Analysis of a Multilevel Iterative Method for Nonlinear Finite Element Equations

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Abstract

The multilevel iterative technique is a powerful technique for solving the systems of equations associated with discretized partial differential equations. We describe how this technique can be combined with a globally convergent approximate Newton method to solve nonlinear PDEs. We show that asymptotically only one Newton iteration per level is required; thus the complexity for linear and nonlinear problems is essentially equal.

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1. Introduction

In this discussion we present an extension of a multilevel iterative method for linear elliptic equations to nonlinear boundary value problems. In particular, we show how to use an approximate-Newton multilevel scheme to solve the discrete nonlinear systems of equations which arise form a standard weak formulation of the nonlinear partial differential equation.

The framework of our analysis combines the multilevel iterative methods for linear finite element equations discussed in Bank and Dupont [2] and Bank [3] with the global approximate Newton setting of Bank and Rose [4], [5]. Under appropriate conditions of elliptic regularity, we show that both the continuous and discrete solutions exist and that our scheme converges to an approximation within the discretization error of the continuous problem in time (and also space) proportional to the largest discrete problem. That is, we can compute in time $O(N_j)$ an approximation which is $O(N_j^{-q})$ accurate, where q is the appropriate exponent for the N_j-dimensional finite element spaces M_j.

In Section 2, we set up the weak (variational) form of the nonlinear boundary value problem. Using this formulation, we then specify, in Section 3, our regularity assumptions on the smoothness of the nonlinear operator. These assumptions are motivated by the generalized Lax-Milgram analysis presented by Babuska and Aziz in [1] and our previous analysis in [5]. Our main result here is that, asymptotically, we need compute only one approximate Newton iteration per level (refinement), provided that the approximate and exact Newton steps agree to some tolerance which is independent of the level. This implies that the total cost of solving a nonlinear problem of size N_j is bounded by $C \cdot F(N_j)$, where $F(N_j)$ is the cost of solving a linear problem of size N_j and $C \cong 1$. $F(N_j) = O(N_j)$ for the linear multigrid methods described in [2], [3].

In Section 4, we consider the case where the linear approximate-Newton equations are solved by the j-level scheme of [2], [3], and complete the analysis for the time bound cited above. We illustrate our analysis with an example boubdary value problem of the form

$$L(u) = 0 \quad \text{in } \Omega \quad R^2,$$

$$\partial u/\partial n = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where

$$L(u) = \nabla \cdot a \nabla u + f(x,u,\nabla u) \ . \tag{1.2}$$

Our approach for extending multilevel methodology to nonlinear operators using an approximate-Newton iterative scheme differs in several respects from other approaches recently reported or under investigation. We discuss briefly the relation of our scheme to those of Brandt and McCormick [8], Hackbusch [10], and Mansfield [11].

A common thread in our approach and those of [8], [10], is the consideration of a sequence of discrete nonlinear problems, say, $L_j(u_j^*) = 0$, where the u_j^* are successively more accurate approximations of the solutions of the nonlinear operator $L(u) = 0$. As a consequence, the representation of u_j^* in the space containing u_{j+1}^* is such that $L_{j+1}(u_j^*)$ is relatively small. This motivates the choice of taking $u_j^{s_j}$, for some iteration index s_j, as the initial guess in an iterative method to solve

$L_{j+1}(u^*_{j+1}) = 0$. The integer s_j is chosen such that the error $\|u^*_j - u_j^{s_j}\|$ is accurate to within the discretization error. Thus $L_{j+1}(u_j^{s_j})$ will also be relatively small, and consequently the iterative method should require $s_j \leq s$ steps (independent of j) for each mesh level j.

Usually the iterative method selected to compute the u^k_j, $1 \leq k \leq s_j$, is a subtle and recursively winds its way through a sequence of coarser mesh levels; the details need not concern us here. However, each choice of such an iterative method leads to a different 'j-level' strategy. The j-level strategy can be based on a nonlinear iteration, such as the nonlinear Gauss-Seidel method advocated in [8], or on a nonlinear Picard type iteration used in [10]. These schemes make no use of Jacobian information.

In contrast, we use a j-level strategy based on a <u>linear</u> iteration after choosing a linear system to represent the Jacobian. Since asymptotically $s_j = 1$ for this procedure, this strategy will usually require substantially fewer function evaluations of the L_j. On the other hand, for problems where the Jacobian is difficult to compute, our method becomes less attractive.

The recent paper by Mansfield [11] takes a different approach. In order to solve $L_j(u^*_j) = 0$, for some fixed mesh index j, she considers a one parameter embedding $h_j(v,\lambda) = 0$, $0 \leq \lambda \leq 1$, such that $h_j(0,0) = 0$, and $h_j(u^*_j,1) = L_j(u^*_j) = 0$. The solution is continued from $v = 0$ to $v = v^i_j$ by solving $h^i_j(v_i,\lambda_i) = 0$, where $0 = \lambda_1 < \lambda_2 \ldots < \lambda_m = 1$. The λ_i are chosen such that v_i can be computed by Newton's method using v_{i-1} as the initial iterate. Mansfield proves that the error $\|u^*_j - u\|$, where $L(u) = 0$, is accurate to the discretization order, and the number of continuation steps, m, is independent of the mesh. Furthermore, by showing that the number of Newton steps, s_i to obtain the computed v_j^i satisfies $s_i \leq s$ independent of the mesh, and by using a linear j-level iterative scheme for the Newton equations, she obtains an $O(N_j)$ time bound. Assuming that the computed approximation to the u_j^* are accurate to the discretization error, this result is analagous to our theorem in section 4. Note that this method may require m·s linear systems be solved on the finest mesh. Our results would suggest an alternative in which one continues from $\lambda = 0$ to $\lambda = 1$ on the coarsest mesh only, thereby obtaining $u_1^{s_1}$. One then refines the mesh for $\lambda = 1$, and obtains the sequence $u_j^{s_j}$ on the finer meshes. This would asympotically require only one linear system be solved on the finest mesh.

Multilevel iteration is a general, powerful technique for solving nonlinear operator equations which can be approximated by an orderly sequence of discrete nonlinear systems. The linear multigrid schemes of Brandt [7], Hackbusch [9], Nicolaides [12] and possibly others could be adapted in a similar manner to the one proposed here and would yield methods with similar properties. We have found our particular procedure to be effective on a variety of nonlinear PDE's; the implementation was a reasonaby straightfoward extension of the one described in [6] for linear problems.

2. Preliminaries

To introduce ideas, we consider a weak form of the example nonlinear elliptic boundary value problem (1.1)-(1.2): find $u \in H^1(\Omega)$ such that

$$a(u,v) = 0 \qquad \text{for all } v \in H^1(\Omega)$$

$$a(u,v) = \int_\Omega a\nabla u\cdot\nabla v + f(x,u,\nabla u)v\,dx \quad . \tag{2.1}$$

Here $H^1(\Omega)$ denotes the usual Sobelev space equipped with the norm

$$\|u\|_1^2 = (u,u)_1$$

$$(u,v)_1 = \int_\Omega \nabla u\nabla v + uv\,dx. \tag{2.2}$$

We will defer our discussion of nonlinear elliptic problems such as (2.1) until Section 4. In this section and the the next, we prefer to deal with a more abstract problem for which (2.1) is a special case.

Let g be a mapping of a Hilbert space H onto itself. Equip H with an inner product (u,v) and norm $\|u\|^2 = (u,u)$. We consider the following problem: find $u^* \in H$ such that

$$(g(u^*),v) = 0 \qquad \text{for all } v \in H\ . \tag{2.3}$$

In the example above, g is defined implicitly via the Reise representation theorem, $H = H^1(\Omega)$, and the norm and inner product are given by (2.2).

We shall (formally) apply an approximate Newton method to (2.3). Starting from some initial guess $u^0 \in H$, we compute a sequence of iterates $u^k \in H$, k=1,2,3..., as follows: find $x^k \in H$ such that

$$(M^k x^k,v) = -\,(g(u^k),v) \qquad \text{for all } v \in H, \tag{2.4}$$

where M^k is a linear mapping from H to H, approximating, in some sense, the derivative $g'(u^k)$. Then we set

$$u^{k+1} = u^k + t^k x^k, \tag{2.5}$$

where t^k ε (0,1] is a scalar damping parameter. Setting $M^k = g'(u^k)$ and $t^k = 1$ corresponds to Newton's method.

Generally, a procedure such as (2.4)-(2.5) is intractable computationally since H may be infinite dimensional. Thus we seek to discretize (2.3)-(2.5). Let $\{M_j\}$ be an indexed family of finite dimensional subspaces dense in H, nested in the sense that M_j M_k for k > j. Let N_j denote the dimension of M_j. We assume the dimensions of the spaces increase geometrically,

$$N_j = \beta\, N_{j-1},$$

$$\beta > 1 \quad , \tag{2.6}$$

since this will be the typical situation arising in practice. The discrete analogue of (2.3) is: find $u_j^* \in M_j$ such that

$$(g(u_j^*), v) = 0 \qquad \text{for all } v \in M_j. \tag{2.7}$$

Once a basis for M_j has been chosen, (2.7) can be formulated as a set of N_j nonlinear algebraic equations.

The analogue of (2.4)-(2.5) proceeds from an initial guess $u_j^0 \in M_j$, and computes $u_j^k \in M_j$ such that

$$(M_j^k x_j^k, v) = -\,(g(u_j^k), v) \qquad \text{for all } v \in M_j. \tag{2.8}$$

Equation (2.8) corresponds to an $N_j \times N_j$ linear algebraic system to be solved. Then set

$$u_j^{k+1} = u_j^k + t_j^k\, x_j^k\ . \tag{2.9}$$

Corresponding to M_j, we define a sequence of semi-norms, $|\cdot|_j$ on H by

$$|u|_j = \sup_{\substack{v \in M_j \\ v \neq 0}} |(u,v)| \,/\, \|v\| \quad . \tag{2.10}$$

In essence, if $u \in H$ and P_j is the orthogonal projector from H to M_j, then $|u|_j = \|P_j(u)\|$; furthermore, since the M_j are dense in H,

$$\|u\| = \sup_j \; |u|_j \; . \tag{2.11}$$

Thus, $|\cdot|_j$ represents a strong norm on M_k, $k \leq j$, and $|u|_j = \|u\|$ for $u \in M_k$, $k \leq j$, while $|\cdot|_j$ is a semi-norm on M_k with $k > j$. In the solution of (2.7), it is the semi-norm $|\cdot|_j$ which is computable, and the solution u_j^* satisfies $|g(u_j^*)|_j = 0$, while $\|g(u_j^*)\| > 0$ in general.

Suppose solutions u^* and u_j^* of (2.3) and (2.7), respectively, exist (this follows from our assumptions below; see Remark 4). Our central assumption is that the discrete solutions u_j^* are increasingly good approximations of u^*. Specifically, we assume there exists a fixed constant $C_1 = C_1(u,g,\{M_j\})$ and a positive number q such that

$$\|u^* - u_j^*\| \leq C_1 \; N_j^{-q} \quad . \tag{2.12}$$

Given (2.12), our stratgedy for computing approximate solutions which satisfy bounds like (2.12) is to sequentially compute approximate solutions of (2.7), using (2.8)-(2.9), and using the final iterate of the j-1-st problem as the initial guess for the jth. We summarize this procedure in

Algorithm I.

(i) for j=1, carry out s_1 iterations of (2.8)-(2.9),

starting from initial guess $u_1^0 \varepsilon M_1$.

(ii) for $j > 1$, carry out s_j iterations of (2.8)-(2.9),

starting from initial guess $u_j^0 = u_{j-1}^{s_{j-1}} \varepsilon M_{j-1} \quad M_j$.

3. Analysis

We begin by stating the underlying assumptions of our analysis. Our presentation is chosen to be consistant with our analysis in [5].

Given u_1^0, let S_j be closed subsets of M_j inductively defined as follows:

$$S_1 = \{ u \varepsilon M_1 \mid |g(u)|_1 \leq |g(u_1^0)|_1 \},$$

$$S_j = \{ u \varepsilon M_j \mid |g(u)|_j \leq \sup_{v \varepsilon S_{j-1}} |g(v)|_j \} . \tag{3.1}$$

Define

$$S_0 = \{ u \varepsilon H \mid \|g(u)\| \leq \sup_{\substack{v \varepsilon S_j \\ j \geq 1}} \|g(v)\| \} \tag{3.2}$$

<u>A1</u>. S_0 is bounded.

<u>Remark 1.</u> For $w \varepsilon M_j$, $z \varepsilon M_{j-1}$, and $v \varepsilon H$,

$$|(g(v),w)| \leq |(g(v),P_{j-1}w)| + |(g(v)-z,(I-P_{j-1})w)| .$$

Hence

$$|g(v)|_j \leq |g(v)|_{j-1} + \inf_{z \varepsilon M_{j-1}} |g(v)-z|_j \quad . \tag{3.3}$$

Typically, the spaces M_j will be such that the second term can be bounded by $C\, N_{j-1}^{-q}$. Thus if

$$\gamma_1 = |g(u_1^0)|_1,$$

$$\gamma_j = \sup_{v \varepsilon S_j} |g(v)|_j,$$

then

$$\gamma_j \leq \gamma_{j-1} + C\, N_{j-1}^{-q} \quad , \qquad j > 1.$$

If (2.6) holds,

$$\gamma_j \leq \gamma_1 + C\, N_1^{-q}\, (1-\beta^{-q})^{-1} \leq C' \quad .$$

Using (2.11), we see that S_0 is contained in the level set

$$S_0' = \{\, u \varepsilon H \mid \|g(u)\| \leq C' \,\}$$

(c.f. A1 of [5]).

<u>A2</u>. We assume g is differentiable on S_0, and for $u \varepsilon S_0$ and $v,w \varepsilon H$:

$$|\, (g'(u)v,w)\, | \leq C_2\, \|v\|\, \|w\| \ ; \tag{3.4}$$

$$\inf_{\|v\|=1}\ \sup_{\|w\|\leq 1} |\, (g'(u)v,w)\, | \geq k_3^{-1} > 0 \ ; \tag{3.5}$$

$$\sup_{v} |\, (g'(u)v,w)\, | > 0, \quad w \neq 0. \tag{3.6}$$

(C_2 is finite and C_2 and k_3 are independent of u).

<u>Remark 2.</u> Equations (3.4)-(3.6) guarantee that a unique solution $v \in H$ will exist for the problem

$$(g'(u)v,w) = (z,w) \quad \text{for all } w \in H,$$

where $z \in H$ and

$$\|v\| \le k_3 \|z\| ; \tag{3.7}$$

see Babuska and Aziz [1], section 5.2.

<u>A3</u>. For $u \in S_j$, $v,w \in M_j$, and M_j^k as in (2.8), assume

$$\inf_{\|v\|=1} \sup_{\|w\|\le 1} | (g'(u)v,w) | \ge k_6^{-1} > 0 , \tag{3.8}$$

$$\inf_{\|v\|=1} \sup_{\|w\|\le 1} | (M_j^k v,w) | \ge k_1^{-1} > 0 . \tag{3.9}$$

(k_1 and k_6 are independent of u and j).

<u>Remark 3.</u> In our particular application (3.8) will follow from A2, and we will show $k_1 \le 2k_6$ (see inequality 4.7).

We embed S_0 in the closed, convex ball

$$S_1 = \{ u \in H \mid \|u\| \le \sup_{v\in S_0} \|v\| + k_1\|g(v)\| \} . \tag{3.10}$$

<u>A4</u>. We assume g' is Lipshitz on S_1 and for $u,v \in S_1$,

$$\|g'(u)-g'(v)\| \le k_2 \|u-v\| . \tag{3.11}$$

Since g is differentiable, we also have

$$\|g(u)-g(v)\| \leq k_5 \|u-v\| \tag{3.12}$$

for $u,v \in S_1$ (as in [5], equation (2.28)).

Remark 4. Assumption A1 above is analagous to A1 in [5]. Equation (3.9) implies a bound as in (3.7), which, in turn, implies A2 of [5]. Finally, A4 above implies A3 of [5]. Thus the argument used to obtain Theorem 1 of [5] implies the existence of each $u_j^* \in M_j$ and also $u^* \in H$.

We define the relative residuals α_j^k for the solutions of (2.7) by

$$\alpha_j^k = |g'(u_j^k)x_j^k + g(u_j^k)|_j \ / \ |g(u_j^k)|_j \ . \tag{3.13}$$

The quantity α_j^k is computable, and measures how well x_j^k approximates the true Newton step ($\alpha_j^k = 0$ for Newton's method). We will chose the damping parameters t_j^k of (2.9) according to the formula

$$t_j^k = (1 + K_j^k \ |g(u_j^k)|_j \ )^{-1} \quad , \tag{3.14}$$

where the K_j^k are nonnegative scalars.

The following result applies Proposition 1 of [5] for each $j \geq 1$.

Proposition 3.1: Let $\delta_0 \in (0,1-\alpha_0)$, $\alpha_j^0 \in (0,\alpha_0)$, $\alpha_0 < 1$, and let t_j^k be chosen as in (3.14), where

$$0 \leq K_j^k \leq K_0,$$

and

$$K_j^k \geq (k_1^2 k_2/2) \ (1-\alpha_j^k-\delta_0)^{-1} - |g(u_j^k)|_j^{-1} \ . \tag{3.15}$$

Assume A1 - A4 and all $\alpha_j^k \leq \alpha_j^0$. Then

(i) all $u_j^k \varepsilon S_j$, the sequence $|g(u_j^k)|_j$ is strictly decreasing, and $|g(u_j^k)|_j \longrightarrow 0$.

Furthermore,

(ii) $|g(u_j^{k+1})|_j \;/\; |g(u_j^k)|_j \longrightarrow 0$ iff $\alpha_j^k \longrightarrow 0$, and for any fixed p ε (0,1],

$$|g(u_j^{k+1})|_j \leq C_3 \; |g(u_j^k)|_j^{1+p}$$

iff

$$\alpha_j^k \leq C_4 \; |g(u_j^k)|_j^p$$

for positive constants C_3 and C_4.

Note that we may consider K_0 as bounded uniformly in j by

$$K_0 \geq (k_1^2 k_2/2) \; (1-\alpha_0-\delta_0)^{-1} \quad . \tag{3.16}$$

Proposition 3.1 states that the approximate Newton method converges and that the rate of convergence is governed by the α_j^k. The parameter δ_0 is a sufficient decrease parameter and can be used in the actual computation to determine if (3.15) is satisfied. In [5] we prove that for u_j^k sufficiently close to u_j^*, we have

$$k_4 \; \|u_j^k-u_j^*\| \leq |g(u_j^k)|_j \leq k_5 \; \|u_j^k-u_j^*\|$$

showing that the rate of convergence of $|g(u_j^k)|_j$ to zero is also the asymptotic rate of convergence of u_j^k to u_j^*.

In our case, however, we are interested in computing u_j^k only insofar as it is an approximation of u^* of (2.3), and not as an approximation of u_j^* (although the two are clearly related). Thus we want to avoid wasting iterations by computing 'too good' an approximation of u_j^*. In Theorem 3.2, we indicate the degree to which we must approximate u_j^* in order to obtain bounds of the form (2.12) for the computed solutions.

<u>Theorem 3.2</u>: Let u_j^* satisfy (2.7) and let u_j^k, $0 \le k \le s_j$, be computed as in Algorithm I, using (2.8), (2.9), and (3.14). Let $\delta \in (0,\beta^{-q})$, and suppose

$$\|u_1^{s_1}-u_1^*\| \le C_1 \varepsilon N_1^{-q} \tag{3.17}$$

where

$$\varepsilon = \delta\ (1+\beta^q)\ (1-\delta\beta^q)^{-1} \quad ,$$

$$\|u_j^{s_j}-u_j^*\| \le \delta\ \|u_j^0-u_j^*\| \quad , \tag{3.18}$$

and $u_j^0 = u_{j-1}^{s_{j-1}}$, $j > 1$. Then

$$\|u_j^{s_j}-u^*\| \le C_1\ (1+\varepsilon)\ N_j^{-q} \quad . \tag{3.19}$$

Proof. Let $e_j = \|u_j^{s_j}-u_j^*\|$. Then by (3.18), (2.12), and (2.6),

$$\begin{aligned} e_j &\le \delta\ \|u_j^0-u_j^*\| \\ &\le \delta\ \{\ \|u_{j-1}^{s_{j-1}}-u_{j-1}^*\| + \|u_{j-1}^*-u^*\| + \|u^*-u_j^*\|\ \} \\ &\le \delta\ \{\ e_{j-1} + C_1\ (1+\beta^q)\ N_j^{-q}\} \quad . \end{aligned}$$

Solution of the majorizing difference equation, and the use of (3.17) shows

$$e_j \le C_1 \varepsilon N_j^{-q} \quad ,$$

and thus

$$\|u_j^{s_j}-u^*\| \leq e_j + \|u_j^*-u^*\| \leq C_1 \ (1+\varepsilon) \ N_j^{-q}.$$

Theorem 3.2 quantifies the advantage of using the stratgedy embodied in Algorithm I. For each problem after the first, one must reduce the error by only a fixed amount, independent of j, in order to obtain a sequence of approximations at the level of discretization error. The central result of this section is that for j sufficiently large $s_j = 1$. Thus, the asymptotic cost of solving the nonlinear systems (2.7) is essentially the cost of computing approximate solutions of linear systems of the form (2.8).

To see this we use a Taylor expansion as in (2.16) of [5] to obtain, for $v \in M_j$,

$$\begin{aligned}
0 &= (g(u_j^*),v) \\
&= (g(u_j^k),v) + (g'(u_j^k)\{u_j^*-u_j^k\}, \ v) \\
&\quad + \int_0^1 (\ \{g'(u_j^k + s\{u_j^*-u_j^k\}) - g'(u_j^k)\} \ \{u_j^*-u_j^k\},v) \ ds \\
&= (1-t_j^k) \ (g(u_j^k),v) \qquad\qquad (3.20) \\
&\quad + t_j^k \ (g'(u_j^k)x_j^k + g(u_j^k),v) \\
&\quad + (g'(u_j^k) \ u_j^*-u_j^{k+1},v) \\
&\quad + \int_0^1 (\ \{g'(u_j^k + s\{u_j^*-u_j^k\}) - g'(u_j^k)\} \ \{u_j^*-u_j^k\},v) \ ds \quad .
\end{aligned}$$

Moving the third term to the left hand side, taking (semi) norms, and using (3.8), (3.11), and (3.13), we have

$$|u_j^{k+1}-u_j^*|_j \le k_6 \{ (1-t_j^k) |g(u_j^k)|_j$$

$$+ t_j^k\alpha_j^k |g(u_j^k)|_j + (k_2/2) |u_j^k-u_j^*|_j^2 \} . \qquad (3.21)$$

Using Proposition 3.1 and (3.15), (3.16), and

$$|g(u_j^k)|_j \le k_5 |u_j^k-u_j^*|_j$$

(an easy consequence of (3.12), noting that $|v|_j \le \|v\|$ with equality for v ε M_j), we obtain

$$|u_j^{k+1}-u_j^*|_j \le k_6 \{(K_0k_5^2 + k_2/2) |u_j^k-u_j^*|_j + k_5\alpha_j^k\} |u_j^k-u_j^*|_j . \qquad (3.22)$$

Consider the case k=0. Then, using Theorem 3.2 inductively,

$$|u_j^0-u_j^*|_j \le \|u_{j-1}^{s_{j-1}}-u^*\| + \|u^*-u_j^*\|$$

$$\le C_1 \{1 + (1+\varepsilon)\beta^q\} N_j^{-q} ,$$

and from (3.22),

$$|u_j^1-u_j^*|_j \le (C_6N_j^{-q} + C_7\alpha_j^k) |u_j^0-u_j^*|_j , \qquad (3.23)$$

where

$$C_6 = C_1k_6 (K_0k_5^2 + k_2/2) \{1 + (1+\varepsilon)\beta^q\} ;$$

$$C_7 = k_6k_5.$$

For example, suppose that j is sufficiently large that

$$C_6N_j^{-q} < \delta/2.$$

Since we can control α_j^0, we may require

$$C_7\alpha_j^0 < \delta/2. \qquad (3.24)$$

Then (3.18) will be satisfied for $s_j=1$. Note that C_6 and C_7 are independent of j.

__Theorem 3.3__: Let the hypothses of Proposition 3.1 hold, and suppose α_j^0 is sufficiently small (α_j^0 satisfies (3.24), for example). Then for j sufficiently large, we may take $s_j=1$ in (3.19).

We will establish (3.24) for the multi-level iterative method in the next section.

__Remark 5.__ In Algorithm I, we obtain linear convergence of $u_j^{s_j}$ to u^* with the rate of convergence being roughly β^{-q}. Since Newton's method is quadratically convergent, one can ask under what circumstances we can have $u_j^{s_j}$ converge to u^* quadratically. Assuming (2.12) is sharp, this can be accomplished if we allow the dimensions of the spaces M_j to square rather than increase geometrically, i.e.,

$$N_j = \beta N_{j-1}^2,$$

$$\beta > 0 , \tag{3.25}$$

rather than (2.6). If we repeat our analysis using (3.25) in place of (2.6), the analogue of Theorem 3.2, equation (3.18) would indicate that we must reduce the initial error by $\delta N_j^{-q/2}$ rather than by a fixed amount. If we require

$$\alpha_j^k \leq C \; |g(u_j^k)|_j,$$

(which is consistent with quadratic convergence on the basis of Proposition 3.1), then (3.22) implies that the first iteration produces an error reduction of the right order of magnitude $O(N_j^{-q/2})$, but the constant may be too large. Two iterations, however, will be more than sufficient; hence $s_j \leq 2$ for j sufficiently large.

4. A Newton-Multi-level Method

We now return to the example problem (1.1). Let $a \in C^1(\bar{\Omega})$ be positive and bounded in $\bar{\Omega}$; i.e.,

$$0 < \underline{a} \leq a(x) \leq \bar{a} \text{ for } x \in \bar{\Omega} .$$

Let $\partial f/\partial u \in C^0(\bar{\Omega})$, and $\partial f/\partial u_{x_i} \in C^1(\bar{\Omega}))$, i=1,2. For $u \in H^1(\Omega)$, define

$$b(u;v,w) = \int_\Omega a\nabla v\cdot\nabla w + b\cdot\nabla v w + cvw \, dx \tag{4.1}$$

where

$$b_i = \frac{\partial f}{\partial u_{x_i}}(x,u,\nabla u) \quad ,$$

and

$$c = \frac{\partial f}{\partial u}(x,u,\nabla u) \quad .$$

If we make a correspondence between $a(u,v)$ and $(g(u),v)$ as in Section 2, then $b(u;v,w)$ corresponds to $(g'(u)v,w)$. Recall that $H = H^1(\Omega)$, and that the norm and inner product for H are given in (2.2).

Let τ_1 be a quasi-uniform, shape regular triangulation of Ω, and let h_1 denote the diameter of the largest triangle in τ_1 (for convenience, assume Ω is a polygon). We inductively construct a nested sequence of triangulations τ_j, j=1,2,..., as follows: for each triangle $t \in \tau_{j-1}$, construct 4 triangles in τ_j by pairwise connecting the midpoints of the edges of t. Each triangulation will then be quasi-uniform and shape regular, and will have $h_j = h_1 2^{1-j}$ (see [3], [2]). Let M_j denote the space of C^0 piecewise linear polynomials associated with τ_j. Then M_j M_k, $k > j$, and $\beta \cong 4$ in (2.6).

The central issue to be addressed in this section is the method of solving the linear systems (2.8) required by Algorithm I. If we were to use Newton's method ($M_j^k = g'(u_j^k)$) then, in the present context, we would solve the problems: find $\bar{x}_j^k \in M_j$ such that

$$b(u_j^k;\bar{x}_j^k,v) = - a(u_j^k,v) \quad \text{for all } v \in M_j \; . \tag{4.2}$$

(In this case $\alpha_j^k = 0$ on (3.13).

However, rather than solve (4.2) exactly, we will compute an approximate solution, x_j^k, using a multi-level iterative method, in particular, one of the j-level schemes described in [3], [2]. In this case, $M_j^k \neq g'(u_j^k)$ in general, but rather M_j^k is defined implicitly in terms of the iteration (see [5], section 4).

If r iterations of the j-level iteration as used, starting from initial guess zero, then the analysis in [3], [2] shows, that under suitable hypotheses

$$\|x_j^k-\bar{x}_j^k\| \leq \gamma^r \|\bar{x}_j^k\| \quad , \tag{4.3}$$

where $\gamma \in [0,1)$ is a fixed constant independent of j. Furthermore, the cost of each iteration is $O(N_j)$ as $j \longrightarrow \infty$.

We assume that for $u \varepsilon S_0$, the boundary value problem: find $v \varepsilon H^1(\Omega)$ such that

$$b(u;v,w) = (z,w) \qquad \text{for all } w \varepsilon H^1(\Omega) \tag{4.4}$$

and its adjoint: find $v \varepsilon H^1(\Omega)$ such that

$$b^*(u;v,w) = b(u;w,v) = (z,w) \quad \text{for all } w \varepsilon H^1(\Omega) \tag{4.5}$$

have unique solutions for each $z \varepsilon H^1(\Omega)$ (This will follow if assumption A2 is satisfied).

If one assumes (4.4)-(4.5) and a modest amount of elliptic regularity, then one can use the argument in Schatz [13] to prove that the problem: find $v \varepsilon M_j$ such that

$$b(u;v,w) = (z,w) \qquad \text{for all } w \varepsilon M_j \tag{4.6}$$

and its adjoint have unique solutions, provided h_1 is sufficiently small.

This in turn can be used to verify assumption A3, equation (3.8) as follows [1]: Let $v \varepsilon H^1(\Omega)$, and choose the scalar λ sufficiently large that

$$b(u;v,v) + \lambda\ (v,v) \geq C\ \|v\|^2.$$

Note λ is independent of v. By arguments given in [13], the problem: find $z \varepsilon M_j$ such that

$$b(u;z,w) = (\lambda v,w) \qquad \text{for all } w \varepsilon M_j$$

has a unique solution satisfying

$$\|z\| \leq C'\ \|\lambda v\|$$

provided h_1 is sufficiently small.

Now let $v \varepsilon M_j$ with $\|v\| = 1$, and let z be defined as above. Take

$$w = (v + z) / (1 + C'\lambda)$$

and note that $\|w\| \leq 1$. Then

$$b(u;v,w) = (b(u;v,v) + b(u;v,z)) / (1 + C'\lambda)$$

$$= b(u;v,v) + \lambda(v,v)) / (1 + C'\lambda)$$

$$\geq C / (1+C'\lambda) \equiv k_6^{-1}$$

Finally, note that on the basis of (4.3),

$$|(M_j^k)^{-1}|_j \leq |g'(u_j^k)^{-1}|_j + |(M_j^k)^{-1}-g'(u_j^k)^{-1}|_j$$

$$\leq (1+\gamma^r)\ |g'(u_j^k)^{-1}|_j$$

$$\leq (1+\gamma^r)\ k_6 \tag{4.7}$$

showing that we may take $k_1 = 2k_6$ in A3, equation (3.9).

We want to chose r such that the hypotheses of Theorem 3.3 will be satisfied and we can take $s_j = 1$ for large enough j. Observe that

$$
\begin{aligned}
|g'(u_j^k)x_j^k + g(u_j^k)|_j &= \sup_{v\varepsilon M_j} \ | \ b(u_j^k;x_j^k,v) + a(u_j^k,v) \ | \ / \ \|v\| \\
&= \sup_{v\varepsilon M_j} \ | \ b(u_j^k;x_j^k-\bar{x}_j^k,v) \ | \ / \ \|v\| \\
&\leq C_2 \ \|x_j^k-\bar{x}_j^k\| \\
&\leq C_2 \ \gamma^r \ \|\bar{x}_j^k\| \\
&= C_2 \ \gamma^r \ |\bar{x}_j^k|_j \\
&\leq C_2 \ \gamma^r \ k_6 \ |g(u_j^k)|_j
\end{aligned}
\tag{4.8}
$$

where we have used (3.4), (3.8), and (4.3). Thus, form (3.13),

$$
\alpha_j^k \leq C_2 \ k_6 \ \gamma^r \quad . \tag{4.9}
$$

To apply Theorem 3.3, we must have α_j^k sufficiently small that an inequality like (3.24) holds. To insure (3.24), we can require that r be sufficiently large that

$$
C_7C_2k_6\gamma^r \leq \delta/2 \quad . \tag{4.10}
$$

Note that r can be chosen independent of j.

Since $s_j = 1$ asympotically, the bulk of the work per level consists of constructing the linear system (4.2), and then carrying out r iterations of the j-level scheme. Since both of these are asymptotically $O(N_j)$ processes, the work per level can be bounded by, say, C_8N_j operations. The cumulative work for levels 1 to j can then be bounded by

$$
\begin{aligned}
\sum_{k\leq j} C_8N_k &\leq C_8N_j \ \{ \ 1 + \beta^{-1} + \beta^{-2} + \ldots \} \\
&\leq C_8N_j \ (1-\beta^{-1})^{-1} \ ,
\end{aligned}
$$

due to (2.6). We summarize in

<u>Theorem</u> <u>4.1</u>: Let Algorithm I be implemented using the j-level iteration and assume that (4.3) and the hypotheses of Theorem 3.3 hold. Then for j sufficiently large and h_1 sufficiently small,

$$\|u_j^1 - u^*\| \leq C_1 \ (1+\varepsilon) \ N_j^{-q}$$

as in equation (3.19). Furthermore, the computation of $u_j^1 \varepsilon M_j$, including all previous computations in M_k, $k \leq j-1$, requires $O(N_j)$ time.

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