In this dissertation we investigate the solution of boundary value problems on polygonal domains for elliptic partial differential equations. We observe that when the problems are formulated as the boundary integral equations of classical potential theory, the solutions are representable by series of elementary functions. In addition to being analytically perspicuous, the resulting expressions lend themselves to the construction of accurate and efficient numerical algorithms. The results are illustrated by a number of numerical examples.

On the Solution of Elliptic Partial Differential Equations on Regions with Corners

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Technical Report YALEU/DCS/TR-1523

April 23, 2016

⋄ This author's work was supported in part under ONR N00014-11-1-0718, ONR N00014-14-1-0797, ONR N00014-10-1-0570, and by the National Defense Science and Engineering Graduate Fellowship

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Keywords: *Boundary Value Problems, Potential Theory, Corners, Singular Solutions*

On the Solution of Elliptic Partial Differential Equations on Regions with Corners

A Dissertation
Presented to the Faculty of the Graduate School
of
Yale University
in Candidacy for the Degree of
Doctor of Philosophy

by
Kirill Serkh

Dissertation Director: Vladimir Rokhlin

May 2016

Acknowledgments

I would like to thank Prof. Jones, Prof. Coifman, Ms. Kavanaugh, my friends, and my family for all their help and support. Finally, I would like to thank my advisor Prof. Rokhlin for his brilliance and his invaluable guidance, without which this dissertation would have been impossible. I am very grateful for his mentorship; I cannot imagine how these past few years could have been more enriching or more enjoyable.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

In classical potential theory, elliptic partial differential equations (PDEs) are reduced to integral equations by representing the solutions as single-layer or double-layer potentials on the boundaries of the regions. The densities of these potentials satisfy Fredholm integral equations of the *second kind*.

There are three essentially separate regimes in which such boundary integral equations have been studied. In the first regime, the boundary of the region is approximated by a smooth curve. It is known that if the curve is smooth, then the solutions to the integral equations are smooth as well (see, for example, [3]). The existence and uniqueness of the solutions follow from Fredholm theory, and the integral equations can be solved numerically using standard tools (see, for example, [15]).

In the second regime, the boundary of the region is approximated by a curve with perfectly sharp corners. In this regime, the kernels of the integral equations have singularities at the corners, and the existence and uniqueness of the solutions in the L^2-sense are also known (see, for example, [30]). The behavior, in the vicinity of the corners, of the solutions of both the integral equations and of the underlying differential equations have been the subject of much study (see [32], [20] for representative examples). Comprehensive reviews of the literature can be found in (for example) [24], [14].

In the third regime, the assumptions on the boundary are of an altogether different nature. The boundary might be a Lipschitz or Hölder continuous curve, or a fractal, etc.

While during the last fifty years, such environments have been studied in great detail (see, for example, [17], [30], [5], [7], [6], [18], etc.), they are outside the scope of this dissertation.

This dissertation deals with the very special case of polygonal boundaries, and is based on several analytical observations related to the classical boundary integral formulations of the Laplace and Helmholtz equations. More specifically, we observe that the solutions to these integral equations in the vicinity of corners are representable by linear combinations of certain non-integer powers in the Laplace case, and linear combinations of Bessel functions of certain non-integer orders in the Helmholtz case. These formulas lead to the construction of accurate and efficient numerical algorithms, which are illustrated by a number of numerical examples.

The structure of the dissertation is as follows. Chapter 2 contains the mathematical apparatus, numerical algorithms, and numerical examples for Laplace's equation. Chapter 3 contains the mathematical apparatus, numerical algorithms, and numerical examples for the Helmholtz equation. Chapter 4 outlines future extensions and generalizations.

Chapter 2

The Laplace Equation

2.1 Overview

This section provides a brief overview of the principal results of this chapter. The following two subsections 2.1.1 and 2.1.2 summarize the Neumann and Dirichlet cases respectively; subsection 2.1.3 summarizes the numerical algorithm and numerical results.

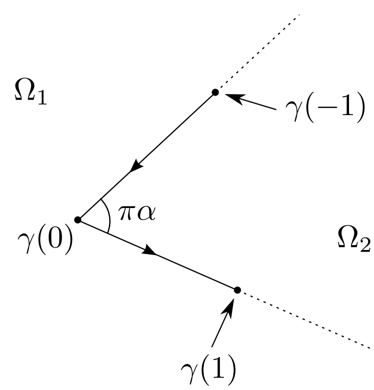

Figure 2.1: A wedge in $\mathbb{R}^2$

2.1.1 The Neumann Case

Suppose that $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge in $\mathbb{R}^2$ with a corner at $\gamma(0)$, and with interior angle $\pi\alpha$. Suppose further that γ is parameterized by arc length, and let $\nu(t)$ denote the inward-facing unit normal to the curve γ at t. Let Γ denote the set $\gamma([-1,1])$. By extending the sides of the wedge to infinity, we divide $\mathbb{R}^2$ into two open sets Ω_1 and Ω_2

(see Figure 2.1).

Let $\phi\colon \mathbb{R}^2 \setminus \Gamma \to \mathbb{R}$ denote the potential induced by a charge distribution on γ with density $\rho\colon [-1,1] \to \mathbb{R}$. In other words, let ϕ be defined by the formula

$$\phi(x) = -\frac{1}{2\pi} \int_{-1}^{1} \log(\|\gamma(t) - x\|)\rho(t)\, dt, \tag{2.1}$$

for all $x \in \mathbb{R}^2 \setminus \Gamma$, where $\|\cdot\|$ denotes the Euclidean norm. Suppose that $g\colon [-1,1] \to \mathbb{R}$ is defined by the formula

$$g(t) = \lim_{\substack{x \to \gamma(t) \\ x \in \Omega_1}} \frac{\partial \phi(x)}{\partial \nu(t)}, \tag{2.2}$$

for all $-1 \le t \le 1$, i.e. g is the limit of the normal derivative of integral (2.1) when x approaches the point $\gamma(t)$ from outside. It is well known that

$$g(s) = \frac{1}{2}\rho(s) + \frac{1}{2\pi}\int_{-1}^{1} K(s,t)\rho(t)\, dt, \tag{2.3}$$

for all $-1 \le s \le 1$, where

$$K(s,t) = \frac{\langle \gamma(t) - \gamma(s), \nu(s) \rangle}{\|\gamma(t) - \gamma(s)\|^2}, \tag{2.4}$$

for all $-1 \le s, t \le 1$, where $\langle \cdot, \cdot \rangle$ denotes the inner product.

The following theorem is one of the principal results of this chapter.

Theorem 2.1. *Suppose that N is a positive integer and that ρ is defined by the formula*

$$\rho(t) = \sum_{n=1}^{N} b_n (\mathrm{sgn}(t))^{n+1} |t|^{\frac{n}{\alpha}-1} + \sum_{n=1}^{N} c_n (\mathrm{sgn}(t))^{n} |t|^{\frac{n}{2-\alpha}-1}, \tag{2.5}$$

for all $-1 \le t \le 1$, where $b_1, b_2, \dots, b_N$ and $c_1, c_2, \dots, c_N$ are arbitrary real numbers. Suppose further that g is defined by (2.3). Then there exist series of real numbers $\beta_0, \beta_1, \dots$

and $\gamma_0, \gamma_1, \ldots$ such that

$$g(t) = \sum_{n=0}^{\infty} \beta_n |t|^n + \sum_{n=0}^{\infty} \gamma_n \operatorname{sgn}(t)|t|^n, \tag{2.6}$$

for all $-1 \leq t \leq 1$. Conversely, suppose that g has the form (2.6). Then, for each positive integer N, there exist real numbers $b_1, b_2, \ldots, b_N$ and $c_1, c_2, \ldots, c_N$ such that ρ, defined by (2.5), solves equation (2.3) to within an error $O(t^N)$.

In other words, if ρ has the form (2.5), then g is smooth on the intervals $[-1, 0]$ and $[0, 1]$. Conversely, if g is smooth, then for each positive integer N there exists a solution ρ of the form (2.5) that solves equation (2.3) to within an error $O(t^N)$.

2.1.2 The Dirichlet Case

Suppose that $\gamma\colon [-1, 1] \to \mathbb{R}^2$ is a wedge in $\mathbb{R}^2$ with a corner at $\gamma(0)$, and with interior angle $\pi\alpha$. Suppose further that γ is parameterized by arc length, and let $\nu(t)$ denote the inward-facing unit normal to the curve γ at t. Let Γ denote the set $\gamma([-1, 1])$. By extending the sides of the wedge to infinity, we divide $\mathbb{R}^2$ into two open sets Ω_1 and Ω_2 (see Figure 2.1).

Let $\phi\colon \mathbb{R}^2 \setminus \Gamma \to \mathbb{R}$ denote the potential induced by a dipole distribution on γ with density $\rho\colon [-1, 1] \to \mathbb{R}$. In other words, let ϕ be defined by the formula

$$\phi(x) = \frac{1}{2\pi} \int_{-1}^{1} \frac{\langle x - \gamma(t), \nu(t) \rangle}{\|x - \gamma(t)\|^2} \rho(t)\, dt, \tag{2.7}$$

for all $x \in \mathbb{R}^2 \setminus \Gamma$, where $\|\cdot\|$ denotes the Euclidean norm and $\langle \cdot, \cdot \rangle$ denotes the inner product. Suppose that $g\colon [-1, 1] \to \mathbb{R}$ is defined by the formula

$$g(t) = \lim_{\substack{x \to \gamma(t) \\ x \in \Omega_2}} \phi(x), \tag{2.8}$$

for all $-1 \leq t \leq 1$, i.e. g is the limit of integral (2.7) when x approaches the point $\gamma(t)$

from inside. It is well known that

$$g(s) = \frac{1}{2}\rho(s) + \frac{1}{2\pi}\int_{-1}^{1} K(t,s)\rho(t)\,dt, \tag{2.9}$$

for all $-1 \le s \le 1$, where

$$K(t,s) = \frac{\langle \gamma(s) - \gamma(t), \nu(t)\rangle}{\|\gamma(s) - \gamma(t)\|^2}, \tag{2.10}$$

for all $-1 \le s, t \le 1$, where $\langle \cdot, \cdot \rangle$ denotes the inner product.

The following theorem is one of the principal results of this chapter.

Theorem 2.2. *Suppose that N is a positive integer and that ρ is defined by the formula*

$$\rho(t) = \sum_{n=0}^{N} b_n (\mathrm{sgn}(t))^{n+1} |t|^{\frac{n}{\alpha}} + \sum_{n=0}^{N} c_n (\mathrm{sgn}(t))^{n} |t|^{\frac{n}{2-\alpha}}, \tag{2.11}$$

for all $-1 \le t \le 1$, where $b_0, b_1, \ldots, b_N$ and $c_0, c_1, \ldots, c_N$ are arbitrary real numbers. Suppose further that g is defined by (2.9). Then there exist series of real numbers $\beta_0, \beta_1, \ldots$ and $\gamma_0, \gamma_1, \ldots$ such that

$$g(t) = \sum_{n=0}^{\infty} \beta_n |t|^n + \sum_{n=0}^{\infty} \gamma_n \,\mathrm{sgn}(t) |t|^n, \tag{2.12}$$

for all $-1 \le t \le 1$. Conversely, suppose that g has the form (2.12). Then, for each positive integer N, there exist real numbers $b_0, b_1, \ldots, b_N$ and $c_0, c_1, \ldots, c_N$ such that ρ, defined by (2.11), solves equation (2.9) to within an error $O(t^{N+1})$.

In other words, if ρ has the form (2.11), then g is smooth on the intervals $[-1, 0]$ and $[0, 1]$. Conversely, if g is smooth, then for each positive integer N there exists a solution ρ of the form (2.11) that solves equation (2.9) to within an error $O(t^{N+1})$.

2.1.3 The Procedure

Recently, progress has been made in solving the boundary integral equations of potential theory numerically (see, for example, [16], [4]). Most such schemes use nested quadratures to resolve the corner singularities. However, the explicit representations (2.5), (2.11) lead to alternative numerical algorithms for the solution of the integral equations of potential theory. More specifically, we use these representations to construct purpose-made discretizations which accurately represent the associated boundary integral equations (see, for example, [23], [21], [31]). Once such discretizations are available, the equations can be solved using the Nyström method combined with standard tools. We observe that the condition numbers of the resulting discretized linear systems closely approximate the condition numbers of the underlying physical problems.

Observation 2.3. While the analysis in this chapter applies only to polygonal domains, a similar analysis carries over to curved domains with corners. A paper containing the analysis, as well as the corresponding numerical algorithms and numerical examples, is in preparation.

Observation 2.4. In the examples in this chapter, the discretized boundary integral equations are solved in a straightforward way using standard tools. However, if needed, such equations can be solved much more rapidly using the numerical apparatus from, for example, [13].

Remark 2.5. Due to the detailed analysis in this chapter, the CPU time requirements of the resulting algorithms are almost independent of the requested precision. Thus, in all the examples in this chapter, the boundary integral equations are solved to essentially full double precision.

The structure of this chapter is as follows. In Section 2.2, we introduce the necessary mathematical preliminaries. Section 2.3 contains the primary analytical tools of this chapter. In sections 2.4 and 2.5, we investigate the Neumann and Dirichlet cases respectively. In Section 2.6, we briefly describe a numerical algorithm and provide several numerical examples.

2.2 Mathematical Preliminaries

2.2.1 Boundary Value Problems

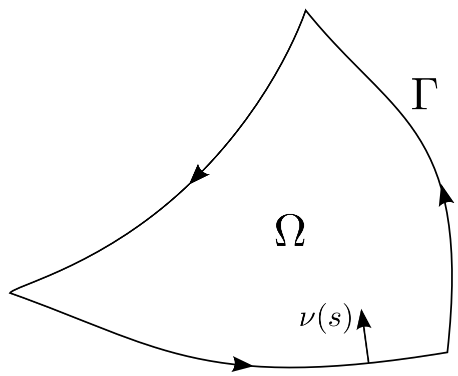

Figure 2.2: A curve in $\mathbb{R}^2$

Suppose that $\gamma : [0, L] \to \mathbb{R}^2$ is a simple closed curve of length L with a finite number of corners. Suppose further that γ is analytic except at the corners. We denote the interior of γ by Ω and the boundary of Ω by Γ, and let $\nu(t)$ denote the normalized internal normal to γ at $t \in [0, L]$. Supposing that g is some function $[0, L] \to \mathbb{C}$, we will solve the following four problems.

1) *Interior Neumann problem* (INP): Find a function $\phi \colon \Omega \to \mathbb{R}$ such that

$$\nabla^2 \phi(x) = 0 \qquad \text{for } x \in \Omega, \tag{2.13}$$

$$\lim_{\substack{x \to \gamma(t) \\ x \in \Omega}} \frac{\partial \phi(x)}{\partial \nu(t)} = g(t) \qquad \text{for } t \in [0, L]. \tag{2.14}$$

2) *Exterior Neumann problem* (ENP): Find a function $\phi \colon \mathbb{R}^2 \setminus \overline{\Omega} \to \mathbb{R}$ such that

$$\nabla^2 \phi(x) = 0 \qquad \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega}, \tag{2.15}$$

$$\lim_{\substack{x \to \gamma(t) \\ x \in \mathbb{R}^2 \setminus \overline{\Omega}}} \frac{\partial \phi(x)}{\partial \nu(t)} = g(t) \qquad \text{for } t \in [0, L]. \tag{2.16}$$

3) *Interior Dirichlet problem* (IDP): Find a function $\phi\colon \Omega \to \mathbb{R}$ such that

$$\nabla^2 \phi(x) = 0 \qquad \text{for } x \in \Omega, \tag{2.17}$$

$$\lim_{\substack{x \to \gamma(t) \\ x \in \Omega}} \phi(x) = g(t) \qquad \text{for } t \in [0, L]. \tag{2.18}$$

4) *Exterior Dirichlet problem* (EDP): Find a function $\phi\colon \mathbb{R}^2 \setminus \overline{\Omega} \to \mathbb{R}$ such that

$$\nabla^2 \phi(x) = 0 \qquad \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega}, \tag{2.19}$$

$$\lim_{\substack{x \to \gamma(t) \\ x \in \mathbb{R}^2 \setminus \overline{\Omega}}} \phi(x) = g(t) \qquad \text{for } t \in [0, L]. \tag{2.20}$$

Suppose that $g \in L^2([0, L])$. Then the interior and exterior Dirichlet problems have unique solutions. If g also satisfies the condition

$$\int_0^L g(t)\, dt = 0, \tag{2.21}$$

then the interior and exterior Neumann problems have unique solutions up to an additive constant (see, for example, [17], [10]).

2.2.2 Integral Equations of Potential Theory

In classical potential theory, boundary value problems are solved by representing the function ϕ by integrals of potentials over the boundary. The potential of a *unit charge* located at $x_0 \in \mathbb{R}^2$ is the function $\psi^0_{x_0}\colon \mathbb{R}^2 \setminus x_0 \to \mathbb{R}$, defined via the formula

$$\psi^0_{x_0}(x) = \log(\|x - x_0\|), \tag{2.22}$$

for all $x \in \mathbb{R}^2 \setminus x_0$, where $\|\cdot\|$ denotes the Euclidean norm. The potential of a *unit dipole* located at $x_0 \in \mathbb{R}^2$ and oriented in direction $h \in \mathbb{R}^2$, $\|h\| = 1$, is the function

$\psi^1_{x_0,h}\colon \mathbb{R}^2 \setminus x_0 \to \mathbb{R}$, defined via the formula

$$\psi^1_{x_0,h}(x) = \frac{\langle h, x_0 - x\rangle}{\|x_0 - x\|^2}, \tag{2.23}$$

for all $x \in \mathbb{R}^2 \setminus x_0$, where $\langle \cdot, \cdot \rangle$ denotes the inner product.

Charge and dipole distributions with density $\rho\colon [0, L] \to \mathbb{R}$ on Γ produce potentials given by the formulas

$$\phi(x) = \int_0^L \psi^0_{\gamma(t)}(x)\rho(t)\,dt, \tag{2.24}$$

and

$$\phi(x) = \int_0^L \psi^1_{\gamma(t),\nu(t)}(x)\rho(t)\,dt, \tag{2.25}$$

respectively, for any $x \in \mathbb{R}^2 \setminus \Gamma$.

Reduction of Boundary Value Problems to Integral Equations

The following four theorems reduce the boundary value problems of Section 2.2.1 to boundary integral equations. They are found in, for example, [26], [30].

Theorem 2.6 (Exterior Neumann problem)**.** *Suppose that $\rho \in L^2([0, L])$. Suppose further that $g\colon [0, L] \to \mathbb{R}$ is defined by the formula*

$$g(s) = -\pi\rho(s) + \int_0^L \psi^1_{\gamma(s),\nu(s)}(\gamma(t))\rho(t)\,dt, \tag{2.26}$$

for any $s \in [0, L]$. Then g is in $L^2([0, L])$, and a solution ϕ to the exterior Neumann problem with right hand side g is obtained by substituting ρ into (2.24). Moreover, for any $g \in L^2([0, L])$, equation (2.26) has a unique solution $\rho \in L^2([0, L])$.

Theorem 2.7 (Interior Dirichlet problem)**.** *Suppose that $\rho \in L^2([0, L])$. Suppose further* *that* $g\colon [0,L] \to \mathbb{R}$ *is defined by the formula*

$$g(s) = -\pi\rho(s) + \int_0^L \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\rho(t)\,dt, \tag{2.27}$$

for any $s \in [0,L]$*. Then* g *is in* $L^2([0,L])$*, and the solution* ϕ *to the interior Dirichlet problem with right hand side* g *is obtained by substituting* ρ *into (2.25). Moreover, for any* $g \in L^2([0,L])$*, equation (2.27) has a unique solution* $\rho \in L^2([0,L])$*.*

The following two theorems make use of the function $\overline{\omega}\colon [0,L] \to \mathbb{R}$, defined as the solution to the equation

$$\int_0^L \overline{\omega}(t)\log(\|x - \gamma(t)\|)\,dt = 1, \tag{2.28}$$

for all $x \in \overline{\Omega}$. In other words, we define the function $\overline{\omega}$ as the density of the charge distribution on Γ when $\overline{\Omega}$ is a conductor.

Theorem 2.8 (Interior Neumann problem)**.** *Suppose that* $\rho \in L^2([0,L])$*. Suppose further that* $g\colon [0,L] \to \mathbb{R}$ *is defined by the formula*

$$g(s) = \pi\rho(s) + \int_0^L \psi^1_{\gamma(s),\nu(s)}(\gamma(t))\rho(t)\,dt, \tag{2.29}$$

for any $s \in [0,L]$*. Then* g *is in* $L^2([0,L])$*, and a solution* ϕ *to the exterior Neumann problem with right hand side* g *is obtained by substituting* ρ *into (2.24).*

Now suppose that g *is an arbitrary function in* $L^2([0,L])$ *such that*

$$\int_0^L g(t)\,dt = 0. \tag{2.30}$$

Then equation (2.29) has a solution $\rho \in L^2([0,L])$*. Moreover, if* ρ_1 *and* ρ_2 *are both solutions to equation (2.29), then there exists a real number* C *such that*

$$\rho_1(t) - \rho_2(t) = C\overline{\omega}(t), \tag{2.31}$$

for $t \in [0, L]$, where $\overline{\omega}$ is the solution to (2.28).

Theorem 2.9 (Exterior Dirichlet problem)**.** *Suppose that $\rho \in L^2([0, L])$. Suppose further that $g\colon [0, L] \to \mathbb{R}$ is defined by the formula*

$$g(s) = \pi\rho(s) + \int_0^L \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\rho(t)\, dt, \tag{2.32}$$

for any $s \in [0, L]$. Then g is in $L^2([0, L])$, and the solution ϕ to the interior Dirichlet problem with right hand side g is obtained by substituting ρ into (2.25).

Now suppose that g is an arbitrary function in $L^2([0, L])$ such that

$$\int_0^L g(t)\overline{\omega}(t)\, dt = 0, \tag{2.33}$$

where $\overline{\omega}$ is the solution to (2.28). Then equation (2.32) has a solution $\rho \in L^2([0, L])$. Moreover, if ρ_1 and ρ_2 are both solutions to equation (2.32), then there exists a real number C such that

$$\rho_1(t) - \rho_2(t) = C, \tag{2.34}$$

for $t \in [0, L]$.

Observation 2.10. Equation (2.26) is the adjoint of equation (2.27), and equation (2.29) is the adjoint of equation (2.32).

Observation 2.11. Suppose that the curve $\gamma\colon [0, L] \to \mathbb{R}^2$ is *not* closed. We observe that if $\rho \in L^2([0, L])$, and g is defined by either (2.26), (2.27), (2.29), or (2.32), then $g \in L^2([0, L])$. Moreover, if $g \in L^2([0, L])$, then equations (2.26), (2.27), (2.29), and (2.32) have unique solutions $\rho \in L^2([0, L])$.

Properties of the Kernels of Equations (2.26), (2.27), (2.29), and (2.32)

The following theorem shows that if a curve γ is has k continuous derivatives, where $k \geq 2$, then the kernels of equations (2.26), (2.27), (2.29), and (2.32) have $k-2$ continuous derivatives. It is found in, for example, [3].

Theorem 2.12. *Suppose that $\gamma\colon [0,L] \to \mathbb{R}^2$ is a curve in $\mathbb{R}^2$ that is parameterized by arc length. Suppose further that $k \geq 2$ is an integer. If γ is C^k on a neighborhood of a point s, where $0 < s < L$, then*

$$\psi^1_{\gamma(s),\nu(s)}(\gamma(t)), \tag{2.35}$$

$$\psi^1_{\gamma(t),\nu(t)}(\gamma(s)), \tag{2.36}$$

are C^{k-2} functions of t on a neighborhood of s and

$$\lim_{t\to s} \psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = \lim_{t\to s} \psi^1_{\gamma(t),\nu(t)}(\gamma(s)) = -\frac{1}{2}k(s), \tag{2.37}$$

where $k\colon [0,L] \to \mathbb{R}$ is the signed *curvature of γ. Furthermore, if γ is analytic on a neighborhood of a point s, where $0 < s < L$, then (2.35) and (2.36) are analytic functions of t on a neighborhood of s.*

When the curve γ is a wedge, the kernels of equations (2.26), (2.27), (2.29), and (2.32) have a particularly simple form, which is given by the following lemma.

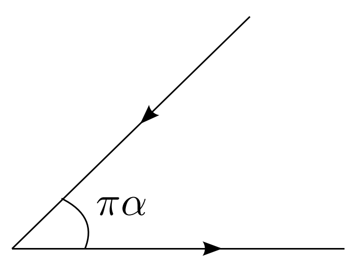

Figure 2.3: A wedge in $\mathbb{R}^2$

Lemma 2.13. *Suppose $\gamma\colon [-1,1] \to \mathbb{R}^2$ is defined by the formula*

$$\gamma(t) = \begin{cases} -t \cdot (\cos(\pi\alpha), \sin(\pi\alpha)) & \textit{if } -1 \leq t < 0, \\ (t, 0) & \textit{if } 0 \leq t \leq 1, \end{cases} \tag{2.38}$$

shown in Figure 2.3. Then, for all $0 < s \leq 1$,

$$\psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = \begin{cases} \dfrac{t\sin(\pi\alpha)}{s^2+t^2+2st\cos(\pi\alpha)} & \text{if } -1 \leq t < 0, \\ 0 & \text{if } 0 \leq t \leq 1, \end{cases} \tag{2.39}$$

and, for all $-1 \leq s < 0$,

$$\psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = \begin{cases} 0 & \text{if } -1 \leq t < 0, \\ \dfrac{-t\sin(\pi\alpha)}{s^2+t^2+2st\cos(\pi\alpha)} & \text{if } 0 \leq t \leq 1. \end{cases} \tag{2.40}$$

Proof. Suppose that $0 < s \leq 1$ and $0 \leq t \leq 1$. Then,

$$\begin{aligned} \psi^1_{\gamma(s),\nu(s)}(\gamma(t)) &= \frac{\langle \nu(s), \gamma(s)-\gamma(t)\rangle}{\|\gamma(s)-\gamma(t)\|^2} \\ &= \frac{\langle (0,1),(s,0)-(t,0)\rangle}{|s-t|^2} \\ &= 0. \end{aligned} \tag{2.41}$$

Now suppose that $0 < s \leq 1$ and $-1 \leq t < 0$. Then,

$$\begin{aligned} \psi^1_{\gamma(s),\nu(s)}(\gamma(t)) &= \frac{\langle \nu(s), \gamma(s)-\gamma(t)\rangle}{\|\gamma(s)-\gamma(t)\|^2} \\ &= \frac{\langle (0,1),(s,0)+t(\cos(\pi\alpha),\sin(\pi\alpha))\rangle}{(s+t\cos(\pi\alpha))^2+(t\sin(\pi\alpha))^2} \\ &= \frac{t\sin(\pi\alpha)}{s^2+t^2+2st\cos(\pi\alpha)}. \end{aligned} \tag{2.42}$$

The proof for the case $-1 \leq s < 0$ is essentially identical. □

Corollary 2.14. *Identities (2.39) and (2.40) remain valid after any rotation or translation of the curve* γ *in* $\mathbb{R}^2$.

Corollary 2.15. *When the curve* γ *is a straight line,* $\psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = 0$ *for all* $-1 \leq s, t \leq 1$.

2.2.3 Several Classical Analytical Facts

In this section we list several classical analytical facts. They can be found in, for example, [22] and [2].

The following theorem describes a property of the zeros of analytic functions.

Theorem 2.16. *If f is a nonzero analytic function on a domain $\Omega \subset \mathbb{C}$, then the zeros of f have no accumulation point in Ω.*

Corollary 2.17 (Analytic continuation)**.** *Suppose that f and g are both analytic functions on a domain $\Omega \subset \mathbb{C}$. Suppose further that f and g are equal on a set a points in Ω that has an accumulation point in Ω. Then f and g are equal on all of Ω.*

The following classical theorem provides a test for the convergence of an infinite series.

Theorem 2.18 (Dirichlet's test)**.** *Suppose $a_1, a_2, \ldots$ is a sequence of real numbers such that*

$$a_n \geq a_{n+1} > 0, \tag{2.43}$$

for each positive integer n, and

$$\lim_{n \to \infty} a_n = 0. \tag{2.44}$$

Suppose further that $b_1, b_2, \ldots$ is a sequence of complex numbers such that, for some real constant M,

$$\left| \sum_{n=1}^{N} b_n \right| \leq M, \tag{2.45}$$

for each positive integer N. Then

$$\sum_{n=1}^{\infty} a_n b_n < \infty. \tag{2.46}$$

The following theorem relates a limit of a power series to the sum of its coefficients.

Theorem 2.19 (Abel's theorem)**.** *Suppose that $a_0, a_1, a_2, \ldots$ is a sequence of real numbers such that*

$$\sum_{n=0}^{\infty} a_n x^n < \infty, \tag{2.47}$$

for all $-1 < x < 1$. Suppose further that

$$\sum_{n=0}^{\infty} a_n < \infty. \tag{2.48}$$

Then

$$\lim_{\substack{x \to 1 \\ x<1}} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n. \tag{2.49}$$

2.3 Analytical Apparatus

The elementary theorems 2.23 and 2.24 in this section are the primary analytical tools of this chapter.

The following theorem provides the value of a certain integral. It is found in, for example, [12], Section 3.252, formula 12. For completeness, a proof is provided in Appendix A.

Theorem 2.20. *Suppose that $-1 < \mu < 1$ and $0 < \alpha < 2$ are real numbers. Then*

$$\int_0^{\infty} \frac{x^{\mu} \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\, dx = \pi a^{\mu-1} \frac{\sin\big(\mu\pi(1-\alpha)\big)}{\sin(\mu\pi)}, \tag{2.50}$$

for all $a > 0$.

The following lemma gives the Taylor series of a certain rational function.

Lemma 2.21. *Suppose that $-1 < p < 1$ and x are real numbers. Then*

$$\frac{p\sin(x)}{1-2p\cos(x)+p^2} = \sum_{n=1}^{\infty} p^n \sin(nx). \tag{2.51}$$

Proof. Let $-1 < p < 1$ and x be real numbers. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} p^n \sin(nx) &= \operatorname{Im}\left(\sum_{n=0}^{\infty} p^n e^{inx}\right) = \operatorname{Im}\left(\frac{1}{1-pe^{ix}}\right) \\
&= \operatorname{Im}\left(\frac{1-pe^{-ix}}{1-2p\cos(x)+p^2}\right) \\
&= \frac{p\sin(x)}{1-2p\cos(x)+p^2}.
\end{aligned} \tag{2.52}$$

□

The following lemma evaluates the integral in (2.50) when it is taken from 0 to 1 instead of from 0 to ∞.

Lemma 2.22. *Suppose that $-1 < \mu < 1$ and $0 < \alpha < 2$ are real numbers. Then*

$$\int_0^1 \frac{x^\mu \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\,dx = \pi a^{\mu-1}\frac{\sin\big(\mu\pi(1-\alpha)\big)}{\sin(\mu\pi)} + \sum_{k=0}^{\infty} \frac{\sin\big((k+1)\pi\alpha\big)}{\mu-k-1} a^k, \tag{2.53}$$

for all $0 < a < 1$.

Proof. Suppose that $0 < a < 1$. Clearly,

$$\int_1^\infty \frac{x^\mu \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\,dx = \int_1^\infty \frac{x^{\mu-1}}{a} \cdot \frac{(\frac{a}{x})\sin(\pi\alpha)}{(\frac{a}{x})^2 - 2(\frac{a}{x})\cos(\pi\alpha) + 1}\,dx. \tag{2.54}$$

Since $\frac{a}{x} < 1$ for all $x \geq 1$, by Lemma 2.21 we observe that

$$\int_1^\infty \frac{x^\mu \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\,dx = \int_1^\infty \frac{x^{\mu-1}}{a} \sum_{n=1}^{\infty} \frac{a^n \sin(n\pi\alpha)}{x^n}\,dx. \tag{2.55}$$

Interchanging the order of integration and summation, we further observe that

$$\int_1^\infty \frac{x^\mu \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\,dx = \sum_{n=1}^\infty \int_1^\infty \frac{a^{n-1}\sin(n\pi\alpha)}{x^{n-\mu+1}}\,dx$$
$$= -\sum_{n=1}^\infty \frac{a^{n-1}\sin(n\pi\alpha)}{\mu - n}$$
$$= -\sum_{k=0}^\infty \frac{\sin((k+1)\pi\alpha)}{\mu - k - 1} a^k. \tag{2.56}$$

Combining (2.50) and (2.56), we find that

$$\int_0^1 \frac{x^\mu \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\,dx = \pi a^{\mu-1}\frac{\sin\big(\mu\pi(1-\alpha)\big)}{\sin(\mu\pi)} + \sum_{k=0}^\infty \frac{\sin\big((k+1)\pi\alpha\big)}{\mu - k - 1} a^k, \tag{2.57}$$

for all $0 < a < 1$.

□

The following two theorems are the primary analytical tools of this chapter.

A simple analytic continuation argument shows that identity (2.53) in lemma 2.22 is also true for all complex μ such that $\text{Re}(\mu) > -1$ and $\mu \neq 1, 2, 3, \ldots$. This observation is summarized by the following theorem.

Theorem 2.23. *Suppose that* $0 < \alpha < 2$ *is a real number and* μ *is complex, so that* $\operatorname{Re}\mu > -1$ *and* $\mu \neq 1, 2, 3, \ldots$. *Then*

$$\int_0^1 \frac{x^\mu \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\,dx = \pi a^{\mu-1}\frac{\sin\big(\mu\pi(1-\alpha)\big)}{\sin(\mu\pi)} + \sum_{k=0}^\infty \frac{\sin\big((k+1)\pi\alpha\big)}{\mu - k - 1} a^k, \tag{2.58}$$

for all $0 < a < 1$.

Proof. Suppose that $0 < a < 1$. We observe that the right and left hand sides of identity (2.53) are both analytic functions of μ, for all μ such that $\text{Re}(\mu) > -1$ and $\mu \neq 1, 2, 3, \ldots$. Therefore, by analytic continuation (Theorem 2.17), it follows that identity (2.53) holds for all complex μ such that $\text{Re}(\mu) > -1$ and $\mu \neq 1, 2, 3, \ldots$. □

The following theorem extends Theorem 2.23 to the case when μ is a positive integer. We prove it by repeated application of L'Hôpital's rule.

Theorem 2.24. *Suppose that* $0 < \alpha < 2$ *is a real number and that* $m = 1, 2, 3, \ldots$. *Then*

$$\int_0^1 \frac{x^m \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\,dx$$
$$= a^{m-1}\pi(1-\alpha)\cos(m\pi\alpha) - a^{m-1}\log(a)\sin(m\pi\alpha) + \sum_{\substack{k\geq 0\\ k\neq m-1}} \frac{\sin\big((k+1)\pi\alpha\big)}{m-k-1}a^k, \tag{2.59}$$

for all $0 < a < 1$.

Proof. Suppose that m is a positive integer and $0 < a < 1$. We observe that the left hand side of identity (2.58) is analytic in μ, for all μ such that $\mathrm{Re}(\mu) > -1$, including the points $\mu = 1, 2, 3, \ldots$. Therefore,

$$\int_0^1 \frac{x^m \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\,dx = \lim_{\mu\to m}\int_0^1 \frac{x^\mu \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\,dx. \tag{2.60}$$

By Theorem 2.23,

$$\int_0^1 \frac{x^m \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\,dx = \lim_{\mu\to m}\left(\pi a^{\mu-1}\frac{\sin\big(\mu\pi(1-\alpha)\big)}{\sin(\mu\pi)} + \sum_{k=0}^{\infty}\frac{\sin\big((k+1)\pi\alpha\big)}{\mu-k-1}a^k\right)$$
$$= \sum_{\substack{k=0\\ k\neq m-1}}^{\infty}\frac{\sin\big((k+1)\pi\alpha\big)}{m-k-1}a^k + \lim_{\mu\to m}\left(\pi a^{\mu-1}\frac{\sin\big(\mu\pi(1-\alpha)\big)}{\sin(\mu\pi)} + \frac{\sin(m\pi\alpha)}{\mu-m}a^{m-1}\right)$$
$$= \sum_{\substack{k=0\\ k\neq m-1}}^{\infty}\frac{\sin\big((k+1)\pi\alpha\big)}{m-k-1}a^k + \lim_{\mu\to m}\frac{p(\mu)}{q(\mu)}, \tag{2.61}$$

where p and q are defined by the formulas

$$p(\mu) = \pi a^{\mu-1}(\mu-m)\sin(\mu\pi(1-\alpha)) + a^{m-1}\sin(\mu\pi)\sin(m\pi\alpha), \tag{2.62}$$

and

$$q(\mu) = (\mu - m)\sin(\mu\pi). \tag{2.63}$$

Clearly,

$$\lim_{\mu \to m} p(\mu) = \lim_{\mu \to m} q(\mu) = 0, \tag{2.64}$$

so we will use L'Hôpital's rule to determine $\lim_{\mu \to m} \frac{p(\mu)}{q(\mu)}$. We observe that

$$\begin{aligned} p'(\mu) &= \pi \log(a) a^{\mu-1}(\mu - m)\sin(\mu\pi(1-\alpha)) + \pi a^{\mu-1}\sin(\mu\pi(1-\alpha)) \\ &\quad + \pi^2(1-\alpha)a^{\mu-1}(\mu - m)\cos(\mu\pi(1-\alpha)) + \pi a^{m-1}\cos(\mu\pi)\sin(m\pi\alpha), \end{aligned} \tag{2.65}$$

and

$$q'(\mu) = \sin(\mu\pi) + (\mu - m)\pi\cos(\mu\pi). \tag{2.66}$$

Since

$$\lim_{\mu \to m} p'(\mu) = \lim_{\mu \to m} q'(\mu) = 0, \tag{2.67}$$

we will use L'Hôpital's rule again. We observe that

$$\begin{aligned} p''(\mu) &= \pi\big(\log(a)\big)^2 a^{\mu-1}(\mu - m)\sin(\mu\pi(1-\alpha) + 2\pi\log(a)a^{\mu-1}\sin(\mu\pi(1-\alpha)) \\ &\quad + 2\pi^2(1-\alpha)\log(a)a^{\mu-1}(\mu - m)\cos(\mu\pi(1-\alpha)) + \pi^2(1-\alpha)a^{\mu-1}\cos(\mu\pi(1-\alpha)) \\ &\quad - \pi^3(1-\alpha)^2 a^{\mu-1}(\mu - m)\sin(\mu\pi(1-\alpha)) - \pi^2 a^{m-1}\sin(\mu\pi)\sin(m\pi\alpha), \end{aligned} \tag{2.68}$$

and

$$q''(\mu) = 2\pi\cos(\mu\pi) - (\mu - m)\pi^2\sin(\mu\pi). \tag{2.69}$$

Since

$$\lim_{\mu\to m} p''(\mu) = 2(-1)^m a^{m-1}\pi^2(1-\alpha)\cos(m\pi\alpha) - 2(-1)^m a^{m-1}\pi\log(a)\sin(m\pi\alpha), \tag{2.70}$$

and

$$\lim_{\mu\to m} q''(\mu) = 2\pi(-1)^m, \tag{2.71}$$

it follows that

$$\begin{aligned}
\int_0^1 \frac{x^m\sin(\pi\alpha)}{a^2-2ax\cos(\pi\alpha)+x^2}\,dx &= \sum_{\substack{k\ge 0\\ k\ne m-1}} \frac{\sin\big((k+1)\pi\alpha\big)}{m-k-1}a^k + \lim_{\mu\to m}\frac{p(\mu)}{q(\mu)} \\
&= \sum_{\substack{k\ge 0\\ k\ne m-1}} \frac{\sin\big((k+1)\pi\alpha\big)}{m-k-1}a^k + \lim_{\mu\to m}\frac{p''(\mu)}{q''(\mu)} \\
&= a^{m-1}\pi(1-\alpha)\cos(m\pi\alpha) - a^{m-1}\log(a)\sin(m\pi\alpha) + \sum_{\substack{k\ge 0\\ k\ne m-1}} \frac{\sin\big((k+1)\pi\alpha\big)}{m-k-1}a^k.
\end{aligned} \tag{2.72}$$

□

The following lemma states that a certain series converges.

Lemma 2.25. *Suppose that m is a positive integer and $0 < \alpha < 2$ is a real number. Then*

$$\sum_{n=1}^{\infty} \frac{\sin(\pi n\alpha)}{m-n\alpha} < \infty. \tag{2.73}$$

Proof. We observe that

$$\frac{1}{n\alpha - m} \ge \frac{1}{(n+1)\alpha - m} > 0, \tag{2.74}$$

for all positive integers n such that $n > m/\alpha$. Moreover,

$$\lim_{n\to\infty} \frac{1}{n\alpha - m} = 0. \tag{2.75}$$

We also observe that, for any positive integer N,

$$\begin{aligned}
\left|\sum_{n=1}^{N} \sin(\pi n\alpha)\right| &\le \left|\sum_{n=0}^{N} e^{in\alpha}\right| = \left|\frac{1 - e^{i(N+1)\alpha}}{1 - e^{i\alpha}}\right| \\
&\le \left|\frac{e^{i(N+1)\alpha/2}}{e^{i\alpha/2}} \cdot \frac{e^{-i(N+1)\alpha/2} - e^{i(N+1)\alpha/2}}{e^{-i\alpha/2} - e^{i\alpha/2}}\right| \\
&= 2\left|\frac{\cos((N+1)\alpha/2)}{\cos(\alpha/2)}\right| \\
&\le \frac{2}{|\cos(\alpha/2)|}.
\end{aligned} \tag{2.76}$$

Hence, (2.73) follows by Dirichlet's test (Theorem 2.18).

□

The following theorem states that a certain Taylor series converges and is bounded on the interval $[0, 1]$.

Theorem 2.26. *Suppose that m and k are positive integers and $0 < \alpha < 2$ is a real number. Suppose further that ϕ is a function $[0, 1] \to \mathbb{R}$ defined by the formula*

$$\phi(t) = \sum_{n=k}^{\infty} \frac{\sin(\pi n\alpha)}{m - n\alpha} t^{n-k}, \tag{2.77}$$

for all $0 \le t \le 1$. Then ϕ is well defined and bounded on the interval $[0, 1]$.

Proof. We observe that

$$\left|\frac{\sin(\pi n\alpha)}{m - n\alpha}\right| = \left|\frac{\sin(\pi(m - n\alpha))}{m - n\alpha}\right| = \pi\left|\frac{\sin(\pi(m - n\alpha))}{\pi(m - n\alpha)}\right| \le \pi, \tag{2.78}$$

for all positive integers n. Therefore,

$$\sum_{n=k}^{\infty} \frac{\sin(\pi n\alpha)}{m - n\alpha} t^{n-k} < \infty, \tag{2.79}$$

for all $0 \leq t < 1$. By Lemma 2.25,

$$\sum_{n=k}^{\infty} \frac{\sin(\pi n\alpha)}{m - n\alpha} < \infty, \tag{2.80}$$

so ϕ is well defined on $[0, 1]$. Furthermore, by Abel's theorem (Theorem 2.19),

$$\lim_{\substack{t\to 1\\ t<1}} \sum_{n=k}^{\infty} \frac{\sin(\pi n\alpha)}{m - n\alpha} t^{n-k} = \sum_{n=k}^{\infty} \frac{\sin(\pi n\alpha)}{m - n\alpha}, \tag{2.81}$$

so ϕ is continuous on the interval $[0, 1]$. Therefore, ϕ is bounded on $[0, 1]$.

□

The following theorem states that a certain matrix is nonsingular.

Theorem 2.27. *Suppose that $0 < \alpha < 2$ is a real number. Suppose further that $A(\alpha)$ is an $n \times n$ matrix defined via the formula*

$$A_{i,j}(\alpha) = \begin{cases} \alpha \cdot \dfrac{\sin(\pi\alpha i)}{j - \alpha i} & \text{if } j \text{ is odd,} \\ (2-\alpha) \cdot \dfrac{\sin(\pi\alpha i)}{j - (2-\alpha)i} & \text{if } j \text{ is even,} \end{cases} \tag{2.82}$$

where $1 \leq i, j \leq n$ are integers. Then $A(\alpha)$ is nonsingular for all but a finite number of $0 < \alpha < 2$.

Proof. We observe that the functions

$$\alpha \cdot \frac{\sin(\pi\alpha i)}{j - 1 - \alpha i}, \tag{2.83}$$

$$(2-\alpha) \cdot \frac{\sin(\pi\alpha i)}{j - (2-\alpha)i}, \tag{2.84}$$

are entire functions of α, where $1 \leq i, j \leq n$ are integers. Therefore, $\det(A(\alpha))$ is an entire function of α. We also observe that

$$A(1) = \pi I, \tag{2.85}$$

where I is the identity matrix, from which it follows that

$$\det(A(1)) = \pi^n. \tag{2.86}$$

Since the interval $[0,2]$ is compact, it follows from Theorem 2.16 that $\det(A(\alpha))$ is equal to 0 at no more than a finite number of points in $[0,2]$. Hence, $A(\alpha)$ is nonsingular for all but a finite number of $0 < \alpha < 2$.

□

2.4 Analysis of the Integral Equation: the Neumann Case

Suppose that the curve $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge defined by (2.38) with interior angle $\pi\alpha$, where $0 < \alpha < 2$ (see Figure 2.3). Let g be a function in $L^2([-1,1])$, and suppose that $\rho \in L^2([-1,1])$ solves the equation

$$-\pi\rho(s) + \int_{-1}^{1} \psi^1_{\gamma(s),\nu(s)}(\gamma(t))\rho(t)\,dt = g(s), \tag{2.87}$$

for all $s \in [-1,1]$.

In this section, we will analyze this boundary integral equation, which is well-posed even though the curve γ is open (see Observation 2.11). We will investigate the behavior of (2.87) for functions $\rho \in L^2([-1,1])$ of the forms

$$\rho(t) = |t|^{\mu-1}, \tag{2.88}$$

$$\rho(t) = \operatorname{sgn}(t)|t|^{\mu-1}, \tag{2.89}$$

where $\mu > \frac{1}{2}$ is a real number and

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0, \end{cases} \tag{2.90}$$

for all real x. If identities (2.39) and (2.40) are substituted into (2.87) and ρ has the forms (2.88) and (2.89), then for most values of μ the resulting g is singular. In Section 2.4.2, we observe that for certain μ, the function g is smooth. In Section 2.4.3, we fix g and view (2.87) as an integral equation in ρ. We then observe that if g is smooth, then the solution ρ is representable by a series of functions of the forms (2.88) and (2.89).

2.4.1 Integral Equations Near a Corner

The following lemma uses a symmetry argument to reduce (2.87) from an integral equation on the interval $[-1,1]$ to two independent integral equations on the interval $[0,1]$.

Theorem 2.28. *Suppose that ρ is a function in $L^2([-1,1])$ and that $g \in L^2([-1,1])$ is given by (2.87). Suppose further that even functions $g_{\text{even}}, \rho_{\text{even}} \in L^2([-1,1])$ are defined via the formulas*

$$g_{\text{even}}(s) = \frac{1}{2}(g(s) + g(-s)), \tag{2.91}$$

$$\rho_{\text{even}}(s) = \frac{1}{2}(\rho(s) + \rho(-s)). \tag{2.92}$$

Then

$$g_{\text{even}}(s) = -\pi \rho_{\text{even}}(s) - \int_0^1 \frac{t \sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)} \rho_{\text{even}}(t)\, dt, \tag{2.93}$$

for all $0 < s \le 1$.

Likewise, suppose that odd functions $g_{\text{odd}}, \rho_{\text{odd}} \in L^2([-1,1])$ are defined via the formulas

$$g_{\text{odd}}(s) = \frac{1}{2}(g(s) - g(-s)), \tag{2.94}$$

$$\rho_{\text{odd}}(s) = \frac{1}{2}(\rho(s) - \rho(-s)). \tag{2.95}$$

Then

$$g_{\mathrm{odd}}(s) = -\pi \rho_{\mathrm{odd}}(s) + \int_0^1 \frac{t \sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)} \rho_{\mathrm{odd}}(t)\, dt, \tag{2.96}$$

for all $0 < s \leq 1$*.*

Proof. By Lemma 2.12,

$$g(s) = -\pi \rho(s) - \int_0^1 \frac{t \sin(\pi\alpha)}{s^2 + t^2 + 2st\cos(\pi\alpha)} \rho(t)\, dt, \tag{2.97}$$

for all $-1 \leq s < 0$, and

$$g(s) = -\pi \rho(s) + \int_{-1}^0 \frac{t \sin(\pi\alpha)}{s^2 + t^2 + 2st\cos(\pi\alpha)} \rho(t)\, dt, \tag{2.98}$$

for all $0 < s \leq 1$. Therefore,

$$g(-s) = -\pi \rho(-s) - \int_0^1 \frac{t \sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)} \rho(t)\, dt, \tag{2.99}$$

for all $0 < s \leq 1$, and

$$g(s) = -\pi \rho(s) - \int_0^1 \frac{t \sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)} \rho(-t)\, dt, \tag{2.100}$$

for all $0 < s \leq 1$.

Adding equations (2.99) and (2.100), we observe that

$$g_{\mathrm{even}}(s) = -\pi \rho_{\mathrm{even}}(s) - \int_0^1 \frac{t \sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)} \rho_{\mathrm{even}}(t)\, dt, \tag{2.101}$$

for all $0 < s \leq 1$.

Likewise, subtracting equation (2.99) from equation (2.100), we observe that

$$g_{\mathrm{odd}}(s) = -\pi \rho_{\mathrm{odd}}(s) + \int_0^1 \frac{t \sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)} \rho_{\mathrm{odd}}(t)\, dt, \tag{2.102}$$

for all $0 < s \leq 1$.

$\square$

2.4.2 The Singularities in the Solution of Equation (2.87)

In this section we observe that for certain functions ρ, the function g is representable by convergent Taylor series on the intervals $[-1, 0]$ and $[0, 1]$.

The Even Case

Suppose that $\rho \in L^2([-1,1])$ is an even function, and suppose that $g \in L^2([-1,1])$ is defined by (2.87). By Theorem 2.28, g is also even and

$$g(s) = -\pi\rho(s) - \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt, \tag{2.103}$$

for all $0 < s \leq 1$.

Suppose further that $\rho(t) = t^{\mu-1}$ for all $0 \leq t \leq 1$. The following theorem shows that for certain values of μ, the function g in (2.103) is representable by a convergent Taylor series on the interval $[0, 1]$.

Theorem 2.29. *Suppose that $0 < \alpha < 2$ is a real number and n is a positive integer. Then*

$$\pi s^{\frac{2n-1}{\alpha}-1} + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)} t^{\frac{2n-1}{\alpha}-1}\,dt = \alpha\sum_{m=1}^{\infty}\frac{\sin(m\pi\alpha)}{2n-1-\alpha m}s^{m-1}, \tag{2.104}$$

$$\pi s^{\frac{2n}{2-\alpha}-1} + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)} t^{\frac{2n}{2-\alpha}-1}\,dt = (2-\alpha)\sum_{m=1}^{\infty}\frac{\sin(m\pi\alpha)}{2n-(2-\alpha)m}s^{m-1}, \tag{2.105}$$

for all $0 < s \leq 1$.

Proof. Suppose that $\frac{2n-1}{\alpha}$ is not an integer. Substituting $\mu = \frac{2n-1}{\alpha}$ into (2.58), we observe that

$$
\begin{aligned}
&\pi s^{\frac{2n-1}{\alpha}-1} + \int_0^1 \frac{t^{\frac{2n-1}{\alpha}} \sin(\pi\alpha)}{s^2 - 2st\cos(\pi\alpha) + t^2}\, dt \\
&= \pi s^{\frac{2n-1}{\alpha}-1} + \pi s^{\frac{2n-1}{\alpha}-1} \frac{\sin\big(\frac{2n-1}{\alpha} \cdot \pi(1-\alpha)\big)}{\sin(\frac{2n-1}{\alpha} \cdot \pi)} + \sum_{k=0}^{\infty} \frac{\sin\big((k+1)\pi\alpha\big)}{\frac{2n-1}{\alpha} - k - 1} s^k \\
&= \pi s^{\frac{2n-1}{\alpha}-1} + \pi s^{\frac{2n-1}{\alpha}-1} \frac{\sin\big(\frac{2n-1}{\alpha} \cdot \pi - (2n-1)\pi\big)}{\sin(\frac{2n-1}{\alpha} \cdot \pi)} + \alpha \sum_{k=0}^{\infty} \frac{\sin\big((k+1)\pi\alpha\big)}{2n-1-\alpha(k+1)} s^k \\
&= \pi s^{\frac{2n-1}{\alpha}-1} - \pi s^{\frac{2n-1}{\alpha}-1} + \alpha \sum_{m=1}^{\infty} \frac{\sin\big(m\pi\alpha\big)}{2n-1-\alpha m} s^{m-1} \\
&= \alpha \sum_{m=1}^{\infty} \frac{\sin\big(m\pi\alpha\big)}{2n-1-\alpha m} s^{m-1},
\end{aligned}
\tag{2.106}
$$

for all $s > 0$.

Now suppose that $\frac{2n-1}{\alpha}$ is an integer. We observe that there is a neighborhood V of α such that $\frac{2n-1}{\alpha}$ is not an integer on $V \setminus \{\alpha\}$. Clearly, the right hand side of (2.106) is bounded and analytic on $V \setminus \{\alpha\}$. Therefore, identity (2.106) extends to the integer case by an application of L'Hôpital's rule.

□

Observation 2.30. Alternatively, identity (2.106) can be proved in the case where $\frac{2n-1}{\alpha}$ is an integer by substituting $m = \frac{2n-1}{\alpha}$ into identity (2.59). We observe that, while a term of the form $a^{m-1}\log(a)$ appears in the right hand side of (2.59), its coefficient is zero when $m = \frac{2n-1}{\alpha}$.

The Odd Case

Suppose that $\rho \in L^2([-1,1])$ is an odd function, and suppose that $g \in L^2([-1,1])$ is defined by (2.87). By Theorem 2.28, g is also odd and

$$
g(s) = -\pi\rho(s) + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)} \rho(t)\, dt, \tag{2.107}
$$

for all $0 < s \le 1$.

Suppose further that $\rho(t) = t^{\mu-1}$ for all $0 \le t \le 1$. The following theorem shows that for certain values of μ, the function g in (2.107) is representable by a convergent Taylor series on the interval $[0, 1]$.

Theorem 2.31. *Suppose that $0 < \alpha < 2$ is a real number and n is a positive integer. Then*

$$-\pi s^{\frac{2n}{\alpha}-1} + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)} t^{\frac{2n}{\alpha}-1}\,dt = \alpha \sum_{m=1}^{\infty} \frac{\sin(m\pi\alpha)}{2n-\alpha m} s^{m-1}, \tag{2.108}$$

$$-\pi s^{\frac{2n-1}{2-\alpha}-1} + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)} t^{\frac{2n-1}{2-\alpha}-1}\,dt = (2-\alpha) \sum_{m=1}^{\infty} \frac{\sin(m\pi\alpha)}{2n-1-(2-\alpha)m} s^{m-1}, \tag{2.109}$$

for all $0 < s \le 1$.

2.4.3 Series Representation of the Solution of Equation (2.87)

Suppose that g is representable by convergent Taylor series on the intervals $[-1, 0]$ and $[0, 1]$. Suppose further that ρ solves equation (2.87). In this section we observe that ρ is representable by certain series of non-integer powers on the intervals $[-1, 0]$ and $[0, 1]$.

The Even Case

Suppose that $g \in L^2([-1, 1])$ is an even function, and suppose that $\rho \in L^2([-1, 1])$ satisfies equation (2.87). By Theorem 2.28, ρ is also even and

$$-\pi\rho(s) - \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)} \rho(t)\,dt = g(s), \tag{2.110}$$

for all $0 < s \le 1$, where $0 < \alpha < 2$.

Let $\lceil x \rceil$ denote the smallest integer n such that $n \ge x$, and let $\lfloor x \rfloor$ denote the largest integer n such that $n \le x$, for all real x. The following theorem shows that if g is representable by a convergent Taylor series on $[0, 1]$, then for any positive integer n there exist unique real numbers $b_1, b_2, \ldots, b_n$ such that the function

$$\rho(t) = \sum_{i=1}^{\lceil n/2 \rceil} b_{2i-1} t^{\frac{2i-1}{\alpha}-1} + \sum_{i=1}^{\lfloor n/2 \rfloor} b_{2i} t^{\frac{2i}{2-\alpha}-1}, \tag{2.111}$$

where $0 \leq t \leq 1$, solves equation (2.110) to within an error $O(t^n)$.

Theorem 2.32. *Suppose that n is a positive integer and $c_1, c_2, \ldots, c_n$ are real numbers. Suppose further that $g\colon [0,1] \to \mathbb{R}$ is defined by the formula*

$$g(t) = \sum_{i=1}^{n} c_i t^{i-1}, \tag{2.112}$$

for all $0 \leq t \leq 1$. Then, for all but a finite number of $0 < \alpha < 2$, there exist unique real numbers $b_1, b_2, \ldots, b_n$ such that

$$\rho(t) = \sum_{i=1}^{\lceil n/2 \rceil} b_{2i-1} t^{\frac{2i-1}{\alpha}-1} + \sum_{i=1}^{\lfloor n/2 \rfloor} b_{2i} t^{\frac{2i}{2-\alpha}-1}, \tag{2.113}$$

for all $0 \leq t \leq 1$, and

$$-\pi\rho(s) - \int_0^1 \frac{t \sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)} \rho(t)\, dt = g(s) + s^n \phi(s), \tag{2.114}$$

for all $0 < s \leq 1$, where $\phi\colon [0,1] \to \mathbb{R}$ is a bounded function representable by a convergent Taylor series of the form

$$\phi(t) = \sum_{i=1}^{\infty} d_i t^{i-1}, \tag{2.115}$$

for all $0 \leq t \leq 1$, where $d_1, d_2, \ldots$ are real numbers.

Proof. By Theorem 2.27, the $n \times n$ matrix $A(\alpha)$ defined by (2.82) is nonsingular for all but a finite number of $0 < \alpha < 2$. Whenever $A(\alpha)$ is nonsingular, there exist unique real numbers $b_1, b_2, \ldots, b_n$ such that

$$-\sum_{j=1}^{n} A(\alpha)_{i,j} b_j = c_i, \tag{2.116}$$

for every $i = 1, 2, \ldots, n$. Suppose that $\rho\colon [0,1] \to \mathbb{R}$ is defined by (2.113). By Theorem 2.29,

$$\begin{aligned}
&-\pi\rho(s) - \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt \\
&= -\alpha \sum_{j=1}^{\lceil n/2\rceil} b_{2j-1} \sum_{i=1}^{\infty} \frac{\sin(\pi\alpha i)}{2j-1-\alpha i} s^{i-1} - (2-\alpha)\sum_{j=1}^{\lfloor n/2\rfloor} b_{2j} \sum_{i=1}^{\infty} \frac{\sin(\pi\alpha i)}{2j-(2-\alpha)i} s^{i-1} \\
&= -\alpha \sum_{j=1}^{\lceil n/2\rceil} b_{2j-1} \sum_{i=1}^{n} \frac{\sin(\pi\alpha i)}{2j-1-\alpha i} s^{i-1} - (2-\alpha)\sum_{j=1}^{\lfloor n/2\rfloor} b_{2j} \sum_{i=1}^{n} \frac{\sin(\pi\alpha i)}{2j-(2-\alpha)i} s^{i-1} + s^n\phi(s),
\end{aligned} \tag{2.117}$$

for all $0 \le s \le 1$, where $\phi\colon [0,1] \to \mathbb{R}$ is defined by the formula

$$\phi(t) = -\alpha \sum_{j=1}^{\lceil n/2\rceil} b_{2j-1} \sum_{i=n+1}^{\infty} \frac{\sin(\pi\alpha i)}{2j-1-\alpha i} t^{i-1-n} - (2-\alpha)\sum_{j=1}^{\lfloor n/2\rfloor} b_{2j} \sum_{i=n+1}^{\infty} \frac{\sin(\pi\alpha i)}{2j-(2-\alpha)i} t^{i-1-n}, \tag{2.118}$$

for all $0 \le t \le 1$. By Theorem 2.26, ϕ is bounded on $[0,1]$. By interchanging the order of summation in (2.117), we observe that

$$\begin{aligned}
&-\pi\rho(s) - \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt \\
&= \sum_{i=1}^{n}\Big(-\alpha \sum_{j=1}^{\lceil n/2\rceil} \frac{\sin(\pi\alpha i)}{2j-1-\alpha i} b_{2j-1} - (2-\alpha)\sum_{j=1}^{\lfloor n/2\rfloor} \frac{\sin(\pi\alpha i)}{2j-(2-\alpha)i} b_{2j}\Big) s^{i-1} + s^n\phi(s) \\
&= \sum_{i=1}^{n}\Big(-\sum_{j=1}^{n} A(\alpha)_{i,j}\, b_j\Big) s^{i-1} + s^n\phi(s) = \sum_{i=1}^{n} c_i s^{i-1} + s^n\phi(s) = g(s) + s^n\phi(s),
\end{aligned} \tag{2.119}$$

for all $0 \le s \le 1$. □

The Odd Case

Suppose that $g \in L^2([-1,1])$ is an odd function, and suppose that $\rho \in L^2([-1,1])$ satisfies equation (2.87). By Theorem 2.28, ρ is also odd and

$$-\pi\rho(s) + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt = g(s), \tag{2.120}$$

for all $0 < s \le 1$, where $0 < \alpha < 2$.

Let $\lceil x \rceil$ denote the smallest integer n such that $n \ge x$, and let $\lfloor x \rfloor$ denote the largest integer n such that $n \le x$, for all real x. The following theorem shows that if g is representable by a convergent Taylor series on $[0,1]$, then for any positive integer n there exist unique real numbers $b_1, b_2, \dots, b_n$ such that the function

$$\rho(t) = \sum_{i=1}^{\lceil n/2 \rceil} b_{2i-1} t^{\frac{2i-1}{2-\alpha}-1} + \sum_{i=1}^{\lfloor n/2 \rfloor} b_{2i} t^{\frac{2i}{\alpha}-1}, \tag{2.121}$$

where $0 \le t \le 1$, solves equation (2.120) to within an error $O(t^n)$.

Theorem 2.33. *Suppose that n is a positive integer and $c_1, c_2, \dots, c_n$ are real numbers. Suppose further that $g\colon [0,1] \to \mathbb{R}$ is defined by the formula*

$$g(t) = \sum_{i=1}^{n} c_i t^{i-1}, \tag{2.122}$$

for all $0 \le t \le 1$. Then, for all but a finite number of $0 < \alpha < 2$, there exist unique real numbers $b_1, b_2, \dots, b_n$ so that

$$\rho(t) = \sum_{i=1}^{\lceil n/2 \rceil} b_{2i-1} t^{\frac{2i-1}{2-\alpha}-1} + \sum_{i=1}^{\lfloor n/2 \rfloor} b_{2i} t^{\frac{2i}{\alpha}-1}, \tag{2.123}$$

for all $0 \le t \le 1$, and

$$-\pi\rho(s) + \int_0^1 \frac{t\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt = g(s) + s^n\phi(s), \tag{2.124}$$

for all $0 \le s \le 1$, where $\phi\colon [0,1] \to \mathbb{R}$ is a bounded function representable by a convergent

Taylor series of the form

$$\phi(t) = \sum_{i=1}^{\infty} d_i t^{i-1}, \tag{2.125}$$

for all $0 \le t \le 1$*, where* $d_1, d_2, \dots$ *are real numbers.*

2.4.4 Summary of Results

We summarize the results of the preceding subsections 2.4.1, 2.4.2, 2.4.3 as follows.

Suppose that the curve $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge defined by (2.38) with interior angle $\pi\alpha$, where $0 < \alpha < 2$ (see Figure 2.3). Let $g \in L^2([-1,1])$, and consider the boundary integral equation

$$-\pi\rho(s) + \int_{-1}^{1} \psi^1_{\gamma(s),\nu(s)}(\gamma(t))\rho(t)\,dt = g(s), \tag{2.126}$$

for all $s \in [-1,1]$, where $\rho \in L^2([-1,1])$.

Suppose that the even and odd parts of g are each representable by convergent Taylor series on the interval $[0,1]$. Then, for each positive integer n, there exist real numbers $b_1, b_2, \dots, b_n$ and $c_1, c_2, \dots, c_n$ such that

$$\rho(t) = \sum_{i=1}^{\lceil n/2 \rceil} b_{2i-1}|t|^{\frac{2i-1}{\alpha}-1} + \sum_{i=1}^{\lfloor n/2 \rfloor} b_{2i}\,\mathrm{sgn}(t)|t|^{\frac{2i}{\alpha}-1} + \sum_{i=1}^{\lceil n/2 \rceil} c_{2i-1}\,\mathrm{sgn}(t)|t|^{\frac{2i-1}{2-\alpha}-1} + \sum_{i=1}^{\lfloor n/2 \rfloor} c_{2i}|t|^{\frac{2i}{2-\alpha}-1}, \tag{2.127}$$

for all $-1 \le t \le 1$, solves equation (2.126) to within an error $O(t^n)$. Moreover, the even and odds parts of this error are also representable by convergent Taylor series on the interval $[0,1]$ (see theorems 2.32 and 2.33).

Observation 2.34. Numerical experiments (see Section 2.6) suggest that, for a certain subclass of functions g, stronger versions of theorems 2.32 and 2.33 are true. Suppose that G is a harmonic function on a neighborhood of the closed unit disc in $\mathbb{R}^2$, and let

$$g(t) = \frac{\partial G}{\partial \nu(t)}(\gamma(t)), \tag{2.128}$$

for all $-1 \le t \le 1$, where $\nu(t)$ is the inward-pointing unit normal vector at $\gamma(t)$. We conjecture that there exist infinite sequences of real numbers $b_1, b_2, \ldots$ and $c_1, c_2, \ldots$ such that

$$\rho(t) = \sum_{i=1}^{\infty} b_{2i-1}|t|^{\frac{2i-1}{\alpha}-1} + \sum_{i=1}^{\infty} b_{2i}\,\mathrm{sgn}(t)|t|^{\frac{2i}{\alpha}-1} + \sum_{i=1}^{\infty} c_{2i-1}\,\mathrm{sgn}(t)|t|^{\frac{2i-1}{2-\alpha}-1} + \sum_{i=1}^{\infty} c_{2i}|t|^{\frac{2i}{2-\alpha}-1}, \tag{2.129}$$

is well defined for all $-1 \le t \le 1$, and (2.129) solves equation (2.126).

Observation 2.35. Numerical experiments (see Section 2.6) indicate that the solution to equation (2.126) is representable by a series of the form (2.127) for a more general class of curves γ. More specifically, suppose that $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge in $\mathbb{R}^2$ with smooth, curved sides, with a corner at 0 and interior angle $\pi\alpha$. Suppose further that all derivatives of γ, 2nd order and higher, are zero at the corner. Then the solution is representable by a series of the form (2.127).

2.5 Analysis of the Integral Equation: the Dirichlet Case

Suppose that the curve $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge defined by (2.38) with interior angle $\pi\alpha$, where $0 < \alpha < 2$ (see Figure 2.3). Let g be a function in $L^2([-1,1])$, and suppose that $\rho \in L^2([-1,1])$ solves the equation

$$g(s) = -\pi\rho(s) + \int_{-1}^{1} \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\rho(t)\,dt, \tag{2.130}$$

for all $s \in [-1,1]$.

In this section, we will analyze this boundary integral equation, which is well-posed even though the curve γ is open (see Observation 2.11). We will investigate the behavior of (2.130) for functions $\rho \in L^2([-1,1])$ of the forms

$$\rho(t) = |t|^\mu, \tag{2.131}$$

$$\rho(t) = \operatorname{sgn}(t)|t|^\mu, \tag{2.132}$$

where $\mu > \frac{1}{2}$ is a real number and

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0, \end{cases} \tag{2.133}$$

for all real x. If identities (2.39) and (2.40) are substituted into (2.130) and ρ has the forms (2.131) and (2.132), then for most values of μ the resulting g is singular. In Section 2.5.2, we observe that for certain μ, the function g is smooth. In Section 2.5.3, we fix g and view (2.130) as an integral equation in ρ. We then observe that if g is smooth, then the solution ρ is representable by a series of functions of the forms (2.131) and (2.132).

The proofs of the theorems in this section are essentially identical to the proofs of the corresponding theorems in Section 2.4, and are omitted.

2.5.1 Integral Equations Near a Corner

The following lemma uses a symmetry argument to reduce (2.130) from an integral equation on the interval $[-1,1]$ to two independent integral equations on the interval $[0,1]$.

Theorem 2.36. *Suppose that ρ is a function in $L^2([-1,1])$ and that $g \in L^2([-1,1])$ is given by (2.130). Suppose further that even functions $g_{\text{even}}, \rho_{\text{even}} \in L^2([-1,1])$ are* *defined via the formulas*

$$g_{\text{even}}(s) = \frac{1}{2}(g(s) + g(-s)), \tag{2.134}$$

$$\rho_{\text{even}}(s) = \frac{1}{2}(\rho(s) + \rho(-s)). \tag{2.135}$$

Then

$$g_{\text{even}}(s) = -\pi\rho_{\text{even}}(s) - \int_0^1 \frac{s\sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)}\rho_{\text{even}}(t)\,dt, \tag{2.136}$$

for all $0 < s \leq 1$.

Likewise, suppose that odd functions $g_{\text{odd}}, \rho_{\text{odd}} \in L^2([-1,1])$ *are defined via the formulas*

$$g_{\text{odd}}(s) = \frac{1}{2}(g(s) - g(-s)), \tag{2.137}$$

$$\rho_{\text{odd}}(s) = \frac{1}{2}(\rho(s) - \rho(-s)). \tag{2.138}$$

Then

$$g_{\text{odd}}(s) = -\pi\rho_{\text{odd}}(s) + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)}\rho_{\text{odd}}(t)\,dt, \tag{2.139}$$

for all $0 < s \leq 1$.

2.5.2 The Singularities in the Solution of Equation (2.130)

In this section we observe that for certain functions ρ, the function g is representable by convergent Taylor series on the intervals $[-1, 0]$ and $[0, 1]$.

The Even Case

Suppose that $\rho \in L^2([-1,1])$ is an even function, and suppose that $g \in L^2([-1,1])$ is defined by (2.130). By Theorem 2.36, g is also even and

$$g(s) = -\pi\rho(s) - \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt, \tag{2.140}$$

for all $0 < s \le 1$.

Suppose further that $\rho(t) = t^\mu$ for all $0 \le t \le 1$. The following theorem shows that for certain values of μ, the function g in (2.140) is representable by a convergent Taylor series on the interval $[0,1]$.

Theorem 2.37. *Suppose that $0 < \alpha < 2$ is a real number and n is a positive integer. Then*

$$\pi + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\,dt = (2-\alpha)\pi - \sum_{m=1}^{\infty}\frac{\sin(m\pi\alpha)}{m}s^m, \tag{2.141}$$

$$\pi s^{\frac{2n-1}{\alpha}} + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}t^{\frac{2n-1}{\alpha}}\,dt = \alpha\sum_{m=1}^{\infty}\frac{\sin(m\pi\alpha)}{2n-1-\alpha m}s^m, \tag{2.142}$$

$$\pi s^{\frac{2n}{2-\alpha}} + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}t^{\frac{2n}{2-\alpha}}\,dt = (2-\alpha)\sum_{m=1}^{\infty}\frac{\sin(m\pi\alpha)}{2n-(2-\alpha)m}s^m, \tag{2.143}$$

for all $0 < s \le 1$.

Proof. Taking the limit $\mu \to 0$ in (2.58) and applying L'Hôpital's rule once, we observe that

$$\int_0^1 \frac{\sin(\pi\alpha)}{a^2-2ax\cos(\pi\alpha)+x^2}\,dx = (1-\alpha)\pi a^{-1} - \sum_{k=0}^{\infty}\frac{\sin\big((k+1)\pi\alpha\big)}{k+1}a^k, \tag{2.144}$$

for all $0 < a \le 1$, from which identity (2.141) clearly follows.

The proofs of identities (2.142) and (2.143) are essentially identical to the corresponding proofs in Theorem 2.29. □

The Odd Case

Suppose that $\rho \in L^2([-1,1])$ is an odd function, and suppose that $g \in L^2([-1,1])$ is defined by (2.130). By Theorem 2.36, g is also odd and

$$g(s) = -\pi\rho(s) + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt, \tag{2.145}$$

for all $0 < s \leq 1$.

Suppose further that $\rho(t) = t^\mu$ for all $0 \leq t \leq 1$. The following theorem shows that for certain values of μ, the function g in (2.145) is representable by a convergent Taylor series on the interval $[0,1]$.

Theorem 2.38. *Suppose that $0 < \alpha < 2$ is a real number and n is a positive integer. Then*

$$-\pi + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\,dt = -\alpha\pi - \sum_{m=1}^{\infty}\frac{\sin(m\pi\alpha)}{m}s^m, \tag{2.146}$$

$$-\pi s^{\frac{2n}{\alpha}} + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}t^{\frac{2n}{\alpha}}\,dt = \alpha\sum_{m=1}^{\infty}\frac{\sin(m\pi\alpha)}{2n-\alpha m}s^m, \tag{2.147}$$

$$-\pi s^{\frac{2n-1}{2-\alpha}} + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}t^{\frac{2n-1}{2-\alpha}}\,dt = (2-\alpha)\sum_{m=1}^{\infty}\frac{\sin(m\pi\alpha)}{2n-1-(2-\alpha)m}s^m, \tag{2.148}$$

for all $0 < s \leq 1$.

2.5.3 Series Representation of the Solution of Equation (2.130)

Suppose that g is representable by convergent Taylor series on the intervals $[-1,0]$ and $[0,1]$. Suppose further that ρ solves equation (2.130). In this section we observe that ρ is representable by certain series of non-integer powers on the intervals $[-1,0]$ and $[0,1]$.

The Even Case

Suppose that $g \in L^2([-1,1])$ is an even function, and suppose that $\rho \in L^2([-1,1])$ satisfies equation (2.130). By Theorem 2.36, ρ is also even and

$$-\pi\rho(s) - \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt = g(s), \tag{2.149}$$

for all $0 < s \le 1$, where $0 < \alpha < 2$.

Let $\lceil x \rceil$ denote the smallest integer n such that $n \ge x$, and let $\lfloor x \rfloor$ denote the largest integer n such that $n \le x$, for all real x. The following theorem shows that if g is representable by a convergent Taylor series on $[0,1]$, then for any positive integer n there exist unique real numbers $b_0, b_1, \dots, b_n$ such that the function

$$\rho(t) = \sum_{i=1}^{\lceil n/2 \rceil} b_{2i-1} t^{\frac{2i-1}{\alpha}} + \sum_{i=0}^{\lfloor n/2 \rfloor} b_{2i} t^{\frac{2i}{2-\alpha}}, \tag{2.150}$$

where $0 \le t \le 1$, solves equation (2.149) to within an error $O(t^{n+1})$.

Theorem 2.39. *Suppose that n is a positive integer and $c_0, c_1, \dots, c_n$ are real numbers. Suppose further that $g\colon [0,1] \to \mathbb{R}$ is defined by the formula*

$$g(t) = \sum_{i=0}^{n} c_i t^i, \tag{2.151}$$

for all $0 \le t \le 1$. Then, for all but a finite number of $0 < \alpha < 2$, there exist unique real numbers $b_0, b_1, \dots, b_n$ so that

$$\rho(t) = \sum_{i=1}^{\lceil n/2 \rceil} b_{2i-1} t^{\frac{2i-1}{\alpha}} + \sum_{i=0}^{\lfloor n/2 \rfloor} b_{2i} t^{\frac{2i}{2-\alpha}}, \tag{2.152}$$

for all $0 \le t \le 1$, and

$$-\pi\rho(s) - \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt = g(s) + s^{n+1}\phi(s), \tag{2.153}$$

for all $0 < s \le 1$, where $\phi\colon [0,1] \to \mathbb{R}$ is a bounded function representable by a convergent

Taylor series of the form

$$\phi(t) = \sum_{i=0}^{\infty} d_i t^i, \tag{2.154}$$

for all $0 \le t \le 1$*, where* $d_0, d_1, \ldots$ *are real numbers.*

The Odd Case

Suppose that $g \in L^2([-1,1])$ is an odd function, and suppose that $\rho \in L^2([-1,1])$ satisfies equation (2.130). By Theorem 2.36, ρ is also odd and

$$-\pi\rho(s) + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2+t^2-2st\cos(\pi\alpha)}\rho(t)\,dt = g(s), \tag{2.155}$$

for all $0 < s \le 1$, where $0 < \alpha < 2$.

Let $\lceil x \rceil$ denote the smallest integer n such that $n \ge x$, and let $\lfloor x \rfloor$ denote the largest integer n such that $n \le x$, for all real x. The following theorem shows that if g is representable by a convergent Taylor series on $[0,1]$, then for any positive integer n there exist unique real numbers $b_0, b_1, \ldots, b_n$ such that the function

$$\rho(t) = \sum_{i=1}^{\lceil n/2 \rceil} b_{2i-1} t^{\frac{2i-1}{2-\alpha}} + \sum_{i=0}^{\lfloor n/2 \rfloor} b_{2i} t^{\frac{2i}{\alpha}}, \tag{2.156}$$

where $0 \le t \le 1$, solves equation (2.155) to within an error $O(t^{n+1})$.

Theorem 2.40. *Suppose that* n *is a positive integer and* $c_0, c_1, \ldots, c_n$ *are real numbers. Suppose further that* $g\colon [0,1] \to \mathbb{R}$ *is defined by the formula*

$$g(t) = \sum_{i=0}^{n} c_i t^i, \tag{2.157}$$

for all $0 \le t \le 1$*. Then, for all but a finite number of* $0 < \alpha < 2$*, there exist unique real* *numbers* $b_0, b_1, \dots, b_n$ *so that*

$$\rho(t) = \sum_{i=1}^{\lceil n/2 \rceil} b_{2i-1} t^{\frac{2i-1}{2-\alpha}} + \sum_{i=0}^{\lfloor n/2 \rfloor} b_{2i} t^{\frac{2i}{\alpha}}, \tag{2.158}$$

for all $0 \le t \le 1$*, and*

$$-\pi\rho(s) + \int_0^1 \frac{s\sin(\pi\alpha)}{s^2 + t^2 - 2st\cos(\pi\alpha)} \rho(t)\, dt = g(s) + s^{n+1}\phi(s), \tag{2.159}$$

for all $0 < s \le 1$*, where* $\phi\colon [0,1] \to \mathbb{R}$ *is a bounded function representable by a convergent Taylor series of the form*

$$\phi(t) = \sum_{i=0}^{\infty} d_i t^i, \tag{2.160}$$

for all $0 \le t \le 1$*, where* $d_0, d_1, \dots$ *are real numbers.*

2.5.4 Summary of Results

We summarize the results of the preceding subsections 2.5.1, 2.5.2, 2.5.3 as follows.

Suppose that the curve $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge defined by (2.38) with interior angle $\pi\alpha$, where $0 < \alpha < 2$ (see Figure 2.3). Let $g \in L^2([-1,1])$, and consider the boundary integral equation

$$-\pi\rho(s) + \int_{-1}^{1} \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\rho(t)\, dt = g(s), \tag{2.161}$$

for all $s \in [-1,1]$, where $\rho \in L^2([-1,1])$.

Suppose that the even and odd parts of g are each representable by convergent Taylor series on the interval $[0,1]$. Then, for each positive integer n, there exist real numbers $b_0, b_1, \dots, b_n$ and $c_0, c_1, \dots, c_n$ such that

$$\rho(t) = \sum_{i=1}^{\lceil n/2 \rceil} b_{2i-1} |t|^{\frac{2i-1}{\alpha}} + \sum_{i=0}^{\lfloor n/2 \rfloor} b_{2i}\, \mathrm{sgn}(t) |t|^{\frac{2i}{\alpha}} + \sum_{i=1}^{\lceil n/2 \rceil} c_{2i-1}\, \mathrm{sgn}(t) |t|^{\frac{2i-1}{2-\alpha}} + \sum_{i=0}^{\lfloor n/2 \rfloor} c_{2i} |t|^{\frac{2i}{2-\alpha}}, \tag{2.162}$$

for all $-1 \leq t \leq 1$, solves equation (2.161) to within an error $O(t^{n+1})$. Moreover, the even and odds parts of this error are also representable by convergent Taylor series on the interval $[0,1]$ (see theorems 2.39 and 2.40).

Observation 2.41. Numerical experiments (see Section 2.6) suggest that, for a certain subclass of functions g, stronger versions of theorems 2.39 and 2.40 are true. Suppose that G is a harmonic function on a neighborhood of the closed unit disc in $\mathbb{R}^2$, and let

$$g(t) = G(\gamma(t)), \tag{2.163}$$

for all $-1 \leq t \leq 1$. We conjecture that there exist infinite sequences of real numbers $b_0, b_1, \ldots$ and $c_0, c_1, \ldots$ such that

$$\rho(t) = \sum_{i=1}^{\infty} b_{2i-1}|t|^{\frac{2i-1}{\alpha}} + \sum_{i=0}^{\infty} b_{2i}\,\mathrm{sgn}(t)|t|^{\frac{2i}{\alpha}} + \sum_{i=1}^{\infty} c_{2i-1}\,\mathrm{sgn}(t)|t|^{\frac{2i-1}{2-\alpha}} + \sum_{i=0}^{\infty} c_{2i}|t|^{\frac{2i}{2-\alpha}}, \tag{2.164}$$

is well defined for all $-1 \leq t \leq 1$, and (2.164) solves equation (2.161).

Observation 2.42. Numerical experiments (see Section 2.6) indicate that the solution to equation (2.161) is representable by a series of the form (2.162) for a more general class of curves γ. More specifically, suppose that $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge in $\mathbb{R}^2$ with smooth, curved sides, with a corner at 0 and interior angle $\pi\alpha$. Suppose further that all derivatives of γ, 2nd order and higher, are zero at the corner. Then the solution is representable by a series of the form (2.162).

2.6 The Algorithm

To solve the integral equations of potential theory on polygonal domains, we use a modification of an algorithm described in [4]; instead of discretizing the corner singularities using nested quadratures, we use the representations (2.127), (2.162) to construct purpose-made discretizations (see, for example, [23], [21], [31]). A detailed description of this part of the procedure will be published at a later date. The resulting linear systems were solved directly using standard techniques.

We illustrate the performance of the algorithm with several numerical examples. The exterior and interior Neumann problems and the exterior and interior Dirichlet problems were solved on each of the domains in figures 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, where the boundary data were generated by charges *inside* the regions for the exterior problems and *outside* the regions for the interior problems. The numerical solution was tested by comparing the computed potential to the true potential at several arbitrary points; Table 2.1 presents the results.

We also solved the interior Dirichlet problem on the domains in figures 2.6, 2.7, where the boundary data were generated by charges *inside* the regions. We computed the numerical solution using both our algorithm and a naive algorithm which used nested quadratures near the corners. The solution produced by our algorithm was then tested by comparing the computed potentials at several arbitrary points; Table 2.2 presents the results.

The following abbreviations are used in tables 2.1 and 2.2 (see Section 2.2.1):

INP Interior Neumann problem

ENP Exterior Neumann problem

IDP Interior Dirichlet problem

EDP Exterior Dirichlet problem

Observation 2.43. Clearly, the curves Γ_1 and Γ_2 are not polygons. However, all derivatives of the curves, 2nd order and higher, are zero at the corners. We observe that in this case, the singularities in the solutions of the boundary integral equations are identical to those in the polygonal case.

Observation 2.44. We observe that if the boundary values are produced by charges *inside* the regions for the exterior problems, or *outside* the regions for the interior problems, then certain terms in the representations of the solutions near the corners vanish.

More specifically, in the exterior Neumann case, the terms $c_1, c_2, \ldots$ in (2.127) vanish. In the interior Dirichlet case, the terms $b_0, b_1, \ldots$ in (2.162) vanish.

Observation 2.45. The boundary integral equations for the interior Neumann problem and exterior Dirichlet problem are not solvable for all boundary functions g (see theorems 2.8 and 2.9); the associated integral operators have one-dimensional null spaces. Thus, the condition numbers of the resulting linear systems for these problems are not reported in Table 2.1.

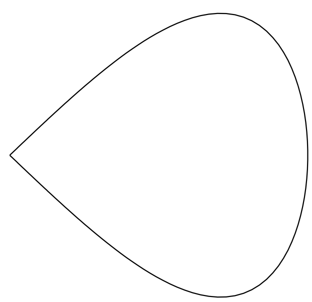

Figure 2.4: Γ_1: A cone in $\mathbb{R}^2$

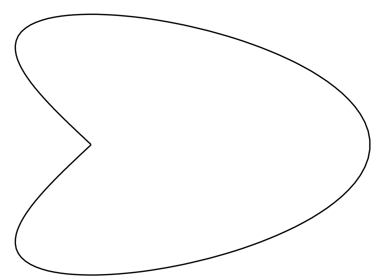

Figure 2.5: Γ_2: A curve in $\mathbb{R}^2$ with an inward-pointing wedge

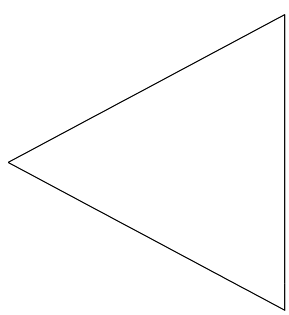

Figure 2.6: Γ_3: An equilateral triangle in $\mathbb{R}^2$

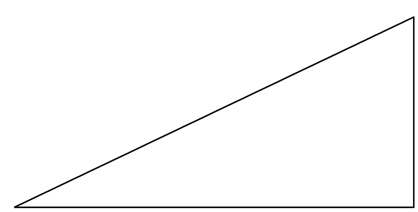

Figure 2.7: Γ_4: A right triangle in $\mathbb{R}^2$

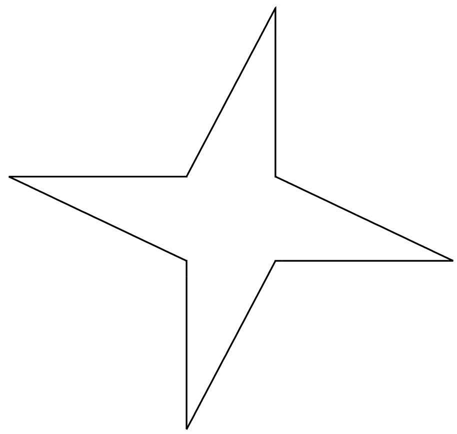

Figure 2.8: Γ_5: A star-shaped polygon in $\mathbb{R}^2$

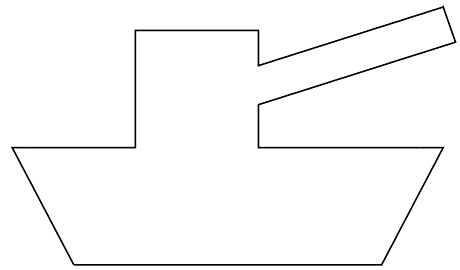

Figure 2.9: Γ_6: A tank-shaped polygon in $\mathbb{R}^2$

	Boundary curve	Number of nodes	Running time	Largest absolute error	Condition number
IDP	Γ_1	187	0.17051E-01	0.29976E-14	0.62087E+01
	Γ_2	212	0.22705E-01	0.40079E-13	0.62316E+01
	Γ_3	287	0.37674E-01	0.94369E-15	0.14004E+02
	Γ_4	362	0.80397E-01	0.38858E-15	0.26151E+02
	Γ_5	720	0.60129E+00	0.36415E-13	0.24935E+02
	Γ_6	1031	0.21035E+01	0.33085E-13	0.51047E+01
EDP	Γ_1	187	0.17309E-01	0.51625E-14	–
	Γ_2	287	0.44541E-01	0.39274E-14	–
	Γ_3	267	0.38770E-01	0.15404E-14	–
	Γ_4	362	0.84188E-01	0.21400E-13	–
	Γ_5	760	0.71015E+00	0.72997E-14	–
	Γ_6	1031	0.17802E+01	0.94369E-14	–
INP	Γ_1	193	0.18507E-01	0.37331E-14	–
	Γ_2	268	0.38332E-01	0.19984E-14	–
	Γ_3	285	0.48853E-01	0.34972E-14	–
	Γ_4	380	0.97122E-01	0.47878E-15	–
	Γ_5	848	0.10073E+01	0.23537E-13	–
	Γ_6	1103	0.22105E+01	0.57732E-14	–
ENP	Γ_1	193	0.19109E-01	0.10353E-13	0.17855E+01
	Γ_2	248	0.34520E-01	0.28866E-14	0.17856E+01
	Γ_3	285	0.46238E-01	0.26715E-15	0.26858E+01
	Γ_4	380	0.95644E-01	0.77716E-15	0.58866E+01
	Γ_5	808	0.88322E+00	0.72164E-14	0.58641E+01
	Γ_6	1148	0.24347E+01	0.44409E-14	0.21538E+01

Table 2.1: Numerical results for the Dirichlet and Neumann problems

	Boundary curve	Number of nodes	Running time	Largest absolute error	Condition number
IDP	Γ_3	267	0.36433E-01	0.31086E-14	0.14004E+02
	Γ_4	362	0.77586E-01	0.16653E-14	0.26151E+02

Table 2.2: Numerical results for the interior Dirichlet problem with the charges *inside* the regions

Chapter 3

The Helmholtz Equation

3.1 Overview

This section provides a brief overview of the principal results of this chapter. The following two subsections 3.1.1 and 3.1.2 summarize the Neumann and Dirichlet cases respectively; subsection 3.1.3 summarizes the numerical algorithm and numerical results.

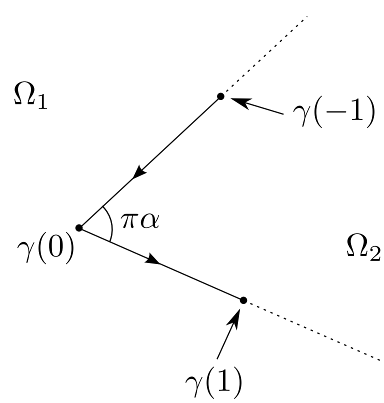

Figure 3.1: A wedge in $\mathbb{R}^2$

3.1.1 The Neumann Case

Suppose that $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge in $\mathbb{R}^2$ with a corner at $\gamma(0)$, and with interior angle $\pi\alpha$. Suppose further that γ is parameterized by arc length, and let $\nu(t)$ denote the inward-facing unit normal to the curve γ at t. Let Γ denote the set $\gamma([-1,1])$. By extending the sides of the wedge to infinity, we divide $\mathbb{R}^2$ into two open sets Ω_1 and Ω_2 (see Figure 3.1).

Let $\phi\colon \mathbb{R}^2 \setminus \Gamma \to \mathbb{C}$ denote the generalized potential (see, for example, [25]) induced by a charge distribution on γ with density $\rho\colon [-1,1] \to \mathbb{C}$. In other words, let ϕ be defined by the formula

$$\phi(x) = \frac{i}{4} \int_{-1}^{1} H_0(k\|\gamma(t) - x\|)\rho(t)\, dt, \tag{3.1}$$

for all $x \in \mathbb{R}^2 \setminus \Gamma$, where $\|\cdot\|$ denotes the Euclidean norm. Suppose that $g\colon [-1,1] \to \mathbb{C}$ is defined by the formula

$$g(t) = \lim_{\substack{x\to\gamma(t)\\ x\in\Omega_1}} \frac{\partial\phi(x)}{\partial\nu(t)}, \tag{3.2}$$

for all $-1 \le t \le 1$, i.e. g is the limit of the normal derivative of integral (3.1) when x approaches the point $\gamma(t)$ from outside. It is well known that

$$g(s) = \frac{1}{2}\rho(s) + \frac{i}{4}\int_{-1}^{1} K(s,t)\rho(t)\, dt, \tag{3.3}$$

for all $-1 \le s \le 1$, where

$$K(s,t) = \frac{k\langle\gamma(s) - \gamma(t), \nu(s)\rangle}{\|\gamma(s) - \gamma(t)\|}\, H_1(k\|\gamma(s) - \gamma(t)\|), \tag{3.4}$$

for all $-1 \le s,t \le 1$, where $\langle\cdot,\cdot\rangle$ denotes the inner product.

The following theorem is one of the principal results of this chapter.

Theorem 3.1. *Suppose that N is a positive integer and that ρ is defined by the formula*

$$\rho(t) = \sum_{n=1}^{N} b_n(\operatorname{sgn}(t))^{n+1}\frac{J_{\frac{n}{\alpha}}(k|t|)}{|t|} + \sum_{n=1}^{N} c_n(\operatorname{sgn}(t))^{n}\frac{J_{\frac{n}{2-\alpha}}(k|t|)}{|t|}, \tag{3.5}$$

for all $-1 \le t \le 1$, where $b_1, b_2, \ldots, b_N$ and $c_1, c_2, \ldots, c_N$ are arbitrary complex numbers. Suppose further that g is defined by (3.3). Then there exist series of complex numbers

$\beta_0, \beta_1, \ldots$ and $\gamma_0, \gamma_1, \ldots$ such that

$$g(t) = \sum_{n=0}^{\infty} \beta_n |t|^n + \sum_{n=0}^{\infty} \gamma_n \operatorname{sgn}(t) |t|^n, \tag{3.6}$$

for all $-1 \le t \le 1$. Conversely, suppose that g has the form (3.6). Then, for each positive integer N, there exist complex numbers $b_1, b_2, \ldots, b_N$ and $c_1, c_2, \ldots, c_N$ such that ρ, defined by (3.5), solves equation (3.3) to within an error $O(t^N)$.

In other words, if ρ has the form (3.5), then g is smooth on the intervals $[-1, 0]$ and $[0, 1]$. Conversely, if g is smooth, then for each positive integer N there exists a solution ρ of the form (3.5) that solves equation (3.3) to within an error $O(t^N)$.

3.1.2 The Dirichlet Case

Suppose that $\gamma\colon [-1, 1] \to \mathbb{R}^2$ is a wedge in $\mathbb{R}^2$ with a corner at $\gamma(0)$, and with interior angle $\pi\alpha$. Suppose further that γ is parameterized by arc length, and let $\nu(t)$ denote the inward-facing unit normal to the curve γ at t. Let Γ denote the set $\gamma([-1, 1])$. By extending the sides of the wedge to infinity, we divide $\mathbb{R}^2$ into two open sets Ω_1 and Ω_2 (see Figure 3.1).

Let $\phi\colon \mathbb{R}^2 \setminus \Gamma \to \mathbb{C}$ denote the generalized potential (see, for example, [25]) induced by a dipole distribution on γ with density $\rho\colon [-1, 1] \to \mathbb{C}$. In other words, let ϕ be defined by the formula

$$\phi(x) = \frac{i}{4} \int_{-1}^{1} \frac{k \langle \gamma(t) - x, \nu(t) \rangle}{\| \gamma(t) - x \|} H_1(k \| \gamma(t) - x \|) \rho(t) \, dt, \tag{3.7}$$

for all $x \in \mathbb{R}^2 \setminus \Gamma$, where $\| \cdot \|$ denotes the Euclidean norm and $\langle \cdot, \cdot \rangle$ denotes the inner product. Suppose that $g\colon [-1, 1] \to \mathbb{C}$ is defined by the formula

$$g(t) = \lim_{\substack{x \to \gamma(t) \\ x \in \Omega_2}} \phi(x), \tag{3.8}$$

for all $-1 \le t \le 1$, i.e. g is the limit of integral (3.7) when x approaches the point $\gamma(t)$

from inside. It is well known that

$$g(s) = \frac{1}{2}\rho(s) + \frac{i}{4}\int_{-1}^{1} K(t,s)\rho(t)\,dt, \tag{3.9}$$

for all $-1 \le s \le 1$, where

$$K(t,s) = \frac{k\langle\gamma(t)-\gamma(s),\nu(t)\rangle}{\|\gamma(t)-\gamma(s)\|}\, H_1(k\|\gamma(t)-\gamma(s)\|), \tag{3.10}$$

for all $-1 \le s,t \le 1$, where $\langle\cdot,\cdot\rangle$ denotes the inner product.

The following theorem is one of the principal results of this chapter.

Theorem 3.2. *Suppose that N is a positive integer and that ρ is defined by the formula*

$$\rho(t) = \sum_{n=0}^{N} b_n(\operatorname{sgn}(t))^{n+1} J_{\frac{n}{\alpha}}(k|t|) + \sum_{n=0}^{N} c_n(\operatorname{sgn}(t))^{n} J_{\frac{n}{2-\alpha}}(k|t|), \tag{3.11}$$

for all $-1 \le t \le 1$, where $b_0, b_1, \ldots, b_N$ and $c_0, c_1, \ldots, c_N$ are arbitrary complex numbers. Suppose further that g is defined by (3.9). Then there exist series of complex numbers $\beta_0, \beta_1, \ldots$ and $\gamma_0, \gamma_1, \ldots$ such that

$$g(t) = \sum_{n=0}^{\infty} \beta_n |t|^n + \sum_{n=0}^{\infty} \gamma_n \operatorname{sgn}(t)|t|^n, \tag{3.12}$$

for all $-1 \le t \le 1$. Conversely, suppose that g has the form (3.12). Then, for each positive integer N, there exist complex numbers $b_0, b_1, \ldots, b_N$ and $c_0, c_1, \ldots, c_N$ such that ρ, defined by (3.11), solves equation (3.9) to within an error $O(t^{N+1})$.

In other words, if ρ has the form (3.11), then g is smooth on the intervals $[-1,0]$ and $[0,1]$. Conversely, if g is smooth, then for each positive integer N there exists a solution ρ of the form (3.11) that solves equation (3.9) to within an error $O(t^{N+1})$.

3.1.3 The Procedure

Recently, progress has been made in solving the boundary integral equations of potential theory numerically (see, for example, [16], [4]). Most such schemes use nested quadratures to resolve the corner singularities. However, the explicit representations (3.5), (3.11) lead to alternative numerical algorithms for the solution of the integral equations of potential theory. More specifically, we use these representations to construct purpose-made discretizations which accurately represent the associated boundary integral equations (see, for example, [23], [21], [31]). Once such discretizations are available, the equations can be solved using the Nyström method combined with standard tools. We observe that the condition numbers of the resulting discretized linear systems closely approximate the condition numbers of the underlying physical problems.

Observation 3.3. While the analysis in this chapter applies only to polygonal domains, a similar analysis carries over to curved domains with corners. A paper containing the analysis, as well as the corresponding numerical algorithms and numerical examples, is in preparation.

Observation 3.4. In the examples in this chapter, the discretized boundary integral equations are solved in a straightforward way using standard tools. However, if needed, such equations can be solved much more rapidly using the numerical apparatus from, for example, [28].

Remark 3.5. Due to the detailed analysis in this chapter, the CPU time requirements of the resulting algorithms are almost independent of the requested precision. Thus, in all the examples in this chapter, the boundary integral equations are solved to essentially full double precision.

The structure of this chapter is as follows. In Section 3.2, we introduce the necessary mathematical preliminaries. Section 3.3 contains the primary analytical tools of this chapter. In sections 3.4 and 3.5, we investigate the Neumann and Dirichlet cases respectively. In Section 3.6, we briefly describe a numerical algorithm and provide several numerical examples.

3.2 Mathematical Preliminaries

3.2.1 Boundary Value Problems

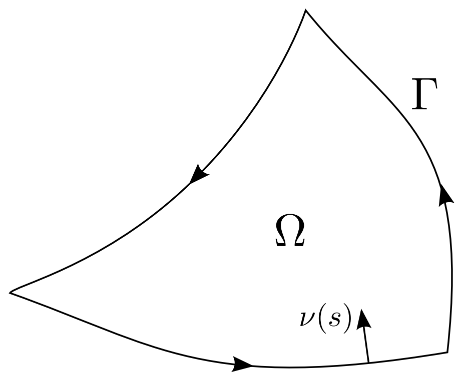

Figure 3.2: A curve in $\mathbb{R}^2$

Suppose that $\gamma : [0, L] \to \mathbb{R}^2$ is a simple closed curve of length L with a finite number of corners. Suppose further that γ is analytic except at the corners. We denote the interior of γ by Ω and the boundary of Ω by Γ, and let $\nu(t)$ denote the normalized internal normal to γ at $t \in [0, L]$. Supposing that g is some function $[0, L] \to \mathbb{C}$, we will solve the following four problems.

1) *Interior Neumann problem* (INP): Find a function $\phi\colon \Omega \to \mathbb{C}$ such that

$$\nabla^2\phi(x) + k^2\phi(x) = 0 \qquad \text{for } x \in \Omega, \tag{3.13}$$

$$\lim_{\substack{x\to\gamma(t)\\ x\in\Omega}} \frac{\partial\phi(x)}{\partial\nu(t)} = g(t) \qquad \text{for } t \in [0, L]. \tag{3.14}$$

2) *Exterior Neumann problem* (ENP): Find a function $\phi\colon \mathbb{R}^2 \setminus \overline{\Omega} \to \mathbb{C}$ such that

$$\nabla^2\phi(x) + k^2\phi(x) = 0 \qquad \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega}, \tag{3.15}$$

$$\lim_{\substack{x\to\gamma(t)\\ x\in\mathbb{R}^2\setminus\overline{\Omega}}} \frac{\partial\phi(x)}{\partial\nu(t)} = g(t) \qquad \text{for } t \in [0, L], \tag{3.16}$$

$$\lim_{|x|\to\infty} \sqrt{|x|}\Big(\frac{\partial u(x)}{\partial|x|} - iku(x)\Big) = 0. \tag{3.17}$$

3) *Interior Dirichlet problem* (IDP): Find a function $\phi\colon \Omega \to \mathbb{C}$ such that

$$\nabla^2\phi(x) + k^2\phi(x) = 0 \qquad \text{for } x \in \Omega, \tag{3.18}$$

$$\lim_{\substack{x\to\gamma(t)\\ x\in\Omega}} \phi(x) = g(t) \qquad \text{for } t \in [0, L]. \tag{3.19}$$

4) *Exterior Dirichlet problem* (EDP): Find a function $\phi\colon \mathbb{R}^2 \setminus \overline{\Omega} \to \mathbb{C}$ such that

$$\nabla^2\phi(x) + k^2\phi(x) = 0 \qquad \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega}, \tag{3.20}$$

$$\lim_{\substack{x\to\gamma(t)\\ x\in\mathbb{R}^2\setminus\overline{\Omega}}} \phi(x) = g(t) \qquad \text{for } t \in [0, L], \tag{3.21}$$

$$\lim_{|x|\to\infty} \sqrt{|x|}\Big(\frac{\partial u(x)}{\partial |x|} - iku(x)\Big) = 0. \tag{3.22}$$

Condition (3.17), (3.22) is known as the Sommerfeld radiation condition (see, for example, [8]).

Suppose that $g \in L^2([0, L])$ and that $\mathrm{Im}(k) \geq 0$. Then the exterior Neumann problem and the exterior Dirichlet problem have unique solutions. Furthermore, the interior Neumann and interior Dirichlet problems have unique solutions except for countably many real k (see, for example, [19]). Such values of k for which either the interior Neumann or interior Dirichlet problems do not have unique solutions are known as resonances.

3.2.2 Integral Equations of Potential Theory

In potential theory for the Helmholtz equation (see, for example, [25]), boundary value problems are solved by representing the solution by integrals of charges and dipoles over the boundary. The potential of a *unit charge* located at $x_0 \in \mathbb{R}^2$ is the function $\psi^0_{x_0}\colon \mathbb{R}^2 \setminus x_0 \to \mathbb{C}$, defined via the formula

$$\psi^0_{x_0}(x) = \frac{i}{4} H_0(k\|x - x_0\|), \tag{3.23}$$

for all $x \in \mathbb{R}^2 \setminus x_0$, where $\|\cdot\|$ denotes the Euclidean norm. The potential of a *unit dipole* located at $x_0 \in \mathbb{R}^2$ and oriented in direction $h \in \mathbb{R}^2$, $\|h\| = 1$, is the function $\psi^1_{x_0,h} \colon \mathbb{R}^2 \setminus x_0 \to \mathbb{C}$, defined via the formula

$$\psi^1_{x_0,h}(x) = \frac{ik\langle x_0 - x, h\rangle}{4\|x_0 - x\|} H_1(k\|x_0 - x\|), \tag{3.24}$$

for all $x \in \mathbb{R}^2 \setminus x_0$, where $\langle \cdot, \cdot \rangle$ denotes the inner product.

Charge and dipole distributions with density $\rho\colon [0, L] \to \mathbb{C}$ on Γ produce potentials given by the formulas

$$\phi(x) = \int_0^L \psi^0_{\gamma(t)}(x)\rho(t)\, dt, \tag{3.25}$$

and

$$\phi(x) = \int_0^L \psi^1_{\gamma(t),\nu(t)}(x)\rho(t)\, dt, \tag{3.26}$$

respectively, for any $x \in \mathbb{R}^2 \setminus \Gamma$.

Reduction of Boundary Value Problems to Integral Equations

The following four theorems reduce the boundary value problems of Section 3.2.1 to boundary integral equations. They are found in, for example, [8].

Theorem 3.6 (Exterior Neumann problem)**.** *Suppose that* $\rho \in L^2([0, L])$ *and* $\mathrm{Im}(k) \geq 0$*. Suppose further that* $g\colon [0, L] \to \mathbb{C}$ *is defined by the formula*

$$g(s) = \frac{1}{2}\rho(s) + \int_0^L \psi^1_{\gamma(s),\nu(s)}(\gamma(t))\rho(t)\, dt, \tag{3.27}$$

for $s \in [0, L]$*. Then* g *is in* $L^2([0, L])$*, and the solution* ϕ *to the exterior Neumann problem with right hand side* g *is obtained by substituting* ρ *into (3.25).*

Suppose now that $g \in L^2([0, L])$ *and* $\mathrm{Im}(k) \geq 0$*. Then equation (3.27) has a unique solution* $\rho \in L^2([0, L])$ *except for countably many real* k*.*

Theorem 3.7 (Interior Dirichlet problem)**.** *Suppose that* $\rho \in L^2([0,L])$ *and* $\operatorname{Im}(k) \geq 0$. *Suppose further that* $g\colon [0,L] \to \mathbb{C}$ *is defined by the formula*

$$g(s) = \frac{1}{2}\rho(s) + \int_0^L \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\rho(t)\,dt, \tag{3.28}$$

for $s \in [0,L]$. *Then* g *is in* $L^2([0,L])$, *and the solution* ϕ *to the interior Dirichlet problem with right hand side* g *is obtained by substituting* ρ *into (3.26).*

Suppose now that $g \in L^2([0,L])$ *and* $\operatorname{Im}(k) \geq 0$. *Then equation (3.28) has a unique solution* $\rho \in L^2([0,L])$ *except for countably many real* k.

Theorem 3.8 (Interior Neumann problem)**.** *Suppose that* $\rho \in L^2([0,L])$ *and* $\operatorname{Im}(k) \geq 0$. *Suppose further that* $g\colon [0,L] \to \mathbb{C}$ *is defined by the formula*

$$g(s) = -\frac{1}{2}\rho(s) + \int_0^L \psi^1_{\gamma(s),\nu(s)}(\gamma(t))\rho(t)\,dt, \tag{3.29}$$

for $s \in [0,L]$. *Then* g *is in* $L^2([0,L])$, *and the solution* ϕ *to the interior Neumann problem with right hand side* g *is obtained by substituting* ρ *into (3.25).*

Suppose now that $g \in L^2([0,L])$ *and* $\operatorname{Im}(k) \geq 0$. *Then equation (3.29) has a unique solution* $\rho \in L^2([0,L])$ *except for countably many real* k.

Theorem 3.9 (Exterior Dirichlet problem)**.** *Suppose that* $\rho \in L^2([0,L])$ *and* $\operatorname{Im}(k) \geq 0$. *Suppose further that* $g\colon [0,L] \to \mathbb{C}$ *is defined by the formula*

$$g(s) = -\frac{1}{2}\rho(s) + \int_0^L \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\rho(t)\,dt, \tag{3.30}$$

for $s \in [0,L]$. *Then* g *is in* $L^2([0,L])$, *and the solution* ϕ *to the exterior Dirichlet problem with right hand side* g *is obtained by substituting* ρ *into (3.26).*

Suppose now that $g \in L^2([0,L])$ *and* $\operatorname{Im}(k) \geq 0$. *Then equation (3.30) has a unique solution* $\rho \in L^2([0,L])$ *except for countably many real* k.

Observation 3.10. Equation (3.28) is the adjoint of equation (3.27), and equation (3.30) is the adjoint of equation (3.29).

Remark 3.11. The exterior Dirichlet and exterior Neumann problems do not have resonances. However, the corresponding integral equations (3.27) and (3.30) have what are known as spurious resonances, which are values of k for which the corresponding *adjoint* equations have resonances. There exist alternative formulations known as combined-potential integral equations which address this problem (see, for example, [8]). A straightforward modification of these techniques makes them compatible with the approach of this dissertation.

Properties of the Kernels of Equations (3.27), (3.28), (3.29), and (3.30)

When the curve γ is a wedge, the kernels of equations (3.27), (3.28), (3.29), and (3.30) have a particularly simple form, which is given by the following lemma.

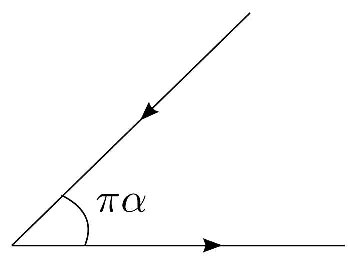

Figure 3.3: A wedge in $\mathbb{R}^2$

Lemma 3.12. *Suppose* $\gamma\colon [-1,1] \to \mathbb{R}^2$ *is defined by the formula*

$$\gamma(t) = \begin{cases} -t \cdot (\cos(\pi\alpha), \sin(\pi\alpha)) & \text{if } -1 \le t < 0, \\ (t, 0) & \text{if } 0 \le t \le 1, \end{cases} \tag{3.31}$$

shown in Figure 3.3. Then, for all $0 < s \le 1$*,*

$$\psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = \begin{cases} -\frac{i}{4}\frac{H_1(k\sqrt{s^2+t^2+2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2+2st\cos(\pi\alpha)}} kt\sin(\pi\alpha) & \text{if } -1 \le t < 0, \\ 0 & \text{if } 0 \le t \le 1, \end{cases} \tag{3.32}$$

and, for all $-1 \le s < 0$,

$$\psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = \begin{cases} 0 & \textit{if } -1 \le t < 0, \\ \dfrac{i}{4}\dfrac{H_1(k\sqrt{s^2+t^2+2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2+2st\cos(\pi\alpha)}}kt\sin(\pi\alpha) & \textit{if } 0 \le t \le 1. \end{cases} \tag{3.33}$$

Corollary 3.13. *Identities (3.32) and (3.33) remain valid after any rotation or translation of the curve* γ *in* $\mathbb{R}^2$.

Corollary 3.14. *When the curve* γ *is a straight line,* $\psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = 0$ *for all* $-1 \le s, t \le 1$.

3.2.3 Fourier Transform

Suppose that $\mathcal{F}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denotes the Fourier transform, so that

$$\mathcal{F}[f](\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx, \tag{3.34}$$

for all $\omega \in \mathbb{R}$ and

$$\mathcal{F}^{-1}[F](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\,d\omega, \tag{3.35}$$

for all $x \in \mathbb{R}$.

Suppose that $-1 < \nu < 0$ is a real number. The following theorem states that, in the distributional sense, the Fourier transform of the function

$$z^{\nu}, \tag{3.36}$$

for $z \in \mathbb{R}$, where z^{ν} has the branch cut $(-i\infty, 0]$, is equal to the function

$$\frac{\sqrt{2\pi}e^{i\pi\nu/2}}{\Gamma(-\nu)}\omega^{-\nu-1}\mathbb{1}_{[0,\infty)}(\omega), \tag{3.37}$$

for $\omega \in \mathbb{R}$, where $\mathbb{1}_A$ is defined by the formula

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{3.38}$$

Lemma 3.15. *Suppose that $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space on $\mathbb{R}$, or the space of all infinitely differentiable functions $f\colon \mathbb{R} \to \mathbb{C}$ such that $f^{(k)}(x)$ decays superalgebraically as $x \to \pm\infty$, for all $k \geq 0$. Suppose further that $-1 < \nu < 0$ is a real number. Then*

$$\int_{-\infty}^{\infty} z^\nu \varphi(z)\, dz = \frac{\sqrt{2\pi} e^{i\pi\nu/2}}{\Gamma(-\nu)} \int_0^\infty \omega^{-\nu-1} \mathcal{F}^{-1}[\varphi](\omega) d\omega, \tag{3.39}$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$, where z^ν has the branch cut $(-i\infty, 0]$ and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, defined by (3.35).

3.2.4 Bessel Functions

The Bessel functions of order ν are solutions to the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0, \tag{3.40}$$

for $0 < x < \infty$. In agreement with standard practices, we denote the Bessel functions of the *first* and *second kinds* by J_ν and Y_ν, respectively, and the Hankel function of the *first kind* by H_ν. We denote the modified Bessel functions of the *first* and *second kinds* by I_ν and K_ν respectively, defined via the formulas

$$I_\nu(z) = e^{-\frac{\pi}{2}\nu i} J_\nu(iz), \tag{3.41}$$

and

$$K_\nu(z) = \frac{\pi i}{2} e^{\frac{\pi}{2}\nu i} H_\nu(iz), \tag{3.42}$$

for all $-\pi < \arg(z) \leq \pi/2$ (see, for example, [1]).

Expansion of an Arbitrary Function into a Series of Bessel Functions

The following theorem states that any analytic function is expressible by a series of Bessel functions of integer order.

Theorem 3.16. *Suppose that f is analytic on the unit disc. Then there exists an infinite sequence of complex numbers $a_0, a_1, \dots, a_n, \dots$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n J_n(z), \tag{3.43}$$

for all $|z| < 1$. Conversely, if a function f is expressible by a series of the form (3.43), for all $|z| < 1$, then f is analytic on the unit disc.

Integral Representations of Bessel Functions

The following theorem provides an integral representation of the function K_ν.

Theorem 3.17. *Suppose that $\nu > -\frac{1}{2}$ is a real number. Then*

$$K_\nu(z) = \frac{(\frac{z}{2})^\nu \Gamma(\frac{1}{2})}{\Gamma(\nu + \frac{1}{2})} \int_1^\infty e^{-zt}(t^2 - 1)^{\nu - \frac{1}{2}} \, dt, \tag{3.44}$$

for all complex z such that $\mathrm{Re}(z) > 0$.

The following lemma uses Theorem 3.17 to evaluate a certain integral.

Lemma 3.18. *Suppose that $\nu > -1$ and $0 < \alpha < 1$ are real numbers. Then*

$$\int_0^\infty \omega^\nu e^{-\sin(\pi\alpha)\sqrt{\omega^2+k^2}} \, d\omega = \frac{\sin(\pi\alpha)}{2\sqrt{\pi}} \Gamma(\tfrac{\nu}{2} + \tfrac{1}{2}) \Big(\frac{2k}{\sin(\pi\alpha)} \Big)^{\frac{\nu}{2}+1} K_{\frac{\nu}{2}+1}(k \sin(\pi\alpha)), \tag{3.45}$$

for all $k > 0$.

Proof. By a change of variables $u = \sqrt{\omega^2 + k^2}$, we obtain the identity

$$\int_0^\infty \omega^\nu e^{-\sin(\pi\alpha)\sqrt{\omega^2+k^2}} \, d\omega = \int_k^\infty (\sqrt{u^2 - k^2})^{\nu - 1} e^{-\sin(\pi\alpha) u} u \, du, \tag{3.46}$$

for all $k > 0$. Integrating (3.46) by parts, we have

$$\begin{aligned}
&\int_0^\infty \omega^\nu e^{-\sin(\pi\alpha)\sqrt{\omega^2+k^2}}\,d\omega \\
&= \Big(\frac{(\sqrt{u^2-k^2})^{\nu+1}}{\nu+1} e^{-\sin(\pi\alpha)u}\Big|_k^\infty + \frac{\sin(\pi\alpha)}{\nu+1}\int_k^\infty (\sqrt{u^2-k^2})^{\nu+1} e^{-\sin(\pi\alpha)u}\,du \\
&= \frac{\sin(\pi\alpha)}{\nu+1}\int_k^\infty (\sqrt{u^2-k^2})^{\nu+1} e^{-\sin(\pi\alpha)u}\,du \\
&= \frac{\sin(\pi\alpha)}{\nu+1} k^{\nu+2}\int_1^\infty (\sqrt{u^2-1})^{\nu+1} e^{-k\sin(\pi\alpha)u}\,du, \qquad (3.47)
\end{aligned}$$

for all $k > 0$. We then observe that, combining (3.47) and (3.44),

$$\begin{aligned}
&\int_0^\infty \omega^\nu e^{-\sin(\pi\alpha)\sqrt{\omega^2+k^2}}\,d\omega = \frac{\sin(\pi\alpha)}{\nu+1} k^{\nu+2}\cdot\frac{\Gamma(\frac{\nu}{2}+\frac{3}{2})}{\sqrt{\pi}}\Big(\frac{2}{k\sin(\pi\alpha)}\Big)^{\frac{\nu}{2}+1} K_{\frac{\nu}{2}+1}(k\sin(\pi\alpha)) \\
&= \frac{\sin(\pi\alpha)}{2\sqrt{\pi}}\Gamma(\tfrac{\nu}{2}+\tfrac{1}{2})\Big(\frac{2k}{\sin(\pi\alpha)}\Big)^{\frac{\nu}{2}+1} K_{\frac{\nu}{2}+1}(k\sin(\pi\alpha)), \qquad (3.48)
\end{aligned}$$

for all $k > 0$.

□

Integrals Involving Bessel Functions

The following theorem is an identity involving the integral of the Bessel function K_0 multiplied by a complex exponential. It is found in, for example, formula 6.616 (3) of [12].

Theorem 3.19. *Suppose that $y > 0$ and $k > 0$ are real numbers. Then*

$$\int_{-\infty}^\infty K_0(k\sqrt{y^2+x^2})e^{i\omega x}\,dx = \frac{\pi e^{-y\sqrt{\omega^2+k^2}}}{\sqrt{\omega^2+k^2}}, \qquad (3.49)$$

for all real ω.

The following corollary follows immediately from differentiating (3.49) with respect to y.

Corollary 3.20. *Suppose that $y > 0$ and $k > 0$ are real numbers. Then*

$$\int_{-\infty}^{\infty} \frac{K_1(k\sqrt{y^2+x^2})}{\sqrt{y^2+x^2}} kye^{i\omega x}\,dx = \pi e^{-y\sqrt{\omega^2+k^2}}, \tag{3.50}$$

for all real ω.

Series Representations of Bessel Functions

The following theorem represents the Bessel function J_ν by a series.

Theorem 3.21. *Suppose that ν is a real number. Then*

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-\frac{z^2}{4})^j}{j!\Gamma(\nu+j+1)}, \tag{3.51}$$

for all complex z such that $|\arg(z)| < \pi$.

Corollary 3.22. *Suppose that ν is a real number. Then*

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(\frac{z^2}{4})^j}{j!\Gamma(\nu+j+1)}, \tag{3.52}$$

for all complex z such that $|\arg(z)| < \pi$.

Miscellaneous Identities Involving Bessel Functions

The following theorem is a straightforward consequence of the Gegenbauer addition theorem, and can be found in, for example, [1].

Theorem 3.23. *Suppose that $0 < y < 1$, $x > 1$, $k > 0$, and $0 < \alpha < 2$ are real numbers. Then*

$$\frac{H_1(k\sqrt{x^2+y^2-2xy\cos(\pi\alpha)})}{\sqrt{x^2+y^2-2xy\cos(\pi\alpha)}} ky\sin(\pi\alpha) = \frac{2}{x}\sum_{j=1}^{\infty} j\sin(\pi\alpha j)H_j(kx)J_j(ky). \tag{3.53}$$

The following theorem provides an indefinite integral involving a product of the Bessel function J_ν with a Hankel function of integer order.

Theorem 3.24. *Suppose that j is a nonnegative integer, and that $k > 0$ and $\nu > 0$ are real numbers. Then*

$$\int \frac{1}{x} H_j(kx) J_\nu(kx)\, dx = \frac{H_j(kx)J_\nu(kx)}{j+\nu} - kx \frac{H_{j+1}(kx)J_\nu(kx) - H_j(kx)J_{\nu+1}(kx)}{j^2 - \nu^2}. \tag{3.54}$$

The following two theorems provide the first terms in the asymptotic expansions of the Bessel function J_ν and the Hankel function H_ν, respectively.

Theorem 3.25. *Suppose that ν is an arbitrary complex number. Then*

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} (\cos(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + e^{|\operatorname{Im}(z)|} O(|z|^{-1})), \tag{3.55}$$

as $|z| \to \infty$, where $|\arg(z)| < \pi$.

Theorem 3.26. *Suppose that ν is an arbitrary complex number. Then*

$$H_\nu(z) = \sqrt{\frac{2}{\pi z}} (\exp(i(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)) + e^{iz} O(|z|^{-1})), \tag{3.56}$$

as $|z| \to \infty$, where $|\arg(z)| < \pi$.

The following theorem evaluates the limit of a certain expressing involving Bessel functions and Hankel functions, and follows from theorems 3.25 and 3.26.

Theorem 3.27. *Suppose that j is a nonnegative integer and that $\nu > 0$ is a real number. Then*

$$\lim_{k\to\infty} \Big(\frac{H_j(k)J_\nu(k)}{j+\nu} - k \frac{H_{j+1}(k)J_\nu(k) - H_j(k)J_{\nu+1}(k)}{j^2-\nu^2} \Big) = \frac{2i}{\pi} \frac{e^{i(\nu-j)\pi/2}}{j^2-\nu^2}. \tag{3.57}$$

The following theorem evaluates the Wronskian of J_ν and Y_ν, and is found in, for example, [1].

Theorem 3.28. *Suppose that* ν *is an arbitrary complex number. Then*

$$Y_\nu(z)J_{\nu+1}(z) - Y_{\nu+1}(z)J_\nu(z) = \frac{2}{\pi z}, \tag{3.58}$$

for all $|\arg(z)| < \pi$*.*

The following theorem uses Theorem 3.28 to evaluate the limit of a certain expression involving products of Bessel functions and Hankel functions.

Theorem 3.29. *Suppose that* n *and* m *are positive integers, and that* k *is an arbitrary complex number. Then*

$$\begin{aligned} &i \cdot n \cdot \lim_{\alpha\to 1} \sin(\pi\alpha n)\Big(\frac{H_n(k)J_{\frac{m}{\alpha}}(k)}{n+\frac{m}{\alpha}} - k\frac{H_{n+1}(k)J_{\frac{m}{\alpha}}(k) - H_n(k)J_{\frac{m}{\alpha}+1}(k)}{n^2-(\frac{m}{\alpha})^2}\Big) \\ &= \begin{cases} (-1)^{m+1} & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases} \end{aligned} \tag{3.59}$$

Proof. Suppose that $n \neq m$. Clearly,

$$\lim_{\alpha\to 1} \sin(\pi\alpha n)\Big(\frac{H_n(k)J_{\frac{m}{\alpha}}(k)}{n+\frac{m}{\alpha}} - k\frac{H_{n+1}(k)J_{\frac{m}{\alpha}}(k) - H_n(k)J_{\frac{m}{\alpha}+1}(k)}{n^2-(\frac{m}{\alpha})^2}\Big) = 0. \tag{3.60}$$

Next, suppose that $n = m$. We observe that

$$\begin{aligned} &i \cdot n \cdot \lim_{\alpha\to 1} \sin(\pi\alpha n)\Big(\frac{H_n(k)J_{\frac{n}{\alpha}}(k)}{n+\frac{n}{\alpha}} - k\frac{H_{n+1}(k)J_{\frac{n}{\alpha}}(k) - H_n(k)J_{\frac{n}{\alpha}+1}(k)}{n^2-(\frac{n}{\alpha})^2}\Big) \\ &= -i \cdot n \cdot k \cdot \lim_{\alpha\to 1} \sin(\pi\alpha n)\Big(\frac{H_{n+1}(k)J_{\frac{n}{\alpha}}(k) - H_n(k)J_{\frac{n}{\alpha}+1}(k)}{n^2-(\frac{n}{\alpha})^2}\Big) \\ &= -i \cdot n \cdot k \cdot \lim_{\alpha\to 1} \Big(\frac{H_{n+1}(k)J_{\frac{n}{\alpha}}(k) - H_n(k)J_{\frac{n}{\alpha}+1}(k)}{n+\frac{n}{\alpha}}\Big)\frac{\sin(\pi\alpha n)}{n-\frac{n}{\alpha}} \\ &= -\frac{ik}{2}\big(H_{n+1}(k)J_n(k) - H_n(k)J_{n+1}(k)\big)\pi(-1)^n \\ &= \frac{k}{2}\big(Y_{n+1}(k)J_n(k) - Y_n(k)J_{n+1}(k)\big)\pi(-1)^n. \end{aligned} \tag{3.61}$$

Combining (3.61) and (3.58), it follows that

$$i \cdot n \cdot \lim_{\alpha \to 1} \sin(\pi \alpha n) \Big(\frac{H_n(k) J_{\frac{n}{\alpha}}(k)}{n + \frac{n}{\alpha}} - k \frac{H_{n+1}(k) J_{\frac{n}{\alpha}}(k) - H_n(k) J_{\frac{n}{\alpha}+1}(k)}{n^2 - (\frac{n}{\alpha})^2} \Big) = (-1)^{n+1}. \tag{3.62}$$

□

Corollary 3.30. *Suppose that n and m are positive integers, and that k is an arbitrary complex number. Then*

$$\begin{aligned} & i \cdot n \cdot \lim_{\alpha \to 1} \sin(\pi \alpha n) \Big(\frac{H_n(k) J_{\frac{m}{2-\alpha}}(k)}{n + \frac{m}{2-\alpha}} - k \frac{H_{n+1}(k) J_{\frac{m}{2-\alpha}}(k) - H_n(k) J_{\frac{m}{2-\alpha}+1}(k)}{n^2 - (\frac{m}{2-\alpha})^2} \Big) \\ &= \begin{cases} (-1)^m & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases} \end{aligned} \tag{3.63}$$

3.3 Analytical Apparatus

3.3.1 The Principal Analytical Observation

Theorem 3.39 in this section is the primary analytical tool of this chapter.

The following lemma evaluates the Fourier transform of a certain expression involving the Bessel function K_1.

Lemma 3.31. *Suppose that $k > 0$ and $0 < \alpha < 1$ are real numbers. Then*

$$\int_{-\infty}^{\infty} \frac{K_1(k\sqrt{z^2 + 1 - 2z\cos(\pi\alpha)})}{\sqrt{z^2 + 1 - 2z\cos(\pi\alpha)}} k \sin(\pi\alpha) e^{i\omega z} \, dz = \pi e^{i\omega \cos(\pi\alpha) - \sin(\pi\alpha)\sqrt{\omega^2 + k^2}}, \tag{3.64}$$

for all real ω.

Proof. By a change of variables $z = u + \cos(\pi\alpha)$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{K_1(k\sqrt{z^2+1-2z\cos(\pi\alpha)})}{\sqrt{z^2+1-2z\cos(\pi\alpha)}} k\sin(\pi\alpha)e^{i\omega z}\,dz \\
&= \int_{-\infty}^{\infty} \frac{K_1(k\sqrt{u^2+(\sin(\pi\alpha))^2})}{\sqrt{u^2+(\sin(\pi\alpha))^2}} k\sin(\pi\alpha)e^{i\omega(u+\cos(\pi\alpha))}\,du \\
&= e^{i\omega\cos(\pi\alpha)} \int_{-\infty}^{\infty} \frac{K_1(k\sqrt{u^2+(\sin(\pi\alpha))^2})}{\sqrt{u^2+(\sin(\pi\alpha))^2}} k\sin(\pi\alpha)e^{i\omega u}\,du \\
&= e^{i\omega\cos(\pi\alpha)} \int_{-\infty}^{\infty} \frac{K_1(k\sin(\pi\alpha)\sqrt{u^2+1})}{\sqrt{u^2+1}} k\sin(\pi\alpha)e^{i\omega\sin(\pi\alpha)u}\,du, \qquad (3.65)
\end{aligned}$$

for all real ω. Combining (3.65) with (3.50), it follows that

$$\int_{-\infty}^{\infty} \frac{K_1(k\sqrt{z^2+1-2z\cos(\pi\alpha)})}{\sqrt{z^2+1-2z\cos(\pi\alpha)}} k\sin(\pi\alpha)e^{i\omega z}\,dz = \pi e^{i\omega\cos(\pi\alpha)-\sin(\pi\alpha)\sqrt{\omega^2+k^2}}, \qquad (3.66)$$

for all real ω.

□

The following lemma evaluates the integral of a product of the Bessel function K_1 and the function z^ν, where ν is a real number.

Lemma 3.32. *Suppose that $k > 0$, $0 < \alpha < 1$, and $-1 < \nu < 0$ are real numbers. Then*

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{K_1(k\sqrt{z^2+1-2z\cos(\pi\alpha)})}{\sqrt{z^2+1-2z\cos(\pi\alpha)}} k\sin(\pi\alpha)z^\nu\,dz \\
&\qquad = \frac{\pi e^{i\pi\nu/2}}{\Gamma(-\nu)} \int_0^{\infty} \omega^{-\nu-1} e^{i\omega\cos(\pi\alpha)-\sin(\pi\alpha)\sqrt{\omega^2+k^2}}\,d\omega, \qquad (3.67)
\end{aligned}$$

where z^ν has the branch cut $(-i\infty, 0]$.

Proof. Identity (3.67) follows immediately from combining (3.64) and (3.39).

□

The following theorem represents the left hand side of (3.67) by a series involving the Bessel functions K_ν, where ν is a real number.

Theorem 3.33. *Suppose that $0 < \alpha < 1$ and $-1 < \nu < 0$ are real numbers, and that* $\mathrm{Re}(k) > 0$. *Then*

$$\int_{-\infty}^{\infty} \frac{K_1(k\sqrt{z^2+1-2z\cos(\pi\alpha)})}{\sqrt{z^2+1-2z\cos(\pi\alpha)}} k\sin(\pi\alpha) z^{\nu}\,dx = \frac{\sqrt{\pi}\sin(\pi\alpha)e^{i\pi\nu/2}}{2\Gamma(-\nu)} \sum_{m=0}^{\infty} \frac{(i\cos(\pi\alpha))^m}{m!} \cdot \Gamma(-\tfrac{\nu}{2}+\tfrac{m}{2})\Big(\frac{2k}{\sin(\pi\alpha)}\Big)^{-\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}} K_{-\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}}(k\sin(\pi\alpha)), \tag{3.68}$$

where z^{ν} has the branch cut $(-i\infty, 0]$.

Proof. Suppose that $k > 0$ is a real number. From (3.67), we observe that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{K_1(k\sqrt{z^2+1-2z\cos(\pi\alpha)})}{\sqrt{z^2+1-2z\cos(\pi\alpha)}} k\sin(\pi\alpha) z^{\nu}\,dz \\
&= \frac{\pi e^{i\pi\nu/2}}{\Gamma(-\nu)} \int_0^{\infty} \omega^{-\nu-1} e^{i\omega\cos(\pi\alpha)-\sin(\pi\alpha)\sqrt{\omega^2+k^2}}\,d\omega \\
&= \frac{\pi e^{i\pi\nu/2}}{\Gamma(-\nu)} \int_0^{\infty} \sum_{m=0}^{\infty} \frac{(i\cos(\pi\alpha))^m}{m!} \omega^{-\nu+m-1} e^{-\sin(\pi\alpha)\sqrt{\omega^2+k^2}}\,d\omega \\
&= \frac{\pi e^{i\pi\nu/2}}{\Gamma(-\nu)} \sum_{m=0}^{\infty} \frac{(i\cos(\pi\alpha))^m}{m!} \int_0^{\infty} \omega^{-\nu+m-1} e^{-\sin(\pi\alpha)\sqrt{\omega^2+k^2}}\,d\omega.
\end{aligned} \tag{3.69}$$

Combining (3.69) and (3.45), it follows that

$$\int_{-\infty}^{\infty} \frac{K_1(k\sqrt{x^2+1-2x\cos(\pi\alpha)})}{\sqrt{x^2+1-2x\cos(\pi\alpha)}} k\sin(\pi\alpha) x^{\nu}\,dx = \frac{\sqrt{\pi}\sin(\pi\alpha)e^{i\pi\nu/2}}{2\Gamma(-\nu)} \sum_{m=0}^{\infty} \frac{(i\cos(\pi\alpha))^m}{m!} \cdot \Gamma(-\tfrac{\nu}{2}+\tfrac{m}{2})\Big(\frac{2k}{\sin(\pi\alpha)}\Big)^{-\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}} K_{-\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}}(k\sin(\pi\alpha)). \tag{3.70}$$

An application of analytic continuation shows that identity (3.70) holds for all complex k such that $\mathrm{Re}(k) > 0$. □

The following theorem evaluates the integral in (3.68) when it is taken from 0 to ∞.

Theorem 3.34. *Suppose that $0 < \alpha < 2$ and $\nu > -1$ are real numbers, and that*

$\mathrm{Re}(k) > 0$. *Then*

$$\int_0^\infty \frac{K_1(k\sqrt{x^2+1-2x\cos(\pi\alpha)})}{\sqrt{x^2+1-2x\cos(\pi\alpha)}} k\sin(\pi\alpha) x^\nu\,dx = \frac{\sqrt{\pi}\sin(\pi\alpha)}{2\sin(\pi\nu)\Gamma(-\nu)} \sum_{m=0}^\infty \frac{(\cos(\pi\alpha))^m}{m!}$$
$$\cdot \sin(\pi(\tfrac{\nu}{2}-\tfrac{m}{2}))\Gamma(-\tfrac{\nu}{2}+\tfrac{m}{2})\Big(\frac{2k}{\sin(\pi\alpha)}\Big)^{-\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}} K_{-\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}}(k\sin(\pi\alpha)). \tag{3.71}$$

Proof. Suppose that $F\colon \mathbb{R}\times(0,1)\to\mathbb{R}$ is defined by the formula

$$F(x,\alpha) = \frac{K_1(k\sqrt{x^2+1-2x\cos(\pi\alpha)})}{\sqrt{x^2+1-2x\cos(\pi\alpha)}} k\sin(\pi\alpha), \tag{3.72}$$

for all real x and all $0<\alpha<1$. We observe that

$$F(-x,\alpha) = F(x,1-\alpha), \tag{3.73}$$

for all real x and all $0<\alpha<1$. We will use the property (3.73) to express the integral

$$\int_0^\infty F(x,\alpha)x^\nu\,dx \tag{3.74}$$

in terms of integrals of the form

$$\int_{-\infty}^\infty F(z,\alpha)z^\nu\,dz. \tag{3.75}$$

We observe that

$$\int_{-\infty}^\infty F(z,\alpha)z^\nu\,dz = \int_0^\infty F(z,\alpha)z^\nu\,dz + \int_{-\infty}^0 F(z,\alpha)z^\nu\,dz$$
$$= \int_0^\infty F(x,\alpha)x^\nu\,dx + e^{i\pi\nu}\int_0^\infty F(-x,\alpha)x^\nu\,dx, \tag{3.76}$$

where z^ν has the branch cut $(-i\infty,0]$. By (3.73), we obtain

$$\int_{-\infty}^\infty F(z,\alpha)z^\nu\,dz = \int_0^\infty F(x,\alpha)x^\nu\,dx + e^{i\pi\nu}\int_0^\infty F(x,1-\alpha)x^\nu\,dx. \tag{3.77}$$

Replacing α with $1-\alpha$ in (3.77), we also have that

$$\int_{-\infty}^{\infty} F(z,1-\alpha)z^{\nu}\,dx = \int_{0}^{\infty} F(x,1-\alpha)x^{\nu}\,dx + e^{i\pi\nu}\int_{0}^{\infty} F(x,\alpha)x^{\nu}\,dx. \tag{3.78}$$

Multiplying (3.77) by $-e^{-i\pi\nu}$ and adding it to (3.78), we obtain

$$\begin{aligned}\int_{0}^{\infty} F(x,\alpha)x^{\nu}\,dx &= \frac{1}{e^{i\pi\nu}-e^{-i\pi\nu}}\Big(-e^{-i\pi\nu}\int_{-\infty}^{\infty} F(z,\alpha)z^{\nu}\,dx + \int_{-\infty}^{\infty} F(z,1-\alpha)z^{\nu}\,dx\Big)\\ &= \frac{e^{-i\pi\nu/2}}{2i\sin(\pi\nu)}\Big(-e^{-i\pi\nu/2}\int_{-\infty}^{\infty} F(z,\alpha)z^{\nu}\,dx + e^{i\pi\nu/2}\int_{-\infty}^{\infty} F(z,1-\alpha)z^{\nu}\,dx\Big). \end{aligned}\tag{3.79}$$

Suppose that $-1<\nu<0$. Combining (3.79) with (3.72) and (3.68), we observe that

$$\begin{aligned}&\int_{0}^{\infty} \frac{K_1(k\sqrt{x^2+1-2x\cos(\pi\alpha)})}{\sqrt{x^2+1-2x\cos(\pi\alpha)}}k\sin(\pi\alpha)x^{\nu}\,dx = \frac{\sqrt{\pi}\sin(\pi\alpha)}{4i\sin(\pi\nu)\Gamma(-\nu)}\sum_{m=0}^{\infty}\frac{(\cos(\pi\alpha))^m}{m!}\\ &\quad\cdot(-e^{-i\pi(\nu/2-m/2)}+e^{i\pi(\nu/2-m/2)})\Gamma(-\tfrac{\nu}{2}+\tfrac{m}{2})\Big(\frac{2k}{\sin(\pi\alpha)}\Big)^{-\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}}K_{-\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}}(k\sin(\pi\alpha))\\ &= \frac{\sqrt{\pi}\sin(\pi\alpha)}{2\sin(\pi\nu)\Gamma(-\nu)}\sum_{m=0}^{\infty}\frac{(\cos(\pi\alpha))^m}{m!}\sin(\pi(\tfrac{\nu}{2}-\tfrac{m}{2}))\Gamma(-\tfrac{\nu}{2}+\tfrac{m}{2})\Big(\frac{2k}{\sin(\pi\alpha)}\Big)^{-\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}}\\ &\quad\cdot K_{-\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}}(k\sin(\pi\alpha)).\end{aligned}\tag{3.80}$$

An application of analytic continuation shows that identity (3.80) holds for all complex ν such that $\mathrm{Re}(\nu)>-1$ and for all complex α such that $0<\mathrm{Re}(\alpha)<2$.

□

The following theorem follows immediately from a change of variables in (3.71).

Theorem 3.35. *Suppose that* $0<\alpha<2$ *and* $\nu>-1$ *are real numbers, and that* $\mathrm{Re}(k)>0$*. Then*

$$\begin{aligned}&\int_{0}^{\infty} \frac{K_1(k\sqrt{x^2+y^2-2xy\cos(\pi\alpha)})}{\sqrt{x^2+y^2-2xy\cos(\pi\alpha)}}ky\sin(\pi\alpha)x^{\nu}\,dx = \frac{\sqrt{\pi}\sin(\pi\alpha)}{2\sin(\pi\nu)\Gamma(-\nu)}\sum_{m=0}^{\infty}\frac{(\cos(\pi\alpha))^m}{m!}\\ &\quad\cdot\sin(\pi(\tfrac{\nu}{2}-\tfrac{m}{2}))\Gamma(-\tfrac{\nu}{2}+\tfrac{m}{2})\Big(\frac{2k}{\sin(\pi\alpha)}\Big)^{-\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}}K_{-\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}}(ky\sin(\pi\alpha))y^{\frac{\nu}{2}+\frac{m}{2}+\frac{1}{2}},\end{aligned}\tag{3.81}$$

for all $y > 0$.

The following theorem evaluates the integral in (3.81) when x^ν is replaced with $I_\nu(kx)$.

Theorem 3.36. *Suppose that* $0 < \alpha < 2$ *and* $\nu > -1$ *are real numbers, and that* $\mathrm{Re}(k) > 0$. *Then*

$$\frac{1}{2\pi}\int_0^\infty \frac{K_1(k\sqrt{x^2+y^2-2xy\cos(\pi\alpha)})}{\sqrt{x^2+y^2-2xy\cos(\pi\alpha)}} ky\sin(\pi\alpha) I_\nu(kx)\,dx = \frac{\sin(\pi\nu(1-\alpha))}{2\sin(\pi\nu)} I_\nu(ky) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{n\sin(\pi n\alpha)}{n^2-\nu^2} I_n(ky), \tag{3.82}$$

for all $y > 0$.

Proof. Suppose that $\mathrm{Re}(k) > 0$. Then a straightforward but tedious calculation combining (3.81) with (3.52) yields identity (3.82). □

The following theorem follows immediately from (3.82), and replaces K_1 with H_1 and I_ν with J_ν.

Theorem 3.37. *Suppose that* $0 < \alpha < 2$ *and* $\nu > -1$ *are real numbers, and that* $\mathrm{Im}(k) > 0$. *Then*

$$\frac{i}{4}\int_0^\infty \frac{H_1(k\sqrt{x^2+y^2-2xy\cos(\pi\alpha)})}{\sqrt{x^2+y^2-2xy\cos(\pi\alpha)}} ky\sin(\pi\alpha) J_\nu(kx)\,dx = \frac{\sin(\pi\nu(1-\alpha))}{2\sin(\pi\nu)} J_\nu(ky) - \frac{1}{\pi}\sum_{n=1}^{\infty} e^{i\pi(\nu-n)/2}\frac{n\sin(\pi n\alpha)}{n^2-\nu^2} J_n(ky), \tag{3.83}$$

for all $y > 0$.

The following theorem evaluates the integral in identity (3.83) when it is taken from 1 to ∞.

Theorem 3.38. *Suppose that* $0 < \alpha < 2$ *and* $\nu > -1$ *are real numbers, and that*

$\mathrm{Im}(k) > 0$. *Then*

$$\int_1^\infty \frac{H_1(k\sqrt{x^2+y^2-2xy\cos(\pi\alpha)})}{\sqrt{x^2+y^2-2xy\cos(\pi\alpha)}} ky\sin(\pi\alpha)J_\nu(kx)\,dx = 2\sum_{n=1}^\infty n\sin(\pi\alpha n) \cdot \Big(\frac{2i}{\pi}\frac{e^{i(\nu-n)\pi/2}}{n^2-\nu^2} - \frac{H_n(k)J_\nu(k)}{n+\nu} + k\frac{H_{n+1}(k)J_\nu(k)-H_n(k)J_{\nu+1}(k)}{n^2-\nu^2}\Big)J_n(ky), \tag{3.84}$$

for all $0 < y < 1$.

Proof. Suppose that $0 < y < 1$ is a real number. By (3.53), we observe that

$$\begin{aligned}&\int_1^\infty \frac{H_1(k\sqrt{x^2+y^2-2xy\cos(\pi\alpha)})}{\sqrt{x^2+y^2-2xy\cos(\pi\alpha)}} ky\sin(\pi\alpha)J_\nu(kx)\,dx \\ &= 2\sum_{n=1}^\infty n\sin(\pi\alpha n)J_n(ky)\int_1^\infty \frac{1}{x}H_n(kx)J_\nu(kx)\,dx.\end{aligned} \tag{3.85}$$

Applying Theorem 3.24 to (3.85), we have that

$$\begin{aligned}&\int_1^\infty \frac{H_1(k\sqrt{x^2+y^2-2xy\cos(\pi\alpha)})}{\sqrt{x^2+y^2-2xy\cos(\pi\alpha)}} ky\sin(\pi\alpha)J_\nu(kx)\,dx \\ &= 2\sum_{n=1}^\infty n\sin(\pi\alpha n)J_n(ky)\Big(\frac{H_n(kx)J_\nu(kx)}{n+\nu} - k\frac{H_{n+1}(kx)J_\nu(kx)-H_n(kx)J_{\nu+1}(kx)}{n^2-\nu^2}\Big|_1^\infty.\end{aligned} \tag{3.86}$$

Combining (3.86) with (3.57), we finally observe that

$$\int_1^\infty \frac{H_1(k\sqrt{x^2+y^2-2xy\cos(\pi\alpha)})}{\sqrt{x^2+y^2-2xy\cos(\pi\alpha)}} ky\sin(\pi\alpha)J_\nu(kx)\,dx = 2\sum_{n=1}^\infty n\sin(\pi\alpha n) \cdot \Big(\frac{2i}{\pi}\frac{e^{i(\nu-n)\pi/2}}{n^2-\nu^2} - \frac{H_n(k)J_\nu(k)}{n+\nu} + k\frac{H_{n+1}(k)J_\nu(k)-H_n(k)J_{\nu+1}(k)}{n^2-\nu^2}\Big)J_n(ky). \tag{3.87}$$

□

The following theorem is the principal result of this section.

Theorem 3.39. *Suppose that* $0 < \alpha < 2$ *and* $\nu > 0$ *are real numbers, and that* k *is an* *arbitrary complex number. Then*

$$\frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{x^2+y^2-2xy\cos(\pi\alpha)})}{\sqrt{x^2+y^2-2xy\cos(\pi\alpha)}} ky\sin(\pi\alpha) J_\nu(kx)\,dx = \frac{\sin(\pi\nu(1-\alpha))}{2\sin(\pi\nu)} J_\nu(ky) + \frac{i}{2}\sum_{n=1}^\infty n\sin(\pi\alpha n)\Big(\frac{H_n(k)J_\nu(k)}{n+\nu} - k\frac{H_{n+1}(k)J_\nu(k)-H_n(k)J_{\nu+1}(k)}{n^2-\nu^2}\Big) J_n(ky), \tag{3.88}$$

for all $0 < y < 1$.

Proof. Suppose that $\mathrm{Im}(k) > 0$. Then identity (3.88) follows immediately from combining (3.83) and (3.84). We then observe that, by analytic continuation, identity (3.88) holds for all complex k.

□

3.3.2 Miscellaneous Analytical Facts

The following technical lemma describes the properties of a certain elementary function.

Theorem 3.40. *Suppose that* $0 < \alpha < 2$ *and that* $f\colon \mathbb{R} \to \mathbb{R}$ *is defined by the formula*

$$f(\mu) = \frac{\sin(\pi\mu(1-\alpha))}{\sin(\pi\mu)}, \tag{3.89}$$

for all $\mu \in \mathbb{R}$. *Then* $f(\mu) = 1$ *if and only if*

$$\mu = \frac{2n}{\alpha}, \tag{3.90}$$

or

$$\mu = \frac{2n-1}{2-\alpha}, \tag{3.91}$$

for some integer n. *Likewise* $f(\mu) = -1$ *if and only if*

$$\mu = \frac{2n-1}{\alpha}, \tag{3.92}$$

or

$$\mu = \frac{2n}{2-\alpha}, \tag{3.93}$$

for some integer n.

The following theorem states that a certain matrix is nonsingular.

Theorem 3.41. *Suppose that $\mathbb{Z}^+$ denotes the positive integers. Suppose further that k is an arbitrary complex number, and that $h_1, h_2 \colon \mathbb{C} \times \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{C}$ are defined by the formulas*

$$\begin{aligned} h_1(\alpha, n, m) &= in\sin(\pi\alpha n)\Big(\frac{H_n(k)J_{\frac{m}{\alpha}}(k)}{n+\frac{m}{\alpha}} - k\frac{H_{n+1}(k)J_{\frac{m}{\alpha}}(k) - H_n(k)J_{\frac{m}{\alpha}+1}(k)}{n^2 - (\frac{m}{\alpha})^2}\Big), \\ h_2(\alpha, n, m) &= in\sin(\pi\alpha n)\Big(\frac{H_n(k)J_{\frac{m}{2-\alpha}}(k)}{n+\frac{m}{2-\alpha}} - k\frac{H_{n+1}(k)J_{\frac{m}{2-\alpha}}(k) - H_n(k)J_{\frac{m}{2-\alpha}+1}(k)}{n^2 - (\frac{m}{2-\alpha})^2}\Big), \end{aligned} \tag{3.94}$$

for all $\alpha \in \mathbb{C}$, where n and m are positive integers. Suppose finally that $A(\alpha)$ is an $N \times N$ matrix defined via the formula

$$A_{n,m}(\alpha) = \begin{cases} h_1(\alpha, n, m) & \text{if } m \text{ is odd}, \\ h_2(\alpha, n, m) & \text{if } m \text{ is even}, \end{cases} \tag{3.95}$$

for all $\alpha \in \mathbb{C}$, where $1 \le n, m \le N$ are integers. Then $A(\alpha)$ is nonsingular for all but a finite number of $0 \le \alpha \le 2$. Moreover,

$$A(1) = I, \tag{3.96}$$

where I is the identity matrix.

Proof. The functions $h_1(\alpha, n, m)$ and $h_2(\alpha, n, m)$ are entire functions of α for all positive integers n and m. Therefore, $\det(A(\alpha))$ is an entire function of α. By (3.59) and (3.63),

$$A(1) = I, \tag{3.97}$$

where I is the identity matrix, from which it follows that

$$\det(A(1)) = 1. \tag{3.98}$$

Since $\det(A(\alpha))$ is not identically zero and the interval $[0,2]$ is compact, $\det(A(\alpha))$ is equal to zero at no more than a finite number of points in $[0,2]$. Hence, $A(\alpha)$ is nonsingular for all but a finite number of points $0 \le \alpha \le 2$.

□

3.4 Analysis of the Integral Equation: the Neumann Case

Suppose that the curve $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge defined by (3.31) with interior angle $\pi\alpha$, where $0 < \alpha < 2$ (see Figure 3.3). Let g be a function in $L^2([-1,1])$, and suppose that $\rho \in L^2([-1,1])$ solves the equation

$$\frac{1}{2}\rho(s) + \int_{-1}^{1} \psi^1_{\gamma(s),\nu(s)}(\gamma(t))\rho(t)\,dt = g(s), \tag{3.99}$$

for all $s \in [-1,1]$. In this section, we analyze this boundary integral equation.

We will investigate the behavior of (3.99) for functions $\rho \in L^2([-1,1])$ of the forms

$$\rho(t) = \frac{J_\mu(k|t|)}{|t|}, \tag{3.100}$$

$$\rho(t) = \operatorname{sgn}(t)\frac{J_\mu(k|t|)}{|t|}, \tag{3.101}$$

where $\mu > \frac{1}{2}$ is a real number and

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0, \end{cases} \tag{3.102}$$

for all real x. If ρ has the forms (3.100) and (3.101), then for most values of μ the resulting function g is singular; in Section 3.4.2, we observe that for certain μ, the function g is smooth. In Section 3.4.3, we fix g and view (3.99) as an integral equation in ρ. We observe that, for sufficiently smooth functions g, the solution ρ is representable by a series of functions of the forms (3.100) and (3.101).

3.4.1 Integral Equations Near a Corner

The following lemma uses a symmetry argument to reduce (3.99) from an integral equation on the interval $[-1, 1]$ to two independent integral equations on the interval $[0, 1]$.

Theorem 3.42. *Suppose that ρ is a function in $L^2([-1,1])$ and that $g \in L^2([-1,1])$ is given by (3.99). Suppose further that the functions $g_{\text{even}}, \rho_{\text{even}} \in L^2([-1,1])$ are defined via the formulas*

$$g_{\text{even}}(s) = \frac{1}{2}(g(s) + g(-s)), \tag{3.103}$$

$$\rho_{\text{even}}(s) = \frac{1}{2}(\rho(s) + \rho(-s)). \tag{3.104}$$

Then

$$g_{\text{even}}(s) = \frac{1}{2}\rho_{\text{even}}(s) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\rho_{\text{even}}(t)\,dt, \tag{3.105}$$

for all $0 < s \leq 1$.

Likewise, suppose that odd functions $g_{\text{odd}}, \rho_{\text{odd}} \in L^2([-1,1])$ are defined via the formulas

$$g_{\text{odd}}(s) = \frac{1}{2}(g(s) - g(-s)), \tag{3.106}$$

$$\rho_{\text{odd}}(s) = \frac{1}{2}(\rho(s) - \rho(-s)). \tag{3.107}$$

Then

$$g_{\text{odd}}(s) = \frac{1}{2}\rho_{\text{odd}}(s) - \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\rho_{\text{odd}}(t)\,dt, \tag{3.108}$$

for all $0 < s \leq 1$.

Proof. By Lemma 3.12, we observe that

$$g(s) = \frac{1}{2}\rho(s) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2+2st\cos(\pi\alpha)}}{\sqrt{s^2+t^2+2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\rho(t)\,dt, \tag{3.109}$$

for all $-1 \leq s < 0$, and

$$g(s) = \frac{1}{2}\rho(s) - \frac{i}{4}\int_{-1}^0 \frac{H_1(k\sqrt{s^2+t^2+2st\cos(\pi\alpha)}}{\sqrt{s^2+t^2+2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\rho(t)\,dt, \tag{3.110}$$

for all $0 < s \leq 1$. Replacing s with $-s$ in (3.109), we have

$$g(-s) = \frac{1}{2}\rho(-s) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\rho(t)\,dt, \tag{3.111}$$

for all $0 < s \leq 1$, and, by a change of variables in (3.110),

$$g(s) = \frac{1}{2}\rho(s) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\rho(-t)\,dt, \tag{3.112}$$

for all $0 < s \leq 1$. Adding equations (3.111) and (3.112), we observe that

$$g_{\text{even}}(s) = \frac{1}{2}\rho_{\text{even}}(s) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\rho_{\text{even}}(t)\,dt, \tag{3.113}$$

for all $0 < s \leq 1$. Likewise, subtracting equation (3.111) from equation (3.112),

$$g_{\text{odd}}(s) = \frac{1}{2}\rho_{\text{odd}}(s) - \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\rho_{\text{odd}}(t)\,dt, \tag{3.114}$$

for all $0 < s \leq 1$.

□

3.4.2 The Singularities in the Solution of Equation (3.99)

In this section we observe that for certain functions ρ, the function g defined by (3.99) is representable by convergent Taylor series on the intervals $[-1, 0]$ and $[0, 1]$.

The Even Case

Suppose that $\rho \in L^2([-1, 1])$ is an even function, and suppose that $g \in L^2([-1, 1])$ is defined by (3.99). By Theorem 3.42, g is also even and

$$g(s) = \frac{1}{2}\rho(s) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\rho(t)\,dt, \tag{3.115}$$

for all $0 < s \leq 1$. Suppose further that

$$\rho(t) = \frac{J_\mu(kt)}{t}, \tag{3.116}$$

for all $0 < t \leq 1$. The following theorem shows that for certain values of μ, the function g in (3.115) is representable by a convergent Taylor series on the interval $[0, 1]$.

Theorem 3.43. *Suppose that $0 < \alpha < 2$ is a real number and m is a positive integer.*

Then

$$\frac{1}{2}\frac{J_{\frac{2m-1}{\alpha}}(ks)}{s}+\frac{i}{4}\int_0^1\frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}kt\sin(\pi\alpha)\frac{J_{\frac{2m-1}{\alpha}}(kt)}{t}\,dx$$
$$=\frac{i}{2}\sum_{n=1}^{\infty}n\sin(\pi\alpha n)\Big(\frac{H_n(k)J_{\frac{2m-1}{\alpha}}(k)}{n+\frac{2m-1}{\alpha}}-k\frac{H_{n+1}(k)J_{\frac{2m-1}{\alpha}}(k)-H_n(k)J_{\frac{2m-1}{\alpha}+1}(k)}{n^2-(\frac{2m-1}{\alpha})^2}\Big)\frac{J_n(ks)}{s},\tag{3.117}$$

and

$$\frac{1}{2}\frac{J_{\frac{2m}{2-\alpha}}(ks)}{s}+\frac{i}{4}\int_0^1\frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}kt\sin(\pi\alpha)\frac{J_{\frac{2m}{2-\alpha}}(kt)}{t}\,dx$$
$$=\frac{i}{2}\sum_{n=1}^{\infty}n\sin(\pi\alpha n)\Big(\frac{H_n(k)J_{\frac{2m}{2-\alpha}}(k)}{n+\frac{2m}{2-\alpha}}-k\frac{H_{n+1}(k)J_{\frac{2m}{2-\alpha}}(k)-H_n(k)J_{\frac{2m}{2-\alpha}+1}(k)}{n^2-(\frac{2m}{2-\alpha})^2}\Big)\frac{J_n(ks)}{s},\tag{3.118}$$

for all $0<s\leq 1$.

Proof. Substituting $\mu=\frac{2m-1}{\alpha}$ into (3.88), we observe that

$$\frac{1}{2}\frac{J_{\frac{2m-1}{\alpha}}(ks)}{s}+\frac{i}{4}\int_0^1\frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}kt\sin(\pi\alpha)\frac{J_{\frac{2m-1}{\alpha}}(kt)}{t}\,dx$$
$$=\frac{1}{2}\frac{J_{\frac{2m-1}{\alpha}}(ks)}{s}+\frac{1}{2}\frac{\sin\big(\frac{2m-1}{\alpha}\cdot\pi(1-\alpha)\big)}{\sin(\frac{2m-1}{\alpha}\cdot\pi)}\frac{J_{\frac{2m-1}{\alpha}}(ks)}{s}$$
$$+\frac{i}{2}\sum_{n=1}^{\infty}n\sin(\pi\alpha n)\Big(\frac{H_n(k)J_{\frac{2m-1}{\alpha}}(k)}{n+\frac{2m-1}{\alpha}}-k\frac{H_{n+1}(k)J_{\frac{2m-1}{\alpha}}(k)-H_n(k)J_{\frac{2m-1}{\alpha}+1}(k)}{n^2-(\frac{2m-1}{\alpha})^2}\Big)\frac{J_n(ks)}{s},\tag{3.119}$$

for all $0<s\leq 1$. By Lemma 3.40,

$$\frac{1}{2}\frac{J_{\frac{2m-1}{\alpha}}(ks)}{s}+\frac{1}{2}\frac{\sin\big(\frac{2m-1}{\alpha}\cdot\pi(1-\alpha)\big)}{\sin(\frac{2m-1}{\alpha}\cdot\pi)}\frac{J_{\frac{2m-1}{\alpha}}(ks)}{s}$$
$$=\frac{1}{2}\frac{J_{\frac{2m-1}{\alpha}}(ks)}{s}-\frac{1}{2}\frac{J_{\frac{2m-1}{\alpha}}(ks)}{s}$$
$$=0,\tag{3.120}$$

for all $0<s\leq 1$. Therefore, combining (3.119) and (3.120), we obtain identity (3.117).

The proof of identity (3.118) is essentially identical.

□

The Odd Case

Suppose that $\rho \in L^2([-1,1])$ is an odd function, and suppose that $g \in L^2([-1,1])$ is defined by (3.99). By Theorem 3.42, g is also odd and

$$g(s) = \frac{1}{2}\rho(s) - \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\rho(t)\,dt, \tag{3.121}$$

for all $0 < s \le 1$. Suppose further that

$$\rho(t) = \frac{J_\mu(kt)}{t}, \tag{3.122}$$

for all $0 < t \le 1$. The following theorem shows that for certain values of μ, the function g in (3.121) is representable by a convergent Taylor series on the interval $[0,1]$.

Theorem 3.44. *Suppose that $0 < \alpha < 2$ is a real number and m is a positive integer. Then*

$$\begin{aligned} &\frac{1}{2}\frac{J_{\frac{2m}{\alpha}}(ks)}{s} - \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\frac{J_{\frac{2m}{\alpha}}(kt)}{t}\,dx \\ &= -\frac{i}{2}\sum_{n=1}^{\infty} n\sin(\pi\alpha n)\Big(\frac{H_n(k)J_{\frac{2m}{\alpha}}(k)}{n+\frac{2m}{\alpha}} - k\frac{H_{n+1}(k)J_{\frac{2m}{\alpha}}(k) - H_n(k)J_{\frac{2m}{\alpha}+1}(k)}{n^2-(\frac{2m}{\alpha})^2}\Big)\frac{J_n(ks)}{s}, \end{aligned} \tag{3.123}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{J_{\frac{2m-1}{2-\alpha}}(ks)}{s} - \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\frac{J_{\frac{2m-1}{2-\alpha}}(kt)}{t}\,dx \\ &= -\frac{i}{2}\sum_{n=1}^{\infty} n\sin(\pi\alpha n)\Big(\frac{H_n(k)J_{\frac{2m-1}{2-\alpha}}(k)}{n+\frac{2m-1}{2-\alpha}} - k\frac{H_{n+1}(k)J_{\frac{2m-1}{2-\alpha}}(k) - H_n(k)J_{\frac{2m-1}{2-\alpha}+1}(k)}{n^2-(\frac{2m-1}{2-\alpha})^2}\Big)\frac{J_n(ks)}{s}, \end{aligned} \tag{3.124}$$

for all $0 < s \le 1$.

3.4.3 Series Representation of the Solution of Equation (3.99)

Suppose that g is a function representable by Taylor series on the intervals $[-1,0]$ and $[0,1]$. Suppose further that the function ρ satisfies equation (3.99). In this section we observe that ρ is representable by certain series of Bessel functions of non-integer order on the intervals $[-1,0]$ and $[0,1]$.

The Even Case

Suppose that $g \in L^2([-1,1])$ is an even function, and suppose that $\rho \in L^2([-1,1])$ satisfies equation (3.99). By Theorem 3.42, ρ is also even and

$$\frac{1}{2}\rho(s) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\rho(t)\,dt = g(s), \tag{3.125}$$

for all $0 < s \leq 1$.

Let $\lceil x \rceil$ denote the smallest integer n such that $n \geq x$, and let $\lfloor x \rfloor$ denote the largest integer n such that $n \leq x$, for all real x. Theorem 3.46 in this section shows that if g is representable by a Taylor series on $[0,1]$, then for any positive integer N there exist unique complex numbers $b_1, b_2, \ldots, b_N$ such that the function

$$\rho(t) = \sum_{n=1}^{\lceil N/2 \rceil} b_{2n-1} \frac{J_{\frac{2n-1}{\alpha}}(kt)}{t} + \sum_{i=1}^{\lfloor N/2 \rfloor} b_{2n} \frac{J_{\frac{2n}{2-\alpha}}(kt)}{t}, \tag{3.126}$$

where $0 < t \leq 1$, solves equation (3.125) to within an error $O(t^N)$.

The following lemma evaluates the left hand side of (3.125) when ρ has the form (3.126).

Lemma 3.45. *Suppose that N is a positive integer and that $b_1, b_2, \ldots, b_N$ are arbitrary complex numbers. Suppose further that $\rho\colon [0,1] \to \mathbb{C}$ is defined by*

$$\rho(t) = \sum_{n=1}^{\lceil N/2 \rceil} b_{2n-1} \frac{J_{\frac{2n-1}{\alpha}}(kt)}{t} + \sum_{i=1}^{\lfloor N/2 \rfloor} b_{2n} \frac{J_{\frac{2n}{2-\alpha}}(kt)}{t}, \tag{3.127}$$

for $0 < t \le 1$*, and that* $g\colon [0,1] \to \mathbb{C}$ *is defined by*

$$g(s) = \frac{1}{2}\rho(s) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\rho(t)\,dt, \tag{3.128}$$

for $0 < s \le 1$*. Suppose finally that* $A(\alpha)$ *is the* $N \times N$ *matrix defined by (3.95). Then there exists a sequence of complex numbers* $c_{N+1}, c_{N+2}, \dots$ *such that*

$$g(t) = \frac{1}{2}\sum_{n=1}^{N}\Big(\sum_{m=1}^{N} A_{n,m}(\alpha)b_m\Big)\frac{J_n(kt)}{t} + \sum_{n=N+1}^{\infty} c_n\frac{J_n(kt)}{t}, \tag{3.129}$$

for all $0 < t \le 1$*.*

Proof. Identity (3.129) follows immediately from Theorem 3.43. The observation that the right hand side of (3.129) is representable by a convergent Taylor series centered at $t = 0$, on the interval $[0,1]$, follows from Theorem 3.16.

□

The following theorem is the principal result of this section.

Theorem 3.46. *Suppose that* $g\colon [0,1] \to \mathbb{C}$ *is representable by a convergent Taylor series centered at zero, on the interval* $[0,1]$*. Suppose further that* N *is a positive integer. Then, for all but a finite number of* $0 < \alpha < 2$*, there exist complex numbers* $b_1, b_2, \dots, b_N$ *and a sequence of complex numbers* $d_N, d_{N+1}, \dots$ *such that if*

$$\rho(t) = \sum_{n=1}^{\lceil N/2\rceil} b_{2n-1}\frac{J_{\frac{2n-1}{\alpha}}(kt)}{t} + \sum_{i=1}^{\lfloor N/2\rfloor} b_{2n}\frac{J_{\frac{2n}{2-\alpha}}(kt)}{t}, \tag{3.130}$$

for all $0 < t \le 1$*, then*

$$\frac{1}{2}\rho(s) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\rho(t)\,dt = g(s) + \sum_{n=N}^{\infty} d_n t^n, \tag{3.131}$$

for all $0 < t \le 1$*.*

Proof. By Theorem 3.16, there exists a sequence of complex numbers $c_1, c_2, \ldots$ such that

$$g(t) = \sum_{n=1}^{\infty} c_n \frac{J_n(kt)}{t}, \tag{3.132}$$

for all $0 \le t \le 1$. By Theorem 3.41, the $N \times N$ matrix $A(\alpha)$ defined by (3.95) is nonsingular for all but a finite number of $0 \le \alpha \le 2$; whenever $A(\alpha)$ is nonsingular, there exist unique complex numbers $b_1, b_2, \ldots, b_N$ such that

$$\frac{1}{2} \sum_{m=1}^{N} A(\alpha)_{n,m} b_m = c_n, \tag{3.133}$$

for all $n = 1, 2, \ldots, N$. Suppose that ρ is defined by (3.130). By Lemma 3.45, there exists a sequence of complex numbers $\tilde{c}_{N+1}, \tilde{c}_{N+2}, \ldots$ such that

$$\begin{aligned} &\frac{1}{2}\rho(s) + \frac{i}{4} \int_0^1 \frac{H_1(k\sqrt{s^2 + t^2 - 2st\cos(\pi\alpha)}}{\sqrt{s^2 + t^2 - 2st\cos(\pi\alpha)}} kt \sin(\pi\alpha) \rho(t)\, dt \\ &= \frac{1}{2} \sum_{n=1}^{N} \Big(\sum_{m=1}^{N} A_{n,m}(\alpha) b_m \Big) \frac{J_n(ks)}{s} + \sum_{n=N+1}^{\infty} \tilde{c}_n \frac{J_n(ks)}{s}, \end{aligned} \tag{3.134}$$

for all $0 < s \le 1$. Combining (3.134) and (3.133), we observe that

$$\begin{aligned} &\frac{1}{2}\rho(s) + \frac{i}{4} \int_0^1 \frac{H_1(k\sqrt{s^2 + t^2 - 2st\cos(\pi\alpha)}}{\sqrt{s^2 + t^2 - 2st\cos(\pi\alpha)}} kt \sin(\pi\alpha) \rho(t)\, dt \\ &= \sum_{n=1}^{N} c_n \frac{J_n(ks)}{s} + \sum_{n=N+1}^{\infty} \tilde{c}_n \frac{J_n(ks)}{s} \\ &= g(s) + \sum_{n=N+1}^{\infty} (\tilde{c}_n - c_n) \frac{J_n(ks)}{s}, \end{aligned} \tag{3.135}$$

for all $0 < s \le 1$. By Theorem 3.16, there exists a sequence of complex numbers $d_N, d_{N+1}, \ldots$ such that

$$\frac{1}{2}\rho(s) + \frac{i}{4} \int_0^1 \frac{H_1(k\sqrt{s^2 + t^2 - 2st\cos(\pi\alpha)}}{\sqrt{s^2 + t^2 - 2st\cos(\pi\alpha)}} kt \sin(\pi\alpha) \rho(t)\, dt = g(s) + \sum_{n=N}^{\infty} d_n s^n, \tag{3.136}$$

for all $0 < s \leq 1$.

□

The Odd Case

Suppose that $g \in L^2([-1,1])$ is an odd function, and suppose that $\rho \in L^2([-1,1])$ satisfies equation (3.99). By Theorem 3.42, ρ is also odd and

$$\frac{1}{2}\rho(s) - \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\rho(t)\,dt = g(s), \qquad (3.137)$$

for all $0 < s \leq 1$.

Let $\lceil x \rceil$ denote the smallest integer n such that $n \geq x$, and let $\lfloor x \rfloor$ denote the largest integer n such that $n \leq x$, for all real x. Theorem 3.47 in this section shows that if g is representable by a convergent Taylor series on $[0,1]$, then for any positive integer N there exist unique complex numbers $b_1, b_2, \ldots, b_N$ such that the function

$$\rho(t) = \sum_{n=1}^{\lceil N/2 \rceil} b_{2n-1} \frac{J_{\frac{2n-1}{2-\alpha}}(kt)}{t} + \sum_{i=1}^{\lfloor N/2 \rfloor} b_{2n} \frac{J_{\frac{2n}{\alpha}}(kt)}{t}, \qquad (3.138)$$

where $0 < t \leq 1$, solves equation (3.137) to within an error $O(t^N)$.

The following theorem is the principal result of this section.

Theorem 3.47. *Suppose that* $g\colon [0,1] \to \mathbb{C}$ *is representable by a convergent Taylor series centered at zero, on the interval* $[0,1]$*. Suppose further that* N *is a positive integer. Then, for all but a finite number of* $0 < \alpha < 2$*, there exist complex numbers* $b_1, b_2, \ldots, b_N$ *and a sequence of complex numbers* $d_N, d_{N+1}, \ldots$ *such that if*

$$\rho(t) = \sum_{n=1}^{\lceil N/2 \rceil} b_{2n-1} \frac{J_{\frac{2n-1}{2-\alpha}}(kt)}{t} + \sum_{i=1}^{\lfloor N/2 \rfloor} b_{2n} \frac{J_{\frac{2n}{\alpha}}(kt)}{t}, \qquad (3.139)$$

for all $0 < t \leq 1$, *then*

$$\frac{1}{2}\rho(s) - \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} kt\sin(\pi\alpha)\rho(t)\,dt = g(s) + \sum_{n=N}^{\infty} d_n t^n, \tag{3.140}$$

for all $0 < t \leq 1$.

3.4.4 Summary of Results

We summarize the results of the preceding subsections 3.4.1, 3.4.2, 3.4.3 as follows. Suppose that the curve $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge defined by (3.31) with interior angle $\pi\alpha$, where $0 < \alpha < 2$ (see Figure 3.3). Suppose further that $g \in L^2([-1,1])$, and let $\rho \in L^2([-1,1])$ be the solution to the boundary integral equation

$$\frac{1}{2}\rho(s) + \int_{-1}^{1} \psi^1_{\gamma(s),\nu(s)}(\gamma(t))\rho(t)\,dt = g(s), \tag{3.141}$$

for all $s \in [-1,1]$.

If N is an arbitrary positive integer and ρ is defined by the formula

$$\rho(t) = \sum_{n=1}^{N} b_n(\mathrm{sgn}(t))^{n+1}\frac{J_{\frac{n}{\alpha}}(k|t|)}{|t|} + \sum_{n=1}^{N} c_n(\mathrm{sgn}(t))^{n}\frac{J_{\frac{n}{2-\alpha}}(k|t|)}{|t|}, \tag{3.142}$$

for all $-1 \leq t \leq 1$, where $b_1, b_2, \ldots, b_N$ and $c_1, c_2, \ldots, c_N$ are arbitrary complex numbers, then g is representable by convergent Taylor series centered at zero, on the intervals $[-1,0]$ and $[0,1]$ (see theorems 3.43 and 3.44).

Conversely, suppose that g representable by convergent Taylor series centered at zero, on the intervals $[-1,0]$ and $[0,1]$. Then, for each positive integer N, there exist complex numbers $b_1, b_2, \ldots, b_N$ and $c_1, c_2, \ldots, c_N$ such that (3.142) solves equation (3.141) to within an error $O(t^N)$. Moreover, this error is representable by convergent Taylor series centered at zero, on the intervals $[-1,0]$ and $[0,1]$ (see theorems 3.46 and 3.47).

Observation 3.48. Numerical experiments (see Section 3.6) suggest that, for a certain subclass of functions g, stronger versions of theorems 3.46 and 3.47 are true. Suppose that G solves the Helmholtz equation

$$\nabla^2 G(x) + k^2 G(x) = 0 \tag{3.143}$$

on a neighborhood of the closed unit disc in $\mathbb{R}^2$, and let

$$g(t) = \frac{\partial G}{\partial \nu(t)}(\gamma(t)), \tag{3.144}$$

for all $-1 \le t \le 1$, where $\nu(t)$ is the inward-pointing unit normal vector at $\gamma(t)$. We conjecture that there exist infinite sequences of complex numbers $b_1, b_2, \dots$ and $c_1, c_2, \dots$ such that

$$\rho(t) = \sum_{n=1}^{\infty} b_n (\mathrm{sgn}(t))^{n+1} \frac{J_{\frac{n}{\alpha}}(k|t|)}{|t|} + \sum_{n=1}^{\infty} c_n (\mathrm{sgn}(t))^{n} \frac{J_{\frac{n}{2-\alpha}}(k|t|)}{|t|}, \tag{3.145}$$

is well defined for all $-1 \le t \le 1$, and solves equation (3.141).

Observation 3.49. Numerical experiments (see Section 3.6) indicate that the solution to equation (3.141) is representable by a series of the form (3.142) for a more general class of curves γ. More specifically, suppose that $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge in $\mathbb{R}^2$ with smooth, curved sides, with a corner at 0 and interior angle $\pi\alpha$. Suppose further that all derivatives of γ, 2nd order and higher, are zero at the corner. Then the solution is representable by a series of the form (3.142).

3.5 Analysis of the Integral Equation: the Dirichlet Case

Suppose that the curve $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge defined by (3.31) with interior angle $\pi\alpha$, where $0 < \alpha < 2$ (see Figure 3.3). Let g be a function in $L^2([-1,1])$, and suppose that $\rho \in L^2([-1,1])$ solves the equation

$$\frac{1}{2}\rho(s) + \int_{-1}^{1} \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\rho(t)\,dt = g(s), \tag{3.146}$$

for all $s \in [-1,1]$. In this section, we analyze this boundary integral equation.

We will investigate the behavior of (3.146) for functions $\rho \in L^2([-1,1])$ of the forms

$$\rho(t) = J_\mu(k|t|), \tag{3.147}$$

$$\rho(t) = \mathrm{sgn}(t) J_\mu(k|t|), \tag{3.148}$$

where $\mu > \frac{1}{2}$ is a real number and

$$\mathrm{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0, \end{cases} \tag{3.149}$$

for all real x. If ρ has the forms (3.147) and (3.148), then for most values of μ the resulting function g is singular; in Section 3.5.2, we observe that for certain μ, the function g is smooth. In Section 3.5.3, we fix g and view (3.146) as an integral equation in ρ. We observe that, for sufficiently smooth functions g, the solution ρ is representable by a series of functions of the forms (3.147) and (3.148).

The proofs of the theorems in this section are essentially identical to the proofs of the corresponding theorems in Section 3.4, and are omitted.

3.5.1 Integral Equations Near a Corner

The following lemma uses a symmetry argument to reduce (3.146) from an integral equation on the interval $[-1,1]$ to two independent integral equations on the interval $[0,1]$.

Theorem 3.50. *Suppose that ρ is a function in $L^2([-1,1])$ and that $g \in L^2([-1,1])$ is given by (3.146). Suppose further that the functions $g_{\mathrm{even}}, \rho_{\mathrm{even}} \in L^2([-1,1])$ are defined via the formulas*

$$g_{\mathrm{even}}(s) = \frac{1}{2}(g(s) + g(-s)), \tag{3.150}$$

$$\rho_{\mathrm{even}}(s) = \frac{1}{2}(\rho(s) + \rho(-s)). \tag{3.151}$$

Then

$$g_{\text{even}}(s) = \frac{1}{2}\rho_{\text{even}}(s) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} ks\sin(\pi\alpha)\rho_{\text{even}}(t)\,dt, \tag{3.152}$$

for all $0 < s \le 1$.

Likewise, suppose that odd functions $g_{\text{odd}}, \rho_{\text{odd}} \in L^2([-1,1])$ *are defined via the formulas*

$$g_{\text{odd}}(s) = \frac{1}{2}(g(s) - g(-s)), \tag{3.153}$$

$$\rho_{\text{odd}}(s) = \frac{1}{2}(\rho(s) - \rho(-s)). \tag{3.154}$$

Then

$$g_{\text{odd}}(s) = \frac{1}{2}\rho_{\text{odd}}(s) - \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} ks\sin(\pi\alpha)\rho_{\text{odd}}(t)\,dt, \tag{3.155}$$

for all $0 < s \le 1$.

3.5.2 The Singularities in the Solution of Equation (3.146)

In this section we observe that for certain functions ρ, the function g defined by (3.146) is representable by convergent Taylor series on the intervals $[-1,0]$ and $[0,1]$.

The Even Case

Suppose that $\rho \in L^2([-1,1])$ is an even function, and suppose that $g \in L^2([-1,1])$ is defined by (3.146). By Theorem 3.50, g is also even and

$$g(s) = \frac{1}{2}\rho(s) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} ks\sin(\pi\alpha)\rho(t)\,dt, \tag{3.156}$$

for all $0 < s \le 1$. Suppose further that

$$\rho(t) = J_\mu(kt), \tag{3.157}$$

for all $0 < t \le 1$. The following theorem shows that for certain values of μ, the function g in (3.156) is representable by a convergent Taylor series on the interval $[0, 1]$.

Theorem 3.51. *Suppose that $0 < \alpha < 2$ is a real number and m is a positive integer. Then*

$$\frac{1}{2}J_0(ks) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} ks\sin(\pi\alpha)J_0(kt)\,dx = \frac{2-\alpha}{2}J_0(ks)$$
$$+ \frac{i}{2}\sum_{n=1}^{\infty} n\sin(\pi\alpha n)\Big(\frac{H_n(k)J_0(k)}{n} - k\frac{H_{n+1}(k)J_0(k) - H_n(k)J_1(k)}{n^2}\Big)J_n(ks), \tag{3.158}$$

and

$$\frac{1}{2}J_{\frac{2m-1}{\alpha}}(ks) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} ks\sin(\pi\alpha)J_{\frac{2m-1}{\alpha}}(kt)\,dx$$
$$= \frac{i}{2}\sum_{n=1}^{\infty} n\sin(\pi\alpha n)\Big(\frac{H_n(k)J_{\frac{2m-1}{\alpha}}(k)}{n+\frac{2m-1}{\alpha}} - k\frac{H_{n+1}(k)J_{\frac{2m-1}{\alpha}}(k) - H_n(k)J_{\frac{2m-1}{\alpha}+1}(k)}{n^2 - (\frac{2m-1}{\alpha})^2}\Big)J_n(ks), \tag{3.159}$$

and

$$\frac{1}{2}J_{\frac{2m}{2-\alpha}}(ks) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} ks\sin(\pi\alpha)J_{\frac{2m}{2-\alpha}}(kt)\,dx$$
$$= \frac{i}{2}\sum_{n=1}^{\infty} n\sin(\pi\alpha n)\Big(\frac{H_n(k)J_{\frac{2m}{2-\alpha}}(k)}{n+\frac{2m}{2-\alpha}} - k\frac{H_{n+1}(k)J_{\frac{2m}{2-\alpha}}(k) - H_n(k)J_{\frac{2m}{2-\alpha}+1}(k)}{n^2 - (\frac{2m}{2-\alpha})^2}\Big)J_n(ks), \tag{3.160}$$

for all $0 < s \le 1$.

The Odd Case

Suppose that $\rho \in L^2([-1,1])$ is an odd function, and suppose that $g \in L^2([-1,1])$ is defined by (3.146). By Theorem 3.50, g is also odd and

$$g(s) = \frac{1}{2}\rho(s) - \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} ks\sin(\pi\alpha)\rho(t)\,dt, \qquad (3.161)$$

for all $0 < s \le 1$. Suppose further that

$$\rho(t) = J_\mu(kt), \qquad (3.162)$$

for all $0 < t \le 1$. The following theorem shows that for certain values of μ, the function g in (3.161) is representable by a convergent Taylor series on the interval $[0,1]$.

Theorem 3.52. *Suppose that $0 < \alpha < 2$ is a real number and m is a positive integer. Then*

$$\frac{1}{2}J_0(ks) - \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} ks\sin(\pi\alpha)J_0(kt)\,dx = \frac{\alpha}{2}J_0(ks)$$
$$-\frac{i}{2}\sum_{n=1}^{\infty} n\sin(\pi\alpha n)\Big(\frac{H_n(k)J_0(k)}{n} - k\frac{H_{n+1}(k)J_0(k) - H_n(k)J_1(k)}{n^2}\Big)J_n(ks), \qquad (3.163)$$

and

$$\frac{1}{2}J_{\frac{2m}{\alpha}}(ks) - \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} ks\sin(\pi\alpha)J_{\frac{2m}{\alpha}}(kt)\,dx$$
$$= -\frac{i}{2}\sum_{n=1}^{\infty} n\sin(\pi\alpha n)\Big(\frac{H_n(k)J_{\frac{2m}{\alpha}}(k)}{n+\frac{2m}{\alpha}} - k\frac{H_{n+1}(k)J_{\frac{2m}{\alpha}}(k) - H_n(k)J_{\frac{2m}{\alpha}+1}(k)}{n^2 - (\frac{2m}{\alpha})^2}\Big)J_n(ks),$$
$$(3.164)$$

and

$$\frac{1}{2}J_{\frac{2m-1}{2-\alpha}}(ks) - \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} ks\sin(\pi\alpha) J_{\frac{2m-1}{2-\alpha}}(kt)\,dx$$
$$= -\frac{i}{2}\sum_{n=1}^{\infty} n\sin(\pi\alpha n)\Big(\frac{H_n(k)J_{\frac{2m-1}{2-\alpha}}(k)}{n+\frac{2m-1}{2-\alpha}} - k\frac{H_{n+1}(k)J_{\frac{2m-1}{2-\alpha}}(k) - H_n(k)J_{\frac{2m-1}{2-\alpha}+1}(k)}{n^2-(\frac{2m-1}{2-\alpha})^2}\Big)J_n(ks), \tag{3.165}$$

for all $0 < s \leq 1$.

3.5.3 Series Representation of the Solution of Equation (3.146)

Suppose that g is a function representable by Taylor series on the intervals $[-1,0]$ and $[0,1]$. Suppose further that the function ρ satisfies equation (3.146). In this section we observe that ρ is representable by certain series of Bessel functions of non-integer order on the intervals $[-1,0]$ and $[0,1]$.

The Even Case

Suppose that $g \in L^2([-1,1])$ is an even function, and suppose that $\rho \in L^2([-1,1])$ satisfies equation (3.146). By Theorem 3.50, ρ is also even and

$$\frac{1}{2}\rho(s) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)})}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} ks\sin(\pi\alpha)\rho(t)\,dt = g(s), \tag{3.166}$$

for all $0 < s \leq 1$.

Let $\lceil x\rceil$ denote the smallest integer n such that $n \geq x$, and let $\lfloor x\rfloor$ denote the largest integer n such that $n \leq x$, for all real x. Theorem 3.53 in this section shows that if g is representable by a convergent Taylor series on $[0,1]$, then for any positive integer N there exist unique complex numbers $b_0, b_1, \ldots, b_N$ such that the function

$$\rho(t) = \sum_{n=1}^{\lceil N/2\rceil} b_{2n-1}J_{\frac{2n-1}{\alpha}}(kt) + \sum_{i=0}^{\lfloor N/2\rfloor} b_{2n}J_{\frac{2n}{2-\alpha}}(kt), \tag{3.167}$$

where $0 < t \leq 1$, solves equation (3.166) to within an error $O(t^{N+1})$.

The following theorem is the principal result of this section.

Theorem 3.53. *Suppose that $g\colon [0,1] \to \mathbb{C}$ is representable by a convergent Taylor series centered at zero, on the interval $[0,1]$. Suppose further that N is a positive integer. Then, for all but a finite number of $0 < \alpha < 2$, there exist complex numbers $b_0, b_1, \dots, b_N$ and a sequence of complex numbers $d_{N+1}, d_{N+2}, \dots$ such that if*

$$\rho(t) = \sum_{n=1}^{\lceil N/2 \rceil} b_{2n-1} J_{\frac{2n-1}{\alpha}}(kt) + \sum_{i=0}^{\lfloor N/2 \rfloor} b_{2n} J_{\frac{2n}{2-\alpha}}(kt), \tag{3.168}$$

for all $0 < t \leq 1$, then

$$\frac{1}{2}\rho(s) + \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} ks\sin(\pi\alpha)\rho(t)\,dt = g(s) + \sum_{n=N+1}^{\infty} d_n t^n, \tag{3.169}$$

for all $0 < t \leq 1$.

The Odd Case

Suppose that $g \in L^2([-1,1])$ is an odd function, and suppose that $\rho \in L^2([-1,1])$ satisfies equation (3.146). By Theorem 3.50, ρ is also odd and

$$\frac{1}{2}\rho(s) - \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} ks\sin(\pi\alpha)\rho(t)\,dt = g(s), \tag{3.170}$$

for all $0 < s \leq 1$.

Let $\lceil x \rceil$ denote the smallest integer n such that $n \geq x$, and let $\lfloor x \rfloor$ denote the largest integer n such that $n \leq x$, for all real x. Theorem 3.54 in this section shows that if g is representable by a convergent Taylor series on $[0,1]$, then for any positive integer N there exist unique complex numbers $b_0, b_1, \dots, b_N$ such that the function

$$\rho(t) = \sum_{n=1}^{\lceil N/2 \rceil} b_{2n-1} J_{\frac{2n-1}{2-\alpha}}(kt) + \sum_{i=0}^{\lfloor N/2 \rfloor} b_{2n} J_{\frac{2n}{\alpha}}(kt), \tag{3.171}$$

where $0 < t \leq 1$, solves equation (3.170) to within an error $O(t^{N+1})$.

The following theorem is the principal result of this section.

Theorem 3.54. *Suppose that $g\colon [0,1] \to \mathbb{C}$ is representable by a convergent Taylor series centered at zero, on the interval $[0,1]$. Suppose further that N is a positive integer. Then, for all but a finite number of $0 < \alpha < 2$, there exist complex numbers $b_0, b_1, \dots, b_N$ and a sequence of complex numbers $d_{N+1}, d_{N+2}, \dots$ such that if*

$$\rho(t) = \sum_{n=1}^{\lceil N/2 \rceil} b_{2n-1} J_{\frac{2n-1}{2-\alpha}}(kt) + \sum_{i=0}^{\lfloor N/2 \rfloor} b_{2n} J_{\frac{2n}{\alpha}}(kt), \tag{3.172}$$

for all $0 < t \le 1$, then

$$\frac{1}{2}\rho(s) - \frac{i}{4}\int_0^1 \frac{H_1(k\sqrt{s^2+t^2-2st\cos(\pi\alpha)}}{\sqrt{s^2+t^2-2st\cos(\pi\alpha)}} ks\sin(\pi\alpha)\rho(t)\,dt = g(s) + \sum_{n=N+1}^{\infty} d_n t^n, \tag{3.173}$$

for all $0 < t \le 1$.

3.5.4 Summary of Results

We summarize the results of the preceding subsections 3.5.1, 3.5.2, 3.5.3 as follows. Suppose that the curve $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge defined by (3.31) with interior angle $\pi\alpha$, where $0 < \alpha < 2$ (see Figure 3.3). Suppose further that $g \in L^2([-1,1])$, and let $\rho \in L^2([-1,1])$ be the solution to the boundary integral equation

$$\frac{1}{2}\rho(s) + \int_{-1}^{1} \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\rho(t)\,dt = g(s), \tag{3.174}$$

for all $s \in [-1,1]$.

If N is an arbitrary positive integer and ρ is defined by the formula

$$\rho(t) = \sum_{n=0}^{N} b_n(\mathrm{sgn}(t))^{n+1} J_{\frac{n}{\alpha}}(k|t|) + \sum_{n=0}^{N} c_n(\mathrm{sgn}(t))^{n} J_{\frac{n}{2-\alpha}}(k|t|), \tag{3.175}$$

for all $-1 \le t \le 1$, where $b_0, b_1, \dots, b_N$ and $c_0, c_1, \dots, c_N$ are arbitrary complex numbers, then g is representable by convergent Taylor series centered at zero, on the intervals

$[-1,0]$ and $[0,1]$ (see theorems 3.51 and 3.52).

Conversely, suppose that g representable by convergent Taylor series centered at zero, on the intervals $[-1,0]$ and $[0,1]$. Then, for each positive integer N, there exist complex numbers $b_0, b_1, \ldots, b_N$ and $c_0, c_1, \ldots, c_N$ such that (3.175) solves equation (3.174) to within an error $O(t^{N+1})$. Moreover, this error is representable by convergent Taylor series centered at zero, on the intervals $[-1,0]$ and $[0,1]$ (see theorems 3.53 and 3.54).

Observation 3.55. Numerical experiments (see Section 3.6) suggest that, for a certain subclass of functions g, stronger versions of theorems 3.53 and 3.54 are true. Suppose that G solves the Helmholtz equation

$$\nabla^2 G(x) + k^2 G(x) = 0 \tag{3.176}$$

on a neighborhood of the closed unit disc in $\mathbb{R}^2$, and let

$$g(t) = \frac{\partial G}{\partial \nu(t)}(\gamma(t)), \tag{3.177}$$

for all $-1 \le t \le 1$, where $\nu(t)$ is the inward-pointing unit normal vector at $\gamma(t)$. We conjecture that there exist infinite sequences of complex numbers $b_0, b_1, \ldots$ and $c_0, c_1, \ldots$ such that

$$\rho(t) = \sum_{n=0}^{\infty} b_n (\mathrm{sgn}(t))^{n+1} J_{\frac{n}{\alpha}}(k|t|) + \sum_{n=0}^{\infty} c_n (\mathrm{sgn}(t))^n J_{\frac{n}{2-\alpha}}(k|t|), \tag{3.178}$$

is well defined for all $-1 \le t \le 1$, and solves equation (3.174).

Observation 3.56. Numerical experiments (see Section 3.6) indicate that the solution to equation (3.174) is representable by a series of the form (3.175) for a more general class of curves γ. More specifically, suppose that $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a wedge in $\mathbb{R}^2$ with smooth, curved sides, with a corner at 0 and interior angle $\pi\alpha$. Suppose further that all derivatives of γ, 2nd order and higher, are zero at the corner. Then the solution is representable by a series of the form (3.175).

3.6 The Algorithm

To solve the integral equations of potential theory on polygonal domains, we use a modification of the algorithm described in [4]; instead of discretizing the corner singularities using nested quadratures, we use the representations (3.142), (3.175) to construct purpose-made discretizations (see, for example, [23], [21], [31]). A detailed description of this part of the procedure will be published at a later date. The resulting linear systems were solved directly using standard techniques.

We illustrate the performance of the algorithm with several numerical examples. The exterior and interior Neumann problems and the exterior and interior Dirichlet problems were solved on each of the domains in figures 3.4, 3.5, 3.6, 3.7, where the boundary data were generated by charges *inside* the regions for the exterior problems and *outside* the regions for the interior problems. The numerical solution was tested by comparing the computed potential to the true potential at several arbitrary points; Table 3.1 presents the results.

We also solved the interior Dirichlet problem on the domains in figures 3.4, 3.5, where the boundary data were generated by charges *inside* the regions. We computed the numerical solution using both our algorithm and a naive algorithm which used nested quadratures near the corners. The solution produced by our algorithm was then tested by comparing the computed potentials at several arbitrary points; Table 3.2 presents the results.

The following abbreviations are used in tables 3.1 and 3.2 (see Section 3.2.1):

INP Interior Neumann problem

ENP Exterior Neumann problem

IDP Interior Dirichlet problem

EDP Exterior Dirichlet problem

Observation 3.57. It is easy to observe from the values of k and figures 3.4, 3.5, 3.6, and 3.7 that the regions are between approximately 1 and 15 wavelengths in size. A discussion of the use of these techniques on large-scale problems will be published at a later date. In the examples in tables 3.1 and 3.2, k has been chosen so that no resonances, real or spurious, were encountered.

Observation 3.58. We observe that if the boundary values are produced by charges *inside* the regions for the exterior problems, or *outside* the regions for the interior problems, then certain terms in the representations of the solutions near the corners vanish. More specifically, in the exterior Neumann case, the terms $c_1, c_2, \ldots$ in (3.142) vanish. In the interior Dirichlet case, the terms $b_0, b_1, \ldots$ in (3.175) vanish.

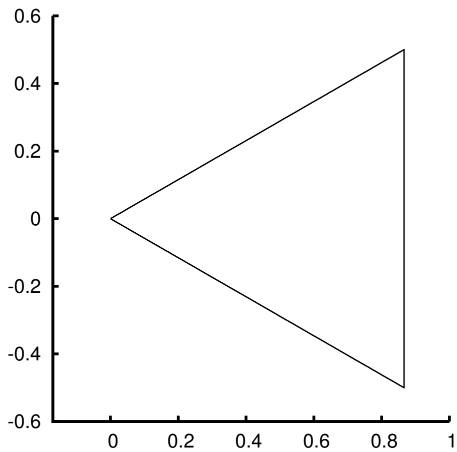

Figure 3.4: Γ_3: An equilateral triangle in $\mathbb{R}^2$

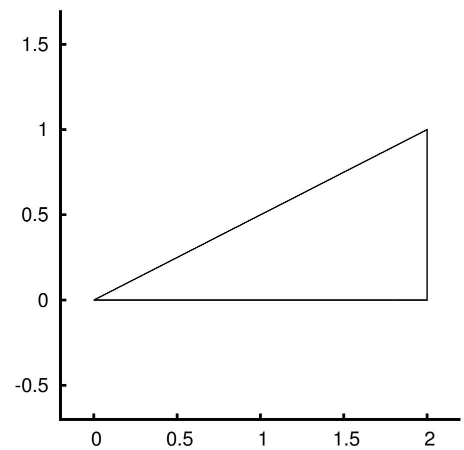

Figure 3.5: Γ_4: A right triangle in $\mathbb{R}^2$

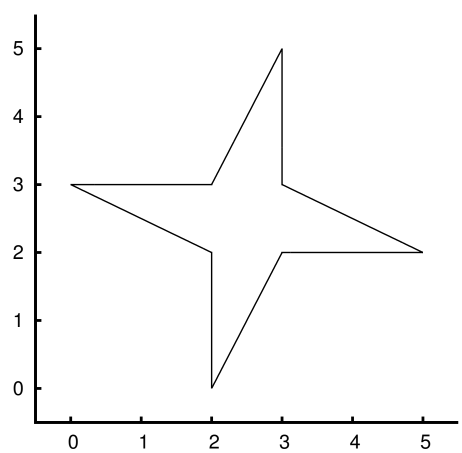

Figure 3.6: Γ_5: A star-shaped polygon in $\mathbb{R}^2$

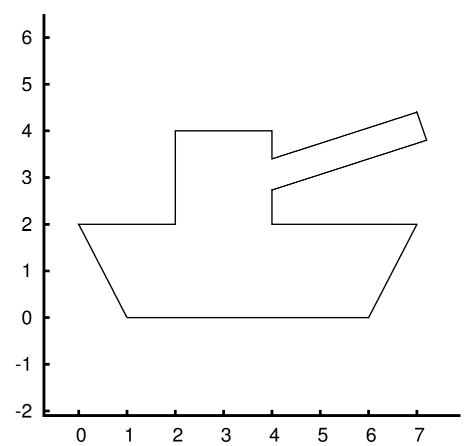

Figure 3.7: Γ_6: A tank-shaped polygon in $\mathbb{R}^2$

	Boundary curve	k	Number of nodes	Running time	Largest absolute error	Condition number
INP	Γ_3	17.0	300	0.21332E+00	0.10637E-12	0.26840E+01
	Γ_4	20.0	380	0.51532E+00	0.40858E-14	0.58822E+01
	Γ_5	5.0	888	0.44069E+01	0.77807E-12	0.58678E+01
	Γ_5	15.0	968	0.54840E+01	0.72651E-12	0.58642E+01
	Γ_6	5.0	1103	0.82281E+01	0.19629E-12	0.21551E+01
	Γ_6	15.0	1233	0.10702E+02	0.85866E-13	0.36966E+01
ENP	Γ_3	17.0	330	0.26114E+00	0.39361E-13	0.26858E+01
	Γ_4	20.0	380	0.47460E+00	0.24299E-13	0.58866E+01
	Γ_5	5.0	768	0.28891E+01	0.21976E-13	0.58642E+01
	Γ_5	15.0	848	0.36247E+01	0.15480E-13	0.58641E+01
	Γ_6	5.0	1233	0.11114E+02	0.26442E-13	0.21544E+01
	Γ_6	15.0	1233	0.10633E+02	0.63324E-12	0.27749E+01
IDP	Γ_3	14.0	252	0.13164E+00	0.11935E-14	0.14267E+02
	Γ_4	20.0	362	0.32332E+00	0.17205E-13	0.27116E+02
	Γ_5	5.0	720	0.22676E+01	0.60295E-13	0.27116E+02
	Γ_5	15.0	800	0.32559E+01	0.40323E-13	0.26809E+02
	Γ_6	5.0	1031	0.63333E+01	0.39089E-12	0.54196E+01
	Γ_6	15.0	1161	0.95445E+01	0.16712E-12	0.89957E+01
EDP	Γ_3	14.0	252	0.12741E+00	0.10749E-12	0.14499E+02
	Γ_4	20.0	362	0.37710E+00	0.59616E-13	0.29318E+02
	Γ_5	5.0	720	0.22715E+01	0.43344E-12	0.52029E+02
	Γ_5	15.0	800	0.32999E+01	0.14326E-12	0.38847E+02
	Γ_6	5.0	1031	0.63988E+01	0.26817E-13	0.73436E+01
	Γ_6	15.0	1161	0.97222E+01	0.12680E-12	0.65323E+01

Table 3.1: Numerical results for the Helmholtz Neumann and Helmholtz Dirichlet problems

	Boundary curve	k	Number of nodes	Running time	Largest absolute error	Condition number
IDP	Γ_3	14.0	252	0.18070E+00	0.17986E-13	0.14267E+02
	Γ_4	20.0	362	0.32343E+00	0.76927E-12	0.27116E+02

Table 3.2: Numerical results for the Helmholtz interior Dirichlet problem with the charges *inside* the regions

Chapter 4

Extensions and Generalizations

4.1 Other Integral Equations

The apparatus of this dissertation generalizes to other boundary integral equations, such as the combined-potential integral equation (see, for example, [8]) and related situations. This line of investigation is being vigorously pursued.

4.2 Curved Boundaries with Corners

While this dissertation only deals with the solution of Laplace's equation and the Helmholtz equation on domains with polygonal boundaries, a similar analysis applies to the case of *curved* boundaries with corners. More specifically, if the boundary is smooth except at corners, the solutions to the associated boundary integral equations of classical potential theory are also representable by series of elementary functions. This analysis, along with the requisite numerical apparatus, will be described in a forthcoming paper.

4.3 Generalization to Three Dimensions

The generalization of the apparatus of this dissertation to three dimensions is fairly straightforward, but the detailed analysis has not been carried out. This line of research is being vigorously pursued.

4.4 Robin and Mixed Boundary Conditions

This dissertation deals with the solution of Laplace's equation and the Helmholtz equation on polygonal domains with either Dirichlet or Neumann boundary conditions. There are two additional boundary conditions that have not yet been analyzed in detail: the Robin condition, which specifies a linear combination of the values of the solution and the values of its derivative on the boundary; and the mixed boundary condition, which specifies Dirichlet boundary conditions on some sides of the polygon and Neumann boundary conditions on others. The results of our pending investigation will be reported at a later date.

Chapter 5

Appendix A

In this section we provide a proof of Theorem 2.20, which is restated here as Theorem 5.4.

The following lemma provides the value of a certain contour integral.

Lemma 5.1. *Suppose that* $-1 < \mu < 1$ *and* $0 < \alpha < 1$ *are real numbers. Then*

$$\int_{-\infty}^{\infty} \frac{z^{\mu}}{1 - 2z\cos(\pi\alpha) + z^2}\,dz = -\pi \frac{e^{i\pi\alpha\mu}}{\sin(\pi\alpha)}. \tag{5.1}$$

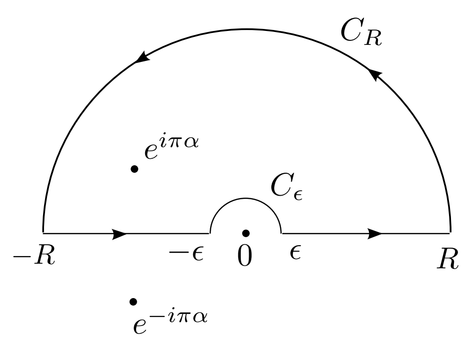

Figure 5.1: A contour in $\mathbb{C}$

Proof. Let ϵ and R be real numbers such that $0 < \epsilon < 1$ and $1 < R$. Suppose that C is a contour consisting of the intervals $[-R, -\epsilon]$ and $[\epsilon, R]$, together with the arcs C_R, defined by the formula $Re^{i\theta}$ for all $0 \le \theta \le \pi$, and C_ϵ defined by the formula $\epsilon e^{i\theta}$ for all

$0 \leq \theta \leq \pi$ (see figure 5.1). We observe that

$$\frac{z^\mu}{1 - 2z\cos(\pi\alpha) + z^2} = \frac{z^\mu}{(e^{i\pi\alpha} - z)(e^{-i\pi\alpha} - z)} \tag{5.2}$$

is a meromorphic function of z inside C, with a simple pole inside C at $e^{i\pi\alpha}$. Since $e^{i\pi\alpha}$ is within the contour, by the residue theorem

$$\oint_C \frac{z^\mu}{1 - 2z\cos(\pi\alpha) + z^2}\, dz = \oint_C \frac{z^\mu}{(e^{i\pi\alpha} - z)(e^{-i\pi\alpha} - z)}\, dz = 2\pi i \frac{(e^{i\pi\alpha})^\mu}{e^{-i\pi\alpha} - e^{i\pi\alpha}} = -\pi \frac{e^{i\pi\alpha\mu}}{\sin(\pi\alpha)}. \tag{5.3}$$

We observe that if $z \in C_R$, then $|z| = R$ and

$$\left|\frac{z^\mu}{1 - 2z\cos(\pi\alpha) + z^2}\right| = \left|\frac{z^\mu}{(e^{i\pi\alpha} - z)(e^{-i\pi\alpha} - z)}\right| \leq \frac{R^\mu}{(R-1)^2}. \tag{5.4}$$

Consequently,

$$\begin{aligned}
&\left|\lim_{R\to\infty} \int_{C_R} \frac{z^\mu}{1 - 2z\cos(\pi\alpha) + z^2}\, dz\right| \leq \lim_{R\to\infty} \int_{C_R} \left|\frac{z^\mu}{1 - 2z\cos(\pi\alpha) + z^2}\right| dz \\
&\leq \lim_{R\to\infty} \int_{C_R} \frac{R^\mu}{(R-1)^2}\, dz \\
&= \lim_{R\to\infty} \pi \frac{R^{\mu+1}}{(R-1)^2} \\
&= 0.
\end{aligned} \tag{5.5}$$

Likewise, we observe that if $z \in C_\epsilon$, then $|z| = \epsilon$ and

$$\left|\frac{z^\mu}{1 - 2z\cos(\pi\alpha) + z^2}\right| = \left|\frac{z^\mu}{(e^{i\pi\alpha} - z)(e^{-i\pi\alpha} - z)}\right| \leq \frac{\epsilon^\mu}{(1-\epsilon)^2}. \tag{5.6}$$

Consequently,

$$\left|\lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{z^\mu}{1-2z\cos(\pi\alpha)+z^2}\,dz\right| \le \lim_{\epsilon\to 0}\int_{C_\epsilon}\left|\frac{z^\mu}{1-2z\cos(\pi\alpha)+z^2}\right|dz$$
$$\le \lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{\epsilon^\mu}{(1-\epsilon)^2}\,dz$$
$$= \lim_{\epsilon\to 0}\pi\frac{\epsilon^{\mu+1}}{(1-\epsilon)^2}$$
$$= 0. \tag{5.7}$$

Therefore,

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\oint_C\frac{z^\mu}{1-2z\cos(\pi\alpha)+z^2}\,dz = \int_{-\infty}^{\infty}\frac{z^\mu}{1-2z\cos(\pi\alpha)+z^2}\,dz. \tag{5.8}$$

Combining formulas (5.3) and (5.8), we observe that

$$\int_{-\infty}^{\infty}\frac{z^\mu}{1-2z\cos(\pi\alpha)+z^2}\,dz = -\pi\frac{e^{i\pi\alpha\mu}}{\sin(\pi\alpha)}. \tag{5.9}$$

□

The following corollary evaluates the integral in formula (5.1) when it is taken from 0 to ∞ instead of from $-\infty$ to ∞.

Corollary 5.2. *Suppose that $-1<\mu<1$ and $0<\alpha<1$ are real numbers. Then,*

$$\int_0^{\infty}\frac{x^\mu\sin(\pi\alpha)}{1-2x\cos(\pi\alpha)+x^2}\,dx = \pi\frac{\sin(\mu\pi(1-\alpha))}{\sin(\mu\pi)}. \tag{5.10}$$

Proof. Lemma 5.1 states that

$$-\pi\frac{e^{i\pi\alpha\mu}}{\sin(\pi\alpha)} = \int_{-\infty}^{\infty}\frac{z^\mu}{1-2z\cos(\pi\alpha)+z^2}\,dz, \tag{5.11}$$

which we rewrite in the form

$$
\begin{aligned}
-\pi\frac{e^{i\pi\alpha\mu}}{\sin(\pi\alpha)} &= \int_0^\infty \frac{z^\mu}{1-2z\cos(\pi\alpha)+z^2}\,dz + \int_{-\infty}^0 \frac{z^\mu}{1-2z\cos(\pi\alpha)+z^2}\,dz \\
&= \int_0^\infty \frac{z^\mu}{1-2z\cos(\pi\alpha)+z^2}\,dz + e^{i\pi\mu}\int_0^\infty \frac{z^\mu}{1+2z\cos(\pi\alpha)+z^2}\,dz. \quad (5.12)
\end{aligned}
$$

Replacing α with $1-\alpha$, we also observe that

$$
-\pi\frac{e^{i\pi(1-\alpha)\mu}}{\sin(\pi\alpha)} = \int_0^\infty \frac{z^\mu}{1+2z\cos(\pi\alpha)+z^2}\,dz + e^{i\pi\mu}\int_0^\infty \frac{z^\mu}{1-2z\cos(\pi\alpha)+z^2}\,dz. \tag{5.13}
$$

Multiplying formula (5.12) by $-e^{-i\pi\mu}$ and adding it to formula (5.13), we observe that

$$
\pi\frac{e^{i\pi(\alpha-1)\mu}-e^{i\pi(1-\alpha)\mu}}{\sin(\pi\alpha)} = (e^{i\pi\mu}-e^{-i\pi\mu})\int_0^\infty \frac{z^\mu}{1-2z\cos(\pi\alpha)+z^2}\,dz. \tag{5.14}
$$

Therefore,

$$
\pi\frac{\sin(\mu\pi(1-\alpha))}{\sin(\mu\pi)} = \int_0^\infty \frac{z^\mu\sin(\pi\alpha)}{1-2z\cos(\pi\alpha)+z^2}\,dz. \tag{5.15}
$$

□

A simple analytic continuation argument shows that identity (5.10) in corollary 5.2 is also true for all real $0<\alpha<2$. This observation is summarized by the following theorem.

Theorem 5.3. *Suppose that $-1<\mu<1$ and $0<\alpha<2$ are real numbers. Then,*

$$
\int_0^\infty \frac{x^\mu\sin(\pi\alpha)}{1-2x\cos(\pi\alpha)+x^2}\,dx = \pi\frac{\sin(\mu\pi(1-\alpha))}{\sin(\mu\pi)}. \tag{5.16}
$$

Proof. We observe that the right and left hand sides of identity (5.10) are both analytic functions of α, for all complex α such that $0<\mathrm{Re}(\alpha)<2$. Therefore, by analytic continuation (Theorem 2.17), it follows that identity (5.10) holds for all complex α such that $0 < \mathrm{Re}(\alpha) < 2$. □

The following theorem is the pricipal result of this section. It follows from a simple change of variables in (5.16).

Theorem 5.4. *Suppose that $-1 < \mu < 1$ and $0 < \alpha < 2$ are real numbers. Then*

$$\int_0^\infty \frac{x^\mu \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\,dx = \pi a^{\mu-1}\frac{\sin\bigl(\mu\pi(1-\alpha)\bigr)}{\sin(\mu\pi)}, \tag{5.17}$$

for all $a > 0$.

Proof. Let $a > 0$. We observe that

$$\int_0^\infty \frac{x^\mu \sin(\pi\alpha)}{1 - 2x\cos(\pi\alpha) + x^2}\,dx = \int_0^\infty \frac{\bigl(\frac{x}{a}\bigr)^\mu \sin(\pi\alpha)}{1 - 2\bigl(\frac{x}{a}\bigr)\cos(\pi\alpha) + \bigl(\frac{x}{a}\bigr)^2}\cdot\frac{dx}{a}$$

$$= a^{1-\mu}\int_0^\infty \frac{x^\mu \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\,dx. \tag{5.18}$$

Therefore, by Theorem 5.3,

$$\int_0^\infty \frac{x^\mu \sin(\pi\alpha)}{a^2 - 2ax\cos(\pi\alpha) + x^2}\,dx = \pi a^{\mu-1}\frac{\sin\bigl(\mu\pi(1-\alpha)\bigr)}{\sin(\mu\pi)}. \tag{5.19}$$

□

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