Path Integrals of Information

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TR-1226

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Abstract: We note that there are at least 14 orders of magnitude difference in scale between observable macroscopic brain activity and synaptic transmission processes. This motivates development of a quantum description of the brain and its activity. We begin by devising a double slit experiment analog for neural transmission that exhibits interference in that transmission. Next the uncertainty principle is demonstrated for neural transmission. Then use of a path integral formalism is motivated, by showing that the conventional model of transmission of information in a neural net has the form of a discrete approximation to the Feynman path integral. To exploit the path integral methodology, we first produce a Lagrangian-type theory for neural nets and a concomitant principle of least action. The latter involves a so-called greedy variation to accommodate the dissipation of neural transmission. Using these tools, we derive the wave function of a neural network by means of the path integral approach. We verify that in the limit as a scale parameter vanishes, the path integral wave formulation leads to the Hopfield equations that form the underlying classical level description. Finally a neural net wave equation (Schrödinger equation) is derived for the wave function by appropriate differentiation of the path integral that defines it.

Keywords: brain, interference, Lagrangian, least action, neural transmission, path integral, Schrödinger equation, uncertainty principle, wave equation, wave function

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1. INTRODUCTION

There are many models of the flow of information in the brain and its neurons, models at various scales of phenomenology (from sub-neuron to neural assembly) Haykin, 1999, McKenna, Davis, and Zornetzer 1992, Valient, 1994. Most such models, involving as they do input-output equations, circuit equations, etc. are expressed in terms of the methods of classical physics. Non classical physical effects seldom find their way into such modeling, since most believe that the scale is wrong for their relevance. Exceptions are found in the work of Penrose, 1989, 1994 and 1997, Hammeroff and Penrose[2], 1996 and in the work of Miranker[3], 1997, 2002.

A key requirement for the meaningfulness of quantum effects in mechanics is the small scale on which such effects prevail. More accurately, a scale small compared to the scale on which we base our experience and observation of the matter that exhibits those effects to us. In fact, quantum effects are taken to be universal, present at all scales, but their magnitude diminishes rapidly with an approach to the macroscopic scale of the observer. Where can such a disparity of scales be found in the brain? The human brain contains about one hundred billion neurons and more than one hundred thousand billion synapses. So observation at the macroscopic scale (say observation of the behavior of a brain compartment) of effects taking place at the neuronal or subneuronal (synaptic) scale may involve a difference of as many as 14 orders of magnitude (and possibly more[4]). This could very well be enough to take as satisfied the requirement for large difference in scale in the study of the brain by quantum observational methods.

Why would a brain model based on quantum methods be of interest? The development of the methods of quantum mechanics in the last century was critical to explain and to gain understanding of physical phenomena that are inaccessible to classical methods, as is well known. There are a variety of phenomena and effects associated with brain function that seem to be impenetrable by conventional approaches of analysis. Among these phenomena are qualia, emotions, intentionality, etc. The potential to frame methods for study and understanding of these neuronal features is the motivation for development of such quantum methodology.

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[2] Penrose and Hammeroff seek quantum effects in the microtubules that comprise the cyto-skeleton of the neurons. The tubulin dimers (about one million per neuron) that compose the microtubule walls are taken to be the units that encode the quantum states.

[3] Miranker demonstrates quantum effects in the information processing itself that is conducted by neuronal assemblies.

[4] The remarks in footnote 2 suggest one way how several more orders of magnitude might be involved.

We shall develop a quantum description of the brain and its activity. We begin by devising a double slit experiment for neural transmission, and we show that interference effects accompany that transmission. The uncertainty principle is demonstrated by showing that an attempt to determine which of the neurons involved in the experiment is transmitting causes the interference to disappear.

We then derive a wave function that characterizes neuronal information transmission. To motivate this, we first show that the conventional model of transmission of information in a neural net has the form of a discrete approximation to the path integral of Feynman. Then we consider the Hopfield model of a neural network. We show that this model corresponds to a classical dynamical system with dissipation. We then produce a Lagrangian-type theory for such systems and a concomitant principle of least action. The latter involves a so-called greedy variation of the action functional, a feature that accommodates the dissipation. With this in hand we have the tools necessary to define the wave function of a neural network by means of the path integral approach, and we give the relevant derivation. We then note (by taking the limit as a scale parameter vanishes) that the path integral formulation delivers the Hopfield equations[5]. (What we may call the classical equations of neural net dynamics.)

The last step is to produce a wave equation, namely the analog of a Schrödinger equation for the newly derived wave function of neural transmission. This is done, as in the case of quantum mechanics, by an appropriate differentiation of the path integral defining the wave function. This completes the derivation of the principle constituents of a quantum theory for the brain.

2. A NEURONAL DOUBLE SLIT EXPERIMENT

Consider the basic model neuron with n input synapses. Let $w = (w_1, \ldots, w_n)$ be the vector of synaptic weights, and let $v^a = (v_1^a, \ldots, v_n^a)$ be the vector of afferent activity (the neuronal inputs). We take the neuronal output, v^e, to be a gain function, g, (with threshold) of the total input. In particular,

(2.1) $$v^e = g(u - \theta),$$

where the total neuronal input u is a sum of the individual weighted synaptic inputs,

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[5] This property is analogous to the delivery of classical mechanics as a limiting case of quantum mechanics as the Planck constant vanishes, as is well known.

(2.2) $$u = \sum_{k=1}^{n} w_k v_k^a ,$$

and θ is a threshold.

This model is a simplification of actual neuronal information processing that is in fact frequency encoded. Namely what is called the neuronal activity here in fact models the frequency of the actual output. It is essential that we take note of this, and so we formally replace the neuronal output by $\exp(iv_k t)$. Here v_k is the output frequency of the action potential of a neuron, t is the time, and k indexes the neurons.

To formulate a double slit experiment, we need three collections of neurons: the source neurons, the interneurons, I_k, and the target neurons, T_k. The source neurons are analogs of the source of electrons in the double slit experiment of quantum mechanics. The interneurons correspond to the slits. The target neurons correspond to locations (targets) on the screen recording the arrival of the electrons. The signal $\exp(iv_k t)$ is generated at I_k, for each k, in response to the afferent activity supplied by the source neurons. Let x_k denote the distance of travel of a signal from I_k toward T. Then at time t and position x_k, such a signal has the form

(2.3) $$\frac{\exp i(v_k t + \frac{2\pi}{\lambda} x_k)}{x_k}.$$

Here the x_k in the denominator represents a dissipation of signal strength. λ is the signal wavelength, that is, $\lambda / 2\pi$ is the spatial frequency of the signal, a universal quantity.

We confine our attention to two interneurons, I_1 and I_2, say (representing the double slit). Let x_{kj} denote the total distance (length of travel via neuronal processes (axons, dendrites...)) from neuron I_k to target neuron T_j. Then the signals emitted by I_k, $k = 1,2$ arrive and are summed to form the input at T_j. The resulting input value is

(2.4) $$B = \sum_{k=1}^{2} \frac{\exp i(v_k t + \frac{2\pi}{\lambda} x_{kj})}{x_{kj}}.$$

Here for simplicity, we have taken the synaptic weights to have unit value. Suppose the outputs of I_1 and I_2 are of equal strength (i.e., $v_1 = v_2$). (This corresponds to the indistinguishability of electrons being fired by the source toward the slits in the case of quantum mechanics.) Then suppressing the subscript j, we have

(2.5) $$|B|^2 = \left| \frac{e^{\frac{2\pi i}{\lambda}x_1}}{x_1} - \frac{e^{\frac{2\pi i}{\lambda}x_2}}{x_2} \right|^2 .$$

Let y be the distance along the assembly T, where $y = 0$ is the target location where $x_1 = x_2$. A plot of $|B|^2$ vs. y is given in Figure 2.1, demonstrating interference as in the double slit experiment of quantum mechanics (Kleinert, 1995, p. 13).

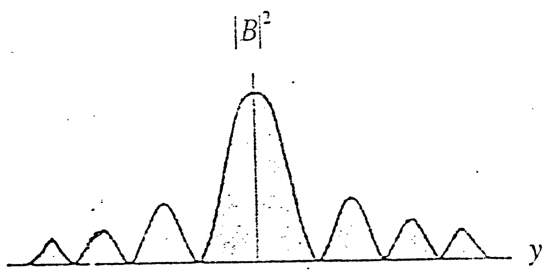

Figure 2.1: Interference pattern of signal arrivals at the cell assembly T

2.1 The uncertainty principle

We shall refer to the uncertainty principle given in the following form (Feynman and Hibbs, 1965, p. 9).

Any determination of the alternative taken by a process capable of following more than one alternative destroys the interference between alternatives.

Now suppose that we seek to determine which of the two neurons I_1 or I_2 fired, say by placing a voltage probe at I_2. In analogy with the case of electrons where the emission is made so weak that only one electron at a time passes through the apparatus, we suppose that the source is so weak that the firing neurons I_1 or I_2 are at most at threshold. In this arrangement we suppose that the probe reduces the net input to I_2, thereby preventing it from firing. In this situation (2.5) becomes

(2.6) $$|B|^2 = 1/x_1^2 ,$$

demonstrating the loss of interference.

3. THE WAVE FUNCTION

We are now going to define a wave function for neural transmission. To do this we adapt the approach of Feynman, employing path integrals. As motivation, we begin by showing that the propagation of information in a neural net has the structure of a discrete approximation to a path integral.

3.1 The neural net as a path integral approximation

Using the neural input-output equations, (2.1), (2.2), and taking the gain g to be linear and homogeneous, we find that the output of a neuron, as it depends on the inputs from $N-1$ preceding layers, may be expressed as follows.

$$g\sum_{x_0}\cdots g\sum_{x_{N-1}}\prod_{k=1}^{N-1} w_{x_k x_{k-1}} v_{x_1 x_0}. \tag{3.1}$$

Note that we have replaced the index label k for neurons in (2.1), (2.2) by the more representative label x_k. x_k indexes neurons in the k-th layer, $k = 1,\ldots,N-1$. Now we replace (3.1) by

$$\sum_{x_0} g\frac{\Delta x}{A}\cdots\sum_{x_{N-1}} g\frac{\Delta x}{A}\exp\left[i\sum_{k=1}^{N-1} w_{x_k x_{k-1}}(t_k - t_{k-1})/h\right], \tag{3.2}$$

so that it expresses the more accurate frequency modulated encoding (corresponding to (2.3)). Note that two time intervals, $\Delta x/A$ and $(t_k - t_{k-1})/h$, have been introduced in (3.2). The first represents the time needed to execute the gain function, and the second represents the time needed to convey the information between neuronal layers. A and h are appropriate scaling factors. (The value of A is specified in (4.6) below, but the value of h is as yet unknown[6].) It's clear that (3.2) has the form of a discrete approximation to a path integral. Namely, it arises by replacing the integrals in the following expression by Riemann sums.

$$\int_{-\infty}^{\infty}\!\!\cdots\!\int e^{\frac{i}{h}S} g\frac{dx_1}{A}\cdots g\frac{dx_{N-1}}{A}, \tag{3.3}$$

where

$$S = \int_0^t w(\xi)d\xi. \tag{3.4}$$

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[6] Of course, in quantum mechanics h is the Planck constant $\hbar$.

((3.3) defines the Feynman path integral in the limit as $N \to \infty$ (Feynman and Hibbs, 1965, Sect. 2-4). In (3.30, (3.4) t is a fixed value of time, $t = N\Delta t$, where $t_k - t_{k-1} = h\Delta t, k = 1,\ldots,N$, and $w(\xi)$ is a continuous version of the synaptic weights. Taking the limit in (3.3) as $N \to \infty$ defines a path integral as a formal expression of propagation in a continuum neural net. Namely,

(3.5) $$\int_{paths} e^{\frac{i}{h}S} \mu(dx),$$

where μ is an appropriate functional measure.

3.2 A Lagrangian formulation of neural net dynamics

The next step in our development is to generate a Lagrangian formulation of neural net dynamics, including a principle of least action. Since these dynamics are dissipative, a novel notion of optimization is required, as we shall see (Mjolsness and Miranker, 1998).

A Lagrangian for a dynamical system with position[7] v and velocity $\dot{v}$ is

(3.6) $$L = K - P,$$

where $K = K(\dot{v})$ is the kinetic energy and $P = P(v)$ is the potential energy. The action S is given by

(3.7) $$S = \int_{-\infty}^{\infty} L dt = \int_{-\infty}^{\infty} L(\dot{v}, v) dt.$$

Let us extremize S by taking a *greedy variation*, namely

(3.8) $$\frac{\delta_G S}{\delta_G v} = \frac{\delta}{\delta v} \int_{-\infty}^{t} L(\dot{v}, v) dt.$$

[7] In correspondence with the use of the variable v to represent the neural output, we use v to express the position variable and $\dot{v}$ the velocity in the Lagrangian formulation here.

This is called a greedy variation since we require optimality, not merely for the trajectory as a whole, but for the trajectory at every instant of time. Then it can be shown (see Appendix 1) that

(3.9) $$\frac{\delta_G S}{\delta_G v} = \frac{\partial L(\dot{v}, v)}{\partial \dot{v}}.$$

This is in contrast to the conventional variation (not greedy) of the action, namely

(3.10) $$\frac{\delta S}{\delta v} = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{v}}\right) - \frac{\partial L}{\partial t}.$$

Suppose the Lagrangian has the form

(3.11) $$L = K + \frac{dE(v)}{dt} = K + \frac{\partial E}{\partial v}\frac{dv}{dt}.$$

Then for the greedy variation of the corresponding action, we would have

(3.12) $$\frac{\partial L}{\partial \dot{v}} = \frac{\partial K}{\partial \dot{v}} + \frac{\partial E}{\partial v}.$$

Let us suppose that in equilibrium (where the velocity $\dot{v}$ vanishes), we have $\partial K(\dot{v} = 0)/\partial \dot{v} = 0$. Then extremization of the action S yields extremal points of E, that is

(3.13) $$\frac{\delta_G S}{\delta_G v} = \frac{\partial L}{\partial \dot{v}} = 0 \quad \Rightarrow \quad \frac{\partial E}{\partial v} = 0.$$

3.3 Application to Hopfield nets, the neural net principle of least action

The relaxation neural net, commonly called the Hopfield net, is associated with an energy (Lyapunov) function $E(v)$. As an example (see Hertz, Krogh and Palmer, 1991 Sect. 3.3), take[8]

(3.14) $$E = E(v) = -\frac{1}{2}\sum_{ij} T_{ij} v_i v_j - \sum_i f_i v_i + \sum_i \Phi_i(v_i),$$

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[8] In (3.14) T_{ij} corresponds to the synaptic weight between neuron i and neuron j, f_i is the exogenous input the network at neuron I, and Φ_i corresponds to the gain function of neuron i.

and take

(3.15) $$K = \frac{1}{2}\sum_i \dot{u}_i^2 g'(u_i),$$

where

(3.16) $$v_i = g(u_i).$$

With the choices (3.14)-(3.16) for the Lagrangian in (3.11), extremization of the action S via greedy variation yields

(3.17) $$\dot{u}_i = -\frac{\partial E}{\partial v_i} = -\sum_j T_{ij}v_j - f_i + \Phi_i'(v_i).$$

We recognize (3.17) as the dynamics of a Hopfield net. (Compare (3.1) with (2.2) and (3.16) with (2.1).) So (3.3), (3.4), and (3.17) are the neural net analogs of the Lagrangian, the action, and the equations of motion in classical mechanics, respectively. From (3.9), (3.13) we see that the *neural net principle of least action* is

(3.18) $$\frac{\delta_G S}{\delta_G v} = 0.$$

3.4 The path integral

Let $a = (v_a, t_a)$ and $b = (v_b, t_b)$ denote points between which a trajectory of the dynamics passes (say from a to b as time increases). Then for h and $A = A(\varepsilon)$ as suitable constants, define the kernel $\Gamma(b,a)$ by means of a path integral as follows.

(3.19) $$\Gamma(b;a) = \lim_{\varepsilon \to 0}\frac{1}{A}\int_{-\infty}^{\infty}\cdots\int e^{\frac{i}{h}S[b,a]}\frac{dv_1}{A}\cdots\frac{dv_{N-1}}{A}.$$

Here $v_1, \ldots, v_{N-1}$ is a uniform partition of (v_a, v_b) with mesh width $\varepsilon = (v_a - v_b)/N$, and

(3.20) $$S[b,a] = \int_{t_a}^{t_b} L(\dot{v}, v)\,dt.$$

To interpret (3.19), (3.20), replace the right hand side of (3.20) by a discrete sum with respect to a partition of (t_a, t_b) also with mesh width ε, in particular by

(3.21) $$\sum_{j=0}^{N-1} L\left(\frac{v_{j+1}-v_j}{\varepsilon}, \frac{v_{j+1}+v_j}{2}\right)\varepsilon.$$

Then the integral dv_i, within (3.19) means

(3.22) $$\int_{-\infty}^{\infty} \exp\frac{i}{h}\left[L\left(\frac{v_{i+1}-v_i}{\varepsilon}, \frac{v_{i+1}+v_i}{2}\right) + L\left(\frac{v_i-v_{i-1}}{\varepsilon}, \frac{v_i+v_{i-1}}{2}\right)\right]dv_i$$

Recall that in the case of Hopfield net dynamics at hand, we have $L = K + \frac{dE}{dt}$, $E = E(v)$, and $K = \frac{1}{2}\sum_i u_i^2 g(u_i)$ with $v_i = g(u_i)$. Now suppose that the limit in (3.19) exists and defines a path integral denoted by

(3.23) $$\int_a^b e^{\frac{i}{h}S[b,a]}\mu(dv),$$

where μ is an appropriate measure.

3.5 The classical limit

Let V denote the potential. Then in the quantum mechanics case, the Lagrangian has the form $L = m\dot{x}^2/2 - V(x,t)$. In this case, h, which appears in the exponential in (3.23), is the Planck constant $\hbar$, and the corresponding path integral is used to define the quantum mechanical wave function. It is shown by a type of stationary phase argument (Feynman and Hibbs (1965), Sect. 2-3.) that in the limit as $\hbar \to 0$, the contributions to the value of the path integral cancel except where the action S is stationary (i.e., where $\delta L/\delta x = 0$). To implement the stationary phase calculation, a variation in the action is performed and using the result in (3.10) the Lagrangian form of the equations of classical mechanics emerge. That is, classical mechanics emerges from quantum mechanics in the limit as $\hbar \to 0$.

The same argument may be applied to the path integral (3.23) except that we make the action stationary (i.e., we conduct the stationary phase calculation) by performing a greedy variation (where $\frac{\delta_G S}{\delta_G v} = \frac{\partial L(\dot{v},v)}{\partial \dot{v}} = 0$ (cf. (3.9), (3.18)). Using L as given in (3.11) and in the Hopfield case (3.14)-(3.16), we see that we recover the Hopfield dynamics (3.17) by this limiting process ($h \to 0$). So we see that the (customary) equations of neural net dynamics emerge from the path integral wave description of the neural net in the limit as $h \to 0$, provided that the action is appropriately defined and extremized (in the greedy sense) as in Section 3.2.

3.6 The wave function

Let $\psi(v,t)$ denote the wave function of the neural net. It is defined by the condition that it have the following property of evolution in time.

(3.24)
$$\psi(v_2,t_2) = \int_{-\infty}^{\infty} \Gamma(v_2,t_2;v_1,t_1)\psi(v_1,t_1)dv_1,$$

where Γ is given in (3.19).

4. THE WAVE EQUATION

We shall now derive an equation that describes the time evolution of the wave function ψ, the neural net Schrödinger equation. (Some details of the derivation are collected in the Appendices 2 and 3.)

Let $\Gamma(i+1;i)$ denote the kernel of (3.19) corresponding to the passage of information in an infinitesimal time interval ε between two locations (two v values) indexed by $i+1$ and i, respectively. Then consider the following approximation for $\Gamma(i+1;i)$.

(4.1)
$$\Gamma(i+1;i) = \frac{1}{A}\exp\left[\frac{i}{h}\varepsilon L\left(\frac{v_{i+1}-v_i}{\varepsilon},\frac{v_{i+1}+v_i}{2}\right)\right].$$

Now consider $\psi(v,t+\varepsilon)$, and using (4.1), take the following approximation for it.

(4.2)
$$\psi(v,t+\varepsilon) = \frac{1}{A}\int_{-\infty}^{\infty} \exp\left[\frac{i}{h}\varepsilon L\left(\frac{v-v_1}{\varepsilon},\frac{v+v_1}{\varepsilon}\right)\right]\psi(v_1,t)dv_1.$$

The Lagrangian (cf. (3.11)) may be analogously approximated:

(4.3)
$$L\left(\frac{v-v_1}{\varepsilon},\frac{v+v_1}{2}\right) = \frac{1}{2g'(u)}\left(\frac{v-v_1}{\varepsilon}\right)^2 + E_v\left(\frac{v+v_1}{2}\right)\frac{v-v_1}{\varepsilon}.$$

The first term on the right in (4.3) uses the second relation in (A2.4) of Appendix 2. Note that $g'(u) = g'(g^{-1}(v)) \approx g'(g^{-1}(\frac{v+v_1}{2}))$. Then inserting (4.3) into (4.2) gives

$$(4.4)\quad \psi(v,t+\varepsilon) = \int_{-\infty}^{\infty} \frac{1}{A}\left\{\exp\left[\frac{i}{2hg'(u)}\frac{(v-v_1)^2}{\varepsilon}\right]\right\}\left\{\exp\left[\frac{i}{h}(v-v_1)E_v\left(\frac{v+v_1}{2}\right)\right]\right\}\psi(v_1,t)dv_1.$$

Setting $v_1 = v + \eta$, (4.4) becomes

$$(4.5)\quad \psi(v,t+\varepsilon) = \int_{-\infty}^{\infty} \frac{1}{A}\exp\left[\frac{i}{2hg'(u)}\frac{\eta^2}{\varepsilon} - \frac{i}{h}\eta E_v\left(\frac{2v+\eta}{2}\right)\right]\psi(v+\eta,t)d\eta.$$

In Appendix 3, we expand (4.5) in Taylor series in ε, and we then equate terms of equal order in ε. The zeroeth order equation determines A. Namely

$$(4.6)\quad A = \left(\frac{2\pi ih\varepsilon}{m(v)}\right)^{\frac{1}{2}},$$

where

$$(4.7)\quad m(v) = 1/g'(g^{-1}(v)).$$

The first order equation is vacuous. The second order equation yields the wave equation we seek. Namely

$$(4.8)\quad \frac{h}{i}\frac{\partial\psi}{\partial t} = \frac{h^2}{2m(v)}\frac{\partial^2\psi}{\partial v^2} + \frac{h}{i}E_v\frac{\partial\psi}{\partial v} - V\psi,$$

a neural net Schrödinger equation. Here

$$(4.9)\quad V = V(v) = \frac{E_v^2(v)}{2m(v)} + \frac{ih}{2m(v)}E_{vv}(v).$$

Setting

$$(4.10)\quad \varphi = \psi\exp\left[\frac{h}{4im(v)}\int^{v}\frac{E_v(v')}{m(v')}dv'\right],$$

to account for the dissipation, a direct calculation shows that φ satisfies the Schrödinger equation of quantum mechanics. In particular,

$$\frac{\partial \varphi}{\partial t} = -\frac{h}{2im(v)}\frac{\partial^2 \varphi}{\partial v^2} + U(v)\varphi, \tag{4.11}$$

where for the potential we have

$$U(v) = \frac{1}{m}\left(E_v^2 + \frac{1}{2}E_{vv}\right) + \frac{i}{2m}\left(\frac{hm_v}{m^2}E_v - \frac{1-4m}{h}E_v^2 - hE_{vv}\right). \tag{4.12}$$

APPENDICES

Appendix 1

In this appendix we shall demonstrate (3.9), namely that

$$\frac{\delta_G S}{\delta_G v} = \frac{\partial L(\dot{v}, v)}{\partial \dot{v}}. \tag{A1.1}$$

Let us take the greedy variation of S about a function v^*. Let $v = v^* + \varepsilon u$, where $\varepsilon > 0$ and u is an arbitrary function. We have

$$\begin{aligned}\frac{\delta_G S}{\delta_G v} &= \frac{\delta}{\delta v}\int_{-\infty}^{t} L(\dot{v}^*, v^*)d\tau \\ &= \lim_{\varepsilon\to 0}\int_{-\infty}^{t}\frac{L(\dot{v}^* + \varepsilon^*\dot{u}, v^* + \varepsilon^* u) - L(\dot{v}^*, v^*)}{\varepsilon}d\tau \\ &= \int_{-\infty}^{t}\left(\frac{\partial L(\dot{v}^*, v^*)}{\partial \dot{v}}\dot{u} - \frac{\partial L(\dot{v}^*, v^*)}{\partial v}u\right)d\tau \\ &= \frac{\partial L}{\partial \dot{v}}u\Big|_{-\infty}^{t} + \int_{-\infty}^{t}\left(\frac{\partial L}{\partial v} - \frac{\partial}{\partial v}\left(\frac{\partial L}{\partial \dot{v}}\right)\right)d\tau.\end{aligned} \tag{A1.2}$$

Choose $u(\tau)$ to be the function $u(\tau) = u(\tau, t; a) = \exp(-(t-\tau)^2/a)$, where $a > 0$. We see that

$$u(t) = 1,\ \lim_{a\to 0} u(\tau) = 0, \text{ and } u(-\infty) = 0. \tag{A1.3}$$

Then (A1.2) gives

(A1.4)
$$\frac{\delta_G S}{\delta_G v} = \frac{\delta}{\delta v}\int_{-\infty}^{t} L(\dot{v}^*, v^*)d\tau$$
$$= \frac{\partial L}{\partial \dot{v}}u(t) - \frac{\partial L}{\partial \dot{v}}u(-\infty) + \int_{-\infty}^{t}\left(\frac{\partial L}{\partial v} - \frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial \dot{v}}\right)\right)u d\tau .$$

Now take the limit in (A1.4) as $a \to 0$. Using (A1.3), we obtain

(A1.5)
$$\frac{\delta_G S}{\delta_G v} = \frac{\delta}{\delta v}\int_{-\infty}^{t} L(\dot{v}^*, v^*)d\tau = \frac{\partial L(\dot{v}^*, v^*)}{\partial \dot{v}},$$

demonstrating (3.9).

Appendix 2

In this appendix, we shall demonstrate (3.17), namely

(A2.1)
$$\dot{u}_i = -\frac{\partial E}{\partial v_i} = -\sum_j T_{ij} v_j - f_i + \Phi_i'(v_i).$$

To begin, differentiate (3.16) with respect to t, obtaining

(A2.2)
$$\dot{v}_i = g'(u_i)\dot{u}_i .$$

Next we note the familiar inverse differentiation relation

(A2.3)
$$\frac{1}{g'(u_i)} = \frac{\partial}{\partial v_i} g^{-1}(v_i).$$

Using (A2.2) and then (A2.3), enables us to write K as follows (cf. 3.15)

(A2.4)
$$K = \frac{1}{2}\sum_j \dot{u}_j^2 g'(u_j)$$
$$= \sum_j \frac{\dot{v}_j^2}{g'(u_j)}$$
$$= \sum_j \dot{v}_j^2 \frac{\partial}{\partial v_j} g^{-1}(v_j).$$

Inserting this into the extremization condition $\frac{\partial L}{\partial \dot{v}_i} = \frac{\partial K}{\partial \dot{v}_i} + \frac{\partial E}{\partial v_i} = 0$ (cf. (3.12)), we obtain

(A2.5) $$\frac{\partial g^{-1}(v_i)}{\partial v_i}\dot{v}_i + \frac{\partial E}{\partial v_i} = 0.$$

Combining (A2.2) and (A2.3) gives

(A2.6) $$\dot{u}_i = \frac{\partial g^{-1}(v_i)}{\partial v_i}\dot{v}_i,$$

which when inserted into (A2.5) yields (A2.1). This demonstrates (3.17).

Appendix 3

Here we give details of the Taylor series argument following (4.5), beginning by noting the following series replacements.

(A3.1) $$\psi(v,t+\varepsilon) = \psi(v,t) + \varepsilon\frac{\partial\psi}{\partial t} + \cdots.$$

(A3.2) $$\psi(v_1,t) = \psi(v,t) + \eta\frac{\partial\psi}{\partial v} + \frac{1}{2}\eta^2\frac{\partial^2\psi}{\partial v} + \cdots.$$

Recalling that $v = v_1 + \eta$ and employing the abbreviation

(A3.3) $$F = F(\eta) = \exp\left[-\frac{i}{h}\eta E\left(\frac{2v+v_1}{2}\right)\right],$$

we find $F(0) = 1$, $F_\eta(0) = -\frac{i}{h}E_v(v)$, and $F_{\eta\eta}(0) = -\frac{1}{h^2}E_v^2(v) - \frac{i}{h}E_{vv}(v)$. Then we have

(A3.4) $$F = 1 - \frac{i}{h}E_v(v)\eta - \left(\frac{1}{h^2}(E_v(v))^2 + \frac{i}{h}E_{vv}(v)\right)\frac{\eta^2}{2} + \cdots.$$

Using these three Taylor expansions in (4.5), it becomes the following expression (to highest order).

(A3.5)
$$\psi(v,t)+\varepsilon\frac{\partial\psi}{\partial t}=\int_{-\infty}^{\infty}\frac{1}{A}\exp\left[\frac{im(v)}{2h}\frac{\eta^2}{\varepsilon}\right]\times\left\{1-\frac{i}{h}E_v(v)\eta-\left(\frac{1}{h^2}E_v^2(v)+\frac{i}{h}E_{vv}(v)\right)\frac{\eta^2}{2}\right\}\times\left\{\psi(v,t)+\eta\frac{\partial\psi}{\partial v}+\frac{1}{2}\eta^2\frac{\partial^2\psi}{\partial v^2}\right\}d\eta.$$

Here we have introduced the quantity (cf. (3.16), (4.7))

(A3.6)
$$m(v)=1/g'(g^{-1}(v))\approx 1/g'(g^{-1}(v+\eta))=1/g'(u).$$

Equating terms in (A3.5) of order zero in η gives the following equation.

(A3.7)
$$\psi(v,t)=\psi(v,t)\int_{-\infty}^{\infty}\frac{1}{A}\exp\left[\frac{im(v)}{2h}\frac{\eta^2}{\varepsilon}\right]d\eta=\psi(v,t)\frac{1}{A}\left(\frac{2\pi ih\varepsilon}{m(v)}\right).$$

(A3.7) gives the result for A shown in (4.6). The terms of order unity in η in (A3.5) cancel out. What remains is

(A3.8)
$$\varepsilon\frac{\partial\psi}{\partial t}=\left(\frac{m(v)}{2\pi ih\varepsilon}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\eta^2\exp\left[i\frac{m(v)}{2\pi h\varepsilon}\eta^2\right]\times\left[\frac{1}{2}\frac{\partial^2\psi}{\partial v^2}-\frac{i}{h}E_v(v)\frac{\partial\psi}{\partial v}-\frac{1}{2}\left(\frac{1}{h^2}E_v^2(v)+\frac{i}{h}E_{vv}(v)\right)\psi\right]d\eta.$$

Since

(A3.9)
$$\int_{-\infty}^{\infty}\frac{\eta^2}{A}\exp\left[\frac{im(v)}{2h\varepsilon}\eta^2\right]d\eta=\frac{ih\varepsilon}{m(v)},$$

(A3.8) may be written as

(A3.10)
$$-\frac{h}{i}\frac{\partial\psi}{\partial t}=-\frac{h^2}{2m(v)}\frac{\partial^2\psi}{\partial v^2}-\frac{h}{im(v)}E_v(v)\frac{\partial\psi}{\partial v}+\left(\frac{E_v^2(v)}{2m(v)}-\frac{h}{2im(v)}E_{vv}(v)\right)\psi,$$

namely, the neural net wave equation ((4.8), (4.9)).

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