Abstract

The convergence of a typical example from the class of highly successful decoupling algorithms for semiconductor simulation, collectively known as Gummel's method, is considered for one, two and three dimensional models. Because a nonlinear equation is solved for the potential u at every step, the considered version corresponds closely to the algorithms used for numerical computation in practice. As opposed to most earlier publications, the dependence of the regularity of the solution on the device geometry and the nature of the boundary conditions for the system of mixed boundary value problems is considered.

From a detailed analysis of the boundary conditions for a typical two dimensional model we conclude that for a physically realistic device geometry the solution may be expected to be sufficiently regular for the algorithm to converge.

On the Dependence of the Convergence of Gummel's Algorithm on the Regularity of the Solution

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1. Introduction

We consider the convergence of a decoupling algorithm for the solution of the steady-state semiconductor simulation problem for the case of constant mobility, zero generation-recombination and under the assumption of Einstein's relations in three or lower dimensions. This algorithm belongs to the class of highly effective decoupling algorithms which are collectively known as Gummel's algorithm [7].

With k_1 being the doping profile in units of the intrinsic density of the semiconductor, the problem above can be stated in dimensionless form as the following system of partial differential equations (pdes)

$$\begin{aligned} h_1(u,\nu,\omega) &= -\nabla^2 u + e^u\nu - e^{-u}\omega - k_1 = 0, \\ h_2(u,\nu,\omega) &= -\nabla.(e^u\nabla\nu) = 0, \\ h_3(u,\nu,\omega) &= -\nabla.(e^{-u}\nabla\omega) = 0, \end{aligned} \tag{1.1}$$

subject to the appropriate boundary conditions as described in a subsequent section.

Following Jerome in [9] we define an iterative decoupling algorithm for the solution of this system by introducing the mapping $T : (\nu,\omega) \rightarrow (\tilde{\nu},\tilde{\omega})$ in which first the potential equation is solved for an intermediate solution $u(\nu,\omega)$ and subsequently the two current continuity equations are solved for new approximate solutions $\tilde{\nu}$ and $\tilde{\omega}$ employing this new estimate u to the potential. Thus T is defined by

$$\begin{aligned} -\nabla^2 u + e^u\nu - e^{-u}\omega - k_1 = 0, \\ \nabla.(e^u\nabla\tilde{\nu}) = 0, \\ \nabla.(e^{-u}\nabla\tilde{\omega}) = 0. \end{aligned} \tag{1.2}$$

Jerome has proven in [9] that the mapping T is well-defined on appropriate function spaces and that it leaves suitable convex subsets invariant. Hence for appropriate boundary data, we can prove global convergence of the mapping T to a unique fixed point, which corresponds to a solution of the equations (1.1) by proving that the mapping T is a contraction [6].

In [11] Mock has dealt with an algorithm which is similar to algorithm (1.2) in the sense that the equations are decoupled. Nevertheless, both the algorithm and the analysis presented in [11] are essentially different from algorithm (1.2) which we consider here, and the analysis thereof. The most important differences between the algorithm in [11] and algorithm (1.2) are the following: In Mock's version of the decoupling algorithm the *linear* equation

$$-\nabla^2\phi + e^{u^{(n)}}\nu^{(n-1)} - e^{-u^{(n)}}\omega^{(n-1)} - k_1 = 0$$

is solved for ϕ at each iteration of the algorithm and hence only the Laplacean $-\nabla^2$ is inverted, while subsequently the next estimate $u^{(n+1)}$ for the potential u is obtained by a relaxation in which $u^{(n+1)} = (1-\omega)u^{(n)} + \omega\phi$ for sufficiently small $\omega \in (0,1)$. In algorithm (1.2), on the other hand, we solve the *nonlinear* equation

$$-\nabla^2 u^{(n)} + e^{u^{(n)}} \nu^{(n)} - e^{-u^{(n)}} \omega^{(n)} - k_1 = 0$$

for $u^{(n)}$ at each iteration of the algorithm. Hence the algorithm considered here coincides more closely with the algorithms actually used for numerical computation as in [2, 16, 3]. Moreover, the convergence analysis in [11] differs essentially from the presented contraction mapping result in that it requires the provision that ω is sufficiently small.

Another major difference between the analysis in [11] and the one presented here is in the regularity assumptions on the solution. Whereas it is assumed in [11] that the maximum norm of the gradient $||\nabla\nu||_{L_\infty}$ and $||\nabla\omega||_{L_\infty}$ (notation as in this report) are finite, we replace these regularity assumptions by weaker ones because of evidence that the L_∞ norms on the gradients are not finite for typical two and three dimensional MOSFET models. See however, [12] where a monotonicity assumption replaces the L_∞ bound on the gradients.

In Section **2** we will introduce the necessary formalism and theorems, in Section **3** we specify the boundary conditions for the typical two dimensional case as we consider in most detail, in Section **4** we discuss the regularity properties of the components of the solution and in Section **5** we finally present a convergence proof.

2. Definitions, Notation and Formalism

By $|x|$ we denote the Euclidian norm of a vector $x \in R^N$. As usual the L_q norm of a function u on a region Ω in N dimensions, is defined for $0 < q < \infty$ by

$$||u||_{L_q} = \left[\int_\Omega |u|^q d^N x\right]^{\frac{1}{q}},$$

and hence the L_q norm of the gradient of a function u on such a region is defined for $0 < q < \infty$ by

$$||\nabla u||_{L_q} = \left[\int_\Omega |\nabla u|^q d^N x\right]^{\frac{1}{q}}.$$

By L_∞ we indicate the maximum norm.

The inner product of the functions u and v on a region Ω is likewise defined by

$$\langle u, v\rangle = \int_\Omega uv \ \ d^N x,$$

and we will bound such inner products several times by employing Hölder's inequality:

$$\langle u, v\rangle \leq \|u\|_{L_p}\|v\|_{L_q} \quad \text{where} \quad \frac{1}{p}+\frac{1}{q}=1.$$

H^1 denotes the space of functions f which are bounded in the L_2 norm and for which the first derivatives $\frac{\partial f}{\partial x_i}$ for $i = 1, \cdots N$ in R^N are bounded in the L_2 sense as well.

In the sequel we will repeatedly need bounds on L_q norms of functions u in terms of L_p norms of their gradients. These bounds are stated in Sobolev's inequalities as given, for instance, in Theorem 7.10 in [5] and which we rephrase here as the following

Theorem 2.1. *For functions u on a space of dimension N with compact support in $|x| < R$ and having first order derivatives in L_p for $p > 1$*

$$\|u\|_{L_q} \leq const * \|\nabla u\|_{L_p} \text{ if } \quad p < N, \quad q = \frac{Np}{N-p},$$
$$\|u\|_{L_\infty} \leq const * R^{\frac{1}{N}-\frac{1}{p}}\|\nabla u\|_{L_p} \text{ if } \quad p > N.$$

Note that the theorem is more restrictive for higher dimesions in the sense that for higher N a certain L_p bound on the gradient implies a weaker (lower q) L_q bound on the function itself.

The proof of convergence will be given with respect to the norm $\rho((\nu_1,\omega_1),(\nu_2,\omega_2))$ which is defined by the sum of the the L_2 norms of the gradients of ν and ω. Thus

$$\rho((\nu_1,\omega_1),(\nu_2,\omega_2)) \equiv \|\nabla(\nu_2-\nu_1)\|_{L_2} + \|\nabla(\omega_2-\omega_1)\|_{L_2}.$$

This ρ defines a norm because we require the ν_i and the ω_i for $i = 1, 2$ to satisfy mixed boundary conditions which include a Dirichlet condition on a non-zero part of the boundary. Thus ρ defines a metric on the space of equivalence classes of functions with respect to ρ. Following Rudin ([13] remark **3.10**,) these equivalence classes will be called functions.

Because Sobolev's Theorem states that for one dimension the L_∞ norm of a function u is bounded by a constant times the L_q norm of ∇u for $q > 1$, the convergence result with respect to the metric defined by ρ implies convergence in the L_∞ norm of ν and ω for one dimension. This is probably comparable with a one dimensional convergence result in the L_∞ norm of an alternative decoupling algorithm, presented by Thomas I. Seidman in [14]. For a two dimensional model the convergence with respect to ρ implies convergence of ν and ω in the L_q norm for any $q < \infty$. For the three dimensional model it just implies convergence of ν and ω in the L_6 sense.

Apart from the several norms mentioned above and the relations between them we will employ the weak formulation of the system of pde's, which is defined because each of the three pdes above is in divergence form ([5], chapter 8). Thus we seek a solution (u,ν,ω) to the problem

$$\int_{\Omega'} \nabla u\cdot \nabla\phi + e^{u}\nu\phi - e^{-u}\omega\phi - k_1\phi \;\; d^i x = 0,$$
$$\int_{\Omega} e^{u}\nabla\nu\cdot \nabla\phi \;\; d^i x = 0,$$
$$\int_{\Omega} e^{-u}\nabla\omega\cdot \nabla\phi \;\; d^i x = 0,$$

where ϕ is a any function on Ω or Ω' as defined in the next section, for which $\nabla\phi$ is square integrable and which satisfies homogeneous Dirichlet conditions at the contacts. Here $i = 1,2$ or 3 because we will consider $1,2$ and 3 dimensions.

For the proof of the convergence result we finally introduce from [6] the following

Definition 2.1. A mapping $T : X \to X$, where X is a subset of a normed space N, is called a **contraction mapping**, or simply a **contraction**, if there is a positive number $C < 1$ such that

$$\|Tx - Ty\| \leq C\|x - y\| \quad \forall x, y \in X.$$

By demonstrating that T is a contraction, the existence of a unique fixed point of T follows by Theorem **5.15** from [6] as stated in the following

Theorem 2.2. (Contraction Mapping Theorem) *If $T : X \to X$ is a contraction mapping of a closed subset X of a Banach space, then there is exactly one $x^{(*)} \in X$ such that $Tx^{(*)} = x^{(*)}$. For any $x^{(0)} \in X$, the sequence $\{x^{(n)}\}$ defined by $x^{(n+1)} = Tx^{(n)}$ converges to $x^{(*)}$.*

Taking the norm defined by ρ in the Contraction Mapping Theorem, we thus can show convergence with respect to this norm.

3. Boundary Conditions

The device geometry for a typical two dimensional model of an n-channel MOSFET is shown in figure 1. The actual semiconductor region of the device in which the charge transport occurs, is given by the major quadrangle Ω with boundary A-B-C-F-G-H. Electric potentials are applied at the source, gate, drain and backgate contacts. As is indicated in the figure, the so-called gate contact is separated from the semiconductor medium by a thin oxide layer. The region with boundary A-B-C-D-E-F-G-H which includes the thin oxide layer on top, is called Ω'.

In (1.1) the coordinate u is the dimensionless electrostatic potential, ν is related to the density of conduction electrons n in units of the intrinsic density of the semiconductor by $n = e^{u}\nu$ and ω is related to the density of holes p in units of the intrinsic density of the semiconductor by $p = e^{-u}\omega$. For a concise discussion of the relevant physics confer [4], which provides further references as well.

The mixed boundary conditions for the three pde's are described below in terms of the current set of dimensionless coordinates, by

- For the potential u

$$u = u_{applied} + \ln\left[\frac{k_1}{2} + \sqrt{(\frac{k_1}{2})^2 + 1}\right],$$

at source, drain and backgate, where $u_{applied}$ is a constant u_{source} at the source, and a different constant u_{drain} at the drain or $u_{backgate}$ at the backgate,

$$u = u_{gate},$$

a constant, at the gate, Because there is no static space-charge in the oxide between the gate and the semiconductor, the potential within the oxide satisfies Laplace's equation:

$$-\nabla \cdot (\epsilon_{ox}\epsilon_0 \nabla u) = 0,$$

where ϵ_{ox} is the relative dielectric constant of the oxide. With $\mathbf{n}$ being the normal vector to the boundary, we have at the sides of the oxide C-D, E-F the homogeneous Neumann condition $\nabla u \cdot \mathbf{n} = 0$, corresponding to no perpendicular electric field. Across the oxide-semiconductor interface we require continuity of the potential u and continuity of the normal electric field:

$$\epsilon_s \frac{\partial}{\partial \mathbf{n}} u_s = \epsilon_{ox} \frac{\partial}{\partial \mathbf{n}} u_{ox},$$

where ϵ_s is the dielectric constant for the semiconductor, while

$$\frac{\partial u}{\partial \mathbf{n}} = 0$$

at the sides A-B, G-H (i.e. no electric field).

- For the variables ν and ω at source, drain and backgate, we have the constant Dirichlet conditions

$$\nu = e^{-u_{applied}},$$

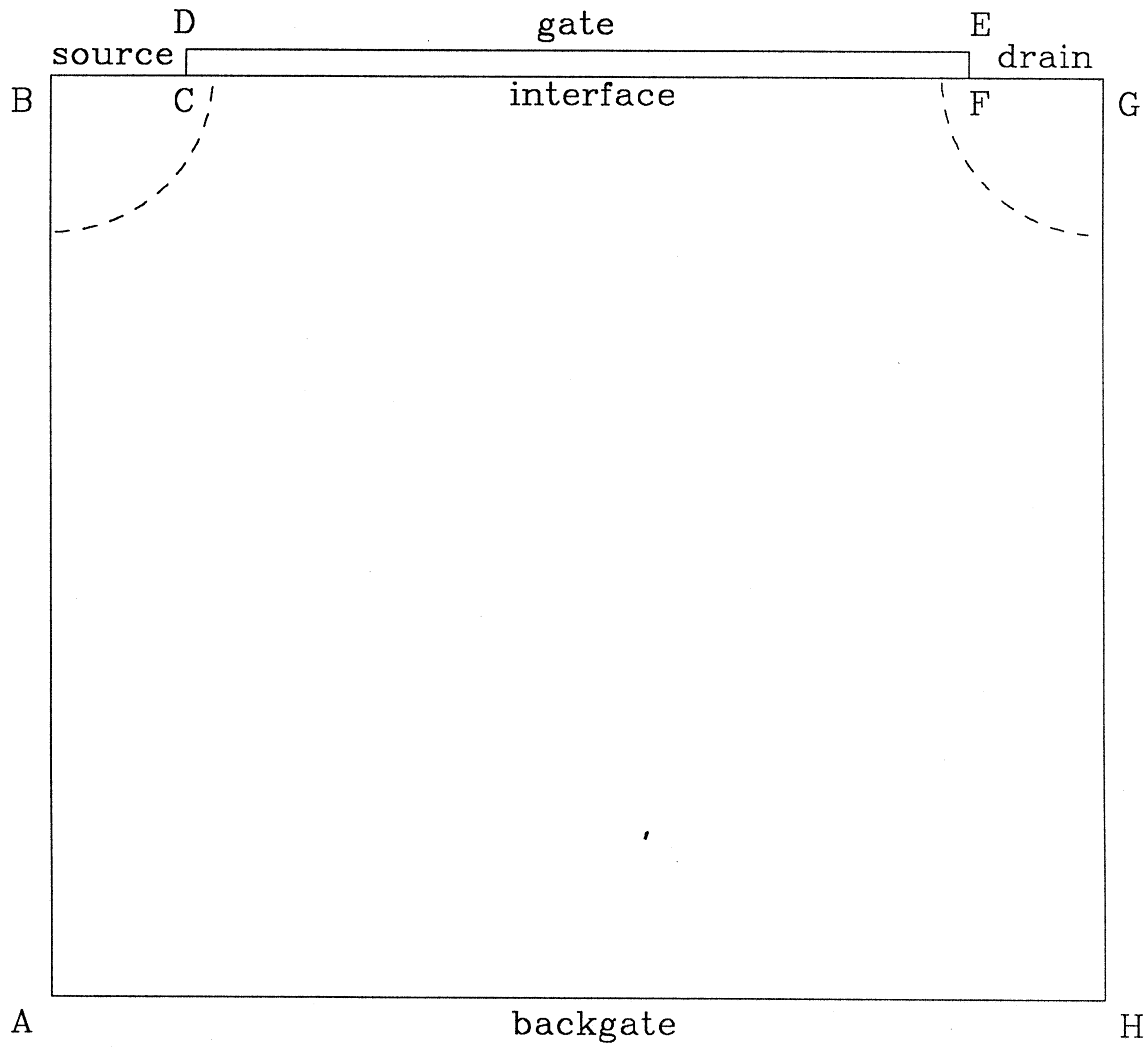
$$\omega = e^{u_{applied}},$$

while

$$\frac{\partial \nu}{\partial \mathbf{n}} = 0,$$

$$\frac{\partial \omega}{\partial \mathbf{n}} = 0.$$

MOSFET geometry



on the rest of the boundary (i.e. no current through the boundary but at the contacts.)

4. Regularity of the solution

For the convergence of algorithm (1.2) the regularity of the components of the solution plays a crucial role. While convergence follows more easily for more regular solutions, we do not want to assume more regularity than is compatible with the physical boundary conditions for the device.

In [9] Jerome has presented a mathematical analysis of the semiconductor simulation problem subject to mixed boundary conditions as are imposed in typical semiconductor models, in which the regularity of the solution to the problem is considered as well. In Theorem **2.1** of this publication the existence of a weak solution (confer [5], chapter 8) for which u, ν and $\omega \in H^1 \cap L_\infty$ is proven for this system of coupled mixed boundary value problems by employing Schauder's fixed point Theorem. This regularity result is obtained by proving that the mapping T as we have introduced in this report is continuous and leaves closed convex subsets of the appropriate function spaces invariant so that a fixed point must exist.

Although Jerome proves L_2 regularity for the gradient of the solution under assumptions on device geometry and boundary regularity which are fulfilled for the boundary conditions as described in the next section, he does not attempt to obtain the strongest regularity result possible. In fact, it is unfortunately not altogether clear how much regularity for the solution to this system of mixed boundary value problems can be inferred from the regularity of the boundary data. For a single pde a number of regularity results are known even for the mixed problem (confer [17],) but for systems with complicated mixed interface conditions as we have at hand, no rigorous analyses appear to be available. A relatively up to date and complete account of the developments with regard to regularity results of this type can be found in [1].

The regularity which is required in the context of a proof of convergence is therefore usually assumed to follow from boundary conditions which are largely left unspecified. In in his earlier publication [11] Mock assumes, for instance, that the boundary data are such that the gradients of the components of the solution are bounded in the uniform L_∞ norm. More recently Seidman examined the boundary conditions more closely in [15], but still made an assumption which coincides with an L_∞ bound on ∇u.

In this report we demonstrate that T is a contraction with respect to the norm as defined by ρ, for bounded variation of the boundary data under the regularity assumption that $||\nabla\nu||_{L_3}$ and $||\nabla\omega||_{L_3}$ are appropriately bounded in the case of a 3 dimensional model and that $||\nabla\nu||_{L_q}$ and $||\nabla\omega||_{L_q}$ are appropriately bounded for $q > 2$ in the case of a 2 dimensional model. Because

Jerome shows in Lemma **4.2** of [9] that for a MOSFET geometry as assumed here the mapping T is compact, T defines a contraction mapping from a complete metric space into itself. Hence it follows by the Contraction Mapping Theorem that the iterates defined by the successive application of the mapping T converge to a fixed point in a closed subset K of $H^1 \cap L_\infty \times H^1 \cap L_\infty$ as defined in [9], up to equivalence with respect to the norm defined by ρ. Hence we still have to asssume more regularity than the class of solutions for which exsistence was proven under physically realistic boundary conditions in [9] possesses. We will argue below, however, that these milder regularity assumptions are likely to be justified for typical physical models whereas assumptions of L_∞ regularity of the gradients, such as made in [11] are not.

The justification for the mentioned regularity assumption is derived from the results for a single pde of Lehman in [10] and Wigley in [17]. In these publications the behaviour of the solution near a corner or a transition from Dirichlet to Neumann boundary conditions is analyzed for the simpler case of a single pde with analytic coefficients instead of a system. From these analyses by Lehman and Wigley it follows that at corners and transitions singular behaviour of the solution can occur, and the nature of the singularity can be accurately predicted.

Each of the three components of the solution to the system (1.1) satisfies mixed boundary conditions. By analogy to the case for a single equation, singular behaviour of the solution may be expected at the corners and the transitions from Neumann to Dirichlet conditions.

In order to describe the singular behaviour near to the discontinuities in boundary conditions we introduce polar coordinates (r, θ), where r is the radial coordinate and θ the azimuthal one, with the transition or the corner at the origin and the boundaries at $\theta = 0$ and $\theta = \pi\alpha$ for some α betweem 0 and 2. From Theorem **3.2** by Wigley in [17] it then follows that for the transitions from source to gate and from gate to drain the most singular behaviour can be expected, because the opening angle $\pi\alpha$ with $\alpha = 1$ is the largest for those transitions.

The cited theorem states for $\alpha = 1$ that the solution can at worst behave like $r^{\frac{1}{2}}$. Thus the gradient can behave like $r^{-\frac{1}{2}}$ and is therefore not bounded in L_∞, although it is bounded in L_q for $q < 4$ in two dimensions.

For the potential u the situation is more complicated because we do not only have a transition from Dirichlet to Neumann conditions at a reentrant corner over an angle of $\frac{3}{2}\pi$, but we have an interface as well. This case can be examined heuristically using techniques as presented in section **2.11** of [8]. Such an analysis which is less rigorous than those given in [10, 17] suggests that for this case the asymptotic behaviour which is to be expected wil be like r^{β}, where $\frac{1}{3} < \beta < \frac{1}{2}$. We will not consider this any further because we do not need higher regularity results for u.

Apart from at the corners and at the transitions no singular behaviour is to be anticipated. Therefore we assume that the most singular behaviour occurring anywhere will not be worse than it would be for a single equation at the transitions form source to gate and from gate to drain.

The assumption in the current proof that the L_3 norms of $\nabla\nu$ and $\nabla\omega$ are bounded by bounded functionals of u for three dimensions and the corresponding assumption for two dimensions do not give rise to unphysical consequences. Because of Sobolev's Theorem the implication of the L_3 or L_q with $q > 2$ bounds on $\nabla\nu$ and $\nabla\omega$ is that ν and ω themselves are limited in L_∞, which ν and ω actually are as explained below. As mentioned above, the asymptotic $O(r^{-\frac{1}{2}})$ bound for the singularities in the gradients of ν and ω results in an L_q bound on the gradients for $q < 4$ which is consistent with the assumed L_3 behaviour as well.

The L_∞ bounds for the components ν and ω mentioned above, follow because of weak maximum principles as described in Theorem **8.1** of [5] by which ν and ω must assume their extrema at the boundary. Likewise it follows by a standard maximum principle argument for u, like given in [9], that

$$\gamma \leq u \leq \delta$$

where γ and δ are given by

$$\gamma = \min\Big\{\inf_{\partial\Omega_D} u, \ln\Big[\frac{k_{1,min}}{2b_\nu} + \sqrt{(\frac{k_{1,min}}{2b_\nu})^2 + \frac{a_\omega}{b_\nu}}\Big]\Big\},$$

$$\delta = \max\Big\{\sup_{\partial\Omega_D} u, \ln\Big[\frac{k_{1,max}}{2a_\nu} + \sqrt{(\frac{k_{1,max}}{2a_\nu})^2 + \frac{b_\omega}{a_\nu}}\Big]\Big\}.$$

respectively, where

$$a_\nu = \inf_{\partial\Omega} \nu,$$
$$b_\nu = \sup_{\partial\Omega} \nu,$$
$$a_\omega = \inf_{\partial\Omega} \omega,$$
$$b_\omega = \sup_{\partial\Omega} \omega,$$

and $\partial\Omega_D$ is the part of the boundary where Dirichlet conditions are prescribed for ν and ω.

Before proceeding it is important to note that the analysis of the algorithm is essentially more complex in two and three dimensions than it is in one dimension. This is due to oversimplification in one dimensional models. In these models one of the most essential parts of the device, the interface between the semiconductor and the oxide, is absent, (see figure.) The singularities as described in [10, 17] for the higher dimensional models can therefore not occur in one dimension. The present analysis of the algorithm reflects this sensitivity to dimensionality.

Resumating the discussion above, we can state that for the one dimensional case L_2 regularity of $\nabla\nu$ and $\nabla\omega$ can be proven and is sufficient for convergence, but that for the higher dimensional cases we still need the following

Assumption: $\nabla\nu$ and $\nabla\omega$ are bounded in the L_q norm by a bounded functional $F(u)$ of u whith

- $q > 2$ for two dimensional models,
- $q = 3$ for three dimensional models.

Because u can be bounded as a function of the boundary conditions only, $F(u)$ is bounded.

The bounds on the L_2 norms of $\nabla\nu$ and $\nabla\omega$ in terms of the boundary data as necessary for the proof of convergence for the one dimensional model, as well as a L_∞ bound, is given in the following

Lemma 4.1. *Let u be a bounded function on $[0, L]$ and let ν be a solution to the weak formulation of the two-point boundary value problem*

$$(e^u \nu_x)_x = 0,$$
$$\nu(0) = \nu_0,$$
$$\nu(L) = \nu_L,$$

on $[0, L]$. Then ν_x can be bounded in the L_∞ norm as given by

$$\|\nu_x\|_{L_\infty} \leq \frac{e^{u_{max}-u_{min}}}{L}|\nu_L - \nu_0|,$$

and in the L_2 norm by

$$\|\nu_x\|_{L_2} \leq \sqrt{\frac{e^{u_{max}-u_{min}}}{L}}|\nu_L - \nu_0|.$$

Proof. From the differential equation it follows directly that the current $I = e^u\nu_x$ is a constant, as it should be in one dimension without generation or recombination of conduction electrons or holes. Therefore

$$\int_0^L e^u \nu_x dx = L * I$$

Moreover, from the differential equation it follows that we have a maximum principle for ν. Hence, by assuming without loss of generality, that $\nu(L) \geq \nu(0)$ it follows that $\nu_x \geq 0$ on $[0, L]$. And from this we can conclude

$$\int_0^L e^u \nu_x dx \leq e^{u_{max}} \int_0^L |\nu_x| dx = e^{u_{max}} \int_0^L \nu_x dx = e^{u_{max}} |\nu(L) - \nu(0)|.$$

But then we find that

$$I \leq e^{u_{max}} \frac{|\nu_L - \nu_0|}{L},$$

and therefore that

$$|\nu_x| \leq e^{u_{max} - u_{min}} \frac{|\nu_L - \nu_0|}{L}.$$

The L_2 result is obtained from the weak formulation of the current continuity equations because

$$||\nu_x||_{L_2}^2 \leq e^{-u_{min}} \int_0^L e^u \nu_x^2 dx =$$

$$e^{-u_{min}} \int_0^L e^u \nu_x [\nu - (\nu_L - \nu_0)\frac{x}{L} - \nu_0]_x dx + e^{-u_{min}} \int_0^L e^u \nu_x [(\nu_L - \nu_0)\frac{x}{L} + \nu_0]_x dx,$$

where the function $\nu - (\nu_L - \nu_0)\frac{x}{L} - \nu_0$ can be taken as the trial function ϕ because it is zero for $x = 0$ and $x = L$. Therefore the first term is zero and we are left with

$$||\nu_x||_{L_2}^2 \leq e^{-u_{min}} \int_0^L e^u \nu_x [(\nu_L - \nu_0)\frac{x}{L} + \nu_0]_x dx =$$

$$e^{-u_{min}} \frac{(\nu_L - \nu_0)}{L} \int_0^L e^u \nu_x dx = e^{-u_{min}} (\nu_L - \nu_0) I.$$

Bounding the current I again we now obtain the result

$$||\nu_x||_{L_2}^2 \leq e^{-u_{min}} \int_0^L e^u \nu_x^2 dx =$$

$$e^{-u_{min}} (\nu_L - \nu_0) I \leq e^{u_{max} - u_{min}} |\nu_L - \nu_0| \frac{|\nu_L - \nu_0|}{L}.$$

And by taking the square root we find the stated L_2 bound.

∎

The L_2 bound for the one dimensional case is somewhat stronger than the corresponding bound which we can prove for the two-dimensional case as in the following

Lemma 4.2. *Let u be a bounded function on $\Omega = [0, L_x] \times [0, L_y]$ and let ν be a solution to the weak formulation of the boundary value problem*

$$\nabla \cdot (e^u \nabla \nu) = 0,$$

$$\nu = \nu_{contact},$$

for $\nu_{contact} = \nu_{source}, \nu_{drain}$ or $\nu_{backgate}$ at the source, drain and backgate respectively,

$$\frac{\partial \nu}{\partial \mathbf{n}} = 0 \text{ elsewhere on } \partial\Omega.$$

Then $\exists\, B_1, B_2 > 0$ *such that* $\nabla\nu$ *can be bounded in the* L_2 *norm as given by*

$$||\nabla\nu||_{L_2} \le e^{u_{max}-u_{min}}\Big[B_1 * |\nu_{drain} - \nu_{source}| + B_2 * |\nu_{backgate} - \nu_{source}|\Big].$$

Proof. By employing the weak formulation of the current continuity equation we see that

$$\int_\Omega e^u|\nabla\nu|^2 d^2x =$$

$$\int_\Omega e^u\nabla\nu\cdot\nabla[\nu - (\nu_{drain} - \nu_{source})g(x) - (\nu_{backgate} - \nu_{source})h(x) - \nu_{source}]d^2x$$
$$+\int_\Omega e^u\nabla\nu\cdot\nabla[(\nu_{drain} - \nu_{source})g(x) + (\nu_{backgate} - \nu_{source})h(x) + \nu_{source}]d^2x$$

where the functions g and h are smooth functions defined on Ω and subject to

- the Dirichlet conditions $g = 0, 1$ and 0 at source, drain and backgate respectively,

and

- the Dirichlet conditions $h = 0, 0$ and 1 at source, drain and backgate respectively,

Hence the function $\nu - (\nu_{drain} - \nu_{source})g(x) - (\nu_{backgate} - \nu_{source})h(x) - \nu_{source}$ can be taken as trial function ϕ because it is zero at source, drain and backgate. Thus

$$||\nabla\nu||^2_{L_2} = \int_\Omega |\nabla\nu|^2 d^2x \le e^{-u_{min}}\int_\Omega e^u|\nabla\nu|^2 d^2x =$$
$$e^{-u_{min}}\int_\Omega e^u\nabla\nu\cdot\nabla\phi d^2x +$$
$$e^{-u_{min}}\int_\Omega e^u\nabla\nu\cdot\nabla[(\nu_{drain} - \nu_{source})g(x) + (\nu_{backgate} - \nu_{source})h(x) + \nu_{source}]d^2x =$$
$$e^{-u_{min}}(\nu_{drain} - \nu_{source})\int_\Omega e^u\nabla\nu\cdot\nabla g(x)d^2x +$$
$$e^{-u_{min}}(\nu_{backgate} - \nu_{source})\int_\Omega e^u\nabla\nu\cdot\nabla h(x)d^2x \le$$
$$e^{u_{max}-u_{min}}\Big[|\nu_{drain} - \nu_{source}| * ||\nabla\nu||_{L_2}||\nabla g||_{L_2} + |\nu_{backgate} - \nu_{source}| * ||\nabla\nu||_{L_2}||\nabla h||_{L_2}\Big].$$

By dividing out the common $||\nabla\nu||_{L_2}$ factor on both sides of the inequality we finally obtain

$$||\nabla\nu||_{L_2} \le e^{u_{max}-u_{min}}\Big[|\nu_{drain} - \nu_{source}| * ||\nabla g||_{L_2} + |\nu_{backgate} - \nu_{source}| * ||\nabla h||_{L_2}\Big].$$

Here $||\nabla g||_{L_2}$ and $||\nabla h||_{L_2}$ are constants independent of both u and ν and hence the desired bound follows.

∎

In order to demonstrate that the mapping T is a contraction for sufficiently bounded variation of the boundary data, the linear dependence of the size of the norms on the boundary data as we have seen in the preceding lemmas will be needed for the case $q > 2$ as well. We will show that this linear dependence follows automatically if the L_q norms are finite. To this end we consider the function ν, defined on Ω and satisfying the current continuity equation in the weak formulation

$$\int e^u \nabla\nu \cdot \nabla\phi \, d^2x = 0, \tag{4.1}$$

subject to

- the constant Dirichlet conditions $\nu = \nu_{source}, \nu_{drain}$ and $\nu_{backgate}$ at source, drain and backgate respectively,
- the homogeneous Neumann condition $\frac{\partial}{\partial \mathbf{n}}\nu = 0$, on the rest of the boundary.

Thus the function ϕ on Ω is such that $\nabla\phi$ is square integrable and that it satisfies homogeneous Dirichlet conditions at the contacts. Then we have the bound as stated in the following

Lemma 4.3. *If $q > 0$ be such that the gradient $\nabla\nu$ is finite in the L_q-norm, then $\|\nabla\nu\|_{L_q}$ is bounded in the sense that*

$$\|\nabla\nu\|_{Lq} = |\nu_{drain} - \nu_{source}| * F(u) + |\nu_{backgate} - \nu_{source}| * G(u),$$

where $F(u)$ and $G(u)$ are bounded functionals of u.

Proof. Let η be a function defined on Ω which satisfies the weak formulation of the pde (4.1) subject to

- the Dirichlet conditions $\eta = 0, 1$ and 0 at source, drain and backgate respectively,
- the homogeneous Neumann condition $\frac{\partial}{\partial \mathbf{n}}\eta = 0$, on the rest of the boundary,

and let μ be a function defined on Ω which satisfies the weak formulation of the pde (4.1) subject to

- the Dirichlet conditions $\mu = 0, 0$ and 1 at source, drain and backgate respectively,
- the homogeneous Neumann condition $\frac{\partial}{\partial \mathbf{n}}\mu = 0$, on the rest of the boundary.

Define the function π by

$$\pi = \nu_{source} + (\nu_{drain} - \nu_{source})\eta + (\nu_{backgate} - \nu_{source})\mu.$$

Then π satisfies the same boundary conditions as ν and by linearity it satisfies the weak formulation of the pde (4.1) as well because

$$\int e^u[\nabla\nu_{source} + (\nu_{drain} - \nu_{source})\nabla\eta + (\nu_{backgate} - \nu_{source})\nabla\mu]\cdot \nabla\phi \; d^2x =$$

$$(\nu_{drain} - \nu_{source}) \int e^u \nabla\eta \cdot \nabla\phi \; d^2x + (\nu_{backgate} - \nu_{source}) \int e^u \nabla\mu \cdot \nabla\phi \; d^2x = 0.$$

Hence by uniqueness of the solution to a boundary value problem defined by the pde (4.1) we must have $\pi = \nu$.

But by the triangle inequality for the L_q norm and because the gradient of a constant is 0 we have that

$$\|\nabla\nu\|_{L_q} = \|\nabla\pi\|_{L_q} \leq |\nu_{drain} - \nu_{source}|\|\nabla\eta\|_{Lq} + |\nu_{backgate} - \nu_{source}|\|\nabla\mu\|_{L_q}.$$

And in this equation $\|\nabla\eta\|_{L_q}$ and $\|\nabla\mu\|_{L_q}$ depend only on u and not on the boundary conditions for ν. Therefore

$$\|\nabla\eta\|_{L_q} = F(u),$$
$$\|\nabla\mu\|_{L_q} = G(u),$$

where $F(u)$ and $G(u)$ are bounded functionals by assumption.

∎

The bounded functionals of u above are understood to be bounded functionals which depend on the function values of u only. Because we have a priori bounds on the function values of u, we hence obtain a priori bounds on these functionals.

Thus for the one dimensional case we can prove L_∞ bounds on $\nabla\nu$ and $\nabla\omega$ which are stronger than is necessary for convergence of the algorithm. For the two dimensional case we can prove an L_2 bound which is just not strong enough, but we can argue that for a physically realistic device the stronger L_q bounds may be expected to hold. For three dimensions we will just show that L_3 bounds are sufficient, but we do not relate this to device geometry because we do not dispose over geometry specifications which are as detailed as necessary.

5. The mapping T as a contraction

We discussed in the section on regularity that Gummel's algorithm defines a mapping T from a closed subset K of $H^1(\Omega) \cap L_\infty \times H^1(\Omega) \cap L_\infty$ into itself. In this section we show that this mapping T is a contraction with respect to the norm defined by ρ, if the variation of the boundary data is sufficiently bounded. Hence it follows by the Contraction Mapping Theorem that the iterative algorithm defined by T converges to a unique fixed point $(\nu^{(*)}, \omega^{(*)})$ under those conditions. With $u^{(*)} = u^{(*)}(\nu^{(*)}, \omega^{(*)})$, the full solution to the problem which coincides with the fixed point for the map T is thus given by $(u^{(*)}, \nu^{(*)}, \omega^{(*)})$.

Following algorithm (1.2) we first need a bound on $u_2 - u_1$ in terms of in $\nu_2 - \nu_1$ and $\omega_2 - \omega_1$. The situation is as described in the following

Lemma 5.1. *For $i = 1, 2$, let u_i satisfy the weak boundary value problem as posed by*

$$\int \nabla u_i \cdot \nabla \phi + e^{u_i} \nu_i \phi - e^{-u_i} \omega_i \phi - k_1 \phi \ d^2x = 0,$$

subject to the boundary conditions for u as given above, where ν_i and ω_i be functions such that $\|\nabla \nu_i\|_{L_2}$ and $\|\nabla \omega_i\|_{L_2}$ are bounded. Then $u_2 - u_1$ is dominated by the differences $\nu_2 - \nu_1$ and $\omega_2 - \omega_1$ as stated in the following inequality:

$$\|\nabla(u_2 - u_1)\|_{L_2} \leq C_1 \|e^{u_1}\|_{L_p} \|\nabla(\nu_2 - \nu_1)\|_{L_2} + C_1 \|e^{-u_1}\|_{L_p} \|\nabla(\omega_2 - \omega_1)\|_{L_2},$$

where $p = \frac{3}{2}$ for three dimensions, $p > 1$ for two dimensions and $p = 1$ for one dimension.

Proof. From the equations of the weak formulations for the u_i as shown in the lemma, we obtain by subtraction and taking $\phi = u_2 - u_1$ (This is possible because $(u_2 - u_1)\nabla(u_2 - u_1) = 0$ on the boundary of the domain of definition of u.)

$$\langle \nabla(u_2 - u_1), \nabla(u_2 - u_1) \rangle + \langle (e^{u_2} - e^{u_1})\nu_2 + (e^{-u_1} - e^{-u_2})\omega_2, u_2 - u_1 \rangle +$$
$$\langle e^{u_1}(\nu_2 - \nu_1) - e^{-u_1}(\omega_2 - \omega_1), u_2 - u_1 \rangle = 0.$$

Using Hölder's inequality, we see that

$$\|\nabla(u_2 - u_1)\|_{L_2}^2 + \langle \nu_2(e^{u_2} - e^{u_1}), (u_2 - u_1) \rangle + \langle \omega_2(e^{-u_1} - e^{-u_2}), u_2 - u_1 \rangle \leq$$
$$\|e^{u_1}\|_{L_{p_1}} \|\nu_2 - \nu_1\|_{L_{q_1}} \|u_2 - u_1\|_{L_{r_1}} + \|e^{-u_1}\|_{L_{p_2}} \|\omega_2 - \omega_1\|_{L_{q_2}} \|u_2 - u_1\|_{L_{r_2}},$$

with $\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} = 1$ and $\frac{1}{p_2} + \frac{1}{q_2} + \frac{1}{r_2} = 1$.

In the expression above

$$\langle \nu_2(e^{u_2} - e^{u_1}), u_2 - u_1 \rangle \geq 0$$

and

$$\langle \omega_2(e^{-u_1} - e^{-u_2}), u_2 - u_1 \rangle \geq 0$$

because $\nu_2 > 0$ and $(e^{u_2} - e^{u_1})(u_2 - u_1) \geq 0$ while $\omega_2 > 0$ and $(e^{-u_1} - e^{-u_2})(u_2 - u_1) \geq 0$ as well. Thus we can drop these terms in the inequality above without changing its nature.

Because we want to obtain convergence in the L_2 norm of the gradient, we want to bound $\|u_2 - u_1\|_{L_r}$ by $\|\nabla(u_2 - u_1)\|_{L_2}$. This is possible using Sobolev's inequalities.

For the one dimensional problem it follows from Sobolev's inequalities that $\|u_2 - u_1\|_{L_\infty}$ can be bounded by $\|\nabla(u_2 - u_1)\|_{L_s}$ for $s > 1$. Thus $\|u_2 - u_1\|_{L_\infty}$ can certainly be bounded by $\|\nabla(u_2 - u_1)\|_{L_2}$.

For the two dimensional problem we obtain from these inequalities that $\|u_2 - u_1\|_{L_q}$ can be bounded by $\|\nabla(u_2 - u_1)\|_{L_p}$ with $p = \frac{q}{q+2}2$, and therefore $p < 2$. Because the L_p norm is dominated by the L_2 norm if $p \leq 2$ it follows that for the two dimensional problem $\|u_2 - u_1\|_{L_q} \leq const * \|\nabla(u_2 - u_1)\|_{L_2}$ for any $q < \infty$.

For the three dimensional problem the L_2 bound on the gradient $\|\nabla(u_2 - u_1)\|_{L_2}$ implies by Sobolev's inequalities at most a bound on the L_6 norm $\|u_2 - u_1\|_{L_6}$. Because the three dimensional case is the most restricted one, we present the details of the proof only for that case.

We obtain for three dimensions

$$\|\nabla(u_2 - u_1)\|_{L_2} \leq C\|e^{u_1}\|_{L_{p_1}}\|\nu_2 - \nu_1\|_{L_{q_1}} + C\|e^{-u_1}\|_{L_{p_2}}\|\omega_2 - \omega_1\|_{L_{q_2}},$$

where $\frac{1}{p_1} + \frac{1}{q_1} = \frac{5}{6}$ and $\frac{1}{p_2} + \frac{1}{q_2} = \frac{5}{6}$.

But we can now use Sobolev's Theorem again and thus we can bound this time $\|\nu_2 - \nu_1\|_{L_q}$ and $\|\omega_2 - \omega_1\|_{L_q}$ by the L_2 norms of their respective gradients if $q = 6$. Thus

$$\|\nabla(u_2 - u_1)\|_{L_2} \leq C\|e^{u_1}\|_{L_{\frac{3}{2}}}\|\nu_2 - \nu_1\|_{L_6} + C\|e^{-u_1}\|_{L_{\frac{3}{2}}}\|\omega_2 - \omega_1\|_{L_6}$$

yields

$$\|\nabla(u_2 - u_1)\|_{L_2} \leq C_1\|e^{u_1}\|_{L_{\frac{3}{2}}}\|\nabla(\nu_2 - \nu_1)\|_{L_2} + C_1\|e^{-u_1}\|_{L_{\frac{3}{2}}}\|\nabla(\omega_2 - \omega_1)\|_{L_2}.$$

∎

In the sequel we will write $\tilde{\nu}_i$ and $\tilde{\omega}_i$ for the respective components corresponding to ν and ω of $T(\nu_i, \omega_i)$.

In the preceding lemma we have employed the potential equation for bounding $\nabla(u_2 - u_1)$ by $\nabla(\nu_2 - \nu_1)$ and $\nabla(\omega_2 - \omega_1)$. In the next two lemmas, the current continuity equations are used for delimiting $\|\nabla(\tilde{\nu}_2 - \tilde{\nu}_1)\|_{L_2}$ and $\|\nabla(\tilde{\omega}_2 - \tilde{\omega}_1))\|_{L_2}$ by $\|\nabla(u_2 - u_1)\|_{L_2}$. It turns out that we need a bound on $\nabla\tilde{\nu}$ and $\nabla\tilde{\omega}$ in the L_q norm, where $q = 3$ for 3D, $q > 2$ for 2D and $q = 2$ for 1D, in order to be able to do this.

We obtain the bounds on $\|\nabla(\tilde{\nu}_2 - \tilde{\nu}_1)\|_{L_2}$ and $\|\nabla(\tilde{\omega}_2 - \tilde{\omega}_1)\|_{L_2}$ by employing the second and third pde as in the following lemma, which is a simplification of Lemma **4.4** by Jerome in [9].

Lemma 5.2. *Let $\tilde{\nu}_1$ be the solution to the boundary value problem $\int e^{u_1}\nabla\tilde{\nu}_1\cdot\nabla\phi\, d^2x = 0$ and $\tilde{\nu}_2$ be the solution to the boundary value problem $\int e^{u_2}\nabla\tilde{\nu}_2\cdot\nabla\phi\, d^2x = 0$, where ϕ is any function on Ω for which $\nabla\phi$ is square integrable, satisfying homogeneous Dirichlet conditions at the contacts. Then*

$$\Big\langle e^{u_1}\nabla(\tilde{\nu}_2-\tilde{\nu}_1),\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\Big\rangle = \Big\langle (e^{u_1}-e^{u_2})\nabla\tilde{\nu}_2,\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\Big\rangle.$$

Proof. By employing the integral identities in the lemma we see because we can take $\tilde{\nu}_2-\tilde{\nu}_1=\phi$ that

$$\begin{gathered}\Big\langle e^{u_1}\nabla(\tilde{\nu}_2-\tilde{\nu}_1),\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\Big\rangle = \\ \Big\langle e^{u_1}\nabla\tilde{\nu}_2,\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\Big\rangle - \Big\langle e^{u_1}\nabla\tilde{\nu}_1,\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\Big\rangle = \\ \Big\langle e^{u_1}\nabla\tilde{\nu}_2,\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\Big\rangle - \Big\langle e^{u_2}\nabla\tilde{\nu}_2,\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\Big\rangle = \\ \Big\langle (e^{u_1}-e^{u_2})\nabla\tilde{\nu}_2,\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\Big\rangle.\end{gathered}$$

∎

Using the above we can prove the following

Lemma 5.3. *Let $\tilde{\nu}_1$ be the solution to the boundary value problem $\int e^{u_1}\nabla\tilde{\nu}_1\cdot\nabla\phi\, d^2x = 0$ and $\tilde{\nu}_2$ be the solution to the boundary value problem $\int e^{u_2}\nabla\tilde{\nu}_2\cdot\nabla\phi\, d^2x = 0$, for ϕ as in the preceding lemma. Then*

$$\|\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\|_{L_2} \le e^{-u_{1,min}}\|e^{u_2}-e^{u_1}\|_{L_p}\|\nabla\tilde{\nu}_2\|_{L_q},$$

where $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$.

Proof. By the Mean Value Theorem for integrals we know that

$$\|\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\|_{L_2}^2 \le e^{-u_{1,min}}\Big\langle e^{u_1}\nabla(\tilde{\nu}_2-\tilde{\nu}_1),\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\Big\rangle.$$

But we can rewrite the inner product on the right hand side using the preceding lemma and then use Hölder's inequality:

$$\begin{aligned}&\|\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\|_{L_2}^2 \le e^{-u_{1,min}}\Big\langle (e^{u_1}-e^{u_2})\nabla\tilde{\nu}_2,\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\Big\rangle \qquad \Rightarrow \\ &\|\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\|_{L_2}^2 \le e^{-u_{1,min}}\|e^{u_1}-e^{u_2}\|_{L_p}\|\nabla\tilde{\nu}_2\|_{L_q}\|\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\|_{L_2},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$. And therefore we obtain the L_2 bound on $\nabla(\tilde{\nu}_2-\tilde{\nu}_1)$

$$\|\nabla(\tilde{\nu}_2-\tilde{\nu}_1)\|_{L_2} \le e^{-u_{1,min}}\|e^{u_2}-e^{u_1}\|_{L_p}\|\nabla\tilde{\nu}_2\|_{L_q},$$

as is asserted in the lemma.

∎

From Lemma **5.3** we see that

$$\|\nabla(\tilde{\nu}_2 - \tilde{\nu}_1)\|_{L_2} \leq e^{-u_{2,min}} \|e^{u_2} - e^{u_1}\|_{L_p} \|\nabla\tilde{\nu}_2\|_{L_q},$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. But by the Mean Value Theorem for derivatives we know that $e^{u_2} - e^{u_1} = e^{u_{int}}(u_2 - u_1)$ for some u_{int} between u_1 and u_2. Hence we have a bound on $e^{u_2} - e^{u_1}$ in terms of $u_2 - u_1$ and $e^{u_{max}}$. We can obtain a more convenient bound in terms of the error of ∇u and $e^{u_{max}}$ by using Sobolev's inequalities again. Thus we have $\|u_2 - u_1\|_{L_p} \leq const * \|\nabla(u_2 - u_1)\|_{L_2}$ for $p = \infty$ in one dimension, for any $p < \infty$ in the case of the two dimensional problem, and for $p = 6$ in the case of the three dimensional problem. As we consider the 3 dimensional case again, we bound $\|u_2 - u_1\|_{L_6}$ by $const * \|\nabla(u_2 - u_1)\|_{L_2}$. This way we can transform the preceding inequality so as to obtain

$$\|\nabla(\tilde{\nu}_2 - \tilde{\nu}_1)\|_{L_2} \leq C_2 e^{u_{max} - u_{1,min}} \|\nabla(u_2 - u_1)\|_{L_2} \|\nabla\tilde{\nu}_2\|_{L_q},$$

where again $q = 2$ in one, $q > 2$ in two and $q = 3$ in three dimensions.

Using the above lemmas and the equivalent results for ω, while δ and γ are the bounds on u from the preceding section, we can now prove the following

Theorem 5.1. *The mapping T as defined by (1.2) is a contraction with respect to the norm ρ with the constant C bounded from above by the expression*

$$F_1(\delta,\gamma) * |\nu_{drain} - \nu_{source}| + F_2(\delta,\gamma) * |\nu_{backgate} - \nu_{source}| +$$
$$F_3(\delta,\gamma) * |\omega_{drain} - \omega_{source}| + F_4(\delta,\gamma) * |\omega_{backgate} - \omega_{source}|,$$

where F_1, F_2, F_3 and F_4 are bounded functions of δ and γ. It therefore follows that C is smaller than 1 if the boundary data are sufficiently bounded, so that T is a contraction, Hence the version of Gummel's algorithm as defined by the mapping T converges for both the one, two and three dimensional problem in the norm defined by ρ to the unique solution $(u^{(*)}, \nu^{(*)}, \omega^{(*)})$ for sufficiently bounded variation of the boundary data.

Proof. We employ the bounds on $\|\nabla(\tilde{\nu}_2 - \tilde{\nu}_1)\|_{L_2}$ and $\|\nabla(\tilde{\omega}_2 - \tilde{\omega}_1)\|_{L_2}$ from Lemma **5.3** and obtain

$$\|\nabla(\tilde{\nu}_2 - \tilde{\nu}_1)\|_{L_2} + \|\nabla(\tilde{\omega}_2 - \tilde{\omega}_1)\|_{L_2} \leq$$
$$C_2 e^{u_{max} - u_{1,min}} \left[\|\nabla\tilde{\nu}_2\|_{L_q} + \|\nabla\tilde{\omega}_2\|_{L_q}\right] \|\nabla(u_2 - u_1)\|_{L_2},$$

where $q = 2, q > 2$ or $q = 3$.

We now apply Lemma **5.1** for eliminating $\nabla(u_2 - u_1)$ on the right hand side. It thus follows with $p = 1$ for 1 dimension, $p > 1$ for 2 dimensions and $p = \frac{3}{2}$ for 3 dimensions that

$$\|\nabla(\tilde{\nu}_2 - \tilde{\nu}_1)\|_{L_2} + \|\nabla(\tilde{\omega}_2 - \tilde{\omega}_1)\|_{L_2} \leq$$
$$C_2 e^{u_{max} - u_{1,min}} \left[\|\nabla\tilde{\nu}_2\|_{L_q} + \|\nabla\tilde{\omega}_2\|_{L_q}\right] *$$
$$\left[C_1\|e^{u_1}\|_{L_p}\|\nabla(\nu_2 - \nu_1)\|_{L_2} + C_1\|e^{-u_1}\|_{L_p}\|\nabla(\omega_2 - \omega_1)\|_{L_2}\right].$$

But for one dimension we have proven in Lemma **4.1** and for two and three dimensions we have subsequently argued that we can assume that

$$\|\nabla\nu\|_{L_q} \leq F_1(u)|\nu_{drain} - \nu_{source}| + G_1(u)|\nu_{backgate} - \nu_{source}|,$$
$$\|\nabla\omega\|_{L_q} \leq F_2(u)|\omega_{drain} - \omega_{source}| + G_2(u)|\omega_{backgate} - \omega_{source}|.$$

Employing these bounds we see that

$$\rho((\tilde{\nu}_1, \tilde{\omega}_1), (\tilde{\nu}_2, \tilde{\omega}_2)) \leq$$
$$C_1 C_2 e^{u_{max} - u_{1,min}} \left[\|e^{u_1}\|_{L_p} + \|e^{-u_1}\|_{L_p}\right]$$
$$\left[F_1(u)|\nu_{drain} - \nu_{source}| + G_1(u)|\nu_{backgate} - \nu_{source}| + F_2(u)|\omega_{drain} - \omega_{source}| + G_2(u)|\omega_{backgate} - \omega_{source}|\right]$$
$$\rho((\nu_1, \omega_1), (\nu_2, \omega_2)).$$

By bounding all functions of u by the using the maximum δ and the minimum γ of u, the asserted bound on the constant C of the mapping T follows.

∎

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