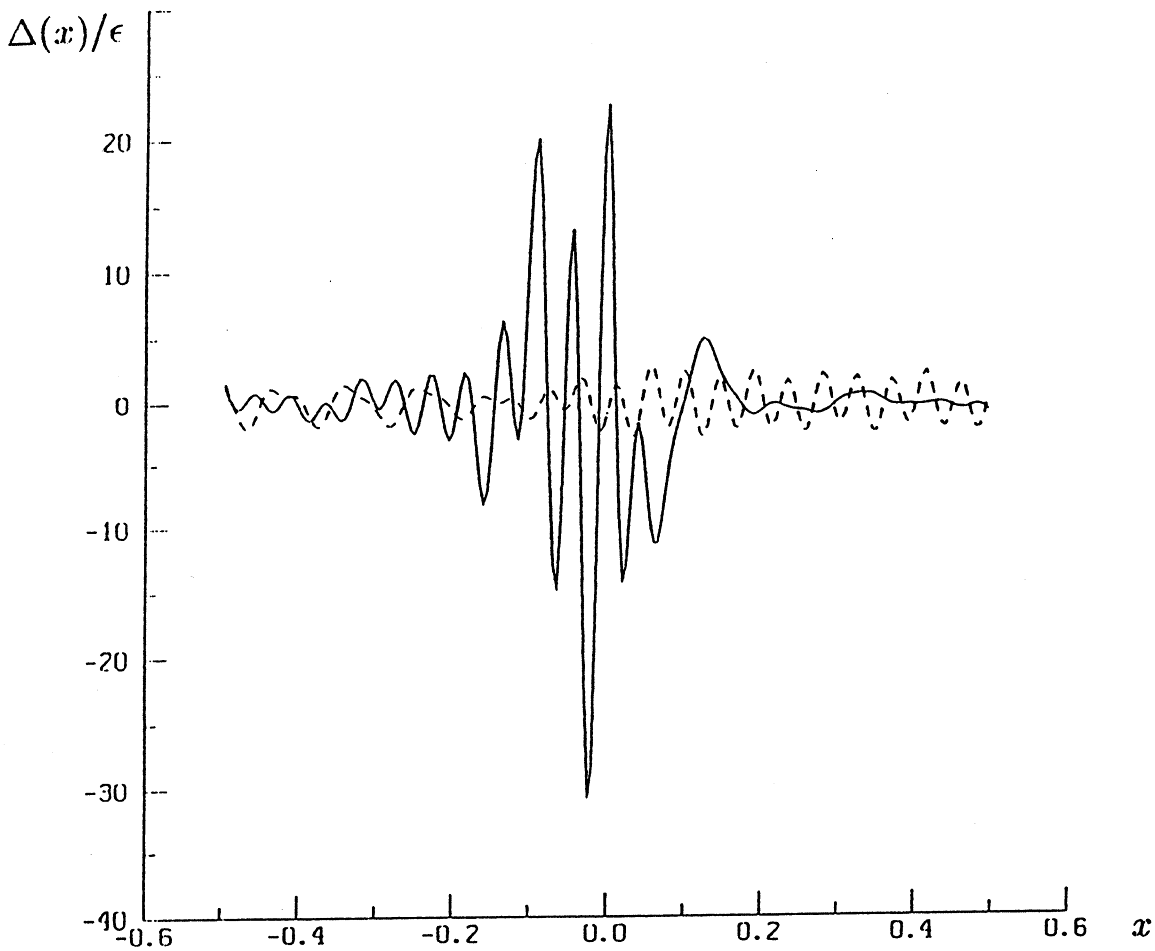

FIG. 4. The function Δ in case of the Gaussian bell (solid line) and the optimized bell (dashed line).

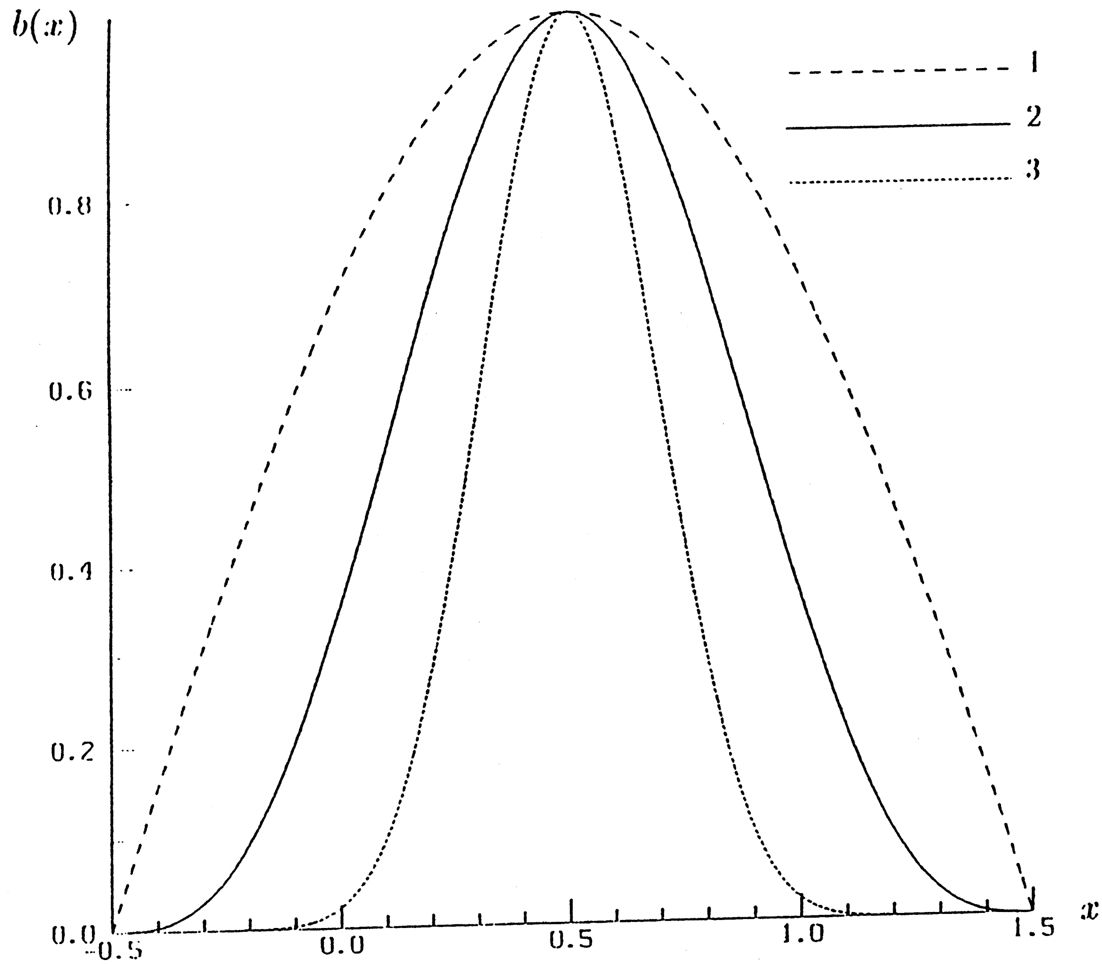

FIG. 5. Unconditionally optimized bells for $K = 1$ (1), $K = 2$ (2), and $K = 6$ (3).